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On the Rigidity of Certain Discrete Groups and Algebraic Varieties

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Dedicated to Friedrich Hirzebruch

In the present note, we shall prove

Theorem 1. *Let D be an irreducible bounded symmetric domain in $\mathbb{C}^n$, $n \geq 2$. Let Γ be a (nonuniform) lattice in* $\mathrm{Aut}(D)$, *i.e. a discrete subgroup of* $\mathrm{Aut}(D)$ *for which $N := D/\Gamma$ (is noncompact and) has finite volume (w.r.t. the locally symmetric metric induced from D). Suppose that a group $\hat{\Gamma}$ isomorphic to Γ (as an abstract group) acts as a discrete automorphism group on a contractible Kähler manifold $\tilde{M}$. Assume that $\tilde{M}/\hat{\Gamma}$ has a finite singularity free cover M (i.e. M is a manifold) which is quasiprojective i.e. admits a compactification as a projective variety $\bar{M}$ and that $\bar{M}\setminus M$ is of codimension at least three in $\bar{M}$. Then $\tilde{M}$ is biholomorphically equivalent to D, and $\hat{\Gamma}$ is conjugate to Γ in* $\mathrm{Aut}(D)$.

Theorem 2. *Let T_g be the Teichmüller space of Riemann surfaces of genus g, Γ_g the modular group and N the moduli space, $g \geq 2$. Let $\hat{\Gamma}$ be isomorphic to Γ_g and act on a contractible Kähler manifold $\tilde{M}$ with a quotient $\tilde{M}/\hat{\Gamma}$ satisfying the same assumptions as in Theorem 1. Then $\tilde{M}$ is biholomorphically equivalent to T_g, and $\hat{\Gamma}$ acts as the modular group.*

The assumption that M can be compactified in such a way that the compactifying divisor is algebraic seems to be (at least in principle) necessary. Namely, a locally Hermitian symmetric variety of finite volume admits nonsingular compactifications, and the compactifying divisor then of course admits topological deformations as a submanifold of the compactified space, and cutting out the deformed submanifold we get a space with a different complex structure homotopically equivalent to the original one.

It is not clear to us whether the restriction is necessary that the compactifying divisor can be blown down to a variety of codimension three.

Our proof uses harmonic maps (cf. [9] as a general reference). We shall first construct a homotopy equivalence of finite energy between $\tilde{M}/\hat{\Gamma}$ and N, then deform it into a harmonic map. We then justify an application of Siu's local analysis [17] to conclude that the harmonic map actually is $\pm$ holomorphic. We show that it is also proper whence it is not difficult to conclude the statement of the theorem.

The restriction on the codimension of $\bar{M}\backslash M$ is needed in the first step, namely the construction of a finite energy homotopy equivalence. This construction depends on a detailed study of complete Kähler metrics on quasiprojective varieties. We expect that our investigation of such metrics (in Sect. 1) will also be useful for other purposes.

The corresponding rigidity theorem in the compact case is due to Siu whose paper [16] pioneered the application of harmonic maps to rigidity questions in algebraic geometry.

Theorem 1 also includes (and generalizes) the celebrated rigidity theorem of Margulis [12] – except for one case, namely quotients of the Siegel upper half plane of degree two. This bounded symmetric domain is of complex dimension 3, and its finite volume quotients require a compactifying variety of dimension 1, i.e. of codimension 2. (Note that since $SO(3,2)$ is locally isomorphic to $\mathrm{Sp}(2,\mathbb{R})$, quotients of the bounded symmetric domain of type IV and dimension 3 are isomorphic to quotients of the Siegel upper half plane.) In all other cases of Margulis' theorem it follows from the study of the boundary components of bounded symmetric domains that the compactifying divisors for finite volume quotients can be blown down to codimension (at least) 3.

Let us also remark that we make use of Margulis' theorem on the arithmeticity of Γ in case rank $D \geqq 2$, because we shall need the Baily-Borel compactification of D/Γ the construction of which depends on the arithmeticity of Γ.

Since the basic idea of our method, however, has nothing to do with the arithmeticity of N, it would be interesting to have an approach to the compactification of finite volume Hermitian symmetric varieties which yields the Baily-Borel result without using arithmeticity. This was carried out for the rank one case in [18], but the higher rank case seems more difficult.

The case rank $D=1$ was already treated in our previous paper [10], and in this case one does not need any assumption on the codimension of the compactifying variety $\bar{M}/M$, due to the fact that in this case the geometric structure of the cusps of D/Γ is much simpler. Therefore, in the present note we shall only deal with the case rank $D \geqq 2$ although the case rank $D=1$ could rather easily be incorporated into our present arguments as well. Also, combining the present note and [10], one can also deal with irreducible group actions on products of bounded symmetric domains.

Concerning Theorem 2, the assumption on the codimension 3 compactification is not vacuous as Baily's compactification of the moduli space of Riemann surfaces of genus $g \geqq 3$ obtained as a subvariety of the Satake compactification of the moduli space of principally polarized Abelian varieties of dimension g via the Jacobian map and Torelli's theorem (cf. [1]) satisfies this requirement (after blowing down a $\mathbb{CP}^1$ factor in the Humbert variety). Namely it contains in its boundary the moduli spaces of surfaces of genus smaller than g, and if the genus drops by 1, the dimension of the moduli space drops by 3. This compactification is dominated by Mumford's compactification of the moduli space of stable curves of genus g where one has a compactifying variety of codimension 1, and both these compactifications lift to finite covers in an explicit way.

In our proof of Theorem 2, we shall make use of some recent results of Tromba and Wolpert on the Weil-Petersson metric of T_g, namely the negativity of the sectional curvature, the negative upper bound for the holomorphic sectional curvature, the strong negativity of the curvature in the sense of Siu as observed by Schumacher, and the convexity of geodesic length functions (cf. [19–21, 15]).

Let us also remark that a rigidity theorem for the moduli space of Riemann surfaces was announced by Schumacher in [15]. Judging from the contents of his paper, it seems that his rigidity theorem should be rather different in spirit from ours.

1. The Construction of the Domain Metric

By assumption, we can consider $\bar{M}$ as an algebraic subvariety of some $\mathbb{CP}^N$, and

$$\Sigma := \bar{M} \backslash M$$

is of (complex) codimension at least three in $\bar{M}$.

Applying Hironaka's theorem [8] and performing repeated blow-ups of the ambient $\mathbb{CP}^N$, we obtain as proper transform $\bar{M}'$ of $\bar{M}$ a compact algebraic manifold and a holomorphic map $\pi: \bar{M}' \to \bar{M}$ with the following properties:

> $\pi: \bar{M}' \backslash \pi^{-1}(\Sigma) \to M$ is bijective for each $z \in \Sigma$, $\pi^{-1}(z)$ is of dimension at least 2. $\Sigma' := \pi^{-1}(\Sigma)$ is a union of smooth hypersurfaces of $\bar{M}'$ with (at worst) normal crossings.

Also, we have a map $\pi: \mathbb{C}\hat{\mathbb{P}}^N \to \mathbb{CP}^N$, where $\mathbb{C}\hat{\mathbb{P}}^N$ is our blow-up of $\mathbb{CP}^N$, which extends $\pi: \bar{M}' \to \bar{M}$.

Let $\Sigma_1, \ldots, \Sigma_\ell$ be the components of Σ' and let σ_λ be a section of $\mathcal{O}(\bar{M}', [\Sigma_\lambda])$ with length $|\sigma_\lambda| < 1$ and $\Sigma_\lambda = \{\sigma_\lambda = 0\}$ $(\lambda = 1, \ldots, l)$. We note that σ_λ has a simple zero along Σ_λ, as Σ_λ has multiplicity one. We put

$$\sigma := \sigma_1 \otimes \ldots \otimes \sigma_l, \qquad \left(\sigma \in \mathcal{O}\left(\bar{M}', \left[\bigotimes_{\lambda=1}^{l} \Sigma_\lambda\right]\right)\right)$$

i.e.

$$|\sigma| = \prod_{\lambda=1}^{l} |\sigma_\lambda| . \tag{1}$$

We let $\bar{\omega}$ be the Kähler form of $\mathbb{CP}^N$. We let (z^α) be local coordinates on $\bar{M}'$, let $(\omega_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}})$ represent $\pi^* \bar{\omega}|_{\bar{M}'}$ in these coordinates and look at the Hermitian matrix given by

$$\frac{i}{2} g_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} = -\frac{i}{2} \frac{\partial}{\partial z^\alpha} \frac{\partial}{\partial z^{\bar{\beta}}} \log\log \frac{1}{|\sigma|^2} dz^\alpha \wedge dz^{\bar{\beta}} + c w_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}} \tag{2}$$

$$\begin{aligned} &= \frac{i}{2} \frac{1}{\left(\log \frac{1}{|\sigma|^2}\right)^2} \frac{\partial}{\partial z^\alpha} \log \frac{1}{|\sigma|^2} dz^\alpha \wedge \frac{\partial}{\partial z^{\bar{\beta}}} \log \frac{1}{|\sigma|^2} dz^{\bar{\beta}} \\ &\quad - \frac{i}{2} \frac{1}{\log \frac{1}{|\sigma|^2}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar{\beta}}} \log \frac{1}{|\sigma|^2} dz^\alpha \wedge dz^{\bar{\beta}} \\ &\quad + c\omega_{\alpha\bar{\beta}} dz^\alpha \wedge dz^{\bar{\beta}}, \end{aligned} \tag{3}$$

where $c > 0$ is a constant.

If G is a finite group acting biholomorphically on M then we can average $g_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}$ under G in order to obtain a Kähler metric (from the considerations below) invariant under the action of G. All our subsequent assertions (Lemmata 1–3) will pertain to this invariant metric, and the action of G will be isometric w.r.t. this metric

Lemma 1. *If $c>0$ is chosen large enough, then*

$$g_{\alpha\bar\beta} dz^\alpha \wedge dz^{\bar\beta}$$

represents a complete Kähler metric on M.

Proof. The Kähler property is clear. Let us first show that $g_{\alpha\bar\beta}$ represents a positive definite matrix. Since the first term in (3) is clearly positive semidefinite, it suffices to show that the sum of the other two terms is positive definite. Actually, both these terms are smooth on $\bar M'$. This is obvious for the third term in (3), and the second term extends smoothly because $-\partial\bar\partial \log|\sigma|^2$ is the curvature form of the line bundle $\left[\bigotimes_{\lambda=1}^{l} \Sigma_\lambda\right]$.

We first investigate a neighborhood of Σ_λ. Let $w \in \Sigma$, $z \in \pi^{-1}(w)$, i.e. $z \in \Sigma_\lambda$ for some $\lambda \in \{1, \ldots, l\}$.

$\bar M'$ sits in $\hat{\mathbb{CP}}^N$, the blow-up of $\mathbb{CP}^N$ along smooth submanifolds, and σ_λ is a restriction of a section defining a hypersurface in $\hat{\mathbb{CP}}^N$. If $\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial z^{\bar\beta}}$ are tangent to $\pi^{-1}(w)$, then they are tangent to a fiber in this hypersurface corresponding to a projective space of suitable dimension. Then

$$-\frac{\partial}{\partial z^\alpha}\frac{\partial}{\partial z^{\bar\beta}} \log\left(\frac{1}{|\sigma_\lambda|^2}\right) dz^\alpha \wedge dz^{\bar\beta}$$

is the curvature of this projective space, hence a positive definite form. $\Big($We note that since we work in the Kähler context, we do not get a contribution from the second fundamental form of $\bar M'$ in $\hat{\mathbb{CP}}^N$, since for the Levi-Civita connection ∇, $\nabla_{\frac{\partial}{\partial z^\alpha}} \frac{\partial}{\partial z^{\bar\beta}} = 0\Big)$.

If on the other hand $\frac{\partial}{\partial z^\alpha}$ and $\frac{\partial}{\partial z^{\bar\beta}}$ are normal to $\pi^{-1}(w)$ (e.g.w.r.t. the Kähler metric on M'), then $\omega_{\alpha\bar\beta}$ is a positive definite form, while $-\frac{1}{\log\frac{1}{|\sigma|^2}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} \log\frac{1}{|\sigma|^2}$ tends to zero when approaching Σ_λ. Finally, cross terms can be handled by the Schwarz inequality and we conclude that (provided $c>0$ is chosen large enough)

$$-\frac{i}{2}\frac{1}{\log\frac{1}{|\sigma|^2}} \frac{\partial^2}{\partial z^\alpha \partial z^{\bar\beta}} \log\frac{1}{|\sigma|^2} + c\omega_{\alpha\bar\beta}$$

is positive semidefinite in a neighborhood of Σ_λ. Away from Σ_λ, $\omega_{\alpha\bar\beta}$ has eigenvalues bounded from below by a positive constant, and hence, if $c>0$ is large enough, our matrix is positive definite everywhere.

In order to show completeness, we shall use the first term in $g_{\alpha\bar{\beta}}$. We let $\gamma(t)$ be a (piecewise smooth) arc running from a level hypersurface $|\sigma|^2=c>0$ into $|\sigma|^2=0$. We calculate

$$\int\limits_{|\sigma|^2=0}^{|\sigma|^2=c}\left(\frac{1}{\left(\log\frac{1}{|\sigma|^2}\right)^2}\frac{\partial}{\partial z^\alpha}\log\frac{1}{|\sigma|^2}\frac{\partial}{\partial z^{\bar{\beta}}}\log\frac{1}{|\sigma|^2}dz^\alpha\wedge dz^{\bar{\beta}}(\dot{\gamma},\dot{\gamma})\right)^{1/2}dt$$

$$=\int\limits_{|\sigma|^2=0}^{|\sigma|^2=c}\frac{1}{\log\frac{1}{|\sigma|^2}}\frac{1}{|\sigma|^2}d|\sigma|^2=\infty\,. \tag{4}$$

Since, as shown above, the other terms in the definition of $g_{\alpha\bar{\beta}}$ are positive semidefinite, it follows that γ has infinite length w.r.t. the metric $g_{\alpha\bar{\beta}}$, and completeness follows.

Lemma 2. *$g_{\alpha\bar{\beta}}$ defines a metric of finite volume on M.*

Proof. This follows since the second and third term in the definition of $g_{\alpha\bar{\beta}}$ are bounded, whereas the first one behaves like the Poincaré metric

$$-\frac{i}{2}\frac{\partial^2}{\partial z\partial\bar{z}}\log\log\frac{1}{|z|^2}dz\wedge d\bar{z}$$

on the punctured unit disk $\{z\in\mathbb{C}:0<|z|<1\}$ which has finite volume in a neighborhood of the origin.

Remark. One can also check that $(g_{\alpha\bar{\beta}})$ has bounded Ricci curvature, cf. e.g. [6].

Lemma 3. *The diameter of $\{|\sigma|^2=c\}$ stays bounded as $c\to 0$.*

Proof. Since the second and third term in the definition of $g_{\alpha\bar{\beta}}$ are bounded, we again have to look only at the first term in (3), namely

$$\left(\frac{1}{\log\frac{1}{|\sigma|^2}}\right)^2\frac{\partial}{\partial z^\alpha}\log\frac{1}{|\sigma|^2}\cdot\frac{\partial}{\partial z^{\bar{\beta}}}\log\frac{1}{|\sigma|^2}dz^\alpha\wedge dz^{\bar{\beta}}$$

We first note that its restriction to the holomorphic tangent directions of $\{|\sigma|^2=c\}$ vanishes. In the remaining real tangent direction, it behaves like the Poincaré metric on the punctured unit disk (cf. the proof of Lemma 2), so that this direction is actually shrinking in diameter.

The claim follows easily.

2. The Construction of a Homotopy Equivalence of Finite Energy

Lemma 4. *Let M be equipped with the metric $(g_{\alpha\bar{\beta}})$ of Lemma 1. Assume that the codimension of $\Sigma(:=\bar{M}\backslash M)$ in $\bar{M}$ is at least 3.*

Let N be a Riemannian manifold homotopically equivalent to M. Then there exists a homotopy equivalence

$$h:M\to N$$

of finite energy:

$$E(h)=\frac{1}{2}\int_M |dh|^2 d\,\mathrm{vol}(g_{\alpha\bar{\beta}})<\infty\,.$$

If a finite group G acts isometrically on M and N and if M and N are equivariantly (w.r.t. the action of G) homotopically equivalent, then h can also be chosen to be equivariant.

Proof. Let

$$\bar{M}\backslash M=\bigcup_{k=1}^{K}\Sigma_k$$

be the disjoint union of its connected components. We split M as

$$M=M_b\cup\bigcup_{k=1}^{K}M_k,$$

where M_b is bounded and M_k is of the form

$$B_k\times\mathbb{R}^+ \text{ (topologically, but not necessarily metrically)},$$

where B_k is the boundary of a neighborhood of Σ_k. Since N is homotopically equivalent to M, we get an induced splitting

$$N=N_b\cup\bigcup_{k=1}^{K}(C_k\times\mathbb{R}^+),$$

where C_k is homotopically equivalent to B_k.

We can actually assume that the given homotopy equivalence between M and N induces homotopy equivalences

$$g_k:B_k\to C_k,$$

and, applying a standard smoothing process, we can assume that each g_k is differentiable and hence in particular has finite energy.

Since M_b is bounded, it now suffices to extend each g_k to a finite energy map $\bar{g}_k$ between $B_k\times\mathbb{R}^+$ and $C_k\times\mathbb{R}^+$.

Of course, it then is sufficient to exhibit the procedure for one end. Hence we drop the index k, and we use the constructions of Sect. 1. The boundary neighborhood B can then be assumed to be of the form $\{|\sigma|^2=c\}$ $(c>0)$.

We then construct a map $\bar{g}$ as follows.

We first take the map p that projects $B\times\mathbb{R}^+$ onto B along the gradient flow curves of $|\sigma|^2$. Here, we take $\mathrm{grad}|\sigma|^2$ w.r.t. the Kähler metric of $\bar{M}'$. Furthermore, let g be the given homotopy between B and C. We put

$$\bar{g}:=g\circ p\,.$$

If G acts isometrically on M and N, we obtain decompositions as above for the quotients M/G and N/G, and lifting these decompositions to M and N, resp. we see that we can construct g and $\bar{g}$ equivariantly.

Since g is a C^1-map between compact manifolds, in order to show that $\bar{g}$ has finite energy, it suffices to show that p has finite energy.

The energy density of p is

$$g^{\alpha\bar{\beta}}(z)\left(g_{i\bar{j}}(p(z))\frac{\partial p^i}{\partial z^\alpha}\frac{\partial p^{\bar{j}}}{\partial z^{\bar{\beta}}}+g_{\bar{j}\bar{i}}(p(z))\frac{\partial p^{\bar{i}}}{\partial z^\alpha}\frac{\partial p^j}{\partial z^{\bar{\beta}}}\right)\det g\, dz^1\wedge dz^{\bar{1}}\wedge\ldots\wedge dz^n\wedge dz^{\bar{n}}. \quad (5)$$

Here, $\det g\, dz^1\wedge dz^{\bar{1}}\wedge\ldots\wedge dz^{\bar{n}}$ is a volume form of the Kähler metric of $\bar{M}'$ (choosing suitable coordinates). Moreover,

$$g^{\alpha\bar{\beta}}=\frac{\hat{g}_{\alpha\bar{\beta}}}{\det g}, \quad (6)$$

where $\hat{g}_{\alpha\bar{\beta}}$ is the corresponding minor of $(g_{\alpha\bar{\beta}})$. Hence the terms with $\det g$ cancel each other.

Also, the terms $g_{i\bar{j}}(p(z))$ and $g_{\bar{j}\bar{i}}(p(z))$ are bounded and can hence be disregarded in our estimates.

Let, after relabeling,

$$z_0\in\bigcap_{\lambda=1}^{l_0}\Sigma_\lambda \qquad (l_0\geqq 1).$$

We want to control the energy of p in a neighborhood of z_0. After performing a C^1-diffeomorphism of this neighborhood, we can assume that (in local coordinates)

$$z_0=0\in\mathbb{C}^n,$$

$$|\sigma|^2=|z^1\cdot\ldots\cdot z^{l_0}|^2$$

and that the flow lines for $\operatorname{grad}|\sigma|^2$ are given by the flow lines for $\operatorname{grad}|z^1\cdot\ldots\cdot z^{l_0}|^2$.

$\operatorname{grad}|\sigma|^2$ determines a holomorphic tangent plane at each $z\in M$. Let this plane be spanned by unit vectors e^1 and $e^{\bar{1}}$. Then

$$g_{i\bar{j}}e^1(p^i)e^{\bar{1}}(p^{\bar{j}})+g_{\bar{j}\bar{i}}e^1(p^{\bar{i}})e^{\bar{1}}(p^j)\leq\frac{\text{const}}{|\sigma|^2}. \quad (7)$$

In the directions normal to this plane, the derivatives of p are uniformly bounded, i.e. if the corresponding unit vectors are denoted by $e^2,\ldots,e^{\bar{n}}$

$$g_{i\bar{j}}e^\mu(p^i)e^{\bar{\mu}}(p^{\bar{j}})+g_{\bar{j}\bar{i}}e^\mu(p^{\bar{i}})e^\mu(p^j)\leqq\text{const} \qquad (\mu=2,\ldots,n). \quad (8)$$

Let us now estimate the corresponding minors $\hat{g}^{\alpha\beta}$.

Let $\hat{g}^{11}$ be the minor corresponding to the direction determined by the vectors e^1 and $e^{\bar{1}}$. We have

$$\hat{g}^{11}\leqq\frac{\text{const}}{\left(\log\frac{1}{|\sigma|^2}\right)^2}. \quad (9)$$

This is seen as follows:

If $w\in\Sigma$, then $\pi^{-1}(w)$ has complex dimension at least 2, since Σ is of codimension 3 in $\bar{M}$.

When approaching $\pi^{-1}(w)$, then in a direction tangent to $\pi^{-1}(w)$, the term

$$-\frac{1}{\log\frac{1}{|\sigma|^2}}\frac{\partial^2}{\partial z^\alpha\partial z^\beta}\log\frac{1}{|\sigma|^2}$$

dominates the term

$$c\omega_{\alpha\bar{\beta}}$$

in the definition of $g_{\alpha\bar{\beta}}$, as the latter tends to zero quadratically, as can be easily seen from the formulae for blow-ups. Since there are always at least two such directions, we get the factor $\left(\log\frac{1}{|\sigma|^2}\right)^{-2}$ in the estimate for $\hat{g}^{11}$.

Likewise, let $\hat{g}^{\mu\mu}$ be the minor corresponding to the $e^{\mu}, e^{\bar{\mu}}$ direction. Then for $\mu \geqq 2$,

$$\hat{g}^{\mu\mu} \leqq \frac{1}{\left(\log\frac{1}{|\sigma|^2}\right)^2} \cdot \frac{1}{|\sigma|^2} \text{const}. \tag{10}$$

This estimate comes from the behavior of the first term in the definition of $g_{\alpha\bar{\beta}}$.

From (5)–(10) we deduce (in local coordinates near z_0)

$$E(p) \leqq \text{const} \int \frac{1}{|\sigma|^2} \frac{1}{\left(\log\frac{1}{|\sigma|^2}\right)^2} dz^1 \wedge dz^{\bar{1}} \wedge \ldots \wedge dz^n \wedge dz^{\bar{n}}.$$

The volume form $dz^1 \wedge dz^{\bar{1}} \wedge \ldots \wedge dz^n \wedge dz^{\bar{n}}$ behaves like

$$|\sigma| \mathrm{d}|\sigma| \mathrm{d}\,\mathrm{vol}(\{|\sigma| = c\})$$

and we conclude that $E(p)$ is finite. q.e.d.

Remark. The homotopy equivalence constructed in Lemma 4 is not proper. Under suitable assumptions on the behavior of the metric of N near the ends (which are satisfied in our applications), one could also construct a proper homotopy equivalence of finite energy.

3. The Harmonic Map and its Properties

a) Existence of a Harmonic Map of Finite Energy

Lemma 5. *Let h be the homotopy equivalence constructed in Lemma 4. Assume that N has nonpositive sectional curvature, that*

a) *there exists a closed loop in N that cannot be homotoped into an end and that*

b) *the universal cover $\tilde{N}$ of N has a convex exhaustion function $f \in C^2$.*

Then h is homotopic to a harmonic homotopy equivalence u of finite energy. If h is equivariant w.r.t. to an isometric action of a finite group G on M and N, then so is u. (In this case we need only a loop on N/G that cannot be homotoped into an end).

Proof. We refer to [9] as a reference for harmonic maps and the associated parabolic problems. We exhaust M by an increasing sequence of bounded smooth domains Ω_i, and look at the parabolic problems

$$\begin{gathered} u_i(x,t): \Omega_i \times [0,\infty) \to N, \\ \frac{\partial u_i}{\partial t} - \tau(u_i) = 0, \\ u_i(x,0) = h(x) \quad \text{for} \quad x \in \Omega_i, \\ u_i(x,t) = h(x) \quad \text{for} \quad x \in \partial\Omega_i \text{ and } t \in [0,\infty). \end{gathered} \tag{11}$$

Here, $\tau(u)$ is the tension field of u, *i.e.* the Euler-Lagrange operator for the energy functional $E(u)=\int_{\Omega}|du|^2 d\Omega$.

We lift u_i to universal covers, obtaining a map

$$\tilde{u}_i(x,t):\tilde{\Omega}_i\times[0,\infty)\to\tilde{N},$$

where $\tilde{\Omega}_i$ is the lift of Ω_i in the universal cover $\tilde{M}$ of M. Since f is convex,

$$\frac{\partial}{\partial t}(f\circ\tilde{u}_i)-\Delta(f\circ\tilde{u}_i)\leqq 0. \tag{12}$$

We shall show that the set of $t\in[0,\infty)$ for which a solution of (11) exists is open and closed. Since it contains 0 it will then follow that a solution of (11) exists for all time $t\in[0,\infty)$. If a solution exists for t_n and $t_n\to t_0$ then a standard normal family argument based on (12) implies that, as $t_n\to t_0$, either $f\circ\tilde{u}_1$ converges uniformly on compact subsets of $\tilde{M}$ to ∞ or to a real-valued limit function $f\circ\tilde{u}_i(\cdot,t_0)$. The first case in impossible since u_i has fixed boundary values on $\partial\Omega$. This, together with a-priori estimates for solutions of (11) shows closeness.

Since Ω_i is bounded and hence $f\circ\tilde{u}_i(\cdot,t)$ is bounded for given t when restricting to the intersection of $\tilde{\Omega}_i$ with a fundamental region for M in $\tilde{M}$, openness follows from the implicit function theorem.

As $t\to\infty$, $u_i(\cdot,t)$ then converges to a harmonic map

$$v_i:\Omega_i\to N.$$

We now let $i\to\infty$.

Since f is convex, again

$$\Delta(f\circ\tilde{v}_i)\geqq 0, \tag{13}$$

where $\tilde{v}_i:\tilde{\Omega}_i\to\tilde{N}$ is the lift to universal covers as before. A normal family argument, based on (13) now shows that either $f\circ\tilde{v}_i$ tends to infinity uniformly on compact subsets or to a real-valued subharmonic function as $i\to\infty$. The first case is impossible because of assumption a).

A-priori estimates then imply that v_i converges together with its derivatives uniformly on compact subsets of M to a harmonic map $u:M\to N$. Since M and hence also N is topologically finite, u is homotopic to h, in particular a homotopy equivalence. Also, by construction

$$E(u)\leqq E(h)$$

and the latter is finite by assumption.

Finally, as in the proof of Lemma 4, the construction can be made equivariant w.r.t. the action of a finite group G of isometries. q.e.d.

The assumptions of Lemma 5 are satisfied in the two cases that interest us here, namely

Corollary 1. *Let M be as before, in particular (equivariantly) homotopically equivalent to N and equipped with the metric of Lemma 1. Then there exists an (equivariant) harmonic homotopy equivalence of finite energy if*

a) *N is a quotient of finite volume of a bounded Hermitian symmetric domain D.*

b) *N is a finite cover of the moduli space of Riemann surfaces of genus* $g \geqq 2$ *equipped with the Weil-Petersson metric, and* $G = \Gamma_g/\pi_1(N)$ *where* Γ_g *is the modular group.*

Proof. In case a), each end of N arises from the action of $\pi_1(N)$ on a boundary component F of D, and since N has finite volume, $\pi_1(N)$ contains elements not leaving F fixed. Such an element represents a closed loop on N that cannot be homotoped into the corresponding end. Taking a product of one such element for each end produces a loop satisfying the requirement of Lemma 5.

Since D is complete of nonpositive sectional curvature, the squared distance function from any point of D yields a strictly convex exhaustion function.

In case b), Wolpert [21] showed that geodesic length functions (i.e. one fixes a homotopy class of closed loops on the underlying surface of genus g and assigns to each point in Teichmüller space the length of the closed geodesic in this class w.r.t. the hyperbolic metric on the Riemann surface represented by this point) are strictly convex w.r.t. the Weil-Petersson metric. Moreover, a suitable sum of geodesic length functions (for example taking all the generators of the fundamental group of the corresponding surface) provides an exhaustion function. Furthermore, Tromba [19] and also Wolpert [20] showed that the Weil-Petersson metric has negative sectional curvature.

Finally, a closed loop on N/G that cannot be homotoped into an end of N/G is constructed as follows: Let $a_1, a_2, \ldots, a_{2g}$ be a set of generators of the fundamental group of our surface of genus g, ordered in such a way that the intersection number between a_{2i-1} and a_{2i} is 1 and the other intersection numbers vanish ($i=1, \ldots, g$). We then let $\gamma \in \Gamma_g$ be the element that interchanges each a_{2i-1} with a_{2i}. It is an easy consequence of the collar lemma for hyperbolic Riemann surfaces that the loop represented by γ cannot be homotoped into the end of N/G.

b) Some Properties of the Harmonic Map

We choose a family $(\phi_\varepsilon)_{\varepsilon>0}$ of cut-off functions with the following properties:

Near each component Σ of $\bar{M}/M$, ϕ_ε is a function of $s := |\sigma|^2$ (defined as in Sect. 1), i.e. $\operatorname{grad}\phi_\varepsilon$ is proportional to $\operatorname{grad}|\sigma|^2$

$$0 \leqq \phi_\varepsilon(s) \leqq 1\,,$$

$$\phi_\varepsilon \to 1 \quad \text{as} \quad \varepsilon \to 0\,,$$

$$\phi_\varepsilon(s) \equiv 0 \quad \text{for} \quad s \cdot \log\frac{1}{s} \leqq \varepsilon\,,$$

$$\varepsilon^2 \phi_\varepsilon'' \text{ is bounded independently of } \varepsilon \text{ as } \varepsilon \to 0\,.$$

Since $\operatorname{grad}|\sigma|^2$ is orthogonal to the hypersurfaces $|\sigma|^2 = \text{const}$, these properties entail

$$|\Delta\phi_\varepsilon| \sim \left| s^2 \left(\log\frac{1}{s}\right)^2 \phi_\varepsilon'' \right| \leqq c\,. \tag{14}$$

We introduce local coordinates u^i on the image N, denote by $(\gamma_{i\bar{j}})$ the corresponding metric tensor, by $R_{i\bar{j}k\bar{l}}$ the curvature tensor, by Γ^i_{jk} the Christoffel symbols, put

$$\partial u^i = \sum_\alpha \frac{\partial u^i}{\partial z^{\bar{\alpha}}} dz^{\bar{\alpha}} \quad \text{etc.}$$

and

$$D\bar{\partial}u^i = \partial\bar{\partial}u^i + \sum_{j,k} \Gamma^i_{jk}\partial u^j \wedge \bar{\partial}u^k .$$

Let ω denote the Kähler form of M, $n = \dim M = \dim N$. Siu's Bochner type identity [16, Sect. 3] then is

$$\begin{aligned}
&\sum_{i,j} \partial\bar{\partial}(\gamma_{i\bar{j}}\bar{\partial}u^i \wedge \partial u^{\bar{j}}) \wedge \omega^{n-2} \\
&\quad = \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}}\bar{\partial}u^i \wedge \partial u^{\bar{j}} \wedge \partial u^k \wedge \overline{\partial u^l} \wedge \omega^{n-2} \\
&\quad\quad - \sum_{i,j} \gamma_{i\bar{j}} D\bar{\partial}u^i \wedge \bar{D}\partial u^{\bar{j}} \wedge \omega^{n-2} .
\end{aligned} \tag{15}$$

Now

$$\begin{aligned}
\left| \int_M \phi_\varepsilon \sum_{i,j} \partial\bar{\partial}(\gamma_{i\bar{j}}\bar{\partial}u^i \wedge \partial u^{\bar{j}}) \wedge \omega^{n-2} \right| &= \left| \int_M \sum_{i,j} \gamma_{i\bar{j}}\bar{\partial}u^i \wedge \partial u^{\bar{j}} \wedge \omega^{n-2} \wedge \partial\bar{\partial}\phi_\varepsilon \right| \\
&\leqq \int_M e(u)\,|\Delta\phi_\varepsilon|\, d\,\mathrm{vol}(M)
\end{aligned}$$

is bounded independently of ε, since the energy of u is finite and because of (14).

Letting $\varepsilon \to 0$, we obtain

$$\left| \int \sum_{i,j} \partial\bar{\partial}(\gamma_{i\bar{j}}\partial u^i \wedge \partial u^{\bar{j}}) \wedge \omega^{n-2} \right| < \infty . \tag{16}$$

On the other hand, in the cases of interest to us (cf. Corollary 1) both terms on the right hand side of (15) are nonpositive. The second term is nonpositive because u is harmonic (cf. [16]). If N is a quotient of a bounded Hermitian symmetric domain, the first term is nonpositive by Siu's analysis of the curvature tensors of such domains, cf. [16]. If N is a quotient of Teichmüller space equipped with the Weil-Petersson metric, then the curvature term is nonpositive by the computation of [15].

(16) then implies that both terms have a finite integral over M.

Moreover, since u is harmonic, the second term of the right hand side of (15) actually controls the complex Hessian of u. Namely, cf. [16], suppose we have normal coordinates at $z \in M$ and $u(z) \in N$ (to facilitate the notation), and that

$$\lambda^i_b \text{ and } \mu^i_b \qquad (b = 1, \ldots, n)$$

are the eigenvalues of $\mathrm{Re}\,\partial\bar{\partial}u^i$ and $\mathrm{Im}\,\partial\bar{\partial}u^i$, resp. Then

$$\begin{aligned}
\sum_{i,j} \gamma_{i\bar{j}} D\bar{\partial}u^i \wedge \bar{D}\partial u^{\bar{j}} \wedge \omega^{n-2} &= \frac{-1}{n^2 - n} \sum_k \sum_{b \neq c} (\lambda^k_b \lambda^k_c + \mu^k_b \mu^k_c)\omega^n \\
&= \frac{1}{n^2 - n} \sum_k \sum_b (\lambda^{k^2}_b + \mu^{k^2}_b)\omega^n
\end{aligned}$$

since $\sum\limits_{b \neq c} \lambda_b \lambda_c = \left(\sum\limits_b \lambda_b\right)^2 - \sum\limits_b \lambda_b^2$, and $\sum\limits_b \lambda_b = 0$, as u is harmonic, and similarly for the μ_b's.

Thus

$$\sum_{i,j} \gamma_{i\bar{j}} D\bar{\partial} u^i \wedge \bar{D}\partial u^{\bar{j}} \wedge \omega^{n-2} = \frac{1}{n^2-n} |D\bar{\partial} u|^2 \omega^n, \tag{17}$$

where $|D\bar{\partial}u|$ denotes the norm of the complex Hessian of u. In particular,

$$\int_M |D\bar{\partial}u|^2 d\,\mathrm{vol}(M) < \infty. \tag{18}$$

Therefore, we can find an exhaustion of M by an increasing sequence (M_j) of smooth subsets for which

$$\begin{aligned} \left| \int_{M_j} \sum_{i,j} \partial\bar{\partial}(\gamma_{i\bar{j}} \bar{\partial} u^i \wedge \partial u^{\bar{j}}) \wedge \omega^{n-2} \right| &= \left| \int_{\partial M_j} \sum_{i,j} \bar{\partial}(\gamma_{i\bar{j}} \bar{\partial} u^i \wedge \partial u^{\bar{j}}) \wedge \omega^{n-2} \right| \\ &\leqq \left(\int_{\partial M_j} e(u) \right)^{1/2} \cdot \left(\int_{\partial M_j} |D\bar{\partial}u|^2 \right)^{1/2} \\ &\to 0 \quad \text{as} \quad j \to \infty. \end{aligned}$$

Hence

$$\int_M \sum_{i,j} \partial\bar{\partial}(\gamma_{i\bar{j}} \bar{\partial} u^i \wedge \partial u^{\bar{j}}) \wedge \omega^{n-2} = 0, \tag{19}$$

hence, since both terms on the right hand side of (15) are nonpositive,

$$\sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} \bar{\partial} u^i \wedge \partial u^{\bar{j}} \wedge \partial u^k \wedge \overline{\partial u^l} \equiv 0 \tag{20}$$

and

$$D\bar{\partial} u^i \equiv 0 \equiv \bar{D}\partial u^{\bar{j}} \quad \text{for} \quad i, j = 1, \ldots, n. \tag{21}$$

Remark. Alternatively, we can use the fact that the Ricci curvature of M is bounded, cf. the remark after Lemma 2, and argue as in [10] to deduce (20) and (21).

c) The Harmonic Map is $\pm$ Holomorphic

We first observe that we can assume w.l.o.g. that N (and consequently also M) has at least two ends. If $N = D/\Gamma$, each end of N arises from the action of Γ on a boundary component F of D, and since $N = D/\Gamma$ has finite volume, Γ contains elements not leaving F fixed. Let γ be such an element. As Γ is residually finite (cf. [4, p. 39]), it contains a normal subgroup Γ' of finite index not containing γ. Then $N' = D/\Gamma'$ has more then one end. Hence, by lifting the harmonic map to a map $u': M' \to N'$ between the corresponding finite coverings of M and N, we can assume that domain and image have at least two ends.

Remark. Actually for $N = D/\Gamma$, Γ' can be explicitly constructed as a congruence subgroup since Γ is arithmetic by Margulis' theorem [11].

If N is a quotient of Teichmüller space there are several ways to see that N has a cover with at least two ends. For example, it can be reduced to the first case by embedding the moduli space of Riemann surfaces of genus g into the moduli space of Abelian varieties of dimension g via the Jacobian and Torelli's theorem, and this latter space is a quotient of the Siegel upper half plane of degree g, i.e. a domain D as above. Cf. [1] for details.

Therefore, we let

$$\bar{M} \backslash M = \bigcup_{k=1}^{K} \Sigma_k$$

with $K \geqq 2$.

The boundary of a neighborhood of any Σ_k then represents a nontrivial $(2n-1)$-dimensional real homology class in M, and since u is a homotopy equivalence, it induces an isomorphism on $(2n-1)$-dimensional homology. Since $H_{2n-1}(M)$ is nontrivial by the preceding observation, the real rank of u has to be $(2n-1)$ at some point.

This implies, taking (20) into account, by Siu's analysis of the curvature tensor of locally symmetric varieties, cf. [17, Theorem 6.7], and by Schumacher's computation [15], resp.

Lemma 6. *u is holomorphic or antiholomorphic.*

d) Global Behaviour of the Harmonic Map and Properness

By Hironaka's theorem [8], we can find a nonsingular compactification $\hat{M}$ dominating the original compactification $\bar{M}$. If $N = D/\Gamma$, we let $\bar{N}$ be the Baily-Borel compactification of N, cf. [2]. The result of Baily-Borel applies since Γ is arithmetic by Margulis' result [11].

If N is the moduli space of Riemann surfaces of genus g, we choose Baily's compactification $\bar{N}$ of N that can be embedded into the above compactification of $D/\mathrm{Sp}(\mathbb{Z}, g)$ where D is the Siegel upper half plane of degree g, cf. [1]. Using Lemma 6, we can therefore consider u in any case as a $\pm$ holomorphic map

$$u: M \to D/\Gamma$$

and by Borel's result [5], u can then be extended as a $\pm$ holomorphic map

$$\bar{u}: \hat{M} \to \overline{D/\Gamma}\,.$$

In particular, $\bar{u}$ is continuous.

For the rest of this section, we can continue to treat both cases simultaneously, namely assume that the image N is of the form D/Γ and use its Baily-Borel compactification.

We shall now obtain some information about the behavior of $\bar{u}$ at the ends of $\hat{M}$.

Let S be a component of $\bar{N} \backslash N$, let $\bar{U}$ be a neighborhood of S in $\bar{N}$, with the property that

$$U := \bar{U} \backslash S$$

is of the form

$$U = \partial U \times \mathbb{R}^+ .$$

In this parametrization, we put

$$U_t = \partial U \times \{t\}$$

(i.e. $U_0 = \partial U \times \{0\} = \partial U$).

For each point $p \in S$, there exists a continuous mapping

$$w: B := \{z \in \mathbb{C} : |z| \leqq 1\} \to \bar{N}$$

with the property that

$$\gamma_t := w(\{|z| = t\}) \subset U_t$$

is a homotopically nontrivial (in U_t) curve, and

$$w(0) = p.$$

It can, e.g., be deduced from Borel's study of the metric behavior of N near the cusps, that we can also assume

$$\text{length}(\gamma_t) \to 0 \quad \text{as} \quad t \to \infty \qquad \text{(cf. [5])}.$$

γ_t then also is homotopically nontrivial in N (cf. Lemma 3 of [10]); also, the homotopy class defined by the curves γ_t uniquely determines a component of $\bar{N} \setminus N$ towards which it shrinks, i.e. if there is a continuous map

$$\tilde{w}: B \to \bar{N} \text{ with } \tilde{w}(\partial B) \text{ homotopic to } \gamma_0,$$

then $\tilde{w}(B) \cap S \neq \emptyset$ where S is our given component of $\bar{N} \setminus N$ (cf. [10, Lemma 4]).

Since u is a homotopy equivalence, there exists a continuous map

$$v: B \to \hat{M}$$

with

$$v(0) \in \hat{\Sigma} := \hat{M} \setminus M, \qquad v(B \setminus \{0\}) \cap \hat{\Sigma} = \emptyset,$$

$$u(v(\partial B)) = \gamma_0.$$

It then follows from the continuity of $\bar{u}$ that

$$\bar{u}(v(0)) \in S.$$

This determines a correspondence between the components of $\hat{M} \setminus M$ and of $\bar{N} \setminus N$, although at this moment we cannot yet conclude that all of $\hat{M} \setminus M$ is mapped into $\bar{N} \setminus N$ (we know $\bar{u}(q) \in \bar{N} \setminus N$ only for those points $q \in \hat{M} \setminus M$ for which there exists a continuous map $v: B \to \hat{M}$ with $v(B) \cap (\hat{M} \setminus M) = v(0)$). In particular, the preimage of any component of $\bar{N} \setminus N$ is nonempty.

Lemma 7. *u is a proper map and induces a well defined correspondence between the components of $\hat{M} \setminus M$ and $\bar{N} \setminus N$.*

Proof. We use again the nonsingular compactification $\hat{M}$ of M. $\hat{M} \setminus M$ then consists of a union of smooth hypersurfaces with normal crossings. In this case, the corresponding complete Kähler metric constructed in Sect. 1 has bounded Ricci curvature (cf. the remark after Lemma 2).

Also, the locally symmetric Kähler metric on N has holomorphic sectional curvature bounded from above by a negative constant.

Therefore, we can apply Royden's Schwarz lemma [13] to conclude the properness of u. Namely, let $p_1 \in \hat{M}\backslash M$ be a point which is mapped to $\bar{N}\backslash N$ under u. Such a point exists in each component of $\hat{M}\backslash M$ by the preceding argument. Let $p_2 \in \hat{M}\backslash M$ lie in the same component of $\hat{M}\backslash M$ as p_1. Let $\gamma_1(t)$ and $\gamma_2(t)$ be curves in M with endpoints p_1 and p_2, resp., and with

$$\operatorname{dist}(\gamma_1(t), \gamma_2(t)) \leqq K$$

where K is a constant independent of $t \in \mathbb{R}^+$ (Lemma 3). The Schwarz lemma then implies that also

$$\operatorname{dist}(u(\gamma_1(t)),\ u(\gamma_2(t)))$$

is bounded independently of t.

Consequently, p_2 is mapped to $\bar{N}\backslash N$, actually to the same component as p_1. q.e.d.

e) Proof of the Theorem

Lemma 7, together with the fact that the restriction of u to the boundary of a neighborhood of the singular set has real rank $2n-1$, implies that u actually has rank $2n$. (This could also have been deduced from the fact that u is $\pm$ holomorphic of real rank at least $2n-1$.)

Since u extends to the compactification, it has a well defined degree, and since it is $\pm$ holomorphic of maximal rank, its degree is nonzero. Let us now show that the absolute value of the degree is at most 1. Namely, let $g: N \to M$ be a proper homotopy equivalence so that $u \circ g$ is homotopic to the identity of N. Passing to a suitable finite cover as above, if necessary, we restrict $u \circ g$ to the boundary B of a neighborhood of an end of N and conclude that $u \circ g$ induces a map of degree ± 1 between B and its image. We then deform $u \circ g$ as above by harmonic replacement on a sequence of open sets exhausting N into a proper harmonic homotopy equivalence h. It is seen from the construction that h also induces a map of degree 1 between B and its image, and hence also that $\deg h = \deg u \circ g$. As before, h is $\pm$ holomorphic of maximal rank and the Schwarz lemma [13] implies

$$|\deg u \cdot \deg g| = |\deg h| \leqq 1 .$$

We conclude

$$|\deg u| = 1 .$$

(If N is a quotient of Teichmüller space, in the preceding argument we equip N with the Weil-Petersson metric and use the fact that this metric has a negative upper bound for its holomorphic sectional curvature, cf. [19] and also [20].)

In order to show that u is bijective, i.e. $\pm$ biholomorphic, we can basically proceed as in [16, p. 110f.]. Let V be the set of points in M, where u is not locally homeomorphic. Since V is defined by the vanishing of the functional determinant of u, it is a complex subvariety of complex codimension 1, unless empty. Since u is of degree 1, $u(V)$ has complex codimension at least 2. In case V is nonempty, the preimage of a generic point of $u(V)$ is a nontrivial compact (since u is proper) analytic subvariety of M, in contradiction to u being a homotopy equivalence. Hence V is empty, proving the theorem.

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