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PARALLELIZABLE MANIFOLDS WITHOUT COMPLEX STRUCTURE

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LET M be a complex manifold which is homotopic to the torus. An interesting question is whether such a manifold is actually biholomorphic to a complex torus or not. In this note, we shall prove the following theorems.

THEOREM 1. *Let M be a compact two dimensional complex manifold with zero Euler number. Suppose there is a basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of the first real cohomology group $H^1(M, R)$ such that the cup product $\alpha_1 \cup \alpha_2 \cup \alpha_3 \cup \alpha_4$ is not zero. Then either M is biholomorphic to the complex torus or M is covered by the euclidean space.*

THEOREM 2. *Let M be a compact two dimensional complex manifold whose tangent bundle is trivial in the topological sense. Then*

(i) *M is a ruled surface of genus 1 (a CP^1-bundle over an elliptic curve).*

(ii) *M is covered by the complex torus or an elliptic fiber bundle over a compact curve of genus > 1.*

(iii) *M is the quotient space of C^2 by some volume-preserving affine transformation group. The first Chern class of M is zero and the first Betti number is three.*

(iv) *The first Betti number of M is one.*

As a corollary of these theorems, we give an example of a compact four dimensional parallelizable manifold which does not admit any complex structure. We note that van de Ven[2] has already given examples of four dimensional almost complex manifolds which do not admit complex structure. However, his method relies on the non-triviality of the Chern numbers and does not seem to extend to cover the parallelizable case.

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§1. PROOF OF THEOREM 1

We need the following two Lemmas of Kodaira[1].

LEMMA 1. *Every holomorphic 1-form on a compact complex surface is closed.*

LEMMA 2. *If the first Betti number b_1 of a compact complex surface is even, then the number of linearly independent homomorphic 1-forms on this surface is equal to $b_1/2$.*

According to the lemmas and our hypothesis, we see that there are two linearly independent holomorphic 1-forms ω_1 and ω_2 such that $\omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2$ is not identically zero on M. It follows easily from this fact that $\omega_1 \wedge \bar{\omega}_1$, $\omega_2 \wedge \bar{\omega}_2$, $\omega_1 \wedge \bar{\omega}_2$, $\omega_2 \wedge \bar{\omega}_1$, $\omega_1 \wedge \omega_2$ and $\bar{\omega}_1 \wedge \bar{\omega}_2$ are linearly independent closed two-forms on M.

Since the Euler number of M is zero and $b_1(M) = 4$, we see that $b_2(M) = 6$ and the above mentioned two-forms actually span the second cohomology group $H^2(M, R)$. From this basis of $H^2(M, R)$, we can compute the cup product structure of $H^2(M, R)$. It turns out that the number of positive eigenvalues of the corresponding symmetric bilinear form is three and the index of M is zero. In particular, the Chern number $C_1^2(M) = 0$ and the geometric genus $p_g = 1$[1].

Let us now observe that the surface M is minimal, i.e. M cannot be obtained by blowing up some other surface $\bar{M}$ at some point. In fact, if this were false, $H^1(\bar{M}, R)$ would enjoy the same property as $H^1(M, R)$. The above argument then shows $b_2(\bar{M}) \geq 6$ and $b_2(M) \geq b_2(\bar{M}) + 1 \geq 7$ which is a contradiction.

Now consider ω_1, ω_2, $\bar{\omega}_1$, $\bar{\omega}_2$ as a basis for $H^1(M, C)$. Let $H_1(M, Z)$ be the first homology group module torsion and h_1, h_2, h_3, h_4 a basis for $H_1(M, Z)$. If $x_0 \in M$ is fixed, then for any $x \in M$, the vector

$$\left(\int_{x_0}^{x} \omega_1, \int_{x_0}^{x} \omega_2\right) \in C^2$$

is determined up to an element of the lattice L in C^2 generated by $(\int_{h_i} \omega_1, \int_{h_i} \omega_2)$ for $i = 1, 2, 3, 4$. Hence there exists a holomorphic map (the Albanese) $A: M \to T^2$ where T^2 is the complex torus C^2/L.

The condition $\omega_1 \wedge \omega_2 \wedge \bar{\omega}_1 \wedge \bar{\omega}_2 \neq 0$ assures us that A is non-degenerate in an open dense set of M. The image $A(M)$ is therefore open in T^2. Since M is compact, this implies $A(M) = T^2$.

Let $\sum_i k_i C_i$ be the divisor defined by $\omega_1 \wedge \omega_2$ such that $k_i > 0$ and C_i are irreducible curves. Then $\sum_i k_i C_i$ is also equal to the canonical divisor of M.

If $A(C_i)$ of each curve C_i is a point, we claim that A is a biholomorphic transformation. In view of the minimality of M, we have only to prove that A is injective outside the C_i's, i.e. the general fiber of A is a point. If this were not true, A will map the fundamental group of the complement of the C_i's into a proper subgroup of L. Hence $A_* \pi_1(M)$ is also a proper subgroup $\bar{L}$ of L. Let $\bar{T}^2$ be the complex torus $C^2/\bar{L}$. Then A can be lifted to a holomorphic map into $\bar{T}^2$ and contradicts the universal property of the Albanese.

Finally, suppose $A(C_1)$ is a curve in T^2. We claim that both C_1 and $A(C_1)$ are non-singular elliptic curves. In fact, according to Lemma 6 of [1] (and its proof), we know that the virtual genus of C_1 is not greater than one and that C_1 does not intersect C_i for all $i > 1$. Since C_1 cannot be a rational curve (otherwise $A(C_1)$ is a point in T^2) C_1 is a non-singular elliptic curve and the Hurwitz formula shows that $A(C_1)$ is also a non-singular elliptic curve.

Projecting T^2 along this curve $A(C_1)$, we obtain a holomorphic map E from M onto a non-singular elliptic curve Δ such that $E(C_1) = a$ is a point. Same argument as before shows that the general fiber of E is a non-singular irreducible curve. We assert that it is again a non-singular elliptic curve. In fact, let $N(C_1)$ be a neighborhood of C_1 such that $C_i \cap N(C_1) = \emptyset$ for $i > 1$. By writing down the local coordinates and shrinking $N(C_1)$ more, one can show $[A|N(C_1)]^{-1}(A(C_1)) = C_1$. Hence C_1 is a connected component of $E^{-1}(a)$ and is therefore $E^{-1}(a)$ itself. If C is an elliptic curve which lies in a small neighborhood of $A(C_1)$, then we see that $A^{-1}(C)$ is disjoint from all the C_i's with $i > 1$. Since A restricted to the complement of the C_i's is a covering map, the general fiber of E has genus one and our assertion is proved.

In conclusion, we have proved that M is an elliptic fiber space over a non-singular elliptic curve Δ (cf. [1]). By semicontinuity of the Euler characteristic of the fiber, we see that the singular fibers are sums of elliptic curves and rational curves. However, the condition $C_2(M) = 0$ excludes the latter case and the singular fibers are multiples of elliptic curves only.

Let $\{a_i\}$ be the image of the singular fibers. Suppose the multiplicity of the fiber at a_i is equal to m_i for each i. Then we form a simply connected covering Riemann surface $\tilde{\Delta}$ of Δ which is unramified over $\Delta - \{a_i\}$ and has branch point of order $m_i - 1$ over each point a_i.

Let $\tilde{M}$ be the fiber space of elliptic curves over $\tilde{\Delta}$ which is induced from M by the projection $\tilde{\Delta} \to \Delta$. Then according to Kodaira's classification of singular fibers, $\tilde{M}$ is free from singular fibers and is an unramified covering manifold of M.

If $m_i = 1$ for all i, then by the formula [1] for the canonical line bundle of M, one sees that M has trivial canonical line bundle and is biholomorphic to the complex torus. (One can also see this by noting that A is then an unramified covering.) If $m_i \neq 1$ for some i, then $\tilde{\Delta}$ is biholomorphic to the disk. Since the fibers of $\tilde{M}$ are all biholomorphic to each other, $\tilde{M}$ is biholomorphic to $C_1 \times D$ where C_1 is an elliptic curve and D is the unit disk.

Let Γ be the group of covering transformations of $C_1 \times D$. It is clear that every element of Γ has the form $(x, y) \to (f_1(y)x + f_2(y), g(y))$. Hence Γ acts on D by linear fractional transformations. Since C_1 is compact, this action of Γ is properly discontinuous. Let Γ_1 be a subgroup of finite index of Γ such that the projected action of Γ_1 on D is free. Then $C_1 \times D/\Gamma_1$ is a complex fiber bundle over D/Γ_1 with fiber C_1. This completes the proof of Theorem 1.

Remark. During a conversation with Clifford Earle, we learned that surfaces of the form $C_1 \times D/\Gamma_1$ can actually appear. In fact, let Γ_1 be the Kleinian group with the presentation

$\{A, B, C, D | ABA^{-1}B^{-1}CD = 1, C^3 = D^3 = 1\}$. Then if $\{1, \omega\}$ is the period of C_1, Γ_1 can act on C_1 in the way such that both A and B act trivially on C_1, C acts by the translation $x \to x + \omega/3$ and D acts by the translations $x \to x + 2\omega/3$. By acting Γ_1 suitably on D, one can verify Γ_1 acts freely and properly discontinuously on $C_1 \times D$ such that $C_1 \times D/\Gamma_1$ satisfies the hypothesis of Theorem 1.

§2. PROOF OF THEOREM 2

In this section, we assume that the tangent bundle of the complex surface M is trivial.

The first information we want to draw from this fact is that M is minimal. In fact, if M were obtained by blowing up $\bar{M}$ at some point, then, by deleting a disk from CP^2, we can imbed it as an open subset of M. By hypothesis, this would mean that the tangent bundle of CP^2 is trivial outside a disk, i.e. CP^2 is almost parallelizable. However, the latter fact is not true because the second Stiefel–Whitney class of CP^2 is not zero.

Now the triviality of the tangent bundle also implies that the Euler number $C_2(M) = 0$ and the Pontryagin number $C_1^2(M) - C_2(M) = 0$. If the first Betti number $b_1(M) \neq 1$, then according to the classification[1], either M is a ruled surface of genus one, the complex torus or an elliptic surface.

It remains to discuss the latter case.

Let $\pi: M \to \Delta$ be a holomorphic map whose generic fibers are elliptic curves. Then as $C_2(M) = 0$, the singular fibers are multiples of a connected elliptic curve and any general fiber is biholomorphically equivalent to a fixed elliptic curve E. As in §1, we know that M has unramified covering $\bar{M}$ so that $\bar{M} = E \times \tilde{\Delta}$ where $\tilde{\Delta}$ is a simply connected Riemann surface.

If $\tilde{\Delta}$ is P^1, then clearly all plurigenera of M are zero. (Otherwise we can lift it to $\bar{M}$.) As $\bar{M}$ is algebraic, by an application of Kodaira's embedding theorem, M is also algebraic so that M is a ruled surface of genus one.

If $\tilde{\Delta}$ is the complex line C, then we claim M has zero first Chern class. In fact, since every holomorphic map from E to C is constant, every automorphism of $E \times C$ has the form $(x, y) \to (f(y)x + g(y), ay + b)$ with $f(y)^{12} = 1$. If this automorphism generates a group that acts properly discontinuous, we must have $|a| = |f(y)| = 1$ so that the automorphism preserves the euclidean volume element. Since M is $E \times C$ quotiented by a subgroup of automorphisms that preserve the euclidean volume element, M admits a volume element which looks like the euclidean volume element locally. Therefore, M has zero first Chern class. The classification of such surfaces is due to Kodaira[1].

Finally, we consider the case when $\tilde{\Delta}$ is biholomorphic to the disk. The same argument as in §1 shows that M is covered by an elliptic fiber bundle over a compact curve of genus >1.

Remark. Kodaira proved that if the fundamental group of M contains an infinite cyclic subgroup of finite index and if the second Betti number of M is zero, then M is a Hopf surface.

§3. AN EXAMPLE

Let T^3 be the three (real) dimensional torus and p^3 be the three dimensional real projective space. Then the connected sum $T^3 \# p^3$ and hence its product with the circle is parallelizable. However, since the resulting manifold is not a $K(\pi, 1)$ either Theorem 1 or Theorem 2 implies that it does not admit any complex structure.

Note added in the proof. A. Sommese has recently constructed a non-standard complex structure on T^3.

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