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A new geometric approach to problems in birational geometry

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1. Introduction

In this paper, we initiate a program to study problems in birational geometry. This approach will be more geometric than other more algebraic approaches. Most of the arguments can, however, be phrased in a purely algebraic way. It is quite likely some of them can be applied to deal with the geometry over different ground fields.

Given a projective variety M, we shall study the geometric information provided by the pluricanonical space $H^0(M, mK_M)$. Note that the minimal model program, as led by Mori, Kawamata, Kollár and others, has achieved great success. While the earlier workers had solved the problem for threefolds completely, the spectacular finite generation question was recently solved by several people, using different approaches: the analytic approach due to Siu [14] and the algebraic approach due to Birkar, Cascini, Hacon and M$^{\mathrm{c}}$Kernan [1]. In our approach, instead of using the full canonical ring, we shall focus our study on the pluricanonical space for a fixed m.

Ideally, we would like to determine the birational type of our algebraic variety based on the information on this space only. Any birational transformations of algebraic manifolds will induce a linear map between the corresponding pluricanonical spaces (for each fixed m). The plurigenera are of course invariant under the birational transformations. But more importantly, there are other finer invariants that are preserved by these transformations. The most important ones are the natural normlike functions (called norms in this introduction) induced by integrating over M the m-th root of the product of a m-pluricanonical form and its conjugate. The norm defines an interesting geometry which was not explored extensively before.

In our work, we shall initiate a program to study this geometry. The first major questions we address to are the following ones:

1. Torelli type theorem. Given two algebraic varieties M and M', suppose there is a linear map that defines an isometry (with respect to the norm mentioned above) between the two normed vector spaces $H^0(M, mK_M)$ and $H^0(M', mK_{M'})$. We claim that with a few exceptional cases of M and M', the linear isometry is induced by a birational map between M and M'. This can be considered as a Torelli type theorem in birational geometry.

We call this kind of theorem a Torelli type theorem because the classical Torelli theorem says that the periods of integrals determine an algebraic curve. This remarkable theorem was generalized to higher dimensional algebraic varieties. The most notable one was the work of Piatetsky-Shapiro and Shafarevich [11] for algebraic K3 surfaces, which was generalized to Kähler K3s by Burns-Rapoport [2], where they proved the injectivity of period maps. The surjectivity of period maps for K3 surfaces was done using Ricci flat metrics by Siu [13] and Todorov [16] following the work of Kulikov [9] and of Perrson and Pinkham [10]. This phenomena of surjectivity is known to be rather generic, and in many cases the period map can be proved to have degree one for hypersurfaces (see e.g. Donagi [5]).

2. Existence. Characterize geometrically and algebraically those normed vector spaces that can be realized as the pluricanonical spaces of some algebraic varieties the way above. Hopefully, there may be some effective way to construct the birational models of these varieties.

3. Computation. In the case of the classical Torelli theorems, the periods can be effectively computed by methods dated back to Picard, Leray, Dwork and others. We hope to calculate these normed spaces effectively too. Some differential geometric methods will be brought in.

4. Relations with questions of GIT and other invariants. Making use of the pluricanonical series, we are able to form new invariant (pseudo-)metrics on the algebraic manifolds. There should be some relationship between these metrics and other well known canonical metrics such as Kähler-Einstein metrics. We hope to build up a link between our approach with other metrical approaches to algebraic geometry.

In this paper, we shall prove that when $|mK_M|$ has no base point and defines a birational map, the normed space is indeed powerful enough to determine the birational type of the algebraic varieties. We can achieve this when m is large enough (depending on the dimension of M only). Indeed we prove a Torelli theorem that is described in **1** under rather general assumptions. We should say that in case the manifold is one dimensional and $m = 2$, the problem was treated by Royden in his study of the biholomorphic transformations of Teichmüller space. We think it is possible to generalize Royden's work to higher dimensional manifolds.

We shall study **2** by using a more differential geometric approach. We here outline what kind of metric we can obtain. At every point $\eta_0 \in H^0(M, mK_M)$ and $\eta_1, \eta_2 \in H^0(M, mK_M)$, viewed as two tangent vectors at η_0 in $H^0(M, mK_M)$, we define a hermitian metric

$$h(\eta_1, \eta_2) = \int_M \frac{\eta_1}{\eta_0} \overline{\left(\frac{\eta_2}{\eta_0}\right)} \langle \eta_0 \rangle_m$$

where $\frac{\eta_1}{\eta_0}$ and $\frac{\eta_2}{\eta_0}$ are viewed as meromorphic functions on M and $\langle \eta_0 \rangle_m$ is a real non-negative continuous (n, n)-form as defined in **2.1**. $H^0(M, mK_M)$ is then given with the structure of a hermitian manifold. This hermitian structure is closely related to the norm we mentioned above. Actually the norm function on $H^0(M, mK_M)$ is the Kähler potential of this hermitian metric in a suitable sense. More details will be given in [4].

2. Pseudonorms on $H^0(M, mK_M)$ and their asymptotic properties

2.1. **The pseudonorm $\langle\langle\ \rangle\rangle_m$.** Let M be a complex manifold of dimension n. To every $\eta \in H^0(M, mK_M)$ we can associate a real nonnegative continuous (n, n)-form on M, denoted as $\langle \eta \rangle_m$, as follows:

let $\mathcal{U} = \{(U, (w_U^j = u_U^j + i v_U^j)_{j=1}^n)\}$ be an open cover of M of coordinate charts. If $\eta_{|_U} = \eta_U (dw_U^1 \wedge \cdots \wedge dw_U^n)^{\otimes m}$ with $\eta_U \in \mathcal{O}_M(U)$, we can define on U a real nonnegative continuous (n, n)-form

$$\langle \eta_{|_U} \rangle_m = |\eta_U|^{\frac{2}{m}} du_U^1 \wedge dv_U^1 \cdots \wedge du_U^n \wedge dv_U^n$$

and can verify that $\{\langle \eta_{|U} \rangle_m\}_{U\in\mathcal{U}}$ does give a globally defined form, denoted as $\langle \eta \rangle_m$. It is routine to see that this definition does not depend on the choice of $\mathcal{U}$.

If M is compact, we define

$$\langle\langle \eta \rangle\rangle_m = \int_M \langle \eta \rangle_m$$

and will abbreviate it as $\langle\langle \eta \rangle\rangle$ if m is clear in the context. Therefore, for a compact complex manifold M we have defined a function

$$\langle\langle\ \rangle\rangle : H^0(M, mK_M) \to \mathbf{R}_{\geq 0}$$

and will call it the *pseudonorm* associated to mK_M.

From the fact that $|a+b|^\alpha \leq |a|^\alpha + |b|^\alpha$ for any $0 < \alpha < 1$ and $a, b \in \mathbf{C}$ we can verify the triangle inequality $\langle\langle \eta_1 + \eta_2 \rangle\rangle \leq \langle\langle \eta_1 \rangle\rangle + \langle\langle \eta_2 \rangle\rangle$ for any $\eta_1, \eta_2 \in H^0(M, mK_M)$. From the definition $\langle\langle \eta \rangle\rangle = 0$ if and only if $\eta = 0 \in H^0(M, mK_M)$. However, $\langle\langle c\eta \rangle\rangle = |c|^{\frac{2}{m}} \langle\langle \eta \rangle\rangle$ for $c \in \mathbf{C}$, which shows that $\langle\langle\ \rangle\rangle$ is not a norm if $m \neq 2$.

We define a metric space structure on $H^0(M, mK_M)$ using $\langle\langle\ \rangle\rangle$ by

$$d(\eta_1, \eta_2) = \langle\langle \eta_1 - \eta_2 \rangle\rangle \text{ for any } \eta_1, \eta_2 \in H^0(M, mK_M).$$

$H^0(M, mK_M)$ so metrized will be denoted as $\big(H^0(M, mK_M), \langle\langle\ \rangle\rangle\big)$.

If $\varphi : M' \dashrightarrow M$ is a birational map, then the induced isomorphism $\Phi : \big(H^0(M, mK_M), \langle\langle\ \rangle\rangle\big) \to \big(H^0(M', mK_{M'}), \langle\langle\ \rangle\rangle\big)$ is an isometry.

2.2. **A local asymptotic expansion.** We will state the main local asymptotic result, whose proof can be found in [3], and then deduce from it the global one in the next section, namely the asymptotic property of $\langle\langle\ \rangle\rangle$.

We first settle the notation as follows:

$n, m \in \mathbf{N}$, $m > 2$, $\Delta_0 = \{(z_1, \ldots, z_n) \in \mathbf{C}^n |\ |z_j| < 1, j = 1 \ldots, n\}$,

$\chi(x_1, y_1, \ldots, x_n, y_n) \in \mathcal{C}^\infty(\overline{\Delta}_0)$, $\phi(z_1, \ldots, z_n) \in \mathcal{O}(\overline{\Delta}_0)$,

$A = (a_1, \ldots, a_n)$, $B = (b_1, \ldots, b_n) \in (\mathbf{N} \cup \{0\})^n$,

$l_j = \frac{b_j + 1}{a_j}$ if $a_j \neq 0$ and $= \infty$ otherwise, $j = 1, \ldots, n$.

$l = \min\{l_j |\ j = 1, \ldots, n\}$,

Assume that $l(A,B) = l_1 = \cdots = l_{\mu(A,B)} < l_{\mu(A,B)+1} \leq \cdots \leq l_n$. Notice that $l(A,B)$ and $\mu(A,B)$ only depend on the multi-indices A and B. If A and B are clear in our arguments we will denote $l(A,B)$ and $\mu(A,B)$ by l and μ respectively.

We abbreviate $(x_1, y_1, \ldots, x_n, y_n)$, $(z_1, \ldots, z_n)$, $z_1^{a_1} \ldots z_n^{a_n}$, $|z_1|^{b_1} \ldots |z_n|^{b_n}$, and $dx_1 dy_1 \ldots dx_n dy_n$ as (X, Y), Z, Z^A, $|Z|^B$, and $dX\ dY$ respectively. Let

$$\Psi(t) = \int_{\overline{\Delta}_0} \chi(X,Y) \big|Z^A + t\phi(Z)\big|^{\frac{2}{m}} \big|Z\big|^{2B} dX dY.$$

Theorem 2.2.1.

$$\Psi(t)-\Psi(0)=\begin{cases} O\left(|t|\left(ln\frac{1}{|t|}\right)^{\mu}\right) & if \;\; 2l+\frac{2}{m}\geq 1; \\ c(A,B,\phi)\ |t|^{2l+\frac{2}{m}}\left(ln\frac{1}{|t|}\right)^{\mu-1} \\ +o\left(|t|^{2l+\frac{2}{m}}\left(ln\frac{1}{|t|}\right)^{\mu-1}\right) & if \;\; 2l+\frac{2}{m}<1, \end{cases}$$

where $c(A,B,\phi)$ is a real number depending on ϕ. In the last case we have $c(A,B,\phi)\geq 0$, and

$$c(A,B,\phi)=0 \iff \phi(0,\ldots,0,z_{\mu+1},\ldots,z_n)\equiv 0.$$

Remark 2.2.2. *More specific information when $2l+\frac{2}{m}\geq 1$ can be given. We actually can show that when $k<2l+\frac{2}{m}<k+1$ where $k\in\mathbf{N}$, $\Psi^{(j)}(0)$ exists for $j\leq k$. In addition, we get in this case an asymptotic expansion for $\Psi(t)-\sum_{j=0}^{k}\frac{\Psi^{j}(0)}{j!}t^j$. We will not need these in this paper so we omit them here. For more detail see* [3].

In **2.4** we will apply this result to obtain the main global result. Let $\eta_0,\eta\in H^0(M,mK_M)$. We hope to describe the asymptotic behavior of $\langle\langle\eta_0+t\eta\rangle\rangle$ as $t\to 0$.

2.3. **The characteristic index and indicatrix.** Before getting into the deduction of the global asymptotic expansion, we introduce several quantities measuring how singular a divisor is at a point in the ambient space.

Let M be a smooth variety, D a nonzero effective divisor on M. (In **2.4** D will be chosen to be $\{\eta_0=0\}$ for the $\eta_0\in H^0(M,mK_M)$ we consider.) We first choose a log resolution $\pi:\widetilde{M}\to M$ for the pair (M,D) and write

$$\pi^*D=\sum_E a_E E \quad \text{and} \quad K_{\widetilde{M}}=\pi^*K_M+\sum_E b_E E,$$

where E runs over all irreducible subvarieties of $\widetilde{M}$ of codimension 1.

Definition 2.3.1. (1) *For every $x\in M$ the log canonical threshold of D at x, denoted as* $\mathrm{lct}(D,x)$, *is given by*

$$\mathrm{lct}(D,x)=\min_{\{E|x\in\pi(E)\}}\frac{b_E+1}{a_E}.$$

We also have the global log canonical threshold of D,

$$\mathrm{lct}(D)=\min_E\ \frac{b_E+1}{a_E}.$$

It is clear that $\mathrm{lct}(D)=\min_{x\in M}\ \mathrm{lct}(D,x)$.

(2) *For every $x\in M$ the log canonical multiplicity of D at x, denoted as $\mu(D,x)$, is given by*

$$\mu(D,x)=\max\left\{ q \;\middle|\; \begin{array}{l} \textit{There exist distinct irreducible divisors } E_1,\ldots,E_q \textit{ in } \widetilde{M} \\ \textit{such that } \frac{b_{E_j}+1}{a_{E_j}}=\mathrm{lct}(D,x) \textit{ for all } j \textit{ and } x\in\pi(\cap E_j). \end{array}\right\}.$$

(3) *The characteristic index of* D *at* x *is the pair* $(\mathrm{lct}(D,x),\mu(D,x))$. *Consider the following total order:*

$$(l_1,\mu_1)>(l_2,\mu_2)\Longleftrightarrow\begin{cases} l_1=l_2 \text{ and } \mu_1>\mu_2\\ \text{or}\\ l_1<l_2\end{cases}$$

The global characteristic index of D*, denoted as* $\big(\mathrm{lct}(D),\mu(D)\big)$*, is given by*

$$\big(\mathrm{lct}(D),\mu(D)\big)=\sup_{p\in M}\big(\mathrm{lct}(D,p),\mu(D,p)\big).$$

(4) *We define the characteristic indicatrix of* (M,D)*, denoted as* $C(D)$*, to be the set of points achieving global characteristic index, i.e.*

$$C(D)=\left\{x\in M \,\middle|\, \big(\mathrm{lct}(D,x),\mu(D,x)\big)=\big(\mathrm{lct}(D),\mu(D)\big)\right\}.$$

Notice that in general $C(D)$ *is different from the minimal log canonical centers of* D.

The total order defined here is adapted to the comparison of the asymptotic order of functions of the form $|t|^l\big(ln\frac{1}{|t|}\big)^\mu$. We have

$$|t|^{l'}\left(ln\frac{1}{|t|}\right)^{\mu'}=o\left(|t|^l\big(ln\frac{1}{|t|}\big)^{\mu}\right) \text{ if } (l,\mu)>(l',\mu'). \tag{1}$$

Let $\mathcal{E}$ be the set of all irreducible divisors E such that $\frac{b_E+1}{a_E}=\mathrm{lct}(D)$,

$$\widetilde{M}_{D,r}=\bigsqcup_{E_1,\dots,E_r:\text{ distinct in }\mathcal{E}} E_1\cap\cdots\cap E_r,$$

and $\iota_r:\widetilde{M}_{D,r}\to\widetilde{M}$ the canonical morphisms induced by inclusions. We have

$$\mu(D)=\sup\{r|\ \widetilde{M}_{D,r}\neq\phi\} \text{ and } C(D)=\pi\iota_{\mu(D)}(\widetilde{M}_{D,\mu(D)}). \tag{2}$$

The log canonical thresholds is well defined, namely it is independent of the choice of log resolutions. In fact

$$\mathrm{lct}(D,x)=\inf\{c>0|\mathcal{J}(M,cD)_x\neq\mathcal{O}_{M,x}\},$$

and the multiplier ideal sheaves $\mathcal{J}(M,cD)$ do not depend on the log resolution we choose.

This also gives the basic inequality

$$\mathrm{lct}(D,x)\le\frac{n}{\mathrm{mult}_x D} \tag{3}$$

(by taking a blow-up $Bl_x(M)\to M$ followed by a log resolution).

Remark 2.3.1. *In the rest of the paper we do not need* $\mu(D,x)$ *and* $C(D)$ *to be independent of the choice of log resolution. For each divisor* D *we can simply choose a fixed resolution to define* $\mu(D,x)$ *and* $C(D)$*. However, they can indeed be defined in terms of some resolution free ideal sheaves, hence are both independent of the choice of resolution (see* [3]*). Instead of giving a formal proof of this independence here, we would like to point out that one can see this, at least analytically, by using Theorem 2.2.1 for the case* $\chi=\phi\equiv 1$ *and that pulling back a differential form by an analytic modification does not change its integral.*

Remark 2.3.2. *The characteristic index is a finer measurement of singularity than* lct *is. For example,* lct *alone can not tell between a reduced non-smooth s.n.c. divisor and a smooth divisor. Higher characteristic indices correspond to worse singularities. In this sense, the characteristic indicatrix $C(D)$ is the set of points at which the pair (M,D) is the most singular.*

2.4. **The asymptotic property of $\langle\langle\ \rangle\rangle_m$.** In this subsection we assume M to be compact. We first give the local setting. As in **2.1**, let $\mathcal{U}=\{(U,(w_U^j)_{j=1}^n)\}$ be a finite open cover of coordinate charts on M. We choose a log resolution $\pi:\widetilde{M}\to M$ for $(M,D_{\eta_0}=\{\eta_0=0\})$ and a finite refinement $\mathcal{V}=\{(V,Z_V=X_V+iY_V)\}$ of $\pi^{-1}\mathcal{U}=\{\pi^{-1}U\}$ formed by charts in $\widetilde{M}$, where Z_V and (X_V,Y_V) abbreviate $(z_V^1,\ldots,z_V^n)$ and $(x_V^1,y_V^1\ldots,x_V^n,y_V^n)$ respectively. Let $\tau:\mathcal{V}\to\mathcal{U}$ be such that $\pi(V)\subset\tau(V)$. Finally, we choose a partition of unity $\{\chi_V(X_V,Y_V)\}$ subordinate to $\mathcal{V}$. $\mathcal{V}$ and $\{\chi_V\}$ can be so chosen that

(i) the image of $Z_V:V\to\mathbf{C}^n$ is $\Delta_0=\{(z_1,\ldots,z_n)\in\mathbf{C}^n|\ |z_j|<1 \text{ for all } j\}$;

(ii) if $U=\tau(V)$, then

$$\pi^*(dw_U^1\wedge\cdots\wedge dw_U^n)=\big(j_V(Z_V)\big)Z_V^{B_V}\ dz_V^1\wedge\cdots\wedge dz_V^n$$

for some nonvanishing $j_V\in\mathcal{O}(\overline{\Delta}_0)$ (hence all its derivatives are bounded) and multi-index $B_V=(b_V^1,\ldots,b_V^n)\in(\mathbf{N}\cup 0)^n$;

(iii) following the notation in (ii), we have

$$\pi^*\eta_0=c_V(Z_V)\big(j_V(Z_V)\big)^m Z_V^{A_V+mB_V}(dz_V^1\wedge\cdots\wedge dz_V^n)^{\otimes m}$$

and

$$\pi^*\eta=c_V(Z_V)\big(j_V(Z_V)\big)^m\phi_V(Z_V)Z_V^{mB_V}(dz_V^1\wedge\cdots\wedge dz_V^n)^{\otimes m}$$

where ϕ_V and $c_V\in\mathcal{O}(\overline{\Delta}_0)$, c_V is nonvanishing, and $A_V=(a_V^1,\ldots,a_V^n)\in(\mathbf{N}\cup 0)^n$;

(iv) for each V we have $l_V^1=\cdots=l_V^{\mu_V}<l_V^{\mu_V+1}\leq\cdots\leq l_V^n$, where $l_V^j=\frac{b_V^j+1}{a_V^j}$.

(v) $\chi_V(0,0)\neq 0$ for every V.

Remark 2.4.1. *In (ii) and (iii) and in the following proof of Theorem 2.4.2 we will have to consider two different kinds of pullbacks via $\pi:\widetilde{M}\to M$ of elements in $H^0(M,K_M)$, and it is important not to mix them up. The first one is $\pi^*:H^0(M,mK_M)\to H^0(\widetilde{M},mK_{\widetilde{M}})$ which acts on K_M as the usual pullback of differential forms via the map π. The second one is $\pi^{**}:H^0(M,mK_M)\to H^0\big(\widetilde{M},\pi^*(mK_M)\big)$, the usual pullback map from the sections of a vector bundle to those of its pullback bundle via a map.*

In terms of the $\mathcal{V}$ and χ_V chosen above we can write

$$\langle\langle\eta_0+t\eta\rangle\rangle=\sum_{V\in\mathcal{V}}\int_{\Delta_0}\big(\chi_V|c_V|^{\frac{2}{m}}|j_V|^2\big)\big|Z_V^{A_V}+t\phi_V\big|^{\frac{2}{m}}\big|Z_V\big|^{2B_V}dX_VdY_V.$$

Our main asymptotic result for $\langle\langle\ \rangle\rangle_m$ is the following

Theorem 2.4.2. *Given $\eta_0, \eta \in H^0(M, mK_M)$, let $(l, \mu) = \big(\mathrm{lct}(D_{\eta_0}), \mu(D_{\eta_0})\big)$ and $C(D_{\eta_0})$ be defined as in* **2.3**. *We have*

$$\langle\langle \eta_0 + t\eta \rangle\rangle - \langle\langle \eta_0 \rangle\rangle = \begin{cases} O\left(|t|\left(ln\frac{1}{|t|}\right)^{\mu}\right) & \text{if } \ 2l + \frac{2}{m} \geq 1; \\ c(\eta_0, \eta)\ |t|^{2l+\frac{2}{m}}\left(ln\frac{1}{|t|}\right)^{\mu-1} \\ +o\left(|t|^{2l+\frac{2}{m}}\left(ln\frac{1}{|t|}\right)^{\mu-1}\right) & \text{if } \ 2l + \frac{2}{m} < 1, \end{cases}$$

where $c(\eta_0, \eta)$ is a real number depending on η_0 and η. In the last case we have $c(\eta_0, \eta) \geq 0$, and

$$c(\eta_0, \eta) = 0 \iff \eta \text{ vanishes on } C(D_{\eta_0}).$$

Proof. Following the notation at the beginning of **2.2**, for each $V \in \mathcal{V}$ we obtain correspondingly a pair (l_V, μ_V). It is clear that $(l, \mu) = \sup_V (l_V, \mu_V)$ according to the total order we introduced in **2.3**(3). For each V, applying Theorem 2.2.1 to the case $\chi = \chi_V |c_V|^{\frac{2}{m}} |j_V|^2$, $\phi = \phi_V$, $A = A_V$ and $B = B_V$, and then summing up the corresponding asymptotic expansions, we obtain the expected expansion.

For the statement about $c(\eta_0, \eta)$, notice that, by (2.1), only those V with $(l_V, \mu_V) = (l, \mu)$ will contribute to $c(\eta_0, \eta)$. More precisely,

$$c(\eta_0, \eta) = \sum_{\{V | (l_V, \mu_V) = (l, \mu)\}} c(A_V, B_V, \phi_V).$$

By Theorem 2.2.1 we know that $c(\eta_0, \eta) \geq 0$ and

$$\begin{aligned} & c(\eta_0, \eta) = 0 \\ \Leftrightarrow\ & c(A_V, B_V, \phi_V) = 0 \text{ for all } V \\ & \text{such that } (l_V, \mu_V) = (l, \mu) \\ \Leftrightarrow\ & \phi_V(0, \ldots, 0, z_V^{\mu+1}, \ldots, z_V^n) \equiv 0 \\ & \text{for all } V \text{ such that } (l_V, \mu_V) = (l, \mu). \end{aligned}$$

We know that $\iota_\mu(\widetilde{M}_{D,\mu})$ (see (2.2)) is defined by $z_V^1 = \cdots = z_V^\mu = 0$ in every such V. Regarding the conditions (ii) and (iii) above satisfied by the $\mathcal{V}$ we choose, the last statement is equivalent to saying that $p^{**}\eta$ vanishes on $\iota_\mu(\widetilde{M}_{D,\mu})$. This is the same as saying that η vanishes on $\pi\iota_\mu(\widetilde{M}_{D,\mu}) = C(D_{\eta_0})$. □

3. Identifying the Images of Rational Maps $\varphi_{|mK_M|}$

We still assume M to be compact. In this section we are going to use Theorem 2.4.2 to study the image of the rational map $\varphi = \varphi_{|mK_M|}$ associated to the linear system $|mK_M|$.

Let $B = Bs|mK_M|$. First we recall the definition of φ. It is given by

$$\begin{aligned} \varphi: \quad M \quad & \dashrightarrow \quad \mathbb{P}H^0(M, mK_M)^* \\ x \quad & \longmapsto \quad \begin{array}{c} \left\{\eta \in H^0(M, mK_M) \;\middle|\; \eta(x) = 0\right\} \text{ viewed} \\ \text{as a hyperplane of } H^0(M, mK_M). \end{array} \end{aligned}$$

Notice that φ is defined only for $x \in M - B$. (Otherwise $\{\eta|\ \eta(x) = 0\} = H^0(M, mK_M)$ is not a hyperplane.) In general, for any hyperplane H in $H^0(M, mK_M)$ we have

$$H \overset{\alpha}{\subseteq} \{\eta|\ \eta_{|_{Bs|H|}} \equiv 0\} = \{\eta|\ \eta_{|_{Bs|H|-B}} \equiv 0\} \overset{\beta}{\subseteq} H^0(M, mK_M)$$

and $Bs|H| - B = \varphi^{-1}(H)$. Therefore

$$\begin{aligned} &H \text{ is in the image of } \varphi \Longleftrightarrow Bs|H| - B \neq \phi \Longleftrightarrow \beta \text{ is } \subsetneq \\ &\Longleftrightarrow \alpha \text{ is an equality } \Longleftrightarrow H = \{\eta|\ \eta_{|_{Bs|H|-B}} \equiv 0\}. \end{aligned} \tag{4}$$

Question: Given H in the image of φ, can we characterize H by a subset of the hyperplane in $H^0(M, mK_M)$ it represents and metrical properties of $\langle\langle\ \rangle\rangle$?

Idea: If we can find $\eta_0 \in H^0(M, mK_M)$ such that $2\text{lct}(D_{\eta_0}) + \frac{2}{m} < 1$ and $\phi \neq C(D_{\eta_0}) - B \subseteq\ Bs|H| - B$, then

$$H \overset{(3.1)}{=} \{\eta|\ \eta_{|_{Bs|H|-B}} \equiv 0\} \subseteq \{\eta|\ \eta_{|_{C(D_{\eta_0})-B}} \equiv 0\}$$

$$\subsetneq H^0(M, mK_M) \text{ since } C(D_{\eta_0}) - B \neq \phi,$$

hence the $\subseteq$ above is actually an equality, and by Theorem 2.4.2

$$H = \{\eta|\ \eta_{|_{C(D_{\eta_0})-B}} \equiv 0\} = \{\eta|\ \eta_{|_{C(D_{\eta_0})}} \equiv 0\} = \{\eta|\ c(\eta_0, \eta) = 0\}.$$

We know that $c(\cdot, \cdot)$ can be read off from $\langle\langle\ \rangle\rangle$.

Definition 3.1. *We say that property* (CS) *(standing for "concentrating singularities") holds for mK_M if for a generic H in the image of φ there exists $\eta_0 \in H^0(M, mK_M)$ such that* $2\text{lct}(D_{\eta_0}) + \frac{2}{m} < 1$ *and* $\phi \neq C(D_{\eta_0}) - B \subseteq Bs|H| - B$.

The following is the main ingredient in using metrical properties of pseudonorms to identify images of rational maps of the form we consider above.

Lemma 3.1. *Let M, M' be compact complex manifolds. If* (CS) *holds for both mK_M and $mK_{M'}$ and*

$$\iota : \left(H^0(M, mK_M), \langle\langle\ \rangle\rangle_m\right) \to \left(H^0(M', mK_{M'}), \langle\langle\ \rangle\rangle_m\right)$$

is a linear isometry, then the isomorphism induced by ι,

$$I : \mathbb{P}H^0(M, mK_M)^* \to \mathbb{P}H^0(M', mK_{M'})^*,$$

maps the closure of the image of $\varphi_{|mK_M|}$ isomorphically onto that of $\varphi_{|mK_{M'}|}$.

Proof. By symmetry, it suffices to prove that I maps a generic point in the image of $\varphi_{|mK_M|}$ into that of $\varphi_{|mK_{M'}|}$.

By (CS), for a generic H in the image of φ we select a section $\eta_0 \in H^0(M, mK_M)$ such that $2\text{lct}(D_{\eta_0}) + \frac{2}{m} < 1$ and $\phi \neq C(D_{\eta_0}) - B = Bs|H| - B$. We already know from **Idea** that $H = \left\{\eta \in H^0(M, mK_M) \middle|\ c(\eta_0, \eta) = 0\right\}$.

By the definition of I and the fact that ι is a linear isometry,

$$\begin{aligned} I(H) = &\ \left\{\iota(\eta) \in H^0(M', mK_{M'}) \middle|\ c(\eta_0, \eta) = 0\right\} \\ = &\ \left\{\iota(\eta) \in H^0(M', mK_{M'}) \middle|\ c(\iota\eta_0, \iota\eta) = 0\right\} \\ = &\ \left\{\eta' \in H^0(M', mK_{M'}) \middle|\ c(\iota\eta_0, \eta') = 0\right\}. \end{aligned}$$

By the first $\Longleftrightarrow$ in (3.1), showing that $I(H)$ is in the image of $\varphi' = \varphi_{|mK_{M'}|}$ is equivalent to showing that $Bs|I(H)| - B' \neq \phi$, where $B' = Bs|mK_{M'}|$.

By Theorem 2.4.2, $C(D_{\iota\eta_0}) \subseteq Bs|I(H)|$, hence it suffices to prove $C(D_{\iota\eta_0}) \nsubseteq B'$. Assume this to be false, i.e. $C(D_{\iota\eta_0}) \subseteq B'$. Since ι is an isometry, $\iota\eta_0$ has the same asymptotic behavior as that of η_0, hence $2\mathrm{lct}(D_{\iota\eta_0}) + \frac{2}{m} = 2\mathrm{lct}(D_{\eta_0}) + \frac{2}{m} < 1$. Theorem 2.4.2 then implies that $\left\{\eta' \middle| \, c(\iota\eta_0, \eta') = 0\right\} = H^0(M', mK_{M'})$, which is also $I(H)$ as shown in last paragraph, a contradiction. $\square$

A more general image identifying result using the pseudonorms can be found in [3] and [4].

Remark 3.1. *The more detailed asymptotic expansions which are mentioned in Remark 2.2.2 actually allow us to remove the condition* $2\mathrm{lct}(D_{\eta_0}) + \frac{2}{m} < 1$ *in the definition of* (CS). *This is useful in getting better uniform bounds for the results in* **4**. *See* [3].

4. Birational Equivalence between Smooth Varieties of General Type

In this section M will be a smooth compact complex manifold such that the rational map $\varphi_{|mK_M|}$ maps M to its image birationally for sufficiently large m. We want to know for which $r \in \mathbf{N}$ (CS) holds for rK_M.

In case rK_M maps M birationally to its image, the condition (CS) admits an equivalent statement in terms of points in M instead of those in the image. It is clear that in this case (CS) can be restated in the following way:

(CS) For a generic point x in M there exists $\eta_0 \in H^0(M, rK_M)$ such that

$$(\mathrm{lct}(D_{\eta_0}, x), \mu(\eta_0, x)) > (\mathrm{lct}(D_{\eta_0}, y), \mu(\eta_0, y))$$

for any $y \neq x$ and $(2\mathrm{lct}(D_{\eta_0}) + \frac{2}{m} =)\ 2\mathrm{lct}(D_{\eta_0}, x) + \frac{2}{m} < 1$.

Definition 4.1. (*i*) *For any* $x \in M$ *we define*

$$V(r,x) = \left\{\eta \in H^0(M, rK_M) \middle| \ \mathrm{mult}_x \eta \geq \frac{2nr}{r-2}\right\}.$$

(*ii*)

$$\mathcal{S}_M = \left\{ 3 \leq r \in \mathbf{N} \,\middle|\, \begin{array}{l} (i)\ Bs|rK_M| = \phi, \text{ and} \\ (ii)\ \text{for a generic } x \in M\ Bs|V(r,x)| = \{x\}. \end{array} \right\}$$

Remark 4.1. (*ii*) *in particular implies that* $\varphi = \varphi_{|rK_M|}$ *maps* M *birationally to its image. Indeed for a generic* $x \in M$

$$\varphi^{-1}\varphi(x) = \varphi^{-1}(\{\eta \in H^0(M, rK_M) | \eta(x) = 0\})$$

$$= Bs\big|\{\eta \in H^0(M, rK_M) | \eta(x) = 0\}\big| \subseteq Bs|V(r,x)| = \{x\}.$$

Lemma 4.1. *If* $r \in \mathcal{S}_M$ *then* (CS) *holds for* rK_M.

Proof. (*ii*) and Bertini's theorem imply that for a generic $x \in M$ there is $\eta_0 \in H^0(M, rK_M)$ such that $\mathrm{mult}_x \eta_0 \geq \frac{2nr}{r-2}$ and $\mathrm{mult}_y \eta_0 \leq 1$ for $y \neq x$. This shows that $\mathrm{lct}(D_{\eta_0}, y) = 1$ or ∞ and by (2.3) that $\mathrm{lct}(D_{\eta_0}, x) \leq \frac{n}{\mathrm{mult}_x \eta_0} < \frac{r-1}{2r} = \frac{1}{2} - \frac{1}{r} < \frac{1}{2}$. It is clear from Definition 2.3.1(3) that

$$(\mathrm{lct}(D_{\eta_0}, x), \mu(\eta_0, x)) > (\mathrm{lct}(D_{\eta_0}, y), \mu(\eta_0, y)).$$

Besides, $2\text{lct}(D_{\eta_0}, x) + \frac{2}{r} < \frac{2n(r-2)}{2nr} + \frac{2}{r} = 1.$ □

Lemma 4.2. *$\mathcal{S}_M$ is a semigroup, i.e. $(r_1 + r_2) \in \mathcal{S}_M$ if r_1, $r_2 \in \mathcal{S}_M$.*

Proof. Condition (i) obviously holds for $(r_1 + r_2)$ if it does for r_1 and r_2. As for condition (ii), for x in some Zariski open subset $U \subseteq M$ we have $Bs|V(r_1, x)| = Bs|V(r_2, x)| = \{x\}$ since r_1 and $r_2 \in \mathcal{S}_M$. We want to show that for $x \in U$, $y \notin Bs|V(r_1 + r_2, x)|$ if $y \neq x$. By Bertini's theorem we can find $\eta_j \in V(r_j, x)$ such that $\eta_j(y) \neq 0$ for $j = 1, 2$. Let $\eta = \eta_1 \otimes \eta_2 \in H^0(M, (r_1 + r_2)K_M)$. We have

$$\text{mult}_x\eta = \text{mult}_x\eta_1 + \text{mult}_x\eta_2 > \frac{2nr_1}{r_1 - 2} + \frac{2nr_2}{r_2 - 2} > \frac{2n(r_1 + r_2)}{r_1 + r_2 - 2}$$

by the fact that $\frac{x+y}{x+y-2} < \frac{x}{x-2} + \frac{y}{y-2}$ if x, $y \geq 3$. Therefore $\eta \in V(r_1 + r_2, x)$ and $\eta(y) \neq 0$. So $y \notin Bs|V(r_1 + r_2, x)|$. □

Lemma 4.3. *Suppose $Bs|mK_M| = \phi$ and $\varphi_{|mK_M|}$ maps M onto its image in $\mathbb{P}H^0(M, mK_M)^*$ birationally. Then $\nu m \in \mathcal{S}_M$ for any integer $\nu \geq 2n + 1$.*

Proof. Condition (i) in the definition of $\mathcal{S}_M$ obviously holds. Only (ii) needs to be verified.

Since $\varphi_{|mK_M|}$ maps M to its image birationally, we can find Zariski open subsets U_0 and U of M and the image of $\varphi_{|mK_M|}$ respectively such that $\varphi_{|mK_M|}$: $U_0 = \varphi^{-1}_{|mK_M|}(U) \xrightarrow{\sim} U$.

We want to show that $y \notin Bs|V(\nu m, x)|$ if $y \neq x$ (i.e. (ii)) for $x \in U_0$. By the choice of U_0 it is clear that $Bs\big|\{\eta \in H^0(M, mK_M)|\ \eta(x) = 0\}\big| = \{x\}$. Therefore, for any $y \neq x$ there exists $\eta \in H^0(M, mK_M)$ such that $\eta(x) = 0$ and $\eta(y) \neq 0$. Taking $\eta_0 = \eta^{\otimes\nu} \in H^0(M, \nu mK_M)$, we have $\eta \in V(\nu m, x)$ since

$$\text{mult}_x\eta_0 = \nu\text{mult}_x\eta \geq \nu > \frac{2n\nu m}{\nu m - 2}$$

when $\nu > 2n + 1$. So $y \notin Bs|V(\nu m, x)|$. □

Lemma 4.4. *Let M be a nonsingular complex projective variety of general type and of dimension n. Let $d \in \mathbf{N}$ be such that $Bs|mdK_M| = \phi$ for $m \geq m_0$. Then there exists $r_0 \in \mathbf{N}$ depending only on m_0 and n such that $rd \in \mathcal{S}_M$ if $r \geq r_0$.*

Proof. It is proved in [6] and [15] that for each $n \in \mathbf{N}$ there exists $m_n \in \mathbf{N}$ such that if M is a smooth projective variety of general type and of dimension n then the rational map $\varphi_{|mK_M|}$ maps M to its image birationally for any $m \geq m_n$.

Choose distinct prime numbers m, m', ν and ν' such that $m, m' \geq \max\{\frac{m_n}{d}, m_0\}$, ν and $\nu' > 2n+1$. Then Lemma 4.3 implies that $m\nu d$ and $m'\nu' d \in \mathcal{S}_M$ and Lemma 4.2 implies the lemma. □

Our main theorem is the following

Theorem 4.1. *Let M and M' be smooth complex projective varieties of general type and of dimension n and $d \in \mathbf{N}$ such that $Bs|mdK_M| = Bs|mdK_{M'}| = \phi$ for $m \geq m_0$. Let $r_0 \in \mathbf{N}$ as given by Lemma 4.4.*

If for some $r \geq r_0$ we have a linear isometry

$$\iota : \big(H^0(M, rdK_M), \langle\langle\ \rangle\rangle\big) \to \big(H^0(M', rdK_{M'}), \langle\langle\ \rangle\rangle\big)$$

then there exists a unique birational map $\psi : M \dashrightarrow M'$ and $c \in \mathbf{C}$ with $|c| = 1$ such that $c\iota = \psi^*$, the isomorphism induced by ψ.

Proof. Lemma 4.1 and Lemma 4.4 together imply the (CS) holds for ρdK_M and $\rho dK_{M'}$ if $\rho \geq r_0$. By Lemma 4.4 and Remark 4.2 $\varphi_{|rdK_M|}$ and $\varphi_{|rdK_{M'}|}$ map M and M' birationally to their images respectively. Denote the isomorphism induced by ι as

$$I : \mathbb{P}H^0(M, rdK_M)^* \to \mathbb{P}H^0(M', rdK_{M'})^*.$$

The assumption and Lemma 3.1 implies that I identifies the images of $\varphi_{|rdK_M|}$ and $\varphi_{|rdK_M|}$. Therefore we obtain a unique birational map ψ making the following diagram of rational maps commutative:

$$\begin{array}{ccc} M & \overset{\psi}{\dashrightarrow} & M' \\ {\scriptstyle \phi_{|rdK_M|}}\downarrow & & \downarrow{\scriptstyle \phi_{|rdK_{M'}|}} \\ \mathbb{P}H^0(M, rdK_M)^* & \xrightarrow{I} & \mathbb{P}H^0(M', rdK_{M'})^* \end{array}$$

Let $\psi^* : H^0(M, rdK_M) \to H^0(M', rdK_{M'})$ be the isomorphism induced by ψ. It is an isometry with respect to $\langle\langle\ \rangle\rangle_{rd}$. Since ψ^* and ι both induce $I : \mathbb{P}H^0(M, rdK_M)^* \to \mathbb{P}H^0(M', rdK_{M'})^*$, there is $c \in \mathbf{C}$ such that $c\iota = \psi^*$. Both ι and ψ^* are isometries with respect to those $\langle\langle\ \rangle\rangle$s, hence $|c| = 1$. □

Using this theorem we can obtain several uniform results. For example, in the case $n = 2$, we can even have r_0 depending only on n. The reason is that it is enough to prove the theorem for M and M' both minimal models. By the classical results due to Bombieri and Kodaira $Bs|mK_M| = Bs|mK_{M'}| = \phi$ if $m \geq 5$. The proof of Lemma 4.4 shows that $\mathcal{S}$, the additive semigroup of $\mathbf{N}$ generated by $\{ab \mid a, b \in \mathbf{N}, a \geq 5, b \geq 6\}$, is contained in $\mathcal{S}_M$. It is not hard to see that $m \in \mathcal{S}$ for any $m \geq 75$, and hence r_0 can be chosen to be 75. Then we can take $d = 1$ and $m_0 = 5$ in Theorem 4.1 and get the following

Theorem 4.2. *Given a linear isometry*

$$\iota : \big(H^0(mK_M), \langle\langle\ \rangle\rangle\big) \to \big(H^0(mK_{M'}), \langle\langle\ \rangle\rangle\big)$$

for some $m \geq 75$, there exists a unique pair of a birational map $\psi : M' \dashrightarrow M$ and a complex number c of unit length such that ψ^*, the isomorphism induced by ψ, is equal to $c\iota$.

For higher dimensions, in the same spirit we obtain the following

Theorem 4.3. *There exists $r_0 \in \mathbf{N}$ which depends on n, such that for any two smooth complex projective varieties M and M' of general type and of dimension n which both admit smooth minimal models, if for some $r \geq r_0$ we have a linear isometry*

$$\iota : \big(H^0(M, 2r(n+2)!K_M), \langle\langle\ \rangle\rangle\big) \to \big(H^0(M', 2r(n+2)!K_{M'}), \langle\langle\ \rangle\rangle\big)$$

then there exists a unique birational map $\psi : M \dashrightarrow M'$ and a unique complex number c of unit length such that ψ^*, the isomorphism induced by ψ, is equal to $c\iota$
.

Proof. As remarked in the paragraph before Theorem 4.2 we may assume that M and M' are both minimal models, i.e. K_M and $K_{M'}$ are both nef.

Kollár's effective base freeness theorem ([8], **1.1 Theorem**) says that if a log pair (X, Δ) is proper and klt of dimension n, L a nef Cartier divisor on X, and $a \in \mathbf{N}$ such that $aL - (K_X + \Delta)$ is nef and big, then $|2(n+2)!(a+n)L|$ is base point free. Applying this to the case $X = M$ (resp. M'), $\Delta = 0$, $L = K_M$ (resp. $K_{M'}$) and $a \geq 2$ we have that $Bs|2m(n+2)!K_M| = Bs|2m(n+2)!K_{M'}| = \phi$ if $m \geq n+2$.

Therefore we may take $d = 2(n+2)!$ and $m_0 = n+2$ in Lemma 4.4 and Theorem 4.1, and then the theorem follows. □

Remark 4.2. *It is shown in* [1] *that every variety of general type admits a minimal model. However in the proof above the smoothness of the minimal models are required.*

Here, in order to illustrate how the main idea goes, we only deal with the case when suitable base point free conditions hold. The presence of base loci is another technical issue. By a careful analysis and modification of the results in **2**, a suitable use of the effective base point freeness, and the existence of minimal models for varieties of general type, we are still able to say something for the general case. The following theorems 4.4 and 4.5 are the precise results whose proofs can be found in [3] and [4].

We first recall some facts about the minimal models. It is known that every projective manifold X of general type admits a minimal model Y with K_Y **Q**-Cartier[1]. The index of Y is defined as $j_Y = \min\{j \mid jK_Y \text{ is Cartier}\}$. It is also known that any two birational minimal models have the same index. Hence we can define the index of a projective manifold to be that of any of its minimal models. We have the following

Theorem 4.4. ([3] *and* [4]) *For every natural number j there exists $r_{n,j}$ which depends only on n and j such that given any two n-dimensional projective manifolds M and M' with indices j, and a linear isometry*

$$\iota : \big(H^0(M, 2r(n+2)!K_M), \langle\langle\ \rangle\rangle\big) \to \big(H^0(M', 2r(n+2)!K_{M'}), \langle\langle\ \rangle\rangle\big)$$

for some $r \geq r_{n,j}$, there exists a unique birational map $\psi : M' \dashrightarrow M$ and a unique complex number c of unit length such that the induced map ψ^* is equal to $c\iota$.

The number $r_{n,j}$ in this theorem depends not only on the dimension n but also on the index of minimal models. To get a uniform result in higher dimensional cases, we need to introduce some objects here. Let

$$V(M, m, r) = \text{image}\big(\text{Sym}^r H^0(M, mK_M) \to H^0(M, rmK_M)\big)$$

for any $m, r \in \mathbf{N}$, where the map is the canonical one. $V(M, m, r)$ inherits from $\big(H^0(M, rmK_M), \langle\langle\ \rangle\rangle_{rm}\big)$ a pseudonorm, still denoted as $\langle\langle\ \rangle\rangle_{rm}$. It is clear that $\big(V(M, m, r), \langle\langle\ \rangle\rangle_{rm}\big)$ is a birational invariant.

Recall also the definition of m_n in the proof of Lemma 4.4, which is a number such that $\phi_{|mK_M|}$ maps M birationally to its image for every $m \geq m_n$. With these notions, we can also prove the following result :

Theorem 4.5. ([3] *and* [4]) *Given a linear isometry*

$$\iota : \big(V(M, m, r), \langle\langle\ \rangle\rangle\big) \to \big(V(M', m, r), \langle\langle\ \rangle\rangle\big)$$

for some $r \geq 2n+1$ and $m \geq m_n$, there exists a unique birational map $\psi : M' \dashrightarrow M$ and a unique complex number c of unit length such that the induced map ψ^* is equal to $c\iota$.

Remark 4.3. *Many of the results in this paper have a more general version for $L + mK_M$ where L is a hermitian line bundle (see* [3]*).*

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