Topology Vol. 31, No. 2, pp. 305-316, 1992.
Printed in Great Britain

0040-9383/92 $5.00 + 0.00

THE TOPOLOGICAL UNIQUENESS OF COMPLETE MINIMAL SURFACES OF FINITE TOPOLOGICAL TYPE

WILLIAM H. MEEKS III† and SHING-TUNG YAU‡

(Received 15 November 1989; in revised form 5 December 1990)

1. INTRODUCTION

IN 1970 Lawson [22] proved that two embedded closed diffeomorphic minimal surfaces in the unit three-dimensional sphere S^3 in $\mathbb{R}^4$ are ambiently isotopic in S^3. Lawson proved this theorem by first proving that an embedded orientable closed minimal surface of genus g in a closed orientable Riemannian three-manifold M^3 with positive Ricci curvature disconnects M^3 into two genus-g handlebodies. A result of Frankel [7] was used to prove this. Lawson then applied a deep result of Waldhausen [33] that states that decompositions of S^3 into two genus-g handlebodies are unique up to ambient isotopy. More precisely, Waldhausen's uniqueness theorem states that whenever a closed surface of genus g in S^3 separates S^3 into handlebodies, then the embedding of the surface is as simple as possible; in other words, the surface is obtained from a two-sphere $S^2 \subset S^3$ by adding handles in an unknotted manner.

Meeks [23] generalized Lawson's argument to the case of orientable closed minimal surfaces in a closed M^3 with nonnegative Ricci curvature. Meek's result and another topological uniqueness result of Waldhausen [33] implies that two closed diffeomorphic minimal surfaces in $S^2 \times S^1$ with the usual product metric are ambiently isotopic. Finally, Meeks–Simon–Yau [25] proved that if Σ is a closed minimal surface in S^3 equipped with a metric of non-negative scalar curvature, then Σ disconnects S^3 into two handlebodies and hence determines a unique ambient class in S^3.

In [23] Meeks considered the problem of the topological uniqueness of minimal surfaces in three-dimensional Euclidean space. In particular, Meeks proved that any two compact diffeomorphic minimal surfaces, with boundary a simple closed curve on the boundary of a smooth convex ball, are ambiently isotopic in the ball. Later Hall [13] showed that there exist two simple closed curves on the unit sphere S^2 that bound a knotted minimal surface of genus one.

In this paper we will use global properties of stable minimal surfaces in $\mathbb{R}^3$ to prove: *A properly embedded minimal surface of finite topological type in* $\mathbb{R}^3$ *is unknotted.* In particular, if Σ_1 and Σ_2 are two such diffeomorphic surfaces, then they are ambiently isotopic. This is the main result of our paper and it is restated in a more precise form in Theorem 5.1. Because of Hall's counterexample to unknottedness for compact minimal

†The research described in this paper was supported by research grant DE-FG02-86ER250125 of the Applied Mathematical Science subprogram of Office of Energy Research, U.S. Department of Energy, and National Science Foundation grant DMS-8611574.
‡Supported in part by the National Science Foundation grant DMS-8711394.

surfaces discussed in the previous paragraph, the proof of Theorem 5.1 depends in an essential way on the completeness of the properly embedded minimal surface in $\mathbb{R}^3$.

The proof of our topological uniqueness theorem further clarifies the geometric description of the ends of a general properly embedded minimal surface Σ in $\mathbb{R}^3$, even when Σ has an infinite number of ends. This geometric description of the ends of a properly embedded minimal surface has been exploited by Choi, Meeks and White [3] to prove that if a properly embedded minimal surface has more than one end, then every intrinsic isometry of the surface extends to an ambient isometry. Hoffman and Meeks [17] have applied our techniques to prove that a properly embedded minimal surface of finite topological type has at most two ends with infinite total curvature. This result implies that the conformal structure of properly embedded minimal surfaces of finite topological type with more than two ends is restricted.

In our proofs we frequently exploit the property that each annular end of a complete embedded minimal surface of finite total curvature is asymptotic to a plane or to a half-catenoid. This property can be proved with the Weierstrass Representation of a minimal surface [31].

Hoffman and Meeks [16, 18] and Callahan, Hoffman and Meeks [1] have constructed properly embedded minimal surfaces in $\mathbb{R}^3$ of finite topological type with every possible genus. All these new examples have either three or four ends. Computer graphics images of these surfaces illustrate that the surfaces have a rather complex geometric appearance. However, our main theorem demonstrates that these surfaces, after composing with a diffeomorphism of $\mathbb{R}^3$, have the simplest possible geometric appearance (see Fig. 1).

Recently Frohman and Meeks [11, 12] have proven a related topological uniqueness theorem for certain properly embedded minimal surfaces with possibly infinite genus (also see [9]). Their Theorem 1.1 states that a properly embedded minimal surface with one end in $\mathbb{R}^3$ is topologically unknotted. Their proof of this result is based in part on some of the technical results in Sections 2 and 3 of our paper. A recent result of Callahan, Hoffman and Meeks (Corollary 2 in [2]) states that every properly embedded nonplanar doubly-periodic minimal surface in $\mathbb{R}^3$ has one end, and hence, it is unknotted by the Frohman–Meeks theorem. See Conjecture 1.2 in [10] for a possible natural necessary and sufficient condition for two properly embedded minimal surfaces in $\mathbb{R}^3$ to be ambiently isotopic.

2. STABLE MINIMAL SURFACES WITH BOUNDARY

In this section our aim is to describe the global geometry of certain minimally immersed surfaces in $\mathbb{R}^3$ that are complete and have compact boundary. It is well-known that a complete minimal surface of finite total curvature in $\mathbb{R}^3$ is conformally diffeomorphic to a closed Riemann surface punctured in a finite number of points (see [20] or [29]). If M has compact boundary and it is a complete minimal surface of finite total curvature, then it remains true that the ends of M are conformally diffeomorphic to punctured disks. In the case where an end of such an M is embedded, then it is asymptotic to a catenoid or to a flat plane in $\mathbb{R}^3$ [31]. The next theorem gives a sufficient condition for a complete minimal surface with compact boundary in $\mathbb{R}^3$ to have finite total curvature.

THEOREM 2.1. *If $f\colon M \to \mathbb{R}^3$ is a complete minimally immersed orientable surface with compact boundary is stable, then M has finite total curvature.*

Proof. This theorem follows immediately from results of Fischer-Colbrie in [5]. In fact, the statements and proofs of Theorem 1 and Corollary 1 in [5] also prove that if M is a complete stable minimal surface with compact boundary in a complete orientable N^3 with nonnegative scalar curvature, then M has finite total curvature. □

3. THE ENDS OF A PROPERLY EMBEDDED MINIMAL SURFACE

The ends of a surface correspond to the different ways to travel to infinity on the surface. We will call a connected proper subsurface E of M an *end* of M if E has compact boundary but E is not compact. A surface is said to have *more than one end* if it contains at least two pairwise disjoint ends. It is straightforward to check that a surface M has more than one end if and only if there is a simple closed curve on M that separates M into two components, each of whose closure is noncompact. The following theorem is the first step in our analysis of the geometry of the ends of a properly embedded minimal surface.

THEOREM 3.1. *Suppose that M is a properly embedded minimal surface in $\mathbb{R}^3$ and suppose δ is a smooth simple closed curve on M that separates M into two components, each of whose closure is noncompact. Let N_1 and N_2 denote the closures of the two components of $\mathbb{R}^3 - M$. Then:*

1. *Either δ is not homologous to zero in N_1 or δ is not homologous to zero in N_2;*
2. *If δ is not homologous to zero in N_1 (resp. N_2), then δ is the boundary of a smooth properly embedded, noncompact, orientable, minimal surface Σ of finite total curvature in N_1 (resp. N_2). Furthermore, Σ is an area-minimizing surface in N_1 (resp. N_2) in the sense that compact subsurfaces of Σ have least area with respect to their boundaries in the region N_1 (resp. N_2).*

Proof. If the curve δ is homologous to zero in N_1, then δ is the boundary of an embedded compact surface K_1 in N_1 such that $K_1 \cap M = \delta$. Similarly, if δ is homologous to zero in N_2, then δ is the boundary of an embedded surface K_2 in N_2 with $K_2 \cap M = \delta$. Thus, if δ is homologous to zero in both N_1 and N_2, then there is a closed surface $K_1 \cup K_2$ that intersects M in the curve δ. Furthermore, one of the components of $M - \delta$ must be contained in the compact region R of $\mathbb{R}^3$ with boundary surface $K_1 \cup K_2$. However, since M is proper, the component of $M - \delta$ contained in R must have compact closure, which contradicts our hypotheses that δ separates M into two noncompact components. Thus, δ can not be homologous to zero in both N_1 and N_2, which proves part 1 of Theorem 3.1. Suppose now that δ is not homologous to zero in N_1.

Let M_+ be the closure of one of the components of $M - \delta$. Let $M_1 \subset \cdots \subset M_k \subset \cdots$ be an exhaustion of M_+ by smooth compact subdomains of M_+ and such that $\delta \subset \partial M_1$. Since ∂N_1 has nonnegative mean curvature, ∂M_i is the boundary of a least-area rectifiable current Δ_i in N_i (see Theorem 1 in [28] and also see [32]). By interior regularity theorems of area-minimizing currents, $\Delta_i - \delta$ is a smooth embedded surface. Since $\Delta_i \cup M_i$ disconnects $\mathbb{R}^3$, the surface Δ_i is orientable and the boundary regularity theorem in [14] shows that Δ_i is a smooth compact least-area minimal surface in N_1 with boundary ∂M_i. The usual compactness theorems for least-area surfaces show that a subsequence $\{\Delta_{i_j}\}$ converges smoothly on compact subsets of $\mathbb{R}^3$ to a properly embedded, stable, minimal surface Σ with $\delta\Sigma = \delta$ (see [32] or the proof of Lemma 3 in [24] for explicit details on this compact property for $\{\Delta_i\}$ in a similar context). Since δ is not homologous to zero in N_1, the surface Σ is noncompact. Since either $\Sigma \cup M_+$ or $\Sigma \cup (M - M_+)$ is a properly embedded surface in $\mathbb{R}^3$, Σ is orientable. Hence, Σ has finite total curvature by Theorem 2.1.

For the sake of completeness we prove the well-known property that any compact smooth subdomain $\hat{\Sigma}$ of Σ has least-area in N_1 (with respect to surfaces in N_1 with boundary equal to $\partial\hat{\Sigma}$). If this condition fails, then we may assume that $\hat{\Sigma} \subset \Sigma$ and there exists a compact surface $\tilde{\Sigma} \subset N_1$ with $\partial\tilde{\Sigma} = \partial\hat{\Sigma}$ and a positive number ε such that $Area(\hat{\Sigma}) = Area(\tilde{\Sigma}) + \varepsilon$. However, $\hat{\Sigma}$ is the smooth limit of least-area surfaces. This means that we can approximate $\hat{\Sigma}$ by a least-area surface $\hat{\Sigma}'$ in N_1 with $Area(\hat{\Sigma}) \le Area(\hat{\Sigma}') + \varepsilon/2$ and such that the boundary curves $\hat{\Sigma}'$ and $\hat{\Sigma}$ form the boundary of surface $\tilde{A}$ of area less than $\varepsilon/2$. However, the surface $\tilde{A} \cup \tilde{\Sigma}$ has the same boundary as $\hat{\Sigma}'$ but has area strictly less than the area of $\hat{\Sigma}'$. This contradicts the supposed least-area property of $\hat{\Sigma}'$ and proves that Σ satisfies the least-area property described in the statement of the theorem. This completes the proof of Theorem 3.1. □

PROPOSITION 3.1. *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$. Suppose that δ is a simple closed curve on M such that δ separates M into two unbounded components E_1 and E_2 where E_1 has one end. If δ is not homologous to zero in the closure N of one of the components of $\mathbb{R}^3 - M$, then δ is the boundary of a stable, orientable, properly embedded, minimal surface Σ in N that has finite total curvature and at most two ends.*

Proof. Suppose N the closure of a component of $\mathbb{R}^3 - M$ where δ is not homologous to zero. Recall the part of the proof of Theorem 3.1 where, from an exhaustion of the end E_1, one produces a least-area orientable surface Σ that is the limit of compact least-area surfaces Σ_i in N with boundary curves consisting of δ and some other curves on the end E_1. We will prove that Σ has at most two ends. Suppose now that Σ has more than two ends and we shall derive a contradition.

Since E_1 has exactly one end, the maximum principle for minimal surfaces implies that the interior of Σ is disjoint from E_1; otherwise, Σ is equal to E_1, which has one end. Hence, $\Sigma \cap E_1 = \delta$. Let W be the region of N with boundary $\Sigma \cup E_1$. Since Σ is an embedded minimal surface with finite total curvature, there exists a positive number R_o such that for $R > R_o$, Σ intersects the sphere $S_R = \{x \in \mathbb{R}^3 \mid |x| = R\}$ in k "parallel" simple closed curves that are almost geodesics on S_R, where k is the number of ends of Σ (see [21]). What this means geometrically is that under the homothety $h(x) \mapsto x/R$, the set $h(S_R \cap \Sigma)$ is a collection of k simple closed curves that are C^2-close to a common great circle in the unit sphere.

Choose an $R > R_o$ such that S_R is transverse to E_1 and such that δ is contained in the ball B_R with boundary S_R. Since E_1 has exactly one end and it is properly embedded in $\mathbb{R}^3$, E_1 can only intersect one of the components of $\mathbb{R}^3 - (\mathrm{Int}(B_R) \cup \Sigma)$ in a noncompact component. Actually we claim that this noncompact component is the only component. Since $\partial E_1 \subset \mathrm{Int}(B_R)$, we conclude that if E_1 intersects some component of $\mathbb{R}^3 - (\mathrm{Int}(B_R) \cup \Sigma)$ in a component that is a compact surface, then the boundary of this compact surface is contained in S_R. The existence of such a compact minimal surface contradicts the convex hull property for minimal surfaces. Hence E_1 can only intersect exactly one component, say C, of $\mathbb{R}^3 - (\mathrm{Int}(B_R) \cup \Sigma)$.

Recall that $\Sigma \neq E_1$ and so for large i the least-area surfaces Σ_i that limit to be Σ intersect E_1 only along $\partial\Sigma_i$. We now prove that the surfaces Σ_i must be contained in the region W. Consider the compact region R_i of $\mathbb{R}^3$ with boundary $M_i \cup \Sigma_i$ where M_i is defined in the proof of Theorem 3.1. Let W' be the closure of $N - W$. If Σ_i is not contained in W, then the region $R_i \cap W'$ gives a homology in N between $\Sigma_i \cap W'$ and a least-area domain $\Sigma_i' \subset \Sigma$. Replacing $\Sigma_i \cap W'$ on Σ_i by Σ_i' yields another least-area surface but this surface is not smooth along $\partial\Sigma_i'$. This contradicts the interior regularity of least-area surfaces and proves $\Sigma_i \subset W$.

Since for large i, $\partial\Sigma_i$ consists of the curve δ and curves contained in $E_1 \cap C$, the convex hull property for minimal surfaces implies that $\Sigma_i \cap (\mathbb{R}^3 - (\text{Int}(B_R) \cup \Sigma)) = \Sigma_i \cap C$. Since the Σ_i limit to be Σ, it follows that $\Sigma \cap (\mathbb{R}^3 - \text{Int}(B_R)) \subset \partial C$. Hence, Σ has either one or two ends, since ∂C has one or two components. □

We now consider the case when the end E has finite topology.

THEOREM 3.2. *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ with more than one end. If E is a smooth annular end of M, then ∂E is the boundary of a stable minimal annulus A of finite total curvature that is contained in the closure of one of the components of $\mathbb{R}^3 - M$.*

Proof. Let $\Sigma \subset N$ be the least-area minimal surface whose existence is guaranteed by Theorem 3.1 and Proposition 3.1 and where N is the closure of one of the components of $\mathbb{R}^3 - M$ where ∂E is not homologous to zero.

Case 1. Σ has one end.

The surface Σ separates N into two components. Let W denote the closure of the component that contains E in its boundary. Let B_R be a ball centered at the origin of radius R where R is chosen large enough so that:

1. $\partial B_r \cap \Sigma$ is a simple closed curve α_r for $r \geqslant R$;
2. $\partial E \subset \text{Int}(B_R)$;
3. ∂B_R is transverse to E.

From the simple topology of the annulus and the maximum principle, one easily checks that $\partial B_R \cap E$ contains a unique component β that is homotopically nontrivial in E. Let C be the component of $\partial B_R \cap W$ that contains β. Notice also that all of the boundary curves of $\partial B_R \cap W$ bound disks in ∂W except the curves α_R and β. Then since neither of the cycles α_R, β is homologous to zero in W, the curve α_R must also be contained in ∂C. It follows that α_R is homotopic to β in W and hence α_R is homotopic to ∂E. By the Geometric Dehn Lemma in [27] and [28], $\partial E \cup \alpha_r$ is the boundary of a least-area embedded minimal annulus A_r in W for $r \geqslant R$. Since the area of $\Sigma \cap B_R$ grows like πR^2, the monotonicity formula implies that a sequence $\{A_{r_i}\}$ have uniformly bounded area in B_R for any R as $r_i \to \infty$. Furthermore, the stable annuli $\{A_{r_i}\}$ have bounded curvature away from ∂E by the curvature estimates of Schoen [30]. Hence the usual compactness theorem shows that the A_{r_i} converge smoothly on compact subsets of $\mathbb{R}^3$, away from ∂E, to a properly embedded stable minimal surface M. Furthermore, the usual argument of lifting curves on M to nearby approximating surfaces shows that M is an annulus. Since the area of A_r is bounded in a neighborhood of ∂E, the proof of Hildebrandt's boundary regularity theorem [15] implies that M is smooth up to and including its boundary.

Case 2. Σ has two ends.

Recall, from the proof of Proposition 3.1, that this case can only occur when the end E is asymptotically contained between the two ends of Σ. The ends of Σ are asymptotic to planes or catenoids whose normals at infinity are parallel. Suppose, after a possible rotation, that the normal vectors to the ends of Σ are vertical. It is then clear that every vertical cone of the form $z^2 = c(x^2 + y^2)$ intersects E in a compact set. Lemma 4 in [17] states that if an annular end of a properly embedded minimal surface intersects a sufficiently flat cone in a compact set, then it must have finite total curvature, Thus, E has finite total curvature and hence has area approximately πr^2 in balls B_r radius r. As in the proof of Case 1, this gives an estimate for the area of a least-area annulus A_r in N with boundary ∂E and a curve in $E \cap \partial B_r$, r large. As before, this estimate leads to the existence of the stable minimal annulus. In fact this estimate also shows that if the least-area surface Σ is obtained by the procedure of Theorem 3.2, then $\Sigma \cap B_r$ has area growth πr^2, which implies Σ has one end. □

4. THE ENDS OF A PROPER MINIMAL SURFACE ARE TOPOLOGICALLY PARALLEL

The first step in the proof of the topological uniqueness theorem for a minimal surface M of finite type in $\mathbb{R}^3$ is to show that there is only one possibility for the placement of M near infinity. The correct notion is captured in the next definition.

Definition. 4.1. A properly embedded surface M of finite type in $\mathbb{R}^3$ will be said to have *topologically parallel ends* if there is a diffeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the intersection of the complement C of the open unit ball in $\mathbb{R}^3$ with $f(M)$ is the intersection of a collection of parallel planes with C.

THEOREM 4.1. *Suppose M is a properly embedded minimal surface in $\mathbb{R}^3$ that has finite topological type. Then the ends of M are topologically parallel.*

Proof. If M has finite total curvature, then the theorem follows directly from the description of M in Theorem 1 in [21]. Originally we found a direct but rather complicated argument to prove Theorem 4.1 when M did not have finite total curvature. These complications can now be avoided by using the Annular End Theorem of Hoffman and Meeks [17]. Their theorem states that M can have at most two annular ends with infinite total curvature and, furthermore, their theorem shows that the ends of M can be well-ordered in the following sense. First suppose that M has exactly two annular ends F_1, F_2 of infinite total curvature and ends $E_1, \ldots, E_n$ of finite total curvature. For large R, if B_R is the ball of radius R, then $\partial B_R \cap (\cup_i E_i)$ is a collection of n simple closed curves that are almost geodesics on ∂B_R, i.e., under the homothety $\mathbf{x} \mapsto \mathbf{x}/R$, these n curves become C^2-close to a common great circle in the unit sphere. Then it follows directly from the proof of the Annular End Theorem that $F_1 \cap \partial B_R$ is contained in one of the disk components, D_1, of $\partial B_R - \cup_i E_i$ and $F_2 \cap \partial B_R$ is contained in the other disk component, D_2, of $\partial B_R - \cup_i E_i$.

The simple topology of annulus and the maximum principle easily imply that $F_i \cap \partial B_R$ contains exactly one curve γ_i that is homotopically nontrivial on F_i, $i = 1, 2$. The Jordan curve theorem shows that the curves $\Gamma = \{\gamma_1, \gamma_2, \partial B_R \cap E_1, \ldots, \partial B_R \cap E_n\}$ disconnect ∂B_R into components consisting of annuli and disks D'_1, D'_2 where $\partial D'_j = \gamma_j, j = 1, 2$. The End Uniqueness Theorem (Theorem 4 in [23]) states that there is a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ that is the identity on Γ, such that $h(M) \cap (\mathbb{R}^3 - \mathrm{Int}(B_R))$ is a cone over Γ. Since Γ consists of topologically parallel curves on ∂B_R, there is a diffeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that the intersection of the complement C of the unit ball with $f(M)$ is equal to the intersection of C with a collection of parallel planes.

The case where M has exactly one end F_1 of infinite total curvature is proved in a similar manner. □

Remark 4.1. Frohman and Meeks recently proved an important generalization of Theorem 4.1 by proving that the ends of an arbitrary properly embedded minimal surface M in $\mathbb{R}^3$ with more than one end can be "ordered" by their height over "the limit tangent plane at infinity" of M. See [10] and [12] for details.

5. THE TOPOLOGICAL UNIQUENESS THEOREM

We now state our main theorem.

THEOREM 5.1. *A properly embedded minimal surface in $\mathbb{R}^3$ that is homeomorphic to a closed surface of genus g with k points removed is ambiently isotopic to the surface obtained by taking the connected sum of k parallel planes along unknotted arcs in the slabs between the planes and then taking the connected sum of this surface with a standardly embedded closed surface of genus g in the standard way.* (See Fig. 1).

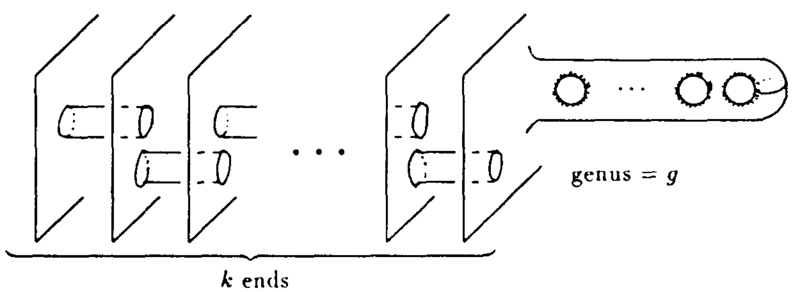

Fig. 1.

Proof. If M has one end, then the theorem was proved in [23]. Assume now that k is greater than one.

Before proceeding with the proof of Theorem 5.1, we give an outline of our approach to the proof. From the statement of Theorem 4.1 we already know that M has a standard appearance in the complement of some ball. In fact, Theorem 4.1 shows that there exists an exhaustion of $\mathbb{R}^3$ by balls, not necessarily round,

$$B_1 \subset B_2 \subset \ldots \subset B_n \subset \ldots \subset \mathbb{R}^3,$$

such that $B_n \cap M = M_n$ is diffeomorphic to M, $M \cap (\mathbb{R}^3 - B_n)$ consists of topologically parallel annuli and the boundary curves of M_i approach infinity as n approaches infinity.

It remains to prove that M_1 is standardly embedded in B_1. Since care must be taken in this step of the proof since P. Hall has proven that there exist knotted minimal surfaces of genus one with two boundary curves on the unit sphere [13]. Hall's example demonstrates that the completeness of the metric on M must be used to show that M_1 is standardly embedded.

One technique that we employ is to minimize area in the ambient isotopy class of M_i rel(∂M_i) in the closure of a fixed component of $\mathbb{R}^3 - M$. By this process we create a stable minimal surface $\tilde{M}_i$ with boundary ∂M_i approaching infinity and we show that $\tilde{M}_i$ must escape the ball B_1 for large values i or $\tilde{M}_i$ converges to a flat plane that intersects B_1 and this plane is disjoint from M. This second possibility is impossible by a result of Meeks and Hoffman that states that a nonplanar minimal surface is not contained in a halfspace of $\mathbb{R}^3$ [19]. During this minimization process, handles on M_i may collapse so that the topology of the stable minimal surface $\tilde{M}_i$ is in general different from the topology of M_i.

The fact that $\tilde{M}_i$ escapes B_1 for large i is then used to get some additional information. This additional information is that there exist certain compressing minimal disks on the closures of the two components of $B_1 - M_1$. (A *compressing disk* is an embedded minimal disk in the closure C of a component of $B_1 - M_1$ and the boundary of this disk is a homotopically nontrivial curve in ∂C.) By cutting along these compressing disks we then reduce the question of uniqueness of M_1 to the question of uniqueness when M_1 has one boundary curve. This case of one boundary curve was proved earlier by Meeks [23]. This completes the outline of the proof.

We first define the required sequence of balls $B_1 \subset B_2 \subset \ldots \subset \mathbb{R}^3$. (For further discussion on the construction of these balls see the proof of Theorem 4 in [23].) Let $E = \{E_1, \ldots, E_k\}$ denote a collection of pairwise disjoint annular end representatives for M. First choose a sequence of round balls $\tilde{B}_1 \subset \tilde{B}_2 \subset \ldots \subset \mathbb{R}^3$ centered at the origin such that $\cup_i \tilde{B}_i = \mathbb{R}^3$, $\partial \tilde{B}_i$ is transverse to M, and $\partial E \subset \tilde{B}_1$. We may choose E so that $\partial E \subset \partial \tilde{B}_1$ and a neighborhood of ∂E is disjoint from $\mathrm{Int}(\tilde{B}_1)$. It was shown in [23] that for $i > 1$, $E \cap \tilde{B}_i$ consists of k compact annuli, one for each end, and a collection of disk components whose union we denote by D_i. Let $\hat{B}_i$ denote the closure of the component of $\tilde{B}_i - D_i$ that contains ∂E.

The balls $\hat{B}_i$ are piecewise smooth balls whose boundaries consist of convex spherical surfaces and minimal disks. By the approximation procedure in the proof of Theorem 1 in [28], we can modify $\hat{B}_i$ in a small neighborhood of $D_i \cap \hat{B}_i$ to obtain for $i > 1$ new smooth balls B_i satisfying:

1. ∂B_i has nonnegative mean curvature;
2. $\partial B_i \cap M = \partial B_i \cap E$ and consists of the k simple closed curves $\partial B_i \cap E_1, \ldots, \partial B_i \cap E_k$;
3. $\mathrm{Int}(B_i) \cap M = M_i$ is diffeomorphic to M;
4. $M - M_i$ consists of k annular end representatives of M.

By Theorem 4.1, we may also assume that the ends E are indexed so that there exists a diffeomorphism $f\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $f(B_1)$ is the ball $\mathscr{B}$ centered at the origin of radius 1 and $f(E_i) = \{(x, y, z) \in (\mathbb{R}^3 - \mathscr{B}) \mid x = -1/i + 1\}$.

Let Y be the closure of the component of $\mathbb{R}^3 - M$ that contains the point $f^{-1}((-1, 0, 0))$ and let W be the closure of the other component. Let $Y_i = Y \cap B_i$, $W_i = W \cap B_i$ and $\alpha_{ij} = E_i \cap \partial B_j$ except that α_{i1} will be denoted by α_i. Let $A_i \subset \partial B_1$ be the annulus with $\partial A_i = \alpha_i \cup \alpha_{i+1}$ (see Fig. 2).

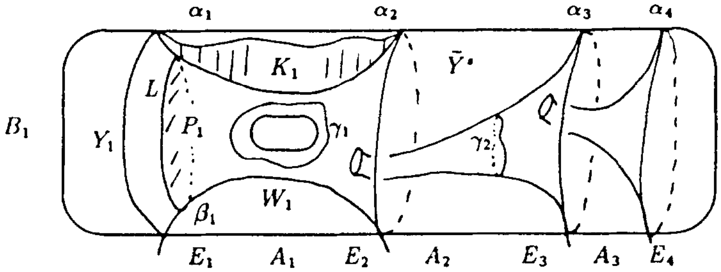

Fig. 2. Picture of M_1 in a convex ball B_1

Let $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ be a collection of pairwise disjoint simple closed curves that generate the kernel of the map $\pi_1(M_1) \to \pi_1(W_1)$ induced by inclusion. The existence of the collection Δ follows from the loop theorem [26]. The Geometric Dehn's Lemma [28] implies that the curves in Δ form the boundary of a collection of pairwise disjoint minimal disks $\{D_1, \ldots, D_n\}$ in W_1 and such that $D_j \cap \partial W_1 = \gamma_j$. Let W^s denote the geodesic closure of $W - \cup_j D_j$ and let M^s denote the surface in W^s that corresponds to the component of

M surgered along $\cup_j D_j$ and that contains the curve α_1. (By geodesic closure of a path connected subset of $\mathbb{R}^3$, we mean the completion of the subset as a metric space in the metric induced by taking the infimum of lengths of curves joining two points in the subset.) Let W_i^s denote the geodesic closure of $W_i - \cup_i D_i$ and similarly define $M_i^s = M^s \cap W_i^s$. It follows that M^s is incompressible in W^s and M_i^s is incompressible in W_i^s. Since W_i^s satisfies the barrier boundary condition in [28] (in other words, ∂W_i^s can be approximated by a smooth surface with nonnegative mean curvature), there exists an embedded minimal surface $\tilde{M}_i$ in W_i^s such that $\partial \tilde{M}_i = \partial M_i^s$, $\tilde{M}_i$ is isotopic to M_i^s relative to ∂M_i^s, and $\tilde{M}_i$ has least area in this isotopy class [25]. Now consider $\tilde{M}_i$ to lie in W_i.

ASSERTION 5.1. *For j sufficiently large, $\partial B_i \cap \tilde{M}_j = \varnothing$.*

Proof. We will show that $\partial B_1 \cap \tilde{M}_j = \varnothing$ for large j. (This argument would also show that $\partial B_i \cap \tilde{M}_j = \varnothing$ for fixed i and large j.) Suppose that $\partial B_1 \cap \tilde{M}_{j_k}$ is nonempty for a subsequence $j_1 < j_2 < \ldots$. Then since the surfaces $\tilde{M}_j$ are stable and the boundaries escape to infinity, a subsequence of the $\{\tilde{M}_{j_1}, \ldots, \tilde{M}_{j_k}, \ldots\}$ converges to a stable orientable surface, one of whose components is a stable surface X that intersects ∂B_1. (See the proof of Proposition 3.1 for this convergence argument.) Since X is a stable complete orientable minimal surface, the surface X contains a component that is a flat plane in W^s ([4] or [6]). However, if X contains a flat plane disjoint from M^s, the surface M lies in a halfspace, which is impossible by the Halfspace Theorem [19]. This contradiction proves the assertion. □

ASSERTION 5.2. *The component of M_1^s that contains α_1 is an annulus in W_1^s and it is topologically parallel to $A_1 = (\partial W_1^s) - M_1^s$ inside W_1^s.*

Proof. Suppose for the moment that M_1^s is an annulus. Then ∂W_1^s is a torus. Propositions 1 and 2 in [23] would then imply W_1^s is a solid torus since ∂W_1^s has nonnegative mean curvature. Since $\alpha_1 \subset \partial M_1^s$ is homotopically nontrivial in W_1^s and α_1 bounds a disk in the closure of the complement of W^s, in fact in ∂B_1, elementary knot theory shows that M_1^s is topologically parallel to A_1 in W_1^s. Thus, it remains to prove that the component T of M^s that contains α_1 is an annulus. Assertion 5.1 implies that for large values of i, $T_i = T \cap W_i^s$ is isotopic rel(∂T_i) to a minimal surface C_i that is disjoint from A_1. Since C_i is contained in a component U of $W^s - W_1^s$ and U has fundamental group isomorphic to $\mathbb{Z}$, C_i can only be incompressible if $\pi_1(C_1) = \mathbb{Z}$. Hence, T_1 must be an annulus. The assertion now follows from these observations. □

It follows from Assertion 5.2 that there is an embedded arc on $M_1 - (\cup_j D_j \cup \mathrm{Int}(A_1))$ and an embedded arc on A_1 such that the union of these two arcs is a simple closed homotopically nontrivial curve δ_1 on ∂W_1 and δ_1 is the boundary of an embedded disk K_1 in W_1. Choose β_1 to be a curve on M_1 such that β_1 is parallel to α_1 and such that the annulus L bounded by β_1 and α_1 is disjoint from $\cup_j D_j$ and β_1 intersects δ_1 transversely in a single point. It follows that β_1 is the boundary of a disk P_1 in Y and we can choose P_1, as well as K_1, to be least area by the Geometric Dehn's Lemma [28]. Let $\tilde{Y}^s$ denote the geodesic closure of the component of $Y - P_1$ that contains the curve α_2 (see Fig. 2). If k equals 2, then skip the following construction.

Assume now that k is greater than 2. Let $\tilde{M}^s = (M \cap \tilde{Y}^s) \cup P_1$. Let $\tilde{M}_i^s$ and $\tilde{Y}_i^s$ be the associated subsets obtained by intersecting with B_i. The kernel of the inclusion $\pi_1(\tilde{M}_1^s) \to \pi_1(\tilde{Y}_1^s)$ is generated by a collection $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ of pairwise disjoint simple closed curves that are disjoint from P_1. (Note we are using the same notation $\Delta = \{\gamma_1, \ldots, \gamma_n\}$ for a similar collection of curves defined earlier in the proof and these curves are not the curves labeled in Fig. 2.). Let $\{D_1, \ldots, D_n\}$ be a collection of pairwise disjoint minimal disks in $\tilde{Y}_1^s$ where $\partial D_i = \gamma_i$ for each integer i. (Note that this notation was also used earlier in the proof to denote a similar collection of disks.) Define Y^s to be the geodesic closure of the component of $\tilde{Y}^s - \cup_j D_j$ that contains the curve α_2. Let M^s be the component in ∂Y^s corresponding to $\tilde{M}^s$ surgered along $\cup_j D_j$ and that contains the curve α_2. Let M_i^s denote the associated compact subdomains of M^s obtained by intersecting with B_i and similarly define Y_i^s. Then the proofs of Assertion 5.1 and 5.2 work in the new case under consideration to show that M_1^s is parallel in Y_1^s to the annulus A_2 on ∂B_1 with boundary curves α_1 and α_2. It is now straightforward to construct simple closed curves δ_2 and β_2 on ∂Y_1^s that satisfy the following properties:

1. δ_2 is the union of an arc on $\tilde{M}_1^s - P_1$ and an arc on A_2;
2. δ_2 is the boundary of an embedded minimal disk K_2 in $\tilde{Y}_1^s$;
3. $\beta_2 \cup \alpha_2$ bounds an annulus L_2 on $\tilde{M}_1^s$ that is disjoint from $\cup_j D_j$. In fact, β_2 can be chosen to be one of the boundary curves of a small annular neighborhood L_2 of $\alpha_2 \cup (\delta_1 \cap \tilde{M}_1^s)$;
4. $(\beta_1 \cup \delta_1) \cap (\beta_2 \cup \delta_2) = \varnothing$;
5. β_2 intersects δ_2 transversely in a single point.

ASSERTION 5.3. *The curve β_2 bounds an embedded minimal disk P_2 in W that is disjoint from the least-area disk K_1 in W (where $\partial K_1 = \delta_1$).*

Proof. Let C be the closure of the component of $B_1 - \tilde{M}^s$ that contains the curve α_1. Since $\beta_2 \cup \alpha_2$ bounds an annulus on $\tilde{M}^s$ that is contained in $C \cap \tilde{M}^s$ and since α_2 bounds a disk component of $\partial B_1 \cap C$, the curve β_2 is the boundary of a disk T in C. Let N_ε be a small regular ε-neighborhood of $K_1 \cup L$ in C where L is the annulus on M with boundary $\beta_1 \cup \alpha_1$. Choose ε small enough so that the boundary of N_ε is disjoint from β_2. By construction of δ_1 and β_1, the boundary of N_ε in the closure of $C - N_\varepsilon$ consists of a single disk component Q that is contained in W_1. Since T and Q are embedded disjoint disks, the usual disk replacement argument, replacing some components of $T - Q$ by components of $Q - T$ and moving slightly, we may assume that T is chosen so that Q is disjoint from T and hence, T is disjoint from $L \cup P_1$. Hence, $T \subset (C - (L \cup P_1))$, which implies that β_2 is the boundary of an embedded disk in W_1. Let P_2 be a least-area disk in W_1 with boundary β_2. By construction, P_2 is a minimal disk in W_1 with boundary β_1 and P_2 is disjoint from the least-area disk K_1 by the disjointness property of least-area minimal disks (see [27]). The assertion is now proved. □

If k is greater than three; then the process of finding pairwise disjoint simple closed curves $\delta_1, \ldots, \delta_j$ and pairwise disjoint simple closed curves $\beta_1, \ldots, \beta_j$ for $j < k$ can be continued such that these curves satisfy the following properties:

1. δ_j consists of an arc on M_1 and an arc on the annulus $A_j \subset \partial B_1$ with boundary α_j and α_{j+1};
2. $\beta_j \subset M_1$ and β_j intersects δ_j transversely in one point when $i = j$ and $\beta_i \cap \delta_j = \varnothing$ when $i \neq j$;
3. δ_j bounds a minimal disk K_j in W when j is odd and in Y when j is even;
4. β_j bounds a minimal disk P_j in W when j is even and in Y when j is odd;
5. The sets $K_i \cap K_j$, $P_i \cap P_j$ and $K_i \cap P_i$ are empty when i is different from j;

6. the curve β_j disconnects M_1 with one component being a planar domain Σ_j with boundary curves $\{\alpha_1, \ldots, \alpha_j, \beta_j\}$;

The same arguments used to prove the existence of the two collections $\{\beta_1, \ldots, \beta_{k-1}\}$, $\{\delta_1, \ldots, \delta_{k-1}\}$ generalize immediately to prove:

ASSERTION 5.4. *When $j = k - 1$, there is a curve β_k on M_1 that satisfies Property 6 given above and such that β_k is the boundary of a minimal disk P_k in the closure one of the components of $B_1 - M_1$ and a minimal disk R in the closure of the other component of the closure of $B_1 - M_1$. Furthermore, R is disjoint from K_i and P_i for $i < k$.*

We now complete the proof of the theorem.

Let S be the sphere with boundary $P_k \cup R$ and let B be the ball with boundary S. Since each component of $S - M_1$ has nonnegative mean curvature, Theorem 2 in [23] implies $M_1 \cap B$ is a disk F with k trivial handles attached. Let $\hat{M} = (M - B) \cup F$. By construction (see properties 1–6 above), $\hat{M} \cap B_1$ is seen to be isotopic to the connected sum of parallel disks in B_1 along unknotted arcs. It follows that the surface M_1 is standardly embedded in B_1 up to ambient isotopy. Since the end structure of M is standard, M is ambiently isotopic to the connected sum of k parallel planes along $k - 1$ unknotted arcs followed by taking the connected sum with a standardly embedded surface of genus g in the standard way (see Fig. 1). The theorem follows from the last statement. □

Remark 5.1. Very recently Frohman [8] has shown that a "real" Heegaard splitting of a compact ball is topologically standard. Theorem 4.1 and Corollary 3.1 in [11] imply rather easily that the surface $M_1 \subset B_1$, defined at the beginning of the proof of Theorem 5.1, is a "real" Heegaard splitting of the ball B_1. Hence, Frohman's topological uniqueness theorem for such splittings of B_1 give an alternative approach to completing the proof of Theorem 5.1.

REFERENCES

1. M. CALLAHAN, D. HOFFMAN and W. H. MEEKS III: Embedded minimal surfaces with four ends. In preparation.
2. M. CALLAHAN, D. HOFFMAN and W. H. MEEKS III: The structure of singly-periodic minimal surfaces. *Invent. Math.*, **99** (1990), 455–481.
3. T. CHOI, W. H. MEEKS III and B. WHITE: A rigidity theorem for properly embedded minimal surfaces in $\mathbb{R}^3$. *J. Diff. Geom.*, **32**(1) (1990), 65–76.
4. M. DO CARMO and C. K. PENG: Stable minimal surfaces in $\mathbb{R}^3$ are planes. *Bull. AMS*, **1** (1979), 903–906.
5. D. FISCHER-COLBRIE: On complete minimal surfaces with finite Morse index in 3-manifolds. *Invent. Math.*, **82** (1985), 121–132.
6. D. FISCHER-COLBRIE and R. SCHOEN: The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature. *Comm. Pure Appl. Math.*, **33** (1980), 199–211.
7. T. FRANKEL: On the fundamental group of a compact minimal submanifold. *Ann. Math.*, **83** (1966), 68–73.
8. C. FROHMAN: Heegaard splittings of the three-ball. Preprint.
9. C. FROHMAN: The topological uniqueness of triply-periodic minimal surface in $\mathbb{R}^3$. *J. Diff. Geom.*, **31** (1990), 277–283.
10. C. FROHMAN and W. H. MEEKS III: The ordering theorem for the ends of properly embedded minimal surfaces. Preprint.
11. C. FROHMAN and W. H. MEEKS III: The topological uniqueness of complete one-ended minimal surfaces and Heegaard surfaces in $\mathbb{R}^3$. Preprint.
12. C. FROHMAN and W. H. MEEKS III: The topology of complete minimal surfaces and Heegaard surfaces in $\mathbb{R}^3$. *Bull. AMS*, **23**(2) (1990), 417–421.
13. P. HALL: Two topological examples in minimal surface theory. *J. Diff. Geom.*, **19** (1984), 475–481.
14. R. HARDT and L. SIMON: Boundary regularity and embedded minimal solutions for the oriented Plateau problem. *Ann. Math.*, **110** (1979), 439–486.

15. S. HILDEBRANDT: Boundary behavior of minimal surfaces. *Archive Rational Mech. Anal.*, **35** (1969), 47–81.
16. D. HOFFMAN and W. H. MEEKS III: A complete embedded minimal surface with genus one, three ends and finite total curvature. *J. Diff. Geom.*, **21** (1985), 109–127.
17. D. HOFFMAN and W. H. MEEKS III: The asymptotic behavior of properly embedded minimal surfaces of finite topology. *J. AMS*, **2**(4) (1989), 667–681.
18. D. HOFFMAN and W. H. MEEKS III: Embedded minimal surfaces of finite topology. *Ann. Math.*, **131** (1990), 1–34.
19. D. HOFFMAN and W. H. MEEKS III: The strong halfspace theorem for minimal surfaces. *Invent. Math.*, **101** (1990), 373–377.
20. A. HUBER: On subharmonic functions and differential geometry in the large. *Comm. Math. Helv.*, **32** (1957), 181–206.
21. L. JORGE and W. H. MEEKS III: The topology of complete minimal surfaces of finite total Gaussian curvature. *Topology*, **22** (1983), 203–221.
22. H. B. LAWSON: The unknottedness of minimal embeddings. *Invent. Math.*, **11** (1970), 183–187.
23. W. H. MEEKS III: The topological uniqueness of minimal surfaces in three-dimensional Euclidean space. *Topology*, **20** (1981), 389–410.
24. W. H. MEEKS III and H. ROSENBERG: The maximum principle at infinity for minimal surfaces in flat three-manifolds. *Comm. Math. Helv.*, **65** (1990), 255–270.
25. W. H. MEEKS III, L. SIMON and S.-T. YAU: The existence of embedded minimal surfaces, exotic spheres and positive Ricci curvature. *Ann. Math.*, **116** (1982), 221–259.
26. W. H. MEEKS III and S.-T. YAU: Topology of three-manifolds and the embedding problems in minimal surface theory. *Ann. Math.*, **112** (1980), 441–484.
27. W. H. MEEKS III and S.-T. YAU: The classical Plateau problem and the topology of three-dimensional manifolds. *Topology*, **21** (1982), 409–442.
28. W. H. MEEKS III and S.-T. YAU: The existence of embedded minimal surfaces and the problem of uniqueness. *Math. Z.*, **179** (1982), 151–168.
29. R. OSSERMAN: *A Survey of Minimal Surfaces*. 2nd edition, Dover Publications, New York, (1986).
30. R. SCHOEN: *Esimates for Stable Minimal Surfaces in Three Dimensional Manifolds*, volume 103 of *Annals of Math. Studies*. Princeton University Press, (1983).
31. R. SCHOEN: Uniqueness, symmetry, and embeddedness of minimal surfaces. *J. Diff. Geom.*, **18** (1983), 791–809.
32. L. SIMON: Lectures on geometric measure theory. In *Proceedings of the Center for Mathematical Analysis*, volume 3, Canberra, Australia, Australian National University (1983).
33. F. WALDHAUSEN: Heegaard-zerlegungen der 3-sphäre. *Topology*, **7** (1968), 195–203.

Department of Mathematics
University of Massachusetts
Amherst, MA 01003,
U.S.A.

Mathematics Department
Harvard University
Cambridge, MA 02138,
U.S.A.