Einstein Manifolds with Zero Ricci Curvature

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The original motivation for solving of the Einstein equation is to understand space-time in the absence of matter; see the essay by Tod in this volume for an overview of the mathematics of general relativity. The equation governs the manner in which space-time is influenced by the sole force of gravity. As is well known, singularities such as black holes can occur. The study of the vacuum Einstein equations is a difficult problem in nonlinear hyperbolic systems; see the essay by Christodoulou in this volume. Complete space-times with nontrivial gravitational radiation are not well understood.

Many exact solutions of the stationary vacuum Einstein equation have been found in the past, including the Schwarzschild solutions and Taub-NUT solutions. It is remarkable that a so-called Wick rotation [**29**] can be performed on these solutions to transform them into complete (positive-definite) Einstein manifolds without singularities. The (Euclidean) Schwarzschild metric is particularly significant as it is a complete non-singular Ricci-flat metric which is not Kähler. (The (Euclidean) Taub-NUT metric is Kähler. For more about these metrics, consult the paper of Lapedes [**48**].) Wick rotation is obtained by replacing t by it in suitable coordinates. Unfortunately there is no systematic theory of Wick rotations on Einstein metrics.

Wick rotation does not work for compact Einstein manifolds. It is not hard to show that up to torus fibrations, Ricci-flat compact manifolds have no continuous group of symmetries. It is therefore difficult to construct such manifolds by exploiting the symmetry group. In fact, for a long time, geometers believed that no compact Ricci-flat manifolds existed beyond these flat space-forms. In particular, it was a general belief that the conjecture of Calabi on Kähler manifolds was too good to be true, as it implied that the only obstruction for the existence of a Ricci-flat Kähler metric comes from the cohomological condition of the vanishing of the first Chern class. This conjecture of Calabi was in fact proved by the author in [**90**] in 1976.

On a compact Kähler manifold, the space of Kähler metrics with zero Ricci curvature is then parameterized by those complex structures with zero first Chern class and their Kähler classes. (It is not known whether every compact Ricci-flat metric on a Kähler manifold is Kähler or not.) This moduli space has played an important role in the effort to compactify string theory.

Until now, nobody has been able to find an explicit formula for any nontrivial Ricci-flat metric on a compact Kähler manifold. It is possible that the recent conjecture of Strominger-Yau-Zaslow [**78**] may solve this question. There are many more explicit examples when the manifold is non-compact. Besides the Taub-NUT space mentioned above, there are also metrics constructed by the twistor method (see the essay by A. Dancer in this volume). Right after I settled the compact case, I was interested in working out the general existence theorem for complete Ricci flat Kähler manifolds. In the International Congress in 1978 in Helsinki, I described a basic conjecture on such manifolds. It is believed that if such a manifold has finite topology, it must be the complement of a divisor D in a compact Kähler manifold M, and D is the anti-canonical divisor of M. (Anderson-Kronheimer-Lebrun [**1**] have found examples of such manifolds with infinite topology.) I could prove some special cases of this conjecture, for example the case of total space of holomorphic bundles over Fano manifolds. (Such examples were worked out at the same time by Calabi [**12**] using more explicit calculations.) More general cases were obtained by Tian and myself, where we assume D supports a non-singular anti-canonical divisor in a Fano manifold [**83**]. The geometry of the metric depends a lot on the multiplicity of the anti-canonical divisor along D. If the multiplicity is one, D is automatically a compact Kähler manifold with trivial first Chern class. No extra assumption is needed in this case and the volume growth of the metric is less than quadratic growth. When the multiplicity is greater than one, we need to make the assumption that there is a Kähler-Einstein metric on D (see the essay by Tian in this volume for criteria for the existence of such a metric). The volume growth of the Ricci-flat metric on the complement of D in this case behaves like standard Euclidean space.

Our construction of such metrics on the complement of D goes as follows.

Let Ω be a holomorphic n-form defined on the complement of D so that $\Omega \wedge \overline{\Omega}$ blows up along D. Then we want to find a complete Kähler metric $g_{i\bar{j}}$ with bounded curvature so that

$$\frac{\det(g_{i\bar{j}})dz^1 \wedge \cdots \wedge dz^n \wedge \overline{dz^1} \wedge \cdots \wedge \overline{dz^n}}{\Omega \wedge \overline{\Omega}}$$

is bounded from above and below by a constant. The idea is to perturb such a metric $g_{i\bar{j}}$ by the Hessian $\frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}$ of some function u so that $g_{i\bar{j}} + \frac{\partial^2 u}{\partial z_i \partial \overline{z_j}}$ is bounded from above and below by a constant multiple of $g_{i\bar{j}}$ and its volume form is $\Omega \wedge \overline{\Omega}$. In this way, we know quite a bit about the Ricci-flat metric near D. We construct $g_{i\bar{j}}$ by taking a Kähler-Einstein metric on D and extending it in a neighborhood of D to satisfy the above conditions. The existence of u is proved by carefully estimating its C^0 norm, which in turn depends on a suitable estimate of Sobolev inequality constants. It will be interesting to find precise asymptotic behavior of the Ricci-flat metric near D. The parameterization of Kähler Ricci-flat metrics is much more difficult than in the compact case. One has to put constraints on the uniform behavior of the metrics near D. It has been known to experts for some time that the Taub-NUT metric provides a family of complete Ricci-flat Kähler metrics on $\mathbb{C}^2$. (See LeBrun [**49**] for an explication and also the essay by A. Dancer in this volume.) Perhaps they are related to various compactifications of $\mathbb{C}^2$ (and the same may hold for more general quasi-projective manifolds).

There have been many attempts to find explicit constructions of Ricci-flat manifolds. See the essays by M. Wang and A. Dancer in this volume for example. These constructions either use group actions to reduce the number of variables (including the concept of a moment map) or build bundles over existing Kähler-Einstein manifolds.

The existence of Ricci-flat metrics gives rise to a new tool to study Kähler manifolds with zero first Chern class (Calabi-Yau manifolds). For example, an immediate corollary is that if for some element ω in the Kähler cone of an n-dimensional Calabi-Yau manifold M and if $C_2(M) \cup \omega^{n-2} = 0$, then M is covered by the flat torus. This was observed by the author at the time when the Calabi Conjecture was solved. In the above statement, if we only assume ω to be on the boundary of the Kähler cone and $C_2(M) \cup \omega^{n-2} = 0$, then depending on the degeneracy of ω, one should be able to make use of differential geometry to classify those manifolds to be fiber spaces with generic fibers as lower-dimensional Calabi-Yau manifolds. P. Wilson was able to study this problem using algebraic methods in [**87**].

A very important consequence of the Ricci-flat metric is that any holomorphic n-form is parallel with respect to the metric. Therefore if it is non-degenerate at one point, it must be non-degenerate everywhere. In particular, a holomorphic symplectic Kähler manifold is automatically hyper-Kähler, and it therefore admits three compatible integrable complex structures. When the manifold has complex two dimensions, a non-vanishing holomorphic two-form Ω will then be parallel and a linear combination of Ω, $\overline{\Omega}$ and the Kähler form then gives rise to a family of parallel two-forms each of which in turn gives rise to an integrable complex structure compatible with the same metric. This family of complex structures is a very powerful tool in the study of hyper-kähler manifolds. For example, a K3 surface may not have rational curves in general. But (see Bryan-Leung [**10**]) this deformation will give rise to a complex structure which does contain rational curves.

The first general theory of Calabi-Yau manifolds was the study of two-dimensional surfaces due to Piatetski-Shapiro and Shafarevich [**67**] (Burns-Rapoport [**11**] for the case of Kähler manifolds). They found that the period map must be injective for the moduli space of K3 surfaces. The question of surjectivity was done much later and was due to Kulikov [**47**] and Pinkham-Persson [**66**]. Both of these papers are deep works and require a great deal of algebraic machinery.

These theorems were drastically simplified by the observations of Todorov [**84**], that the author's existence theorem for Ricci-flat metric can be applied. The key point is an observation of Hitchin [**38**] that the metric provides an S^2 family of complex structures. This rational curve of complex structures provides a way to move in the moduli space. (Much more rigorous and detailed treatments were then given by Siu [**76**].) There were expectations to generalize these methods to higher-dimensional Calabi-Yau manifolds. While this has not been carried out, the famous theorem of Bogomolov [**8**] on the unobstructedness of holomorphic symplectic Kähler manifolds was generalized to general Calabi-Yau manifolds by Tian [**80**] in his thesis and by Todorov [**85**] independently. This basic theorem played an important role on the later development of Calabi-Yau manifolds. (The analog of the formula for proving unobstructedness is being using by Kontsevich, Fukaya, and others to construct higher products in their attempts to work out the algebraic formulation of mirror symmetry [**5, 25**].)

Since the revolutions of string theory in theoretical physics, the theory of Kähler manifolds with zero Ricci curvature (i.e. Calabi-Yau manifolds) has gone through a vigorous change. The fundamental paper of Candelas, Horowitz, Strominger and Witten [**14**] studied the Kaluza-Klein model, where one wants to compactify a ten-dimensional space-time to a four-dimensional space-time by using compact six-dimensional manifolds with nontrivial parallel spinors. The final analysis shows that the compactification is given by a Calabi-Yau manifold of three complex dimensions. This famous paper immediately called for a great deal of work on constructing such manifolds, especially those with Euler number equal to ± 6 and with nontrivial fundamental group. At the beginning, physicists thought that there are only a couple of Calabi-Yau manifolds with three dimensions. During the first major conference on string theory [**91**], the author described many ways to construct these manifolds, and the physicists were rather surprised to find out that there should be at least on the order of ten thousand such manifolds. The author proposed to construct a large class of these manifolds by taking complete intersections of hypersurfaces in products of weighted projective spaces. The first important example is the complete intersection of two cubics in $\mathbb{CP}^3 \times \mathbb{CP}^3$ and a bidegree $(1,1)$ hypersurface. This manifold has Euler number equal to -18. I was able to find a group of order three which acts on it with no fixed point. The quotient manifold then has Euler number -6 and non-trivial fundamental group. Tian and the author [**82**] then found more examples in a similar way. The idea of taking complete intersections in products of weighted projective spaces was soon taken up by Candelas and his group to produce many examples for later study. It was observed by B. Greene that all these constructions lead to manifolds with the same topology. Greene and his coauthors even discussed the phenomenological implication of those manifolds [**2**].

String theory demands extensive calculations on the moduli space of Calabi-Yau manifolds. Since the local Torelli theorem holds, the period of the top-dimensional holomorphic form determines the local geometry of the moduli space. It was observed by Tian [**80**] and the physicists that the Kähler potential can be written as $\log ||\Omega||^2$ where Ω is a local holomorphic family of top-dimensional holomorphic forms. The fact that the holomorphic n-form defines a sub-line bundle of the (flat) bundle of n-dimensional cohomology classes gives a way to calculate the Weil-Petersson geometry with extra data. The quotient of this flat bundle by the line bundle describes the infinitesimal deformation of complex structures and hence gives the tangent bundle of the moduli space.

Two groups studied this kind of geometry (Candelas et al [**13**] and Strominger [**77**]). Strominger coined the name special geometry for it (he originally called it Kähler geometry of restricted type and the author suggested changing it to special geometry). Special geometry turns out to play an important role in later calculations of mirror symmetry.

The works of Gepner [**28**] and Greene-Vafa-Warner [**34**] show heuristically how to attach a conformal field theory and a path integration to certain Calabi-Yau manifolds. Soon after, Dixon [**21**] and Lerche-Vafa-Warner [**50**] made the prediction of mirror symmetry, which asserts that for any Calabi-Yau manifold M, one can associate another Calabi-Yau manifold M' so that by going from M to the mirror M', two three-point correlation functions (one associated to the complex deformations and the other associated to the Kähler deformations) are mapped to one another. The correlation function for complex deformations of M is simply the natural triple product of $H^1(T_M)$ (this works since $\Lambda^3 T$ is trivial). The correlation function for Kähler deformations is much more complicated. Besides the classical topological cup product on $H^1(T_M^*)$, one needs to add corrections due to integration over rational curves! B. Greene and the author called the last triple product the quantum cup product during the first conference on mirror manifolds in 1990 in Berkeley. Vafa called the cohomology arising from such a ring structure quantum cohomology.

For the important example of the quintic in $\mathbb{CP}^4$, Greene-Plesser [**33**] demonstrated the existence of the mirror based on arguments from conformal field theory. Immediately afterwards, Candelas et al [**15**] carried out the complete detailed calculation of the correlation functions based on the mirror statements. The identification of the special geometry on both the Kähler and the complex sides plays an important role. The calculation of such an identification is a spectacular piece of work in mathematics. It depends on studying the periods of holomorphic three-forms which satisfy a Picard-Fuchs equation and on understanding the monodromy associated to the degeneration of complex structure. This work of Candelas et al has greatly influenced the development of Calabi-Yau manifolds in the past ten years. In particular, it provides a beautiful formula to calculate the number of rational curves (which needs to be defined suitably) on the quintic. Even the existence of this formula was not expected in mathematics literature. Later developments due to many mathematicians are all basically reinterpretations of Candelas' formula in various forms.

Candelas' method of calculation was immediately carried out by many groups of mathematicians when the complex deformation space is one-dimensional. When the deformation space is multidimensional, the calculation requires a new method and this was carried out independently by Hosono-Klemm-Thiesen-Yau [**40**] and by Candelas-de la Ossa-Font-Katz-Morrison [**16**]. A further generalization was also done by Hosono-Lian-Yau [**41**]. In the former paper the Frobenius method and the hypergeometric system of Gelfand-Kapranov-Zelevinsky [**26, 27**] were extensively used. The formal parameter in the Frobenius method was later replaced by the hyperplane class in equivariant geometry. This gives the right interpretation of Candelas' formula in terms of equivariant geometry.

It makes sense to talk about the quantum cohomology ring structure for any Kähler manifolds. For manifolds with positive first Chern class, the associativity of quantum cohomology is sometimes enough to determine the instanton sum. This statement comes from the WDVV equations, which are due to a group of physicists (see [**88, 23**]). For these manifolds, mathematicians were able to exploit the associativity of the quantum cohomology to calculate the instanton sums. The concept of a Frobenius manifold was developed to understand these calculations, which in turn led to formulas for counting curves in homogeneous manifolds. On the other hand, it took a much longer time to actually prove the associativity of quantum cohomology.

The first attempt to prove this associativity was due to Ruan-Tian [**71**]. First of all, one needs to define the meaning of the instanton sum. Ruan [**70**] defined it for symplectic manifolds when the curve has genus zero. Then Ruan-Tian [**72**] generalized it to curves of arbitrary genus. The definition is modeled after Donaldson's definition of his gauge invariants for four-dimensional manifolds. A basic ingredient is the compactness argument for pseudo-holomorphic curves essentially due to Sacks-Uhlenbeck [**73**]. It was observed by Gromov [**35**] that pseudo-holomorphic curves can be used to study the rigidity of symplectic manifolds. Ruan-Tian's definition and proof of associativity for quantum cohomology works only for pseudo-holomorphic curves with respect to a generic choice of almost complex structure.

However integrable complex structures are far away from being generic, and therefore the instanton sum needs to be defined differently if we restrict ourselves to projective manifolds only.

Based on the works of Sacks-Uhlenbeck [**73**], Gromov [**35**], Parker-Wolfson [**65**] and others, Kontsevich [**46**] defined the concept of the compactification of the moduli space of rational maps from pointed rational curves to a projective manifold. When the projective manifold is a complete intersection in a certain homogeneous space, there is a way to define a certain obstruction bundle over the above compactified space. If the obstruction bundle has the same rank as the moduli space of maps, we can take the Euler number of the bundle. In general, however, one has to use the construction of the virtual cycle first done by Li-Tian [**51**] to define such a number. For a generic choice of projective hypersurface, the "number" of curves in a fixed topology can be defined in terms of these Euler numbers. For the quintic, we get

$$K_d = \sum_{k|d} n_{d/k} k^{-3},$$

where K_d is the Euler number and $n_{d/k}$ is the expected number of rational curves. This formula, called the covering formula, was discovered by Candelas et al [**17**] and rigorously justified by Aspinwall-Morrison [**4**] and Manin [**59**]. The number n_i is a projective invariant and should be called differently from the symplectic invariant mentioned above. A natural name should be the Schubert invariant to honor the fundamental work begun by Schubert a century ago.

In many important cases, Li-Tian [**52**] and Siebert [**75**] were able to prove these Schubert invariants are the same as the one defined by Ruan-Tian-Gromov. In particular, this demonstrates that the associativity law is valid for these Schubert invariants.

Candelas' formula for the quintic threefold is the following equation of formal power series in T:

$$\frac{5T^3}{6} + \sum_{d>0} K_d e^{dT} = \frac{5}{2}\left(\frac{f_1}{f_0}\frac{f_2}{f_0} - \frac{f_3}{f_0}\right),$$

where $T = \frac{f_1}{f_0}$, K_d is the Euler number as above, and for $i = 0, 1, 2, 3$,

$$f_i = \frac{1}{i!}\left(\frac{d}{dH}\right)^i\Big|_{H=0} \sum_{d\geq 0} e^{d(t+H)} \frac{\prod_{m=1}^{5d}(5H+m)}{\prod_{m=1}^{d}(H+m)^5}.$$

The f_i form a basis for the solution space of $L(f) = 0$, where L is the hypergeometric differential operator

$$L = \left(\frac{d}{dt}\right)^4 - 5e^t\left(5\frac{d}{dt}+1\right)\cdots\left(5\frac{d}{dt}+4\right).$$

Many people have made serious attempts to prove this formula. Witten [**88**] defined the concept of a linear sigma model, and Plesser-Morrison [**62**] made an (unsuccessful) attempt to use this concept to justify Candelas' formula. However, they did demonstrate the importance of the linear sigma model. Soon after, Kontsevich made a serious attempt to apply the Atiyah-Bott localization to prove Candelas' formula [**46**]. While he succeeded in computing the degree-four invariant for the quintic, his formulation is too complicated to be carried out in general. It is important to note that the above H used in the Frobenius method (see [**40**]) is interpreted as the equivariant hyperplane class. Following Kontsevich, Givental [**30**] made another attempt, using ideas of Witten and others and introducing quantum differential equations (these are just equations for determining a flat section of a certain canonically defined connection). However, his claimed proof is not complete. Finally, based on the works of Witten, Kontsevich, Li-Tian, and some new ideas on the concept of Euler data, Lian-Liu-Yau [**53**] gave the first complete proof of Candelas' formula in 1997. Some six months after the publication of [**53**], two works attempting to complete Givental's program appeared. The first one was due to Procesi et al [**7**] and the other one to Pandharipande [**64**]. The first paper did not claim to prove Candelas' formula in its final form, and the second used some ideas of Lian-Liu-Yau.

While the work of Lian-Liu-Yau does not give a construction of the mirror manifold, it does raise many interesting mathematical questions. One should interpret this theory as a theory of characteristic classes or K-theory over a mapping-space sigma model of algebraic manifolds. One advantage of such sigma models is that they allow us to restrict the maps to those from curves of a fixed topology, resulting in a finite-dimensional mapping space.

An important question involved in the theory of Lian-Liu-Yau is the following: Given an algebraic bundle V over an algebraic manifold M and the stable moduli space of maps $\mathcal{M}(g,k)$ from curves of genus g to M with homology class in $k \in H_2(M,\mathbb{Z})$, one can form a virtual bundle $\widetilde{V}$ over $\mathcal{M}(g,k)$ by looking at $H^0(C, f^*V) - H^1(C, f^*V)$, where $f : C \to M$ is a map in $\mathcal{M}(g,k)$. Given a theory of characteristic classes, i.e. a map b from the ring of holomorphic vector bundles to homology classes (which can be refined to algebraic cycles), one can then consider $b(\widetilde{V})$ and consider several numbers related to $b(\widetilde{V})$. For example, we can evaluate $b(\widetilde{V})$ over the Li-Tian class [**51**], which was defined by Li-Tian as a virtual moduli cycle (and subsequently understood by Behrend-Fantechi [**6**] using a somewhat different method), or we can consider a product of $b(\widetilde{V})$ with the Chern class of the tautological line bundle of $\mathcal{M}(g,k)$ and then evaluate this product over the Li-Tian cycle. The method of Lian-Liu-Yau can be used to compute these numbers for a large class of bundles V and M. This class includes, for example, convex and concave bundles over toric varieties or balloon manifolds. The computation of $b(\widetilde{V})$ can be considered as part of the K-theory over sigma models of algebraic manifolds.

It is important to carry out the computations of Lian-Liu-Yau in the most general possible setting. Equally important is to interpret the geometric meaning of the numbers computed. When b is the Euler class and $H^1(C, f^*V) = 0$, the number is interpreted to be related to the counting of the "number" of curves of genus g. This is how one computes the number of rational curves in a generic quintic in $\mathbb{CP}^4$. In that case, one takes V to be the line bundle $\mathcal{O}(5)$ over $\mathbb{CP}^4$. When V is $\mathcal{O}(-3)$ over $\mathbb{CP}^2$, we are dealing with numbers which arise in "local mirror symmetry," i.e. the "number" of rational curves in $\mathbb{CP}^2$ embedded as a hypersurface in a Calabi-Yau manifold (see the works of Vafa et al, e.g. [**45**], and the recent work of Chiang-Klemm-Lian-Roth-Yau-Zaslow [**18**]). The set of all these characteristic numbers over sigma models is very much related to the hypergeometric series of Gelfand-Kapranov-Zelevinsky [**26, 27**]. It would be very interesting to understand the internal structure of these numbers as a map from the K-groups of M.

When b is the Euler class, it is a remarkable theorem of Li-Tian that it is the same as counting the number (up to sign) of pseudo-holomorphic curves of a generic almost-complex structure compatible with the given symplectic structure. Using the proof of Lian-Liu-Yau, one should be able to extend the methods of Li, Ruan and Tian to show that the coefficients n_d of the generating function are integers. This should have deep interest for both combinatoricists and number theorists. The transformation from the hypergeometric series to the generating function is called the mirror transformation. It is also a remarkable fact that by choosing the right coordinates, the mirror transformation has a good q-expansion whose coefficients are integers (as was computed experimentally by Hosono-Klemm-Thiesen-Yau [**40**] and publicized by the authors). When the deformation of the mirror manifold is one-dimensional, this integral condition was verified by Lian-Yau [**57**]. This is a very important fact, as it was used by Lian-Yau to prove divisibility properties of the number of rational curves. For example, it was proved that n_i, the number of rational curves in a quintic, is divisible by 125 in the case i is not divisible by 5. However, such integral properties of the mirror map are not known for the multi-variable case and pose a challenging problem. Note that when the Calabi-Yau manifold has one or two dimensions, the mirror map is related to the j-function. In fact, Lian-Yau [**54, 55, 56**] observed that when the Calabi-Yau manifold is the K3 surface or when the Calabi-Yau manifold contains a pencil of K3 surfaces, the mirror map should be related to the automorphic form which appears in the moonshine conjecture related to the monster group. In his Harvard thesis, Chuck Doran made remarkable progress on this question, as he studied the Painlevé VI equation and its algebraic solutions extensively [**22**].

Duality. conjectures in the recent progress of string theory have clear implications in number theory as was indicated by works of Moore-Witten [**61**]. Also G. Moore has questions on the values of the mirror maps on certain special points on the moduli space determined by a variational principle [**60**]. All these questions imply that a very rich structure of number theory is hidden in the theory of mirror symmetries. Klemm-Lian-Roan-Yau [**44**] developed a generalization of the Schwarzian equation for the mirror map. It was based on such equations that the divisibility properties of number of rational curves were found.

While the theory of Lian-Liu-Yau is able to tackle many important questions in enumerative geometry, it does not explain the geometric meaning of mirror manifolds. The construction of Strominger-Yau-Zaslow [**78**], however, does provide such a framework. In a Calabi-Yau threefold, we look at the space of special Lagrangian (real) tori in the manifold. The moduli space of pairs of such a torus coupled with a $U(1)$ connection over it has a natural complex structure. This is conjectured to be the mirror manifold. Vafa [**86**] has recently extend the SYZ conjecture to include vector bundles in the picture. While Gross [**36, 37**] and Hitchin [**39**] have made significant progress on the SYZ conjecture, a full understanding of this theory is still far away. Key missing ingredients are explicit constructions of special Lagrangian submanifolds in general Calabi-Yau manifolds and holomorphic disks whose boundaries lie on given Lagrangian submanifolds. In any case, the SYZ picture is likely to be correct and it will be very interesting to combine the rigorous treatment of Lian-Liu-Yau with the picture of SYZ. It predicts a construction of a Ricci-flat metric and hopefully can be carried out by understanding the instanton corrections to the semi-flat metric.

Many years ago, Mukai [**63**] observed that the moduli space of $SU(n)$ bundles over a K3 surface has natural hyper-kähler structure. (This can be generalized to other hyper-kähler manifolds.) He introduced the concept of the Mukai transform, which is clearly related to the above theories. Hopefully, a complete mathematical theory encompassing all these ideas can be found soon.

Another important problem is to classify all three-dimensional Calabi-Yau manifolds and those four-dimensional ones that are elliptic fiber spaces. A very much related question is the understanding of construction of manifolds with G_2 and Spin(7) holonomy groups.

Only recently Joyce [**42, 43**] was able to construct non-trivial examples of such manifolds (see Joyce's essay in this volume). They were obtained by singular perturbation which is similar to the construction of C. Taubes on self-dual $SU(2)$ connections over four-manifolds [**79**]. While these manifolds clearly play an important role in the recent progress of string theory, their global structure is still hard to be understood. How do we parameterize them? Are they related to Kähler manifolds in a systematic way? How can we understand the moduli space of bundles with special holonomy groups over these manifolds or Calabi-Yau manifolds?

A recent development of string theory demands that a given Calabi-Yau manifold can be deformed to another. Since these manifolds may have different topology, one must go through singular manifolds to achieve such a goal. One is also allowed to identify manifolds which give rise to the same conformal field theory. Aspinwall-Greene-Morrison [**3**] has studied the deformation of conformal field theory of Calabi-Yau manifolds when these is a "flop" construction which changes the topology of the manifolds. Greene-Morrison-Strominger [**32**] also discussed how the quantum field theory changes when the manifold is deformed to acquire conifold points. These theories demonstrate the possibility of good physical theories even when the target space has singularities. This should mean that we can develop a good geometric theory even when the manifolds acquire singularities. This includes a good metric, a good Hodge theory, a good bundle theory, and a good enumerative geometry on such singular manifolds. Such geometries should reflect the quantum field theory mentioned above. In particular, one would like to see new geometric quantities to capture the limit of the "quantum" geometry when a smooth manifold approaches a singular one. Super-symmetric cycles which represent cycles collapsing to the singularities should play an important role in all these discussions.

In the discussion of connecting different Calabi-Yau manifolds, a particularly important process was suggested by M. Reid [**68**] (some initial ideas date back to Clemens [**20**]). We can destroy the second cohomology of a Calabi-Yau manifold by blowing down rational curves with negative normal bundle. There are theorems by Clemens [**19**], Friedman [**24**] and Tian [**81**] on how to deform the complex structure of the resulting singular manifold to that of a smooth complex manifold. These manifolds need not be algebraic (although they are birational to such manifolds). By passing through this kind of process, Reid suggests to connect all Calabi-Yau three-folds together. It is a rather tempting conjecture. However, since the manifold obtained by smoothing is not Kähler, a canonical Hermitian metric has to be defined to account for properties similar to those given by the Ricci-flat metric. A Weil-Petersson metric on the moduli space based on such canonical metrics would be important because it should help to identify the mirror map.

A few years ago Zaslow and I [**92**] demonstrated the relation between counting singular rational curves with nodes in a K3 surface and automorphic forms. Motivated by the formula, Göttsche [**31**] made the following conjecture for a more general Kähler surface X:

Let C be a sufficiently ample divisor on X, and K be the canonical divisor. Then the number of curves of genus g in $|C|$ passing through $r = -KC + g - \chi(\mathcal{O}_X)$ points is given by the coefficient of $q^{\frac{1}{2}C(C-K)}$ in the following power series in q:

$$B_1^{K^2} B_2^{CK} (DG_2)^r \frac{D^2 G_2}{(\Delta(D^2 G_2))^{\chi(\mathcal{O}_X)/2}},$$

where $D = q\frac{d}{dq}$, G_2 is the Eisenstein series

$$G_2(q) = -\frac{1}{24} + \sum_{k>0} (\sum_{d|k} d) q^k,$$

Δ is the discriminant

$$\Delta(q) = q \prod_{k>0} (1 - q^k)^{24},$$

and the $B_i(q)$ are certain universal power series.

Bryan-Leung [**10**] made the first step to give rigorous proof of the Yau-Zaslow formula for K3 surfaces when the cohomology class is primitive. It is remarkable that A.-K. Liu [**58**] was recently able to obtain the formula for general Kähler surfaces. (Some special cases were obtained with T.-J. Li jointly.) Using the idea of a family of Seiberg-Witten invariants, he is also able to study a similar question for algebraic three-folds which are elliptic fiber spaces. It is rather mysterious that the generating function for counting curves is related to automorphic forms. Perhaps some generalized theory of those forms will be developed in the near future.

Coming back to the classification of Calabi-Yau manifolds, it may be interesting to understand geometric cobordism among such manifolds. When do two Calabi-Yau three-folds bound a seven-dimensional manifold with G_2 holonomy? For G_2-manifolds, one can of course look for a Spin(7) manifold to be the total space.

Recent developments in string theory give rise to the following interesting question. If a manifold M is a metric cone over a compact manifold N such that M has special holonomy group, what conditions does this place on N? (See the chapter in this volume by Boyer-Galicki for details.) In the interesting case when M is Calabi-Yau, this is equivalent to N having an Einstein-Sasaki structure. Recently Boyer-Galicki [**9**] have studied such structures, but do not seem to construct new examples in real dimension 5.

Another interesting development in due to Sawon [**74**]. He uses techniques of Rozansky-Witten [**69**] to develop relations between the Chern numbers of hyperkähler manifolds. In particular, if the real dimension is 8, Sawon finds an upper bound of c_4. The Chern numbers of any Kähler-Einstein manifold satisfy the inequality [**89**]

$$(-1)^n 2(n+1) c_1^{n-2} c_2 \geq (-1)^n n c_1^n.$$

Of course, for a Calabi-Yau n-fold ($n \geq 3$), this inequality is trivial, but perhaps some of the relations developed by Sawon can be extended to more general Calabi-Yau manifolds.

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