

# POSITIVITY OF THE BONDI MASS IN BONDI'S RADIATING SPACETIMES

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ABSTRACT. We find two conditions related to the *news functions* of the Bondi's radiating vacuum spacetimes. We provide a complete proof of the positivity of the Bondi mass by using Schoen-Yau's method under one condition and by using Witten's method under another condition.

## 1. INTRODUCTION

Gravitational waves are predicted by Einstein's general relativity. They are time dependent solutions of the Einstein field equations which radiate or transport energy. Although they have not been detected yet, the existence of gravitational waves has been proved indirectly from observations of the pulsar PSR 1913+16. This rapidly rotating binary system should emit gravitational radiation, hence lose energy and rotate faster. The observed relative change in period agrees remarkably with the theoretical value .

A fundamental conjecture is that gravitational waves can not carry away more energy than they have initially in an isolated gravitational system. It is usually referred as the positive mass conjecture at null infinity. In the Bondi's radiating vacuum spacetime, this conjecture is equivalent to the positivity of the Bondi mass. In the pioneering work of Bondi, van der Burg, Metzner and Sachs on the gravitational waves in vacuum spacetimes, the Bondi mass associated to each null cone is defined and their main result asserts that this Bondi mass is always non-increasing with respect to the retarded time [2, 15, 18]. Therefore, the Bondi mass can be interpreted as the total mass of the isolated physical system measured after the loss due to the gravitational radiation up to that time. The proof of the positivity of the Bondi mass was outlined by Schoen-Yau modifying their arguments in the proof of the positivity of the ADM mass [17]. It was also outlined by physicists applying Witten's spinor method, eg, see [11, 9, 1, 13, 14, 10]. The

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2000 *Mathematics Subject Classification.* 53C50, 83C35.

*Key words and phrases.* Gravitational radiation; Bondi mass; positivity.

main goal of this paper is to find a complete proof of the positivity. Indeed, we find certain conditions related to the *news functions* of the system. The Bondi mass is nonnegative under these conditions.

It is an open problem whether vacuum Einstein field equations always develop logarithmic singularities at null infinity. In [7], the authors studied the polyhomogeneous Bondi expansions. The  $u$ -evolution equations actually indicate that the logarithmic singularities at null infinity can be removed in the axisymmetric case (Appendix D of [7]) if the free function  $\gamma_2(u, x^\alpha)$  is chosen to be zero and  $\gamma_{3,1}(u_0, x^\alpha)$  is chosen to be zero for some  $u_0$ . It is quite possible that the Bondi's radiating vacuum spacetime does not develop any logarithmic singularity at null infinity after suitable "gauge fixing". Therefore we do not consider the polyhomogeneous Bondi expansions [3] in the present paper.

The paper is organized as follows. In Section 2, we state some well-known formulas and results of Bondi, van der Burg, Metzner and Sachs. We employ two fundamental assumptions: **Condition A** and **Condition B**. We also derive a generalized Bondi mass loss formula under these two conditions. In Section 3, we study some basic geometry of the asymptotically null spacelike hypersurfaces. We compute the asymptotic behaviors of the induced metric and the second fundamental form of an asymptotically null spacelike hypersurface given by a certain graph. In Section 4, we use Schoen-Yau's method to prove, under **Condition A** and **Condition B**, if there is a retarded time  $u_0$  such that  $\mathcal{M}(u_0, \theta, \psi)$  defined in Section 2 is constant, then the Bondi mass is nonnegative in the region  $\{u \leq u_0\}$ , and the Bondi mass is zero at  $u \in (-\infty, u_0]$  if and only if the spacetime is flat in certain neighbourhood of spacelike hypersurface  $\{u = u_0 + \sqrt{1 + r^2} - r\}$ . In Section 5, we use Witten's method to prove, under **Condition A** and **Condition B**, if there is a retarded time  $u_0$  such that  $c(u_0, \theta, \psi) = d(u_0, \theta, \psi) = 0$ , then the Bondi mass is nonnegative in the region  $\{u \leq u_0\}$ , and the Bondi mass is zero at  $u \in (-\infty, u_0]$  if and only if the spacetime is flat in certain neighbourhood of spacelike hypersurface  $\{u = u_0 + \sqrt{1 + r^2} - r\}$ . In Section 6, we modify the definition of the Bondi energy-momentum and prove its positivity without **Condition B**.

## 2. BONDI'S RADIATING SPACETIMES

We assume that  $(\mathbb{L}^{3,1}, \tilde{g})$  is a vacuum spacetime (possible with black holes) and  $\tilde{g}$  takes the following Bondi's radiating metric

$$\begin{aligned}
 \tilde{g} = & \left( \frac{V}{r} e^{2\beta} + r^2 e^{2\gamma} U^2 \cosh 2\delta + r^2 e^{-2\gamma} W^2 \cosh 2\delta \right. \\
 & \left. + 2r^2 U W \sinh 2\delta \right) du^2 - 2e^{2\beta} dudr \\
 & - 2r^2 \left( e^{2\gamma} U \cosh 2\delta + W \sinh 2\delta \right) dud\theta \\
 & - 2r^2 \left( e^{-2\gamma} W \cosh 2\delta + U \sinh 2\delta \right) \sin\theta dud\psi \\
 & + r^2 \left( e^{2\gamma} \cosh 2\delta d\theta^2 + e^{-2\gamma} \cosh 2\delta \sin^2\theta d\psi^2 \right. \\
 & \left. + 2 \sinh 2\delta \sin\theta d\theta d\psi \right)
 \end{aligned} \tag{2.1}$$

in coordinates  $(u, r, \theta, \psi)$  ( $u$  is retarded time) where

$$-\infty < u < \infty, \quad r > 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \psi \leq 2\pi,$$

Denote

$$x^0 = u, \quad x^1 = r, \quad x^2 = \theta, \quad x^3 = \psi.$$

We suppose that  $\beta, \gamma, \delta, U, V$  and  $W$  are smooth functions of  $u, r, \theta, \psi$ . Denote  $f_{,\nu} = \frac{\partial f}{\partial x^\nu}$  for  $\nu = 0, 1, 2, 3$  throughout the paper. The metric (2.1) was studied by Bondi, van der Burg, Metzner and Sachs in the theory of gravitational waves in general relativity [2, 15, 18]. They proved that the following asymptotic behavior holds for  $r$  sufficiently large if the spacetime satisfies the outgoing radiation condition [18]

$$\begin{aligned}
 \gamma &= \frac{c(u, \theta, \psi)}{r} + \frac{C(u, \theta, \psi) - \frac{1}{6}c^3 - \frac{3}{2}cd^2}{r^3} + O\left(\frac{1}{r^4}\right), \\
 \delta &= \frac{d(u, \theta, \psi)}{r} + \frac{H(u, \theta, \psi) + \frac{1}{2}c^2d - \frac{1}{6}d^3}{r^3} + O\left(\frac{1}{r^4}\right), \\
 \beta &= -\frac{c^2 + d^2}{4r^2} + O\left(\frac{1}{r^4}\right), \\
 U &= -\frac{l(u, \theta, \psi)}{r^2} + \frac{p(u, \theta, \psi)}{r^3} + O\left(\frac{1}{r^4}\right), \\
 W &= -\frac{\bar{l}(u, \theta, \psi)}{r^2} + \frac{\bar{p}(u, \theta, \psi)}{r^3} + O\left(\frac{1}{r^4}\right), \\
 V &= -r + 2M(u, \theta, \psi) + \frac{\bar{M}(u, \theta, \psi)}{r} + O\left(\frac{1}{r^2}\right),
 \end{aligned}$$

where

$$\begin{aligned}
l &= c_{,2} + 2c \cot \theta + d_{,3} \csc \theta, \\
\bar{l} &= d_{,2} + 2d \cot \theta - c_{,3} \csc \theta, \\
p &= 2N + 3(cc_{,2} + dd_{,2}) + 4(c^2 + d^2) \cot \theta \\
&\quad - 2(c_{,3}d - cd_{,3}) \csc \theta, \\
\bar{p} &= 2P + 2(c_{,2}d - cd_{,2}) + 3(cc_{,3} + dd_{,3}) \csc \theta, \\
\bar{M} &= N_{,2} + \cot \theta + P_{,3} \csc \theta - \frac{c^2 + d^2}{2} \\
&\quad - [(c_{,2})^2 + (d_{,2})^2] - 4(cc_{,2} + dd_{,2}) \cot \theta \\
&\quad - 4(c^2 + d^2) \cot^2 \theta - [(c_{,3})^2 + (d_{,3})^2] \csc^2 \theta \\
&\quad + 4(c_{,3}d - cd_{,3}) \csc \theta \cot \theta + 2(c_{,3}d_{,2} - c_{,2}d_{,3}) \csc \theta.
\end{aligned}$$

Here  $M$  is referred as the *mass aspect* and  $c_{,0}$ ,  $d_{,0}$  are referred as the *news functions*. The u-derivatives of certain functions are

$$\begin{aligned}
C_{,0} &= \frac{c^2 c_{,0}}{2} + cdd_{,0} - \frac{c_{,0}d^2}{2} + \frac{cM}{2} + \frac{d\lambda}{4} \\
&\quad - \frac{N_{,2} - N \cot \theta - P_{,3} \csc \theta}{4}, \\
H_{,0} &= -\frac{c^2 d_{,0}}{2} + cc_{,0}d + \frac{d_{,0}d^2}{2} + \frac{dM}{2} - \frac{c\lambda}{4} \\
&\quad - \frac{P_{,2} - P \cot \theta + N_{,3} \csc \theta}{4}, \\
M_{,0} &= -[(c_{,0})^2 + (d_{,0})^2] + \frac{1}{2}(l_{,2} + l \cot \theta + \bar{l}_{,3} \csc \theta)_{,0}, \\
3N_{,0} &= -M_{,2} - \frac{\lambda_{,3} \csc \theta}{2} - (c_{,0}c_{,2} + d_{,0}d_{,2}) \\
&\quad - 3(cc_{,02} + dd_{,02}) - 4(cc_{,0} + dd_{,0}) \cot \theta \\
&\quad + (c_{,0}d_{,3} - c_{,3}d_{,0} + 3c_{,03}d - 3cd_{,03}) \csc \theta, \\
3P_{,0} &= -M_{,3} \csc \theta + \frac{\lambda_{,2}}{2} + (c_{,2}d_{,0} - c_{,0}d_{,2}) \\
&\quad + 3(cd_{,02} - c_{,02}d) + 4(cd_{,0} - c_{,0}d) \cot \theta \\
&\quad - (c_{,0}c_{,3} + d_{,0}d_{,3} + 3cc_{,03} + 3dd_{,03}) \csc \theta
\end{aligned}$$

where

$$\lambda = \bar{l}_{,2} + \bar{l} \cot \theta - l_{,3} \csc \theta.$$

Denote

$$\begin{aligned}
 \mathcal{M}(u, \theta, \psi) &= M(u, \theta, \psi) - \frac{1}{2}(l_{,2} + l \cot \theta + \bar{l}_{,3} \csc \theta) \\
 &= M(u, \theta, \psi) - \frac{1}{2} \left[ -2c(u, \theta, \psi) + c_{,22}(u, \theta, \psi) \right. \\
 &\quad \left. - \csc^2 \theta c_{,33}(u, \theta, \psi) + 2 \csc \theta d_{,23}(u, \theta, \psi) \right. \\
 &\quad \left. + 3 \cot \theta c_{,2}(u, \theta, \psi) + 2 \cot \theta \csc \theta d_{,3}(u, \theta, \psi) \right].
 \end{aligned} \tag{2.2}$$

Its  $u$ -derivative is

$$\mathcal{M}_{,0} = -[(c_{,0})^2 + (d_{,0})^2]. \tag{2.3}$$

There are some physical conditions [2, 15, 18] ensuring the regularity of (2.1). In this paper, however, we assume

**Condition A:** Each of the six functions  $\beta, \gamma, \delta, U, V, W$  together with its derivatives up to the second orders are equal at  $\psi = 0$  and  $2\pi$ .

**Condition B:** For all  $u$ ,

$$\int_0^{2\pi} c(u, 0, \psi) d\psi = 0, \quad \int_0^{2\pi} c(u, \pi, \psi) d\psi = 0.$$

Let  $\mathbb{N}_{u_0}$  be a null hypersurface which is given by  $u = u_0$ . The Bondi energy-momentum of  $\mathbb{N}_{u_0}$  is defined by [2, 5]:

$$m_\nu(u_0) = \frac{1}{4\pi} \int_{S^2} M(u_0, \theta, \psi) n^\nu dS \tag{2.4}$$

where  $\nu = 0, 1, 2, 3$ ,  $n^0 = 1$ ,  $n^i$  the restriction of the natural coordinate  $x^i$  to the unit sphere, i.e.,

$$n^0 = 1, \quad n^1 = \sin \theta \cos \psi, \quad n^2 = \sin \theta \sin \psi, \quad n^3 = \cos \theta.$$

Under **Condition A** and **Condition B**, we have [2, 15, 22]

$$\frac{d}{du} m_\nu = -\frac{1}{4\pi} \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^\nu dS \tag{2.5}$$

for  $\nu = 0, 1, 2, 3$ . When  $\nu = 0$ , this is the famous Bondi mass loss formula.

The following proposition can be viewed as generalized Bondi mass loss formula. However, it does not seem to appear in any literature before.

**Proposition 2.1.** *Let  $(\mathbb{L}^{3,1}, \tilde{g})$  be a vacuum Bondi's radiating space-time with metric  $\tilde{g}$  given by (2.1). Suppose that **Condition A** and **Condition B** hold. Then*

$$\frac{d}{du} \left( m_0 - \sqrt{\sum_{1 \leq i \leq 3} m_i^2} \right) \leq 0. \quad (2.6)$$

*Proof :* Denote  $|m| = \sqrt{m_1^2 + m_2^2 + m_3^2}$ . We assume  $|m| \neq 0$  otherwise it reduces to the Bondi mass-loss formula. We have

$$\begin{aligned} \frac{d}{du} (m_0 - |m|) &= \frac{dm_0}{du} - \frac{1}{|m|} \sum_{1 \leq i \leq 3} \frac{dm_i}{du} m_i \\ &= -\frac{1}{4\pi} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] dS \right. \\ &\quad \left. - \frac{1}{|m|} \sum_{1 \leq i \leq 3} m_i \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i dS \right\}. \end{aligned}$$

Thus  $\frac{d}{du} (m_0 - |m|) \leq 0$  is equivalent to

$$\sum_{1 \leq i \leq 3} m_i \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i dS \leq |m| \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] dS.$$

Using  $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$ , we obtain

$$\begin{aligned} &\sum_{1 \leq i \leq 3} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i dS \right\}^2 \\ &\leq \sum_{1 \leq i \leq 3} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] dS \right\} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] (n^i)^2 dS \right\} \\ &= \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] dS \right\}^2. \end{aligned}$$

Then by Cauchy-Schwarz inequality,

$$\begin{aligned} &\sum_{1 \leq i \leq 3} m_i \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i dS \\ &\leq |m| \sqrt{\sum_{1 \leq i \leq 3} \left\{ \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] n^i dS \right\}^2} \\ &\leq |m| \int_{S^2} [(c_{,0})^2 + (d_{,0})^2] dS. \end{aligned}$$

Therefore (2.6) holds.

Q.E.D.

## 3. ASYMPTOTICALLY NULL SPACELIKE HYPERSURFACES

The hypersurface  $u = \sqrt{1+r^2} - r$  in the Minkowski spacetime is a hyperbola equipped with the standard hyperbolic 3-metric  $\check{g}$ . Let  $\{\check{e}_i\}$  be the frame

$$\check{e}_1 = \sqrt{1+r^2} \frac{\partial}{\partial r}, \quad \check{e}_2 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad \check{e}_3 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \psi}.$$

Let  $\{\check{e}^i\}$  be the coframe. Denote  $\check{\nabla}_i = \check{\nabla}_{\check{e}_i}$ , etc., where  $\check{\nabla}$  is the Levi-Civita connection of  $\check{g}$ . The connection 1-forms  $\{\check{\omega}_{ij}\}$  are given by  $d\check{e}^i = -\check{\omega}_{ij} \wedge \check{e}^j$  or  $\check{\nabla} \check{e}_i = -\check{\omega}_{ij} \otimes \check{e}_j$ . They are

$$\check{\omega}_{12} = -\frac{\sqrt{1+r^2}}{r} \check{e}^2, \quad \check{\omega}_{13} = -\frac{\sqrt{1+r^2}}{r} \check{e}^3, \quad \check{\omega}_{23} = -\frac{\cot \theta}{r} \check{e}^3.$$

Let  $\mathbb{X}$  be a spacelike hypersurface in vacuum Bondi's radiating spacetime  $(\mathbb{L}^{3,1}, \tilde{g})$  with metric (2.1), which is given by the inclusion:

$$\begin{aligned} i : \mathbb{X}^3 &\longrightarrow \mathbb{L}^{3,1} \\ (y^1, y^2, y^3) &\longmapsto (x^0, x^1, x^2, x^3) \end{aligned}$$

for  $r$  sufficiently large, where

$$x^0 = u(y^1, y^2, y^3), \quad x^1 = y^1 = r, \quad x^2 = y^2 = \theta, \quad x^3 = y^3 = \psi.$$

Let  $g = i^* \tilde{g}$  be the induced metric of  $\mathbb{X}$  and  $h$  be the second fundamental form of  $\mathbb{X}$ . Let  $\tilde{\nabla}$  be the Levi-Civita connection of  $\mathbb{L}^{3,1}$ . For any tangent vectors  $Y_i, Y_j \in T\mathbb{X}$ ,  $i_* Y_i, i_* Y_j$  are the tangent vectors along  $\mathbb{X}$ ,

$$g(Y_i, Y_j) = \tilde{g}(i_* Y_i, i_* Y_j).$$

Let  $e_n$  be the downward unit normal of  $\mathbb{X}$ . The second fundamental form is defined as

$$h(Y_i, Y_j) = \tilde{g}(\tilde{\nabla}_{i_* Y_i} i_* Y_j, e_n).$$

Now it is a straightforward computation that

$$i_* \frac{\partial}{\partial y^i} = \frac{\partial}{\partial x^\alpha} \frac{\partial x^\alpha}{\partial y^i} = \frac{\partial}{\partial x^0} \frac{\partial x^0}{\partial y^i} + \frac{\partial}{\partial x^i}.$$

Denote  $e_i = i_* \check{e}_i$ . Then

$$e_1 = \sqrt{1+r^2} u_{,1} \partial_0 + \check{e}_1, \quad e_2 = \frac{u_{,2}}{r} \partial_0 + \check{e}_2, \quad e_3 = \frac{u_{,3}}{r \sin \theta} \partial_0 + \check{e}_3. \quad (3.1)$$

**Definition 3.1.** *A spacelike hypersurface  $(\mathbb{X}, g, h)$  in an asymptotically flat spacetime is asymptotically null of order  $\tau > 0$  if, for  $r$  sufficiently large,  $g(\check{e}_i, \check{e}_j) = \delta_{ij} + a_{ij}$ ,  $h(\check{e}_i, \check{e}_j) = \delta_{ij} + b_{ij}$ , where  $a_{ij}, b_{ij}$  satisfy*

$$\{a_{ij}, \check{\nabla}_k a_{ij}, \check{\nabla}_l \check{\nabla}_k a_{ij}, b_{ij}, \check{\nabla}_k b_{ij}\} = O\left(\frac{1}{r^\tau}\right). \quad (3.2)$$

Let  $(\mathbb{X}, g, h)$  be an asymptotically null spacelike hypersurface with the induced metric  $g$  and the second fundamental form  $h$  in vacuum Bondi's radiating spacetime  $(\mathbb{L}^{3,1}, \tilde{g})$ , which is given by

$$u = \sqrt{1+r^2} - r + \frac{(c^2 + d^2)_{u=0}}{12r^3} + \frac{a_3(\theta, \psi)}{r^4} + a_4 \quad (3.3)$$

where  $a_4(r, \theta, \psi)$  is a smooth function which satisfies: In the Euclidean coordinate systems  $\{\check{z}^i\}$ ,  $|\check{z}| = r$ ,

$$a_4 = o\left(\frac{1}{r^4}\right), \quad \partial_k a_4 = o\left(\frac{1}{r^5}\right), \quad \partial_k \partial_l a_4 = o\left(\frac{1}{r^6}\right)$$

as  $r \rightarrow \infty$ . We will compute asymptotic behaviors of the induced metric and the second fundamental form of  $\mathbb{X}$ . The induced metric can be obtained by substituting  $du$  into (2.1). Let  $X_n$  be the downward normal vector

$$X_n = -\frac{\partial}{\partial x^0} - \varrho^i \frac{\partial}{\partial x^i}.$$

Let  $e_i$  be given by (3.1). Since  $X_n$  is orthogonal to  $e_i$ , we obtain

$$\tilde{g}(e_i, X_n) = 0.$$

This implies that  $\varrho^i$  satisfies the following equations

$$(u_{,i} \tilde{g}_{00} + \tilde{g}_{0i}) + \varrho^j (u_{,i} \tilde{g}_{0j} + \tilde{g}_{ij}) = 0$$

for  $i = 1, 2, 3$ . Therefore  $\varrho^i$  can be found by solving linear algebraic equations and the unit normal vector is

$$e_n = \frac{X_n}{\sqrt{-\tilde{g}(X_n, X_n)}}.$$

The second fundamental form is then given by

$$h(\check{e}_i, \check{e}_j) = \tilde{g}(\tilde{\nabla}_{e_i} e_j, e_n)$$

for  $1 \leq i, j \leq 3$ . Now we define  $a \approx b$  if and only if  $a = b + o\left(\frac{1}{r^3}\right)$ . For  $r$  sufficiently large, we expand  $c$ ,  $d$  and  $M$  at  $u = 0$  by Taylor series

$$c(u, \theta, \psi) \approx c(0, \theta, \psi) + c_{,0}(0, \theta, \psi)u + \frac{c_{,00}(0, \theta, \psi)}{2}u^2, \quad (3.4)$$

$$d(u, \theta, \psi) \approx d(0, \theta, \psi) + d_{,0}(0, \theta, \psi)u + \frac{d_{,00}(0, \theta, \psi)}{2}u^2, \quad (3.5)$$

$$M(u, \theta, \psi) \approx M(0, \theta, \psi) + M_{,0}(0, \theta, \psi)u + \frac{M_{,00}(0, \theta, \psi)}{2}u^2, \quad (3.6)$$



with the help of Mathematica 5.0, we obtain the asymptotic behaviors of the metric  $g$

$$\begin{aligned}
 g(\check{e}_1, \check{e}_1) &\approx 1 + \frac{16a_3 + M - cc_{,0} - dd_{,0}}{2r^3}, \\
 g(\check{e}_1, \check{e}_2) &\approx -\frac{l}{2r^2} + \frac{12N - 3l_{,0} + 4(cc_{,2} + dd_{,2})}{12r^3}, \\
 g(\check{e}_1, \check{e}_3) &\approx -\frac{\bar{l}}{2r^2} + \frac{12P - 3\bar{l}_{,0} + 4 \csc \theta (cc_{,3} + dd_{,3})}{12r^3}, \\
 g(\check{e}_2, \check{e}_2) &\approx 1 + \frac{2c}{r} + \frac{2(c^2 + d^2) + c_{,0}}{r^2} \\
 &\quad + \frac{c^3 + cd^2 + 2C + 2(cc_{,0} + dd_{,0}) + \frac{c_{,00}}{4}}{r^3}, \\
 g(\check{e}_2, \check{e}_3) &\approx \frac{2d}{r} + \frac{d_{,0}}{r^2} + \frac{c^2d + d^3 + 2H + \frac{d_{,00}}{4}}{r^3} \\
 g(\check{e}_3, \check{e}_3) &\approx 1 - \frac{2c}{r} + \frac{2(c^2 + d^2) - c_{,0}}{r^2} \\
 &\quad + \frac{-c^3 - cd^2 - 2C + 2(cc_{,0} + dd_{,0}) - \frac{c_{,00}}{4}}{r^3}. \\
 h(\check{e}_1, \check{e}_1) &\approx 1 + \frac{c^2 + d^2}{r^2} + \frac{16a_3 - M}{r^3}, \\
 h(\check{e}_1, \check{e}_2) &\approx \frac{l}{2r^2} + \frac{1}{2r^3} \left[ \frac{l_{,0}}{2} - 2(c^2 + d^2) \cot \theta - 4N \right. \\
 &\quad \left. (-cd_{,3} + c_{,3}d) \csc \theta - \frac{13}{3}(cc_{,2} + dd_{,2}) \right], \\
 h(\check{e}_1, \check{e}_3) &\approx \frac{\bar{l}}{2r^2} + \frac{1}{2r^3} \left[ \frac{\bar{l}_{,0}}{2} + cd_{,2} - c_{,2}d - 4P \right. \\
 &\quad \left. - \frac{13}{3}(cc_{,3} + dd_{,3}) \csc \theta \right], \\
 h(\check{e}_2, \check{e}_2) &\approx 1 + \frac{c}{r} + \frac{c_{,0}}{r^2} + \frac{1}{4r^3} \left[ 3M - 16a_3 - 4C - 2l_{,2} \right. \\
 &\quad \left. - 2c(c^2 + d^2) + 5(cc_{,0} + dd_{,0}) + \frac{3}{2}c_{,00} \right], \\
 h(\check{e}_2, \check{e}_3) &\approx \frac{d}{r} + \frac{d_{,0}}{r^2} + \frac{1}{4r^3} \left[ -2d(c^2 + d^2) + 2d \cot^2 \theta \right. \\
 &\quad \left. + 2d \csc^2 \theta - 4c_{,3} \cot \theta \csc \theta - d_{,33} \csc^2 \theta \right. \\
 &\quad \left. - d_{,2} \cot \theta - d_{,22} - 4H + \frac{3}{2}d_{,00} \right], \\
 h(\check{e}_3, \check{e}_3) &\approx 1 - \frac{c}{r} - \frac{c_{,0}}{r^2} + \frac{1}{4r^3} \left[ 3M - 16a_3 + 4C \right. \\
 &\quad \left. + 2c(c^2 + d^2) + 5(cc_{,0} + dd_{,0}) - \frac{3}{2}c_{,00} \right. \\
 &\quad \left. - 2l \cot \theta - 2\bar{l}_{,3} \csc \theta \right].
 \end{aligned}$$

Here all functions in the right hand sides take value at  $u = 0$  and all derivatives with respect to  $x^2$  and  $x^3$  are taken after substituting  $u = 0$ . Therefore  $(\mathbb{X}, g, h)$  is asymptotically null of order 1.

#### 4. POSITIVITY - SCHOEN-YAU'S METHOD

In this section, we will complete the argument in [17]. Denote by  $(\mathbb{X}, g, h)$  the asymptotically null spacelike hypersurface which is given by (3.3) for  $r$  sufficiently large. In [17], Schoen-Yau solved the following Jang's equation on  $\mathbb{X}$ :

$$\left(g^{ij} - \frac{f^i f^j}{1 + |\nabla f|^2}\right) \left(\frac{f_{,ij}}{\sqrt{1 + |\nabla f|^2}} - h_{ij}\right) = 0 \quad (4.1)$$

under the suitable boundary condition

$$f \rightarrow f_0 \quad (4.2)$$

as  $r \rightarrow \infty$  such that the metric

$$\bar{g} = g + \nabla f \otimes \nabla f \quad (4.3)$$

is asymptotically flat. Denote by  $J(f)$  the left hand side of Jang's equation (4.1). Note that in the standard hyperbolic 3-space, (4.1) has a solution  $f = \sqrt{1 + r^2}$ . Therefore it is reasonable to set

$$f_0 = \sqrt{1 + r^2} + o(r).$$

Let  $f$  be a function on  $\mathbb{X}$  which has asymptotic expansion

$$f = \sqrt{1 + r^2} + p(\theta, \psi) \ln r + q(r, \theta, \psi), \quad (4.4)$$

for  $r$  sufficiently large, where  $p(\theta, \psi)$  is a smooth function on  $S^2$  and  $q$  is a smooth function on  $\mathbb{R}^3$  which satisfies the following asymptotic conditions: In the Euclidean coordinate systems  $\{\check{z}^i\}$ ,  $|\check{z}| = r$ ,

$$q = o(1), \quad \partial_k q = o\left(\frac{1}{r}\right), \quad \partial_k \partial_l q = o\left(\frac{1}{r^2}\right), \quad \partial_k \partial_l \partial_j q = o\left(\frac{1}{r^3}\right)$$

as  $r \rightarrow \infty$ .

Let the standard metric of  $S^2$  be  $d\theta^2 + \sin^2 \theta d\psi^2$ . The Laplacian operator of this metric is

$$\Delta_{S^2} = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \csc^2 \theta \frac{\partial^2}{\partial \psi^2}.$$

The *spherical harmonics*  $w_j$  are the eigenfunctions of  $\Delta_{S^2}$ , i.e.,  $\Delta_{S^2} w_j = j(j-1)w_j$  for  $j = 1, 2, \dots$ .

**Proposition 4.1.** *If Jang's equation (4.1) has a solution  $f$  which has the asymptotic expansion (4.4) for  $r$  sufficiently large, then  $p(\theta, \psi)$  and  $\mathcal{M}(0, \theta, \psi)$  must be constant.*

*Proof :* A lengthy computation with the help of Mathematica 5.0 shows that

$$J(f) \approx \frac{\ln r}{r^3} \Delta_{S^2} p + \frac{p - 2\mathcal{M}(0, \theta, \psi)}{r^3}$$

for  $r$  sufficiently large. That  $J(f) = 0$  implies

$$\Delta_{S^2} p = 0, \quad p - 2\mathcal{M} = 0.$$

As there is no nonconstant harmonic function on  $S^2$ , the proposition follows. Q.E.D.

The existence of (4.1) under the boundary condition (4.2) with

$$f_0(r) = \sqrt{1 + r^2} + p \ln r \tag{4.5}$$

for certain constant  $p$  can be established as follows: We extend  $f_0$  to the whole  $\mathbb{X}$  and denote as  $f_0$  also. Denote  $B_R$  as the ball of radius  $R$  in  $\mathbb{R}^3$ . If  $\mathbb{X}$  has no apparent horizon, the existence theorem for the Dirichlet problem [20] indicates that there exists a (smooth) solution  $\bar{f}_R$  of (4.1) on  $B_R$  such that

$$\bar{f}_R|_{\partial B_R} = 0$$

for sufficiently large  $R$ . By the translation invariance of (4.1) in the vertical direction, we find that

$$f_R = \bar{f}_R + f_0(R)$$

is a solution of (4.1) which is  $f_0(R)$  on  $\partial B_R$ . Now the estimates in [16] show that

$$f_R \longrightarrow f$$

on any compact subset of  $\mathbb{X}$ , where  $f$  is a (smooth) solution of (4.1). Write  $f = f_0 + f_1$  where  $\lim_{r \rightarrow \infty} f_1 = 0$ . Substitute it into Jang's equation (4.1) and obtain an equation for  $f_1$ . Then use the similar argument as the proof of Proposition 3 in [16], we can show that for any  $\varepsilon \in (0, 1)$ , there is a constant  $C(\varepsilon)$  depending only on  $\varepsilon$  and the geometry of  $\mathbb{X}$  such that

$$|f_1(\check{z})| + |\check{z}| |\partial f_1(\check{z})| + |\check{z}|^2 |\partial \partial f_1(\check{z})| + |\check{z}|^3 |\partial \partial \partial f_1(\check{z})| \leq C(\varepsilon) |\check{z}|^\varepsilon.$$

Therefore  $f$  has asymptotic behaviors (4.4), (4.5) for  $r$  sufficiently large.

By adding one point compactification, the existence of (4.1) can be extended to  $\mathbb{X}$  with apparent horizons. See [16] for detail.

The following lemma was proved in [22].

**Lemma 4.1.** *Let  $(\mathbb{L}^{3,1}, \tilde{g})$  be a vacuum Bondi's radiating spacetime with metric  $\tilde{g}$  given by (2.1). Suppose that **Condition A** and **Condition B** hold. Then*

$$\int_{S^2} (l_{,2} + l \cot \theta + \bar{l}_{,3} \csc \theta) n^\nu dS = 0$$

for  $\nu = 0, 1, 2, 3$ .

**Theorem 4.1.** *Let  $(\mathbb{L}^{3,1}, \tilde{g})$  be a vacuum Bondi's radiating spacetime with metric  $\tilde{g}$  given by (2.1). Suppose that **Condition A** and **Condition B** hold. If there exists a constant  $u_0$  such that  $\mathcal{M}(u_0, \theta, \psi)$  is constant, then*

$$m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)}$$

for all  $u \leq u_0$ . If the equality holds for some  $u \in (-\infty, u_0]$ ,  $\mathbb{L}^{3,1}$  is flat in the region foliated by all spacelike hypersurfaces which are given by

$$u = u_0 + \sqrt{1 + r^2} - r + o\left(\frac{1}{r^4}\right)$$

for  $r$  sufficiently large. In particular, if the equality holds for all  $u \leq u_0$ ,  $\mathbb{L}^{3,1}$  is flat in the region  $\{u \leq u_0\}$ .

*Proof :* Suppose  $\mathcal{M}(u_0, \theta, \psi) = \frac{p}{2}$ . By the translation invariance of Jang's equation, we can assume that  $u_0 = 0$ . The assumption of the theorem ensures that there exists a smooth solution  $f$  of Jang's equation (4.1) under the boundary condition (4.2) with  $f_0$  given by (4.5). It is obvious that the metric  $\bar{g}$  given by (4.3) is asymptotic flat. Now we show its ADM total energy is  $p$ . Denote by  $g_0$  the flat metric of  $\mathbb{R}^3$  in polar coordinates. Let  $\{e_i^0\}$  be the frame of  $g_0$

$$e_1^0 = \frac{\partial}{\partial r}, \quad e_2^0 = \frac{1}{r} \frac{\partial}{\partial \theta}, \quad e_3^0 = \frac{1}{r \sin \theta} \frac{\partial}{\partial \psi}.$$

Let  $\{e_0^i\}$  be the coframe of  $g_0$ . Denote  $\alpha_{ij} = \bar{g}(e_i^0, e_j^0) - \delta_{ij}$ . Now we use the ADM energy expression in polar coordinates

$$E(\bar{g}) = \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} [(\nabla^0)^j \alpha_{1j} - (\nabla^0)_1 \text{tr}_{g_0}(\alpha)] e_0^2 \wedge e_0^3$$

where  $\nabla^0$  is the Levi-Civita connection of  $g_0$ . Since

$$(\nabla^0)^j \alpha_{1j} - (\nabla^0)_1 tr_{g_0}(\alpha) = \frac{\ln r}{r^2} \Delta_{S^2} p + \frac{4p}{r^2} + o\left(\frac{1}{r^2}\right),$$

we obtain

$$E(\bar{g}) = p.$$

Since it satisfies vacuum Einstein field equations, the Bondi's radiating metric satisfies the dominant energy condition automatically. Therefore the scalar curvature  $\bar{R}$  of  $\bar{g}$  satisfies

$$\bar{R} \geq 2|Y|_{\bar{g}}^2 - 2div_{\bar{g}} Y$$

for certain vector field in  $\bar{\mathbb{X}}$ . Therefore a standard positive mass argument [16, 12] shows that

$$E(\bar{g}) = p \geq 0.$$

And  $p = 0$  if and only if the metric  $\bar{g}$  is flat which implies that  $(\bar{\mathbb{X}}, g, h)$  can be embedded into the Minkowski spacetime as a spacelike hypersurface with the induced metric  $g$  from the Minkowski metric and the second fundamental form  $h$ .

Integrating  $\mathcal{M}(0, \theta, \psi) = \frac{p}{2}$  over unit  $S^2$  and using Lemma 4.1, we obtain the Bondi energy-momentum of slice  $u = 0$

$$m_0(0) = \frac{p}{2}, \quad m_1(0) = m_2(0) = m_3(0) = 0.$$

Thus the theorem follows from Proposition 2.1.

Q.E.D.

## 5. POSITIVITY - WITTEN'S METHOD

In this section, we will use Witten's method [19] and the positive mass theorem near null infinity proved by the third author [21, 23] to study the positivity of the Bondi mass. Let  $(\mathbb{X}, g, h)$  be an asymptotically null spacelike hypersurface. Denote

$$\begin{aligned} E_\nu(\mathbb{X}) &= \frac{1}{16\pi} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{E} n^\nu r \check{e}^2 \wedge \check{e}^3, \\ P_\nu(\mathbb{X}) &= \frac{1}{8\pi} \lim_{r \rightarrow \infty} \int_{S_r} \mathcal{P} n^\nu r \check{e}^2 \wedge \check{e}^3 \end{aligned}$$

where

$$\begin{aligned} \mathcal{E} &= \check{\nabla}^j a_{1j} - \check{\nabla}_1 tr_{\check{g}}(a) - [a_{11} - \delta_{11} tr_{\check{g}}(a)], \\ \mathcal{P} &= b_{11} - \delta_{11} tr_{\check{g}}(b). \end{aligned}$$

Theorem 4.1 in [21] indicates if  $(\mathbb{X}, g, h)$  is asymptotically null spacelike hypersurface of order  $\tau > \frac{3}{2}$  in vacuum Bondi's radiating spacetime (2.1), then

$$E_0(\mathbb{X}) - P_0(\mathbb{X}) \geq \sqrt{\sum_{1 \leq i \leq 3} [E_i(\mathbb{X}) - P_i(\mathbb{X})]^2} \quad (5.1)$$

and the equality implies the spacetime is flat over  $\mathbb{X}$ . (Theorem 4.1 was proved for  $\tau = 3$ . However, the argument goes through if  $c|_{u=0} = d|_{u=0} = 0$  for the above  $(\mathbb{X}, g, h)$  in the Bondi's radiating spacetimes. See also Theorem 3.1 and Remark 3.1 in [23]. The sharp order  $\tau > \frac{3}{2}$  together with certain integrable conditions was also given in [8, 6] to ensure the argument to work.) In general, the hyperbolic mass of an asymptotically null spacelike hypersurface is different from the Bondi mass of the null cone. For instance, if  $c|_{u=0}$  or  $d|_{u=0}$  is nonzero,  $E_0(\mathbb{X}) - P_0(\mathbb{X})$  may not be finite.

**Lemma 5.1.** *Let  $(\mathbb{L}^{3,1}, \tilde{g})$  be a vacuum Bondi's radiating spacetime with metric  $\tilde{g}$  given by (2.1). Let  $(\mathbb{X}, g, h)$  be a spacelike hypersurface  $u$  which is given by (3.3) for  $r$  sufficiently large. Denote  $L(\phi, \psi) = l(0, \phi, \psi)$ ,  $\bar{L}(\phi, \psi) = \bar{l}(0, \phi, \psi)$ . Then*

$$\begin{aligned} \mathcal{E} &\approx \frac{12}{r^2}(c^2 + d^2)_{u=0} \\ &\quad + \frac{1}{r^3}(M + 16a_3 + 15cc_0 + 15dd_0)_{u=0} \\ &\quad - \frac{1}{2r^3}(L_{,2} + L \cot \theta + \bar{L}_{,3} \csc \theta), \\ \mathcal{P} &\approx -\frac{1}{2r^3}(3M - 16a_3 + 5cc_0 + 5dd_0)_{u=0} \\ &\quad + \frac{1}{2r^3}(L_{,2} + L \cot \theta + \bar{L}_{,3} \csc \theta). \end{aligned}$$

*Proof :* Note that

$$\begin{aligned} a_{22} + a_{33} &\approx \frac{4}{r^2}(c^2 + d^2)_{u=0} + \frac{4}{r^3}(cc_{,0} + dd_{,0})_{u=0}, \\ b_{22} + b_{33} &\approx \frac{1}{2r^3}(3M - 16a_3 + 5cc_{,0} + 5dd_{,0})_{u=0} \\ &\quad - \frac{1}{2r^3}(L_{,2} + L \cot \theta + \bar{L}_{,3} \csc \theta). \end{aligned}$$

Using the formula

$$\check{\nabla}_k a_{ij} = \check{e}_k(a_{ij}) - a_{jl}\check{\omega}_{li}(\check{e}_k) - a_{il}\check{\omega}_{lj}(\check{e}_k),$$

we obtain

$$\begin{aligned}
 \mathcal{E} &= \check{e}_j(a_{1j}) - a_{jl}\check{\omega}_{l1}(\check{e}_j) - a_{1l}\check{\omega}_{lj}(\check{e}_j) - \check{\nabla}_1 \text{tr}_{\check{g}}(a) \\
 &\quad + a_{22} + a_{33} \\
 &\approx -\frac{1}{2r^3}(L_{,2} + L \cot \theta + \bar{L}_{,3} \csc \theta) \\
 &\quad + \frac{1}{r^3}(M + 16a_3 - cc_{,0} - dd_{,0})_{u=0} \\
 &\quad + \frac{8\sqrt{1+r^2}}{r^3}(c^2 + d^2)_{u=0} \\
 &\quad + \frac{12\sqrt{1+r^2}}{r^4}(cc_{,0} + dd_{,0})_{u=0} \\
 &\quad + \frac{4}{r^3}(c^2 + d^2)_{u=0} + \frac{4}{r^4}(cc_{,0} + dd_{,0})_{u=0} \\
 &\approx -\frac{1}{2r^3}(L_{,2} + L \cot \theta + \bar{L}_{,3} \csc \theta) \\
 &\quad + \frac{1}{r^3}(M + 16a_3 + 15cc_{,0} + 15dd_{,0})_{u=0} \\
 &\quad + \frac{12}{r^2}(c^2 + d^2)_{u=0} + O\left(\frac{1}{r^4}\right), \\
 \mathcal{P} &= -b_{22} - b_{33} \\
 &\approx -\frac{1}{2r^3}(3M - 16a_3 + 5cc_{,0} + 5dd_{,0})_{u=0} \\
 &\quad + \frac{1}{2r^3}(L_{,2} + L \cot \theta + \bar{L}_{,3} \csc \theta).
 \end{aligned}$$

Q.E.D.

**Theorem 5.1.** *Let  $(\mathbb{L}^{3,1}, \tilde{g})$  be a vacuum Bondi's radiating spacetime with metric  $\tilde{g}$  given by (2.1). Suppose that **Condition A** and **Condition B** hold and  $c|_{u=u_0} = d|_{u=u_0} = 0$  for some  $u_0$ . Then*

$$m_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} m_i^2(u)}$$

for all  $u \leq u_0$ . If the equality holds for some  $u \in (-\infty, u_0]$ ,  $\mathbb{L}^{3,1}$  is flat in the region foliated by all spacelike hypersurfaces which are given by

$$u = u_0 + \sqrt{1+r^2} - r + o\left(\frac{1}{r^4}\right)$$

for  $r$  sufficiently large. In particular, if the equality holds for all  $u \leq u_0$ ,  $\mathbb{L}^{3,1}$  is flat in the region  $\{u \leq u_0\}$ .

*Proof* : By translation, we can assume that  $u_0 = 0$ . Choose an asymptotically null spacelike hypersurface  $\mathbb{X}$  which is given by (3.3) with  $a_3 = 0$  for  $r$  sufficiently large. By Lemma 5.1, we obtain

$$\mathcal{E} - \mathcal{P} \approx -\frac{1}{r^3}(L_{,2} + L \cot \theta + \bar{L}_{,3} \csc \theta) + \frac{5M(0, \theta, \psi)}{2r^3}.$$

Then Lemma 4.1 implies that

$$E_\nu(\mathbb{X}) - P_\nu(\mathbb{X}) = \frac{5}{8}m_\nu(0).$$

Therefore the first part of the theorem follows from (5.1) and Proposition 2.1. For the second part, if the equality holds for some  $u \in (-\infty, u_0]$ , then the equality holds for  $u = u_0$  by Proposition 2.1. Thus  $\mathbb{L}^{3,1}$  is flat over  $\mathbb{X}$  and it follows. Q.E.D.

## 6. MODIFIED BONDI ENERGY-MOMENTUM

We can modify the definition of the Bondi energy-momentum to remove **Condition B**. Define the modified Bondi energy-momentum as

$$\mathbf{m}_\nu(u_0) = \frac{1}{4\pi} \int_{S^2} \mathcal{M}(u_0, \theta, \psi) n^\nu dS \quad (6.1)$$

for  $\nu = 0, 1, 2, 3$ . Then we can prove that

$$\frac{d}{du} \mathbf{m}_\nu = -\frac{1}{4\pi} \int_{S^2} \left( (c_{,0})^2 + (d_{,0})^2 \right) n^\nu dS \quad (6.2)$$

for  $\nu = 0, 1, 2, 3$ , and

$$\frac{d}{du} \left( \mathbf{m}_0 - \sqrt{\sum_{1 \leq i \leq 3} \mathbf{m}_i^2} \right) \leq 0 \quad (6.3)$$

under **Condition A** only.

Now choosing the spacelike asymptotically null hypersurface  $\mathbb{X}$  given by (3.3) with  $a_3 = -\frac{M(0, \theta, \psi)}{16}$ . If  $c|_{u=0} = 0$ ,  $d|_{u=0} = 0$ , then

$$\mathcal{E} - \mathcal{P} \approx \frac{2\mathcal{M}(0, \theta, \psi)}{r^3}.$$

Therefore the following theorems are a direct consequence.



**Theorem 6.1.** *Let  $(\mathbb{L}^{3,1}, \tilde{g})$  be a vacuum Bondi's radiating spacetime with metric  $\tilde{g}$  given by (2.1). Suppose that **Condition A** holds. If (i) either  $\mathcal{M}(u_0, \theta, \psi)$  is constant, (ii) or  $c|_{u=u_0} = d|_{u=u_0} = 0$  for some  $u_0$ , then*

$$\mathbf{m}_0(u) \geq \sqrt{\sum_{1 \leq i \leq 3} \mathbf{m}_i^2(u)}$$

for all  $u \leq u_0$ . If the equality holds for some  $u \in (-\infty, u_0]$ ,  $\mathbb{L}^{3,1}$  is flat in the region foliated by all spacelike hypersurfaces which are given by

$$u = u_0 + \sqrt{1 + r^2} - r - \frac{M(u_0, \theta, \psi)}{16r^4} + o\left(\frac{1}{r^4}\right)$$

for  $r$  sufficiently large. In particular, if the equality holds for all  $u \leq u_0$ ,  $\mathbb{L}^{3,1}$  is flat in the region  $\{u \leq u_0 - \frac{M(u_0, \theta, \psi)}{16r^4}\}$ .

*Acknowledgements.* Xiao Zhang is partially supported by National Natural Science Foundation of China under grants 10231050, 10421001 and the Innovation Project of Chinese Academy of Sciences. This work was partially done when Wenling Huang visited the Morningside Center of Mathematics, Chinese Academy of Sciences, and she would like to thank the center for its hospitality. Part of the main results was announced in [23].

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