# Transformation of algebraic Gromov-Witten invariants of three-folds under flops and small extremal transitions, with an appendix from the stringy and the symplectic viewpoint 

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#### Abstract

We study how Gromov-Witten invariants of projective 3 -folds transform under a standard flop and a small extremal transition in the algebro-geometric setting from the recent development of algebraic relative Gromov-Witten theory and its applications. This gives an algebro-geometric account of Witten's wall-crossing formula for correlation functions of the descendant nonlinear sigma model in adjacent geometric phases of a gauged linear sigma model and of the symplectic approach in an earlier work of An-Min Li and Yongbin Ruan on the same problem. A terse account from the stringy and the symplectic viewpoint is given in the appendix to complement and compare to the discussion in the main text.


Key words: Gromov-Witten invariant, stable relative map, degeneration formula, standard flop, small extremal transition, Calabi-Yau 3-fold, superstring, gauged linear sigma model, phase.

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> C.-H.L. dedicates this review
> to the numerous teachers who educated him and to Ling-Miao Chou for her tremendous love.

A reflection on string $/ M / F / \cdots$ ？－theory，cf．Sec．A．1：
玄之又玄, 衆妙之門。
（ Mystery and beyond mystery，door to all magics．）
大方無隅, 大器晚成, 大音希聲, 大象無形。
（So large that it has no bounds；
so big that it takes a long time to make；
so harmonious that it fits no tunes；
so beautiful that it assumes no shapes．）
$\sim$ Lao－Tzu（600 B．C．），Tao－te Ching
（The Scripture on the Way and its Virtue）， excerpt from Chapter 1 and Chapter 41.

English translation by Ling－Miao Chou．

## Gromov-Witten Invariants under Flops and Extremal Transitions

## 0. Introduction and outline.

Surgeries on and transitions of manifolds/varieties are important issues in classifications of Kähler manifolds and varieties. Once the Gromov-Witten theory for smooth manifolds/varieties are understood (see [Be], [Si1] for reviews and [L-T2], [Si2] for comparisons of the symplectic and the algebro-geometric construction), it is very natural that one wants to extend this understanding to transformations of Gromov-Witten invariants when the manifold/variety changes. Indeed, this theme lies on the cross-road of superstring theory, symplectic geometry, and algebraic geometry.

The generating function of Gromov-Witten invariants of a Calabi-Yau manifold $X$ corresponds to the correlation functions of the A-model for $X$, [Wit4]. The transition of the Calabi-Yau manifolds corresponds to moving from one phase to another of the $d=2, N=(2,2)$ superconformal field theory (SCFT) with a specified number of chiral multiplets depending on the dimension of $X$; and the associated transformation of GromovWitten invariants corresponds to the transformation of the correlation functions of the associated A-models in the different phases of SCFT. In the case of flops of Calabi-Yau 3folds realizable as complete-intersection subvarieties of a toric variety, Witten [Wit5] (year 1993) has put this picture manifestly as a transition between adjacent geometric phases of a $(d=2, N=(2,2))$ gauged linear sigma model (GLSM; [Wit5], see also [M-P], [H-V] and [H-I-V]) and conjectured a wall-crossing formula (cf. [Wit5: Sec. 5.5 and Eq.(5.48)]) by isolating the effect of the rigid curve involved in the flop to the full correlation function (expressed as an instanton sum). The reason that this remained only a conjecture as stated in [Wit5] (even assuming the multiple cover formula at that time) is that it is not clear that the effect of the original rigid curve to the instanton sum of the flopped CalabiYau 3-fold can be really isolated out in such a simple way since curves can break under deformations and there are also possible contributions to Gromov-Witten invariants from reducible curves with some but not all component wrapping the rigid curve.

A such technicality needed to understand the exact details was later fulfilled in the work of A.-M. Li and Y. Ruan in [L-R] (year 1998) within the program of a general study of Gromov-Witten invariants in birational geometry. There they developed a symplectic relative Gromov-Witten theory and derived a gluing formula for symplectic cuts, (see also [I-P1] and [I-P2] of E.-N. Ionel and T.H. Parker for related works). This enabled them to isolate the common summand and the different summand in the instanton sum of 3 -folds on the two sides of a standard flop or a small extremal transition and to understand their changes. How the curves that are involved in the surgery affect the transformation of Gromov-Witten invariants under flops can be extracted by using the gluing formula as a medium; a complete statement of Witten's formula can then be spelled out and justified ([L-R: Theorem A, Corollary A.2, Theorem B, Corollary B.2]).

In the history of the development of Gromov-Witten theory, the symplectic construction and the algebro-geometric construction have been going side-by-side with each other. It is thus desirable to also have a pure algebro-geometric construction parallel to the symplectic relative Gromov-Witten theory in [L-R], [I-P1], and [I-P2]. This was realized by
$\mathrm{J} . \mathrm{Li}$ in $[\mathrm{Li} 1]$ and [Li2], who constructed an algebraic relative Gromov-Witten theory and derived a degeneration formula with respect to a relative ample line bundle for a projective family over a smooth curve with the degenerate fiber $Y_{1} \cup_{D} Y_{2}$ a gluing of two smooth variety-divisor pairs $\left(Y_{i}, D\right)$. His work was later used in [L-Y1] and [L-Y2] to understand Gromov-Witten invariants associated to a curve class for smooth projective varieties under blow-ups along a smooth locus and for a projective conifold that fits in a degeneration with smooth total space. These are the algebro-geometric counterparts of the main tools of $[L-R]$ in symplectic category. The remaining discussions needed to lead to the same statements of [L-R] is given in this note. Subject to the fact that choices of cycles in the algebraic geometry are more restrictive than in symplectic geometry, this puts Witten's wall-crossing formula and the general results of $[\mathrm{L}-\mathrm{R}]$ in the algebro-geometric category.

Finally, it should be noted that there are more general flip-flops and extremal transitions in the study of birational geometry and classifications of Kähler manifolds. We conclude this introduction with a remark from Y. Ruan on a comparison of the symplectic and the algebro-geometric setting for transformations of Gromov-Witten invariants under flops and small extremal transitions:

Remark [symplectic vs. algebraic]. ([Ru3].) In algebro-geometric category there are many flops other than the standard flop. How to decompose them into a composition of blowups and blow-downs can be complicated. In contrast, in the symplectic category one can perform a local holomorphic deformation to deform extremal rational curves into a disjoint union of rational curves of $\mathcal{O}_{\mathbb{P}^{1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-1)$-type curve neighborhood. These curve neighborhood can be patched with the complex structure on the complement of these curves to form a tamed almost complex structure. In such a fashion, one can decompose a general flop into a disjoint union of standard flops in the almost complex category. In this way, the results of $[L-R]$ work for any flops instead of just the standard flop of 3-folds. Similarly for small extremal transitions.

Readers are referred to [L-R: Sec. 1], [Mo2], and [Ru2] for an overview of Gromov-Witten theory in this direction. It can be challenging to give a complete algebro-geometric treatment for Gromov-Witten invariants under general flips/flops/extremal-transitions as well.

For the contents of the note, in Sec. 1 and Sec. 2, we recall the basic definitions and results from [Li1], [Li2], [L-Y1], and [L-Y2] needed for the discussions. In Sec. 3, we derive from these the transformation formula of Gromov-Witten invariants of projective 3-folds under a standard flop and a small extremal transition. A terse account of the problem from the stringy and the symplectic viewpoint is given in the Appendix to compare to and to complement the algebro-geometric viewpoint in the main text.

Convention. Standard notations, terminology, and operations in algebraic geometry can be found in $[\mathrm{Ha}],[\mathrm{Kol}-\mathrm{M}]$, and $[\mathrm{Fu}]$. All schemes are over $\mathbb{C}$. The word "relative" has two different but complementary meanings: one from topology, e.g. a relative map to a smooth variety-divisor pair $(X, D)$ means a map to $X$ with image intersecting $D$ properly; and the
other from algebraic geometry in Grothendieck's formulation meaning relative to a base scheme, e.g. a relative cycle on $W / \mathbb{A}^{1}$ in this note means a formal linear combination of subvarieties on $W$ that are flat over $\mathbb{A}^{1}$.

## Outline.

1. Algebraic relative Gromov-Witten theory and degeneration formulas.
2. Degenerations associated to blow-ups and conifolds.
3. Transformation of GW-invariants of 3-folds.
3.1 Gromov-Witten invariants of 3 -folds under a standard flop.
3.2 Gromov-Witten invariants of 3 -folds under a small extremal transition.

Appendix. The stringy and the symplectic aspect.
A. 1 Transformation of GW-invariants from the stringy viewpoint.
A. 2 Transformation of GW-invariants from the symplectic viewpoint.

## 1 Algebraic relative Gromov-Witten theory and degeneration formulas.

We recall the basic ingredients of the algebraic relative Gromov-Witten theory needed for the discussion. See [Li1], [Li2], and [G-V] for more details.

## Algebraic relative Gromov-Witten theory.

Let $(Y, D)$ be a smooth variety-divisor pair with $Y$ projective. A relative map $f: C \rightarrow$ $(Y, D)$ is a morphism from a prestable curve $C$ to $Y$ such that $f$ meets $D$ properly (i.e. $f^{-1}(D)$ is a divisor on $\left.C\right)$. Fix a topological type $\Gamma=(g, n ; \beta)$ where $g$ is the genus of a prestable curve, $n$ is the number of marked points on that curve and $\beta \in H_{2}(Y ; \mathbb{Z})$, the moduli stack $\mathcal{M}_{g, n}(Y, D ; \beta)$ of relative maps of topological type $\Gamma$ and with finite automorphism group is a Deligne-Mumford stack, from the Hilbert-scheme construction in $[\mathrm{F}-\mathrm{P}]$. Its compactification in the usual moduli stack of $\overline{\mathcal{M}}_{g, n}(Y ; \beta)$ of stable maps to $Y$ of topological type $\Gamma$ can contain maps $f: C \rightarrow Y$ that send a subcurve of $C$ to $D$. When such degeneracy occurs in an $\mathbb{A}^{1}$-family of stable maps to $Y$ with generic fiber a relative map, it can be removed by a finite base change, of degree bounded above by a number determined by $\Gamma$, and then refilling both the domain curve and the target space of the bad fibers. This procedure extends the target space of relative maps in question to the fibers of the universal family of the stack $\mathfrak{Y}^{\text {rel }}$ of expanded relative pairs associated to $(Y, D)$. $\mathfrak{Y}^{\text {rel }}$ with its universal family is the descent of the local standard models of expanded relative pairs $(Y[m], D[m]) \rightarrow \mathbb{A}^{m}$ constructed in [Li1: Sec. 4.1], (see also [G-V: Sec. 2.6]). As an abstract stack, $\mathfrak{Y}^{\text {rel }}$ is isomorohic to the open substack $\mathcal{T}$ of $\overline{\mathcal{M}}_{0,3}$ that parameterizes prestable curves $\left(\mathbb{P}_{[m]}^{1} ; 0,1, \infty\right)$ of arithmetic genus 0 with all nodes separating $\infty$ from
$\{0,1\}$. The fiber of the universal family of $\mathfrak{Y}^{\text {rel }}$ over the $\mathbb{C}$-point $\left(\mathbb{P}_{[m]}^{1} ; 0,1, \infty\right)$ of $\mathfrak{Y}^{\text {rel }}$ is given by the gluing of $Y$ with an $m$-chain of ruled varieties $\Delta_{i}:=\mathbb{P}\left(\mathcal{O}_{D} \oplus \mathcal{N}_{Z / Y}\right)$ over $D$, $i=1, \ldots, m$ :

$$
Y_{[m]}:=Y \cup_{D=D_{1,0}} \Delta_{1} \cup_{D_{1, \infty}=D_{2,0}} \cdots \cup_{D_{m-1, \infty}=D_{m, 0}} \Delta_{m}
$$

with

$$
D_{[m]}=D_{m, \infty} \subset \Delta_{m}
$$

Here, $\mathcal{N}_{D / Y}$ is the normal bundle of $Z$ in $Y, D_{i, 0}\left(\right.$ resp. $\left.D_{i, \infty}\right)$ is the section of $\Delta_{i} \rightarrow$ $D$ corresponding to $\mathcal{N}_{D / Y}$ (resp. $\mathcal{O}_{D}$ ), and the gluing $D_{i, \infty}=D_{i+1,0}$ is given by the projection $\Delta_{i} \rightarrow D$, cf. Figure 1-1. By construction, there is a tautological morphism $\varphi:(Y[m], D[m]) \rightarrow(Y, D)$ that pinches all the $\Delta_{i}$ in fibers of $Y[m]$ over $\mathbb{A}^{m}$ to $D$.


Figure 1-1. A local model $(Y[m], D[m]) / \mathbb{A}^{m}$ for the universal family of the stack $\mathfrak{Y}^{\text {rel }}$ of expanded relative pairs associated to $(Y, D)$ and its tautological morphism to $(Y, D)$. The fiber over the origin $0 \in \mathbb{A}^{m}$ is $\left(Y_{[m]}, D_{[m]}\right)$. Illustrated here is the case $m=2$.

Definition 1.1 [admissible weighted graph]. ([Li1: Definition 4.6].) Given a relative pair $(Y, D)$, an admissible weighted graph $\Gamma$ for $(Y, D)$ is a graph without edges together with the following data:
(1) an ordered collection of legs, an ordered collection of weighted roots, and two weight functions on the vertex set $g: V(\Gamma) \rightarrow \mathbb{Z}_{\geq 0}$ and $b: V(\Gamma) \rightarrow H_{2}(Y ; \mathbb{Z})$,
(2) $\Gamma$ is relatively connected in the sense that either $|V(\Gamma)|=1$ or each vertex in $V(\Gamma)$ has at least one root attached to it.

Given an admissible weighted graph $\Gamma$ for $(Y, D)$ with $l$ vertices, $n$ legs, and $r$ roots, a stable relative map of type $\Gamma$ to the fibers of the universal family of $\mathfrak{Y}^{\text {rel }}$ is locally modelled
on a diagram: (in notation, $f: \mathcal{C} / S \rightarrow Y[m] / \mathbb{A}^{m}$ )

where

- $\mathcal{C}$ is a disjoint union of flat families $\mathcal{C}_{i}, i=1, \ldots, l$, of prestable curves over $S$ of arithmetic genus $g\left(v_{i}\right)$;
- $p_{i}: S \rightarrow \mathcal{C}, i=1, \ldots, n$, and $q_{j}: S \rightarrow \mathcal{C}, j=1, \ldots, r$ are disjoint sections away from the singular locus of fibers of $\mathcal{C}$ over $S$ such that $p_{i}(S) \subset \mathcal{C}_{k}$ (resp. $q_{j}(S) \subset \mathcal{C}_{k}$ ) if $i$-th leg (resp. $j$-th root) is attached to the vertex $v_{k}$ of $\Gamma$;
- $f$ is a nondegenerate predeformable morphism to $Y[m]$ such that $f^{-1}(D[m])=$ $\sum_{j=1}^{r} \mu_{j} q_{j}(S)$, where $\mu_{j}$ is the weight associated to the $j$-th root, and that at each closed point $s \in S, \varphi_{*}\left(f\left(\mathcal{C}_{i s}\right)\right)=b\left(v_{i}\right)$ in $H_{2}(Y ; \mathbb{Z})$;
- at each closed point $s \in S, f_{s}$ has only a finite automorphism group,
([Li1: Sec. 2.2, Sec. 3.1, Sec. 4.2]). We remark that in the above definition, an automorphsim of $f: C \rightarrow Y_{[m]}$ is a pair $(a, b) \in \operatorname{Aut}(C) \times \operatorname{Aut}\left(Y_{[m]} / Y\right)$ (i.e. $b$ is an automorphism on $Y_{[m]}$ that descends to the identity map on $Y$ via $\varphi$ ) such that $f \circ a=b \circ f$, (cf. [Li1: beginning of Sec. 3.1] for the definition of isomorphism classes and automorphisms of relative maps to an extended relative pair, see also [G-V: Sec. 2]). The space of all such stable relative maps of type $\Gamma$ is a Deligne-Mumford stack $\mathfrak{M}(Y, D ; \Gamma)\left(=\mathfrak{M}\left(\mathfrak{Y}^{\text {rel }}, \Gamma\right)\right.$ in [Li1], [Li2], [L-Y1], [L-Y2]; here we adopt a notation change to make what is involved explicit), [Li1: Definition 4.9, Theorem 4.10].

Suppose that $\Gamma$ has $n$ legs and $r$ roots, then there is an evaluation map

$$
e v: \mathfrak{M}(Y, D ; \Gamma) \longrightarrow Y^{n}
$$

associated to the ordinary $n$ marked points on the domain curve of a relative stable morphism and a distinguished evaluation map

$$
\mathbf{q}: \mathfrak{M}(Y, D ; \Gamma) \longrightarrow D^{r}
$$

associated to the $r$ distinguished marked points that are required to be the only points that are mapped to $D[m]$ in a local model.

The obstruction theory associated to the deformation problems related to the stack $\mathfrak{M}(Y, D ; \Gamma)$ are studied in [Li2: Sec. 1 and Sec. 5], see also [G-V: Sec. 2]. A cohomological description of the deformations of the separate constituents of a stable relative map and the natural clutching morphisms that relate the various separate deformation of the constituents are given there. The perfectness of the obstruction theory and hence virtual fundamental class on $[\mathfrak{M}(Y, D ; \Gamma)]^{\text {virt }}$ are proved and constructed in [Li2: Sec. 2]. (See also
[G-V: Sec. 2.8 and Sec. 2.9].) The relative Gromov-Witten invariants of the pair ( $Y, D$ ) associated to $\Gamma$ are then defined by

$$
\begin{array}{ccc}
\Psi_{\Gamma}^{(Y, D)}: H^{*}(Y)^{\times n} \times H^{*}\left(\mathfrak{M}_{\Gamma}\right) & \longrightarrow & H_{*}\left(D^{r}\right) \\
(\alpha, \zeta) & \longmapsto \mathbf{q}_{*}\left(e v^{*}(\alpha) \cup \pi_{\Gamma}^{*}(\zeta)[\mathfrak{M}(Y, D ; \Gamma)]^{\text {virt }}\right),
\end{array}
$$

where $\mathfrak{M}_{\Gamma}$ is the moduli stack of (possibly disconnected) stable nodal curves of the topological type specified by $\Gamma$ and $\pi_{\Gamma}: \mathfrak{M}(Y, D ; \Gamma) \rightarrow \mathfrak{M}_{\Gamma}$ is the forgetful morphism.

## The degeneration formula with respect to a relative ample line bundle.

Let $\pi:\left(W, W_{0}\right) \rightarrow\left(\mathbb{A}^{1}, \mathbf{0}\right)$ be a degeneration with the total space $W$ smooth, $\pi$ projective, the fiber $W_{t}$ over $t \neq \mathbf{0}$ smooth, and the fiber $W_{0}$ over $\mathbf{0} \in \mathbb{A}^{1}$ the gluing $Y_{1} \cup_{D} Y_{2}$ of smooth varieties $Y_{i}$ along isomorphic smooth divisor $D_{i} \simeq D$. Fix a relative ample line bundle $H$ on $W / \mathbb{A}^{1}$. Let $(g, n ; d)$ be a triple of integers. Then a moduli stack $\mathfrak{M}(\mathfrak{W J},(g, n ; d))$ of stable maps from (connected) prestable curves of topological type ( $g, n$ ) $(=($ arithmetic, number of marked points) $)$ to fibers of $\pi$ with $H$-degree $d$ is constructed in [Li1]. By construction, $\mathfrak{M}(\mathfrak{W},(g, n ; d))$ fibers over $\mathbb{A}^{1}$ such that the fiber $\mathfrak{M}(\mathfrak{W},(g, n ; d))_{t}$ over $t \neq \mathbf{0}$ is (a disjoint union over curve classes of $H$-degree $d$ of) the usual moduli stack of stable maps to the smooth $W_{t}$ while the fiber $\mathfrak{M}(\mathfrak{W J},(g, n ; d))_{0}$ over $\mathbf{0}$ gives the moduli stack of stable maps to the singular fiber $W_{0}$, (which is new in algebro-geometric category).

The deformation-obstruction theory of this moduli problem and the perfectness of the tangent-obstruction complex on $\mathfrak{M}(\mathfrak{W},(g, n ; d))$ is studied in [Li2]. The virtual fundamental class $[\mathfrak{M}(\mathfrak{W},(g, n ; d))]^{\text {virt }}$ thus constructed fibers over $\mathbb{A}^{1}$ as well, with the property that its restriction to each fiber $\mathfrak{M}(\mathfrak{W},(g, n ; d))_{t}$ gives the virtual fundamental class $\left[\mathfrak{M}(\mathfrak{W},(g, n ; d))_{t}\right]^{\text {virt }}$ identical to the one constructed directly on the stack $\mathfrak{M}(\mathfrak{W},(g, n ; d))_{t}$. (With appropriate cycles inserted via evaluation maps), this gives a constancy over $\mathbb{A}^{1}$ of Gromov-Witten invariants of fibers $W_{t}$ of $W \rightarrow \mathbb{A}^{1}$ for all $t$. The Gromov-Witten invariants of $W_{0}=Y_{1} \cup_{D} Y_{2}$ can be further expressed in terms of the relative Gromov-Witten invariants of $\left(Y_{1}, D\right)$ and $\left(Y_{2}, D\right)$. The upshot is a degeneration formula that expresses a summation of the usual Gromov-Witten invariants of a fiber $W_{t}, t \neq \mathbf{0}$ in terms of a combination of relative Gromov-Witten invariants of pairs $\left(Y_{1}, D\right)$ and $\left(Y_{2}, D\right)$.

To state the degeneration formula precisely, recall the following definition:
Definition 1.2 [admissible triple]. ([Li1: Definition 4.11].) Given a gluing $Y_{1} \cup_{D} Y_{2}$ of relative pairs, let $\Gamma_{1}$ and $\Gamma_{2}$ be a pair of admissible weighted graphs for $\left(Y_{1}, D\right)$ and $\left(Y_{2}, D\right)$ respectively. Suppose that $\Gamma_{1}$ and $\Gamma_{2}$ have identical number $r$ of roots and $n_{1}$-many and $n_{2}$-many legs respectively. Let $n=n_{1}+n_{2}$ and $I \subset\{1, \ldots, n\}$ be a set of $n_{1}$ elements. Then $\left(\Gamma_{1}, \Gamma_{2}, I\right)$ is called an admissible triple if the following conditions hold:
(1) the weights on the roots of $\Gamma_{1}$ and $\Gamma_{2}$ coincide: $\mu_{1, i}=\mu_{2, i}, i=1, \ldots, r$;
(2) after connecting the $i$-th root of $\Gamma_{1}$ and the $i$-th root of $\Gamma_{2}$ for all $i$, the resulting new graph with $n$ legs and no roots is connected.

Given an admissible triple $\eta=\left(\Gamma_{1}, \Gamma_{2}, I\right)$ as above with $Y_{1} \cup_{D} Y_{2}=$ the degenerate fiber $W_{0}$ of $W / \mathbb{A}^{1}$, one has the genus function

$$
g(\eta):=r+1-\left|V\left(\Gamma_{1} \amalg \Gamma_{2}\right)\right|+\sum_{v \in V\left(\Gamma_{1}\right) \cup V\left(\Gamma_{2}\right)} g(v)
$$

and the $H$-degree function

$$
d(\eta):=\left.\sum_{v \in V\left(\Gamma_{1}\right)} b_{\Gamma_{1}}(v) \cdot H\right|_{Y_{1}}+\left.\sum_{v \in V\left(\Gamma_{2}\right)} b_{\Gamma_{2}}(v) \cdot H\right|_{Y_{2}} .
$$

Denote by $|\eta|$ the triple of integers $\left(g(\eta), n=n_{1}+n_{2} ; d(\eta)\right)$. For each $\eta$, one has a gluing morphism

$$
\Phi_{\eta}: \mathfrak{M}\left(\mathfrak{Y}_{1}^{\text {rel }}, \Gamma_{1}\right) \times_{E^{r}} \mathfrak{M}\left(\mathfrak{Y}_{2}^{\text {rel }}, \Gamma_{2}\right) \rightarrow \mathfrak{M}(\mathfrak{W},(g, n ; d)),
$$

which is finite étale of pure degree $|\operatorname{Eq}(\eta)|$ to its image $\mathfrak{M}\left(\mathfrak{Y}_{1}^{\text {rel }} \sqcup \mathfrak{Y}_{2}^{\text {rel }}, \eta\right)$ in and topologically isomorphic to $\mathfrak{M}\left(\mathfrak{W}_{0}, \eta\right)$. [Li1: Sec. 4.2] and [Li2: Sec. 3.2]. (Here $\operatorname{Eq}(\eta)$ is the set of permutations of the $r$-many roots in $\Gamma_{1}$ that leaves $\eta$ unchanged.)

Given $(g, n ; d)$, let $\Omega_{(g, n ; d)}^{H}$ be the set of admissible triples $\eta$ for the gluing $Y_{1} \cup_{D} Y_{2}$ such that $|\eta|=(g, n ; d), \bar{\Omega}_{(g, n ; d)}^{H}$ be the set of equivalence classes in $\Omega_{(g, n ; d)}^{H}$ from re-ordering of roots, and $\mathbf{m}(\eta)$ be the product of the weight of roots of $\Gamma_{1}$ in $\eta \in \bar{\Omega}_{(g, n ; d)}^{H}$. For $\eta \in \bar{\Omega}_{(g, n ; d)}^{H}$, assume that $G_{\eta}^{*}(\zeta)=\sum_{j \in K_{\eta}} \zeta_{\eta, 1, j} \boxtimes \zeta_{\eta, 2, j}$, where $G_{\eta}: \mathfrak{M}_{\Gamma_{1}} \times \mathfrak{M}_{\Gamma_{2}} \rightarrow \mathfrak{M}_{g, n}$ is the natural morphism between the related moduli stacks of nodal curves. Then J. Li's degeneration formula of Gromov-Witten invariants in numerical form reads: ([Li2: Sec. 3 and Sec. 4])

$$
\begin{aligned}
& \Psi_{(g, n ; d)}^{W_{t}}(\alpha(t), \zeta) \\
& \quad=\sum_{\eta \in \bar{\Omega}_{(g, n ; d)}^{H}} \frac{\mathbf{m}(\eta)}{|\operatorname{Eq}(\eta)|} \sum_{j \in K_{\eta}}\left[\Psi_{\Gamma_{1}}^{\left(Y_{1}, D\right)}\left(j_{1}^{*} \alpha(0), \zeta_{\eta, 1, j}\right) \bullet \Psi_{\Gamma_{2}}^{\left(Y_{2}, D\right)}\left(j_{2}^{*} \alpha(0), \zeta_{\eta, 2, j}\right)\right]_{0},
\end{aligned}
$$

where $\Psi_{(g, n ; d)}^{W_{t}}(\alpha(t), \zeta), t \neq \mathbf{0}$, is the summation of the usual algebraic Gromov-Witten invariants of $W_{t}$ over curves classes on $W_{t}$ of $H$-degree $d ; \alpha \in H_{c}^{0}\left(R^{*} \pi_{3 *} \mathbb{Q}_{W}\right)^{\times n} ; j_{i}$ is the inclusion map $Y_{i} \hookrightarrow W_{0} ; \bullet$ the intersection product in $H_{*}\left(D^{r}\right)$, where $r$ is the number of roots in either of the admissible graphs in the admissible triple $\eta$; and $[\cdot]_{0}$ is the degree-0 component of elements in $H_{*}\left(D^{r}\right)$.

## 2 Degenerations associated to blow-ups and conifolds.

For our applications, the degeneration formula in Sec. 1 has to be refined to one with respect to curve classes. This was done in [L-Y1] and [L-Y2] for the two situations: blowup along a smooth subvariety and the conifold degeneration of 3 -folds. These are closely related to flops and small extremal transitions of 3 -fold. We recall the main results to be used in Sec. 3 and leave readers to [L-Y1] and [L-Y2] for more details.

Remark. 2.1 [homological vs. numerical equivalence]. For a smooth projective variety $X$ over $\mathbb{C}$, the group of homological equivalence classes of (algebraic) curve classes in $H_{2}(X ; \mathbb{Z})$ embeds in the group $N_{1}(X)$ of numerical equivalence classes of (complex) 1-cycles. Since our focus is on curve classes, the discussion and statements in [L-Y1] and [L-Y2] via $N_{1}(X)$ can be converted to ones on the set of curve classes in $H_{2}(X ; \mathbb{Z})$. ([Fu] and [Lie].) For the same reason, all the torsion classes in $H_{2}(X ; \mathbb{Z})$ can be ignored.

## Blow-ups.

Let $X$ be a smooth projective variety and $Z \subset X$ be a smooth subvariety of $X$. Associated to $(X, Z)$ is the degeneration $\pi: W \rightarrow X \times \mathbb{A}^{1}$ from the blow-up of $X \times \mathbb{A}^{1}$ along $Z \times \mathbf{0}$, where $\mathbf{0}$ is a point on $\mathbb{A}^{1}$. By construction, $W / \mathbb{A}^{1}$ has the degenerate fiber $W_{0}=Y_{1} \cup_{E} Y_{2}$ over $\mathbf{0} \in \mathbb{A}^{1}$, where $p_{1}: Y_{1}=B l_{Z} X \rightarrow X$ is the blow-up of $X$ along $Z, p_{2}: Y_{2}=$ $\mathbb{P}\left(\mathcal{N}_{Z / X} \oplus \mathcal{O}_{Z}\right) \rightarrow X$ is a projective space bundle over $Z$, and $E=\mathbb{P} \mathcal{N}_{Z / X}$ with $\mathcal{N}_{Z / X}$ being the normal bundle of $Z$ in $X$ is the exceptional divisor of $p_{1}$.

For a fixed $\pi$-ample line bundle $\mathcal{L}$ on $W$ and a topological type $(g, n ; d)$, then the degeneration formula in Sec. 1 applies to this family. Using the canonical morphism $W / \mathbb{A}^{1} \rightarrow X$, the moduli stack $\mathfrak{M}(\mathfrak{W},(g, n ; d))$ of stable maps from prestable curves into fibers of $W / \mathbb{A}^{1}$ constructed in [Li1] is decomposed into a disjoint union of collections of irreducible componensts of $\mathfrak{M}(\mathfrak{W},(g, n ; d))$ with each collection labelled by an element in $C_{(\mathcal{L}, d)}:=\left\{\beta \in H_{2}(X ; \mathbb{Z}): \mathcal{L} \cdot \beta=d\right\}$. The construction of [Li2] applies to each collection and gives rise to a degeneration formula with respect to a curve class $\beta \in C_{(\mathcal{L}, d)} \subset$ $H_{2}(X ; \mathbb{Z})$ :

$$
\begin{aligned}
& \Psi_{(g, n ; \beta)}^{X}(\alpha, \zeta) \\
& \quad=\sum_{\eta \in \bar{\Omega}_{(g, n ; \beta)}^{\mathcal{L}}} \frac{\mathbf{m}(\eta)}{|\operatorname{Eq}(\eta)|} \sum_{j \in K_{\eta}}\left[\Psi_{\Gamma_{1}}^{\left(Y_{1}, E\right)}\left(j_{1}^{*} \bar{\alpha}(0), \zeta_{\eta, 1, j}\right) \bullet \Psi_{\Gamma_{2}}^{\left(Y_{2}, E\right)}\left(j_{2}^{*} \bar{\alpha}(0), \zeta_{\eta, 2, j}\right)\right]_{0},
\end{aligned}
$$

where $\bar{\alpha}$ is any flat extension of $\alpha \in A_{*}(X)^{\oplus n}$ to $A_{*}\left(W / \mathbb{A}^{1}\right)^{\oplus n}$ and $\Omega_{(g, n ; \beta)}^{\mathcal{L}}=\{\eta=$ $\left.\left(\Gamma_{1}, \Gamma_{2}, I\right) \in \Omega_{(g, n ; d)}^{\mathcal{L}} \mid p_{1 *} b\left(\Gamma_{1}\right)+p_{2 *} b\left(\Gamma_{2}\right)=\beta\right\}$ is the $\beta$-compatible subset of $\Omega_{(g, n ; d)}^{\mathcal{L}}$.

To remove the possible $\mathcal{L}$-dependence on the right-hand side of the above identity, one can choose $\mathcal{L}$ on $W$ to be associated to a sufficiently very ample line bundle on $X$. For such $\mathcal{L}$, the set $\Omega_{(g, n ; \beta)}^{\mathcal{L}}$ of admissible triples in the above identity is stabilized to an $\mathcal{L}$-independent set $\Omega_{(g, n ; \beta)}$ and the degeneration formula above becomes intrinsic to the topological type $(g, n ; \beta)$ in the definition of Gromov-Witten invariants of $X$. We now describe $\Omega_{(g, n ; \beta)}$, which will be needed in Sec. 3.

Let $\gamma$ denote both the unique curve class in $H_{2}\left(Y_{i} ; \mathbb{Z}\right)$ that is contracted by $p_{i}$, (they corresponds to the same class on $E$ ). It has the property that $\gamma \cdot E=-1$ on $Y_{1}$ while $\gamma \cdot E=+1$ on $Y_{2}$. A curve class $\beta^{\prime} \in H_{2}(X ; \mathbb{Z})$ has a unique lifting to a curve class $\tilde{\beta}^{\prime 0}$ in $p_{1 *}^{-1}\left(\beta^{\prime}\right)$ such that all the curve classses in $p_{1 *}^{-1}\left(\beta^{\prime}\right)$ can be written in the form $\tilde{\beta}^{\prime 0}+l \gamma$
for some $l \in \mathbb{Z}_{\geq 0}$. Call $\tilde{\beta}^{\prime 0}$ the minimal lifting of $\beta^{\prime}$ to $Y_{1}$. When $\beta^{\prime}$ is a curve class in $H_{2}(Z ; \mathbb{Z}), \beta^{\prime}$ also has a minimal lifting to $Y_{2}$ with the similar property. Define

$$
\begin{aligned}
\left(H_{2}\left(Y_{1}\right) \times H_{2}\left(Y_{2}\right)\right)_{\beta}^{0}:= & \text { the set of pairs of curve classes }\left(\tilde{\beta}_{1}^{0}, \tilde{\beta}_{2}^{0}\right) \in H_{2}\left(Y_{1} ; \mathbb{Z}\right) \times H_{2}\left(Y_{2} ; \mathbb{Z}\right) \\
& \text { such that } p_{i *} \tilde{\beta}_{i}^{0} \neq 0, \tilde{\beta}_{i}^{0} \text { is the minimal lifting of } p_{i *} \tilde{\beta}_{i}^{0} \text { to } Y_{i}, \\
& i=1,2, \text { and } p_{1 *} \tilde{\beta}_{1}^{0}+p_{2 *} \tilde{\beta}_{2}^{0}=\beta ;
\end{aligned} \quad \begin{aligned}
\left(H_{2}\left(Y_{1}\right)\right)_{\beta}^{0}:= & \text { the set of the minimal lifting } \tilde{\beta}^{0} \text { of } \beta \text { such that } E \cdot \tilde{\beta}^{0} \geq 0 ; \\
\left(H_{2}\left(Y_{2}\right)\right)_{\beta}^{0}:= & \text { the set of the minimal lifting } \tilde{\beta}^{0} \text { of } \beta \text { to } Y_{2} .
\end{aligned}
$$

(By definition, $H_{2}\left(Y_{2}\right)_{\beta}^{0}$ is non-empty only when $\beta$ is representable by a curve in $Z$. Either of $H_{2}\left(Y_{1}\right)_{\beta}^{0}$ and $H_{2}\left(Y_{2}\right)_{\beta}^{0}$ is either empty or a singleton.) With these notations,

$$
\begin{aligned}
& \amalg \coprod_{\tilde{\beta}^{0} \in\left(H_{2}\left(Y_{1}\right)\right)_{\beta}^{0}}\left\{\begin{array}{l|l}
\Gamma_{1}: \text { admissible } & \bullet b\left(\Gamma_{1}\right)=\tilde{\beta}^{0}+\left(E \cdot \tilde{\beta}^{0}\right) \gamma ; \\
\text { weighted graph } & \text { • } g\left(\Gamma_{1}\right)=g, n \text {-many legs; } \\
\text { for }\left(Y_{1}, E\right) & \bullet \text { no roots. }
\end{array}\right\} \\
& \coprod_{\tilde{\beta}^{0} \in\left(H_{2}\left(Y_{2}\right)\right)_{\beta}^{0}}\left\{\left.\begin{array}{l|l}
\Gamma_{2}: \text { admissible } & \bullet b\left(\Gamma_{2}\right)=\tilde{\beta}^{0} ; \\
\text { weighted graph } \\
\text { for }\left(Y_{2}, E\right)
\end{array} \right\rvert\, \begin{array}{ll}
\bullet g\left(\Gamma_{2}\right)=g, n \text {-many legs } ; \\
\bullet \text { no roots } .
\end{array}\right\} .
\end{aligned}
$$

## Conifold degenerations.

Conifold degenerations have played some important roles in stringy dualities and in understanding how different phases of string theory may be connected. Consider a conifold degeneration given by a projective family $\pi: W \rightarrow \mathbb{A}^{1}$ of 3 -varieties with the total space $W$ and fibers $W_{t}, t \neq \mathbf{0}$, smooth and $Y:=W_{0}$ over $\mathbf{0} \in \mathbb{A}^{1}$ a conifold with an isolated singularity whose local analytic germ is modelled on Spec $\mathbb{C}[[x, y, z, w]] /(x y-z w)$. After a base change of degree 2 and resolution of resulting singularities, one obtains a semi-stable reduction $\pi^{s s}: W^{s s} \rightarrow \mathbb{A}^{1}$ of $\pi$ with $W_{t}^{s s}=W_{t^{2}}, t \neq \mathbf{0}$ and $p:=\pi_{0}^{s s}=p_{1} \cup_{E} p_{2}: W_{0}^{s s}=$ $\widetilde{Y} \cup_{E} Q \rightarrow Y$, where $p_{1}: \widetilde{Y} \rightarrow Y$ is the blow-up of $Y$ at the conifold singularity, $Q$ is the quadric hypersurface in $\mathbb{P}^{4}, p_{2}$ contracts the whole $Q$ to the singulaity of $Y$, and $E$ sits in $\widetilde{Y}$ as the exceptional divisor of the blow-up $\widetilde{Y} \rightarrow Y$ and in $Q$ from the intersection with a hyperplane of $\mathbb{P}^{4}$.

Fix a $\pi^{s s}$-ample line bundle $\mathcal{L}$ on $W^{s s}$ and a topological type $(g, n ; d)$, then the degeneration formula in Sec. 1 applies to the family $\pi^{s s}: W^{s s} \rightarrow \mathbb{A}^{1}$. Since $Y$ is topologically a deformation retract of $W_{t}$ for any $t \in \mathbf{0}, R^{\bullet} \pi_{*} \mathbb{Z}_{W}$ has a direct summand that is the trivial
local system on $\mathbb{A}^{1}$ whose fiber is canonically isomorphic to $H_{2}\left(W_{t} ; \mathbb{Z}\right)$ for any $t \in \mathbb{A}^{1}$. Using the tautological morphism $W^{s s} / \mathbb{A}^{1} \rightarrow W / \mathbb{A}^{1}$ from the semi-stable reduction, it follows that for any fixed $t_{0} \neq \mathbf{0}$, the stack $\mathfrak{M}\left(\mathfrak{W}^{s s},(g, n ; d)\right)$ of stable morphisms from prestable curves of genus $g$, with $n$ marked points, into fibers of the universal family of the stack $\mathfrak{W}^{s s}$ of expanded degenerations associated to $W^{s s} / \mathbb{A}^{1}$ with $\mathcal{L}$-degree $d$ can be decomposed into a disjoint union of collections of irreducible componensts of $\mathfrak{M}\left(\mathfrak{W}^{s s},(g, n ; d)\right)$ with each collection labelled by an element of $C_{(\mathcal{L}, d)}:=\left\{\beta \in H_{2}\left(W_{t_{0}} ; \mathbb{Z}\right): \mathcal{L} \cdot \beta=d\right\}$. (Here we identify $W_{t_{0}}$ with $W_{\sqrt{t_{0}}}^{s s}$ to define $\mathcal{L} \cdot \beta$.) The construction of [Li2] applies to each collection and gives rise to a degeneration formula with respect to a curve class $\beta \in C_{(\mathcal{L}, d)} \subset H_{2}\left(W_{t_{0}} ; \mathbb{Z}\right)$ :

$$
\begin{aligned}
& \Psi_{(g, n ; \beta)}^{W_{t_{0}}}(\alpha, \zeta) \\
& \quad=\sum_{\eta \in \bar{\Omega}_{(g, n ; \beta)}^{\mathcal{C}}} \frac{\mathbf{m}(\eta)}{|\mathrm{Eq}(\eta)|} \sum_{j \in K_{\eta}}\left[\Psi_{\Gamma_{1}}^{(\widetilde{Y}, E)}\left(j_{1}^{*} \bar{\alpha}(0), \zeta_{\eta, 1, j}\right) \bullet \Psi_{\Gamma_{2}}^{(Q, E)}\left(j_{2}^{*} \bar{\alpha}(0), \zeta_{\eta, 2, j}\right)\right]_{0},
\end{aligned}
$$

where $\bar{\alpha}$ is any flat extension of $\alpha \in A_{*}\left(W_{t_{0}}\right)^{\oplus n}=A_{*}\left(W_{\sqrt{t_{0}}}^{s s}\right)^{\oplus n}$ to $A_{*}\left(W^{s s} / \mathbb{A}^{1}\right)^{\oplus n}$ and $\Omega_{(g, n ; \beta)}^{\mathcal{L}}:=\left\{\eta=\left(\Gamma_{1}, \Gamma_{2}, I\right) \in \Omega_{(g, n ; d)}^{\mathcal{L}} \mid p_{1 *} b\left(\Gamma_{1}\right)=\beta\right\}$ is the $\beta$-compatible subset of $\Omega_{(g, n ; d)}^{\mathcal{L}}$.

The possible $\mathcal{L}$-dependence on the right-hand side of the above identity can be removed by choosing $\mathcal{L}$ appropriately very ample on $W^{s s}$. For such $\mathcal{L}$, the set $\Omega_{(g, n ; \beta)}^{\mathcal{L}}$ of admissible triples in the above identity is stabilized to an $\mathcal{L}$-independent set $\Omega_{(g, n ; \beta)}^{(g, n, \beta)}$, described as follows. Let $\gamma_{2,1}$ and $\gamma_{2,2}$ be the curve classes in $H_{2}(\widetilde{Y} ; \mathbb{Z})$ represented by the two classes of rational curves from the two rulings of $E$ and $\gamma_{2}$ be the curve class that generates $H_{2}(Q ; \mathbb{Z})(\simeq \mathbb{Z})$. (Both $\gamma_{2,1}$ and $\gamma_{2,2}$ become $\gamma_{2}$ when passing from $\widetilde{Y}$ to $Q$ via $E$.) It has the property that $\gamma_{2,1} \cdot E=\gamma_{2,2} \cdot E=-1$ on $\widetilde{Y}$ while $\underset{\sim}{\gamma} \cdot E=+1$ on $Q$. Let $\tilde{\beta}^{0}$ be the minimal lifting of $\beta$ to the set of curve classes in $H_{2}(\widetilde{Y} ; \mathbb{Z})$. It is characterized by the property that any curve classes in $p_{1 *}^{-1}(\beta)$ is in the set $\tilde{\beta}^{0}+\mathbb{Z}_{\geq 0} \gamma_{2,1}+\mathbb{Z}_{\geq 0} \gamma_{2,2}$. With these notations,

$$
\Omega_{(g, n ; \beta)}=\left\{\begin{array}{l|l}
\eta=\left(\Gamma_{1}, \Gamma_{2}, I\right) & \begin{array}{l}
\bullet b\left(\Gamma_{1}\right)=\tilde{\beta}^{0}+l_{1,1} \gamma_{2,1}+l_{1,2} \gamma_{2,2}, b\left(\Gamma_{2}\right)=l_{2} \gamma_{2}, \\
\left(l_{1,1}+l_{1,2}\right)+l_{2}=D \cdot \tilde{\beta}^{0}, l_{1,1}, l_{1,2}, l_{2} \in \mathbb{Z}_{\geq 0} ; \\
\text { admissible } \\
\text { tripl } \\
\text { for } \tilde{Y} \cup_{E} Q
\end{array} \\
\begin{array}{l}
\text { • } g(\eta)=g, n_{1}+n_{2}=n ; \\
\bullet \sum_{i} \mu_{1, i}=l_{2} ; \\
\bullet I \subset\{1, \ldots, n\},|I|=n_{1} .
\end{array}
\end{array}\right\} .
$$

And the degeneration formula above becomes intrinsic to the topological type $(g, n ; \beta)$ in the definition of Gromov-Witten invariants of $W_{t_{0}}$.

Remark 2.2 [upshot]. For our applications, we write the set $\Omega_{(g, n ; \beta)}$ of admissible triples ( $\Gamma_{1}, \Gamma_{2}, I$ ) for the degeneration associated to blow-ups and to conifold degenerations very explicitly, making it look messy. However, it should be noted that the essential conditions are only $p_{1 *} b\left(\Gamma_{1}\right)+p_{2 *} b\left(\Gamma_{2}\right)=\beta$ and $b\left(\Gamma_{1}\right) \cdot E=b\left(\Gamma_{2}\right) \cdot E$ on curve classes.

## 3 Transformation of GW-invariants of 3-folds.

We now employ Sec. 1 and Sec. 2 to study the transformation of Gromov-Witten invariants of projective 3 -folds under standard flops and small extremal transitions. The discussion here is in algebro-geometric parallel to that of [L-R: Sec. 6] in the symplectic/differentialtopological category. Together this gives an algebro-geometric account of [Wit5: Sec. 5.5] from the stringy viewpoint and [L-R] from the symplectic viewpont.

Recall the following two definitions, which will be used in this section:
Definition 3.0.1 [isomorphic power series]. ([L-R].) Two power series $F$ and $G$ are called isomorphic if there exist decompositions $F=F_{1}+F_{2}$ and $G=G_{1}+G_{2}$ such that $F_{1}=G_{1}$ and that $F_{2}$ and $G_{2}$ are related to each other by analytic continuations.

Definition 3.0.2 [3-point function]. (cf. [C-K].) Given a smooth projective variety $X$, the 3 -point (correlation) function

$$
\Psi^{X}: A_{*}(X)_{\mathbb{Q}} \times A_{*}(X)_{\mathbb{Q}} \times A_{*}(X)_{\mathbb{Q}} \longrightarrow \mathbb{Q}\left\{\left\{H_{2}(X ; \mathbb{Z})\right\}\right\}
$$

of cycle classes on $X$ from Gromov-Witten theory is defined by

$$
\Psi^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)=\sum_{\beta \in H_{2}(X ; \mathbb{Z})} \Psi_{(0,3 ; \beta)}^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) q^{\beta}
$$

### 3.1 Gromov-Witten invariants of 3-folds under a flop.

Transformation of Gromov-Witten invariants of projective 3-folds under a standard flop in the algebro-geometric category is explained in this subsection.

## The standard flop of 3 -folds and the associated degenerations.

Consider the following diagram of a standard flop of projective 3 -folds:

where $X$ (resp. $X^{\prime}$ ) is a smooth projective 3 -fold with an embedded smooth curve $C$ (resp. $\left.C^{\prime}\right) \simeq \mathbb{P}^{1}$ whose normal sheaf $\mathcal{N}_{C / X}\left(\right.$ resp. $\left.\mathcal{N}_{C^{\prime} / X^{\prime}}\right)$ is isomorphic to $\mathcal{O}_{\mathbb{P}^{1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-1)$; $(\tilde{X}, E)$ is the common blow-up of $X$ along $C$ and of $X^{\prime}$ along $C^{\prime}$ with the exceptional divisor $E \simeq \mathbb{P}^{1} \times \mathbb{P}^{1} ; X \rightarrow \underline{X}$ (resp. $X^{\prime} \rightarrow \underline{X}$ ) is the morphism that contracts exactly $C$
(resp. $C^{\prime}$ ); the birational map $\phi: X \rightarrow X^{\prime}$ is the composition of blowing up $X$ along $C$ and then blowing down the ruling of $E$ not contracted by $\widetilde{X} \rightarrow X, \phi$ (or simply $X^{\prime}$ when the diagram is understood implicitly) is called the flop of $X \rightarrow \underline{X}$ (or simply $X$ ).

There are two degenerations associated to the above diagram:

$$
\pi: W:=B l_{C \times \mathbf{0}} X \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1} \quad \text { and } \quad \pi^{\prime}: W^{\prime}:=B l_{C^{\prime} \times \mathbf{0}} X^{\prime} \times \mathbb{A}^{1} \rightarrow \mathbb{A}^{1}
$$

Both $\pi$ and $\pi^{\prime}$ are constant families except over $\mathbf{0}$, where the fiber becomes

$$
W_{0}=\widetilde{X} \cup_{E} \mathbb{P}\left(\mathcal{N}_{C / X} \oplus \mathcal{O}_{C}\right)=: \widetilde{X} \cup_{E} Y \quad \text { for } \pi
$$

and

$$
W_{0}^{\prime}=\widetilde{X} \cup_{E} \mathbb{P}\left(\mathcal{N}_{C^{\prime} / X^{\prime}} \oplus \mathcal{O}_{C^{\prime}}\right)=: \widetilde{X} \cup_{E} Y^{\prime} \quad \text { for } \pi^{\prime}
$$

Here $Y$ and $Y^{\prime}$ are isomorphic but the gluing to $\widetilde{X}$ along $E$ differs by an exchange of the two $\mathbb{P}^{1}$ factors in $E \simeq \mathbb{P}^{1} \times \mathbb{P}^{1}$. Such an automorphism on $E$ as a subvariety in $Y$ or $Y^{\prime}$ does not extend to an automorphism on the whole $Y$ or $Y^{\prime}$; thus in general $W_{0}$ and $W_{0}^{\prime}$ are not isomorphic.

The birational map $\phi: X \rightarrow X^{\prime}$ induces degree-preserving homomorphisms on Chow groups

$$
\phi_{*}: A_{*}(X) \rightarrow A_{*}\left(X^{\prime}\right) \quad \text { and } \quad \phi_{*}^{-1}: A_{*}\left(X^{\prime}\right) \rightarrow A_{*}(X)
$$

via intersection with the (closed) graph $\Gamma_{\phi}$ of $\phi$ in $X \times X^{\prime}$. Since $\phi$ is a curve-surgery on a 3 -fold, $\phi_{*}: A_{3}(X) \rightarrow A_{3}\left(X^{\prime}\right), A_{2}(X) \rightarrow A_{2}\left(X^{\prime}\right)$ are isomorphisms. For a curve class $\gamma \in A_{1}(X)$ with $\gamma \neq[C]$, using Chow's moving lemma, one can represent $\gamma$ by a linear combination of 1-cycles that are disjoint from $C$. This gives a representation of $\phi_{*}(\gamma)$ by a linear combination of 1 -cycles disjoint with $C^{\prime}$. Since $X^{\prime}$ is projective, $C^{\prime}$ is the only locus contracted by $X^{\prime} \rightarrow \underline{X}$, and the preperty of being contracted is a numerical property, one concludes that $\phi_{*}([C]) \neq\left[C^{\prime}\right]$. The same reasoning implies that $\phi_{*}([C])=k\left[C^{\prime}\right]$ for some $k \in \mathbb{Z}$. To compute $k$, observe that, though $\Gamma_{\phi}$ and $C \times X^{\prime}$ have excess intersection along $E$ of dimension 1 larger than expected in $X \times X^{\prime}$, both the embeddings $\Gamma_{\phi} \cap\left(C \times X^{\prime}\right) \simeq E \hookrightarrow X \times X^{\prime}$ and $C \times X^{\prime} \hookrightarrow X \times X^{\prime}$ are regular. The excess intersection formula gives then $\Gamma_{\phi} \cdot\left(C \times X^{\prime}\right)=c_{1}\left(\mathcal{N}_{\left(C \times X^{\prime}\right) /\left(X \times X^{\prime}\right)} / \mathcal{N}_{E / \tilde{X}}\right) \cap[E]$, which is the class $[C]-\left[C^{\prime}\right]$ on $E \simeq C \times C^{\prime}$. This class projects to $-\left[C^{\prime}\right]$ on $X^{\prime}$. Thus, $\phi_{*}([C])=-\left[C^{\prime}\right]$. This argument shows also that $\phi_{*}: A_{1}(X) \rightarrow A_{1}\left(X^{\prime}\right)$ is an isomorphism. Finally, one has a homomorphism $\phi_{*}: A_{0}(X) \rightarrow A_{0}\left(X^{\prime}\right)$.

For the degenerate fiber $W_{0}=\widetilde{X} \cup_{E} Y$ in $W / \mathbb{A}^{1}$, the Chow rings $A_{*} \widetilde{X}$ are determined by $A_{*}(X)$ via blow-up formulas. The Chow ring $A_{*}(Y)$ is generated by $w:=[E]$ and the fiber class $v:=\left[\mathbb{P}^{2}\right]$ of $Y \rightarrow C$. Explicitly, $A_{*}(Y) \simeq \mathbb{Z}[v, w] /\left(v^{2}, w^{3}-2 v w^{2}\right)$ by either the projective space bundle or the toric computation. A class $\left[\zeta_{1}, \zeta_{2}\right]$ in $A_{*}\left(W_{0}\right)$ (with $\zeta_{1}$ a linear combination of cycles in $\widetilde{X}$ and $\zeta_{2}$ a linear combination of cycles in $Y$ ) that comes from a relative class $\alpha \in H^{0}\left(R^{\bullet} \pi_{*} \mathbb{Q}_{W}\right)$ must satisfy the predeformability condition $\zeta_{1} \cdot E=\zeta_{2} \cdot E$.

Consider now the classes in $H^{0}\left(R^{\bullet} \pi_{*} \mathbb{Q}_{W}\right)$ that are representable by linear combinations of relative cycles on $W / \mathbb{A}^{1}$. We will call these classes algebraic. For an algebraic class $\alpha$
in $H^{0}\left(R^{6} \pi_{*} \mathbb{Q}_{W}\right)$ and $H^{0}\left(R^{4} \pi_{*} \mathbb{Q}_{W}\right)$, since $W-W_{0}$ is a trivial family with fiber $X$ over $\mathbb{A}^{1}-\mathbf{0}, \alpha$ determines a unique class $\alpha_{t}:=\left.\alpha\right|_{W_{t}}(t \neq \mathbf{0})$ in $A_{*}(X)$. Using Chow's moving lemma and taking completion, $\alpha$ determines a class $\alpha^{\prime} \in H^{0}\left(R^{4} \pi_{*} \mathbb{Q}_{W}\right)$ that coincides with $\alpha$ over $\mathbb{A}^{1}-\mathbf{0}$ and is represented by a linear combination of relative cycles that intersect $W_{0}$ only at the $\widetilde{X}$ component. For an algebraic class $\alpha \in H^{0}\left(R^{2} \pi_{*} \mathbb{Q}_{W}\right)$, same argument concludes that $\alpha$ determines a class $\alpha^{\prime} \in H^{0}\left(R^{2} \pi_{*} \mathbb{Q}_{W}\right)$ that coincides with $\alpha$ over $\mathbb{A}^{1}-\mathbf{0}$ and can be represented by $p_{1}^{-1}{ }_{*}\left(\alpha_{t}\right)+\left(\alpha_{t} \cdot C\right)\left[\mathbb{P}^{2}\right]$ in $A^{1}\left(\widetilde{X} \cup_{E} Y\right)$. Finally, $H^{0}\left(R^{0} \pi_{*} \mathbb{Q}_{W}\right)$ is generated by $[W]$.

When one is restricted to the $\widetilde{X}$ part of cycles, the above homomorphism coincides with $p_{1}^{-1}{ }_{*}: A_{*}(X) \rightarrow A_{*}(\widetilde{X})$. Together with a similar construction for the degeneration $\pi^{\prime}: W^{\prime} \rightarrow \mathbb{A}^{1}$, one has a commutative diagram:

\[

\]

There is a cycle class map $A_{*}(\cdot) \rightarrow H_{*}(\cdot)$ and the above diegram is compatibility with the topological results on homomorphisms among the associated $H_{*}(\cdot)$ in [L-R].

## GW-invariants of $X, X^{\prime}$ in terms of relative GW-invariants of $(\tilde{X}, E),(Y, E)$.

Recall the moduli stack $\mathfrak{M}(\mathfrak{W},(g, n ; \beta))$ of stable maps of topological type $(g, n ; \beta)$ to fibers of the universal family of the stack $\mathfrak{W}$ of expanded degenerations associated to $W / \mathbb{A}^{1}$ and the set $\Omega_{(g, n ; \beta)}$ of admissible triples that appears in the degeneration formula and is intrinsic to $(g, n ; \beta)$.

For a general projective degeneration $W \rightarrow \mathbb{A}^{1}$ with $W_{0}=Y_{1} \cup_{D} Y_{2}$ discuessed in [Li1] and with the extra condition that $H_{2}\left(W_{t} ; \mathbb{Z}\right)$ is canonically isomorphic to $H_{2}\left(W_{t_{0}} ; \mathbb{Z}\right)$ for some $t_{0} \neq \mathbf{0}$, similar discussions as in [L-Y1] always give a refined degeneration formula from [Li2] for curve classes $\beta \in H_{2}\left(W_{t_{0}} ; \mathbb{Z}\right)$. In such cases, suppose that $\beta$ is decomposed to $\beta_{1}+\beta_{2}$ in $H_{2}\left(W_{0} ; \mathbb{Z}\right)$ (when represented in $H_{2}\left(Y_{1} ; \mathbb{Z}\right) \oplus H_{2}\left(Y_{2} ; \mathbb{Z}\right)$, $\left(\beta_{1}, \beta_{2}\right)$ may not be unique, simply choose any representative) and that ( $\left.\Gamma_{1}, \Gamma_{2}, I\right)$ is an admissible triple in $\Omega_{(g, n ; \beta)}$ such that $b\left(\Gamma_{i}\right)=\beta_{i}$ and that $\Gamma_{i}$ has $n_{i}$ legs and $r$ roots, $i=1,2$. Let $g_{i}:=g\left(\Gamma_{i}\right)$. Then the following additivity relation from the (complex) dimension of the virtual fundamental classes $\left[\overline{\mathcal{M}}_{g, n}\left(W_{t_{0}}, \beta\right)\right]^{\text {virt }}$ and $\left[\mathfrak{M}\left(\left(Y_{i}, D ; \Gamma_{i}\right)\right)\right]^{\text {virt }}, i=1,2$, holds:

$$
\begin{aligned}
& (1-g)\left(\operatorname{dim} W_{t_{0}}-3\right)-\beta \cdot K_{W_{t_{0}}}+n \\
& \quad=\left(1-g_{1}\right)\left(\operatorname{dim} Y_{1}-3\right)-\beta_{1} \cdot K_{Y_{1}}+n_{1}+\left(r-\beta_{1} \cdot D\right) \\
& \quad+\left(1-g_{2}\right)\left(\operatorname{dim} Y_{2}-3\right)-\beta_{2} \cdot K_{Y_{2}}+n_{2}+\left(r-\beta_{2} \cdot D\right)-r(\operatorname{dim} D)
\end{aligned}
$$

By construction, $\operatorname{dim} W_{t}=\operatorname{dim} Y_{1}=\operatorname{dim} Y_{2}=\operatorname{dim} D+1, g=g_{1}+g_{2}+r+1-\left|V\left(\Gamma_{1} \amalg \Gamma_{2}\right)\right|$, $n=n_{1}+n_{2}, \beta_{1} \cdot D=\beta_{2} \cdot D \geq 0$, and $0 \leq r \leq \beta_{1} \cdot D$.

For the degeneration $W / \mathbb{A}^{1}$ with $W_{0}=\widetilde{X} \cup_{E} Y$ from the standard flop of 3-folds, $p_{1}: \widetilde{X} \rightarrow X$ is the blow-up of $X$ along $C$ with exceptional divisor $E$, hence $K_{\tilde{X}}=p_{1}^{*} K_{X}+E$
by computing the Jacobian of $p_{1}$ at a general point of $E ; Y$ is a toric 3 -fold, whose fan structure gives $K_{Y}=-3 E$. The additivity relation thus simplifies to

$$
-\beta \cdot K_{X}+n=\left(-\beta_{1} \cdot p_{1}^{*} K_{X}+n_{1}+r-2 \beta_{1} \cdot E\right)+\left(2 \beta_{2} \cdot E+n_{2}+r\right)-2 r .
$$

There are two cases.
Case ( $a$ ) : the virtual dimension $\operatorname{vdim} \overline{\mathcal{M}}_{g, 0}(X, \beta)=0$. In this case, one has the identity

$$
0=\left(-\beta_{1} \cdot p_{1}^{*} K_{X}+r-2 \beta_{1} \cdot E\right)+\left(2 \beta_{2} \cdot E+r\right)-2 r,
$$

which implies that $\beta_{1} \cdot p_{1}^{*} K_{X}=0$. Thus,

$$
v \operatorname{dim} \mathfrak{M}\left(\widetilde{X}, E ; \Gamma_{1}\right)=r-2 \beta_{1} \cdot E \quad \text { and } \quad v \operatorname{dim} \mathfrak{M}\left(Y, E ; \Gamma_{2}\right)=2 \beta_{2} \cdot E+r .
$$

For $\beta$ not a multiple of $[C]$, it always holds that $\beta_{1} \neq 0$. In this case, the only situation that $\operatorname{vdim} \overline{\mathcal{M}}_{g_{1}, 0}\left(\widetilde{X}, E ; \beta_{1}, r\right) \geq 0$ is when $\beta_{1} \cdot E=r=0$. This implies that $\beta_{2}=0$. It follows from Sec. 2 that in this case elements in $\Omega_{(g, 0 ; \beta)}$ correspond to a unique curve class $\tilde{\beta}$ in $H_{2}(\widetilde{X} ; \mathbb{Z})$, characterized by the conditions $p_{1 *} \tilde{\beta}=\beta$ and $\tilde{\beta} \cdot E=0$. This class coincides with the class associated to $\beta$ from Chow's moving lemma. Thus, for $\beta$ not a multiple of $[C]$, the Gromov-Witten invariant $\Psi_{(g, 0 ; \beta)}^{X}$ of $X$ is equal to the relative Gromov-Witten invariant $\Psi_{(g, 0 ; \tilde{\beta})}^{(\widetilde{X}, E)}$ of $(\widetilde{X}, E)$.

If $\beta$ is a positive multiple of $[C]$, then if $\beta_{1} \neq 0$, it must lie in $E$ with $\beta_{1} \cdot E<0$. Such $\beta_{1}$ does not occur in any admissible triple in $\Omega_{g, 0 ; \beta}$. It follows from Sec. 2 that in this case elements in $\Omega_{(g, 0 ; \beta)}$ correspond to a unique curve class $\tilde{\beta}$ in $H_{2}(Y ; \mathbb{Z})$, characterized by the conditions $p_{2 *} \tilde{\beta}=\beta$ and $\tilde{\beta} \cdot E=0$. Recall that $Y$ is a toric variety. Such $\tilde{\beta}$ on $Y$ is represented by a multiple of the unique toric invariant curve $\bar{C}:=\mathbb{P}^{1}$ not intersecting $E$ and with normal bundle $\mathcal{O}_{\mathbb{P}^{1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-1)$. The related Gromov-Witten invariant $\Psi_{(g, 0 ; \beta)}^{X}$ of $X$ is identical to the absolute Gromov-Witten invariant $\Psi_{(g, 0 ; \tilde{\beta})}^{Y}$.

Case $(b): v \operatorname{dim} \overline{\mathcal{M}}_{g, 0}(X, \beta)>0$. In this case, one considers the Gromov-Witten invariants $\Psi_{(g, n ; \beta)}^{X}\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ with the summation of the codimension of the cycle $\alpha_{i}$ on $X$ equal to $v \operatorname{dim} \overline{\mathcal{M}}_{g, n}(X, \beta)$. Since our purpose is to compare Gromov-Witten invariants under a standard flop, by the Fundamental Axiom and the Divisor Class Axiom, one only needs to consider $\alpha_{i} \in A_{0}(X)$ or $A_{1}(X)$. From the earlier discussion in this section, each such class determines a class in $H^{0}\left(R^{6} \pi_{*} \mathbb{Q}_{W}\right)+H^{0}\left(R^{4} \pi_{*} \mathbb{Q}_{W}\right)$ representable by a relative cycle on $W / \mathbb{A}^{1}$ that intersects $W_{0}$ only at $\widetilde{X}$. The only contribution to $\Psi_{(g, n ; \beta)}^{X}\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ from the summands of the degeneration formula comes from those labelled by admissible triples with $\beta_{2}=0$. As in Case (a), when $\beta$ is not a multiple of $[C]$, there is a unique such admissible triple $\left(\Gamma_{1}, \emptyset, \emptyset\right)$ in $\Omega_{(g, n ; \beta)}$ with $b\left(\Gamma_{1}\right)=\tilde{\beta}$. This $\tilde{\beta}$ is again characterized by $p_{1 *} \tilde{\beta}=\beta$ and $\tilde{\beta} \cdot E=0$. Consequently, $\Psi_{(g, n ; \beta)}^{X}\left(\alpha_{1}, \cdots, \alpha_{n}\right)=\Psi_{(g, n ; \tilde{\beta})}^{(\tilde{X}, E)}\left(p_{1}^{-1}{ }_{*} \alpha_{1}, \cdots, p_{1}^{-1}{ }_{*} \alpha_{n}\right)$. If $\beta$ is a multiple of $[C]$, there is no such admissible triple and $\Psi_{(g, n ; \beta)}^{X}\left(\alpha_{1}, \cdots, \alpha_{n}\right)=0$, which is consistent with the vanishing for the dimensional reason in this situation.

The discussion for $X^{\prime}$ is identical.

## Gromov-Witten invariants under the standard flop.

Comparing Gromov-Witten invariants of $X$ and $X^{\prime}$ both to the relative Gromov-Witten invariants of $(\widetilde{X}, E)$ or the absolute Gromov-Witten invariants of $Y$ from the discussion above, recalling the commutative diagram:

\[

\]

and together with the axioms of Gromov-Witten invariants and Chow's moving lemma gives the following relations:

Theorem 3.1.1 [GW-invariant under flop]. ([L-R: Theorem A and Corollary A.1].) Let $\beta \in H_{2}(X ; \mathbb{Z})$ and $\alpha_{1}, \cdots, \alpha_{n} \in A_{*}(X)_{\mathbb{Q}}$. If $\beta$ is not a multiple of $[C]$, then

$$
\Psi_{(g, n ; \beta)}^{X}\left(\alpha_{1}, \cdots, \alpha_{n}\right)=\Psi_{\left(g, n ; \phi_{*}(\beta)\right)}^{X^{\prime}}\left(\phi_{*} \alpha_{1}, \cdots, \phi_{*} \alpha_{n}\right)
$$

while $\Psi_{(g, 0 ; m[C])}^{X}=\Psi_{\left(g, 0 ; m\left[C^{\prime}\right]\right)}^{X^{\prime}}$. If $X$ is a Calabi-Yau 3-fold, then the first identity is reduced to $\Psi_{(g, 0 ; \beta)}^{X}=\Psi_{\left(g, 0 ; \phi_{*}(\beta)\right)}^{X^{\prime}}$.

This theorem implies the transformation law of 3-point functions under a standard flop. The discussion below in rational cycle classes is similar to $[L-R]$ in differential forms and their Poincaré dual.

The 3-point function can be decomposed into three parts:

$$
\begin{aligned}
& \Psi^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) \\
& \quad=\left(\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}\right)_{0}+\sum_{\beta \neq 0, m[C]} \Psi_{(0,3 ; \beta)}^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) q^{\beta}+\sum_{\beta=m[C], m>0} \Psi_{(0,3 ; \beta)}^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) q^{\beta},
\end{aligned}
$$

where the first term takes only the 0-dimensional component of the triple intersection product. The power series is invariant under permutations of $\alpha_{i}$ 's and we will assume that $\operatorname{codim} \alpha_{1} \geq \operatorname{codim} \alpha_{2} \geq \operatorname{codim} \alpha_{3}$. Theorem 3.1.1 implies that the second term is invariant under $q^{\beta} \rightarrow q^{\phi_{*}(\beta)}$.

For dimensional reason, the third term vanishes except for $\alpha_{i} \in A^{1}(M)$. When there are no insertions, $\Psi_{(0,0 ; m[C])}^{X}=\Psi_{(0,0 ; m[\bar{C}])}^{Y}=1 / m^{3}$ by a localization computation, (see [C-K] for a review and references therein on this result by several independent authors; in $[\mathrm{L}-\mathrm{L}-\mathrm{Y}]$ this is a by-product of the Mirror Principle framework, in which a linearized moduli space for stable maps from [Wit5] is integrated into both the Gromov-Witten theory and the localization computations). Consequently, the third term is given by

$$
\sum_{\beta=m[C], m>0} \Psi_{(0,3 ; \beta)}^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) q^{\beta}=\left([C] \cdot \alpha_{1}\right)\left([C] \cdot \alpha_{2}\right)\left([C] \cdot \alpha_{3}\right) \frac{q^{[C]}}{1-q^{[C]}} .
$$

The first term vanishes except for $\left(\operatorname{codim} \alpha_{1}, \operatorname{codim} \alpha_{2}, \operatorname{codim} \alpha_{3}\right)=(1,1,1),(2,1,0)$, or $(3,0,0)$. Compare now the first and the third term on the $X$ and the $X^{\prime}$ side.

For $\left(\operatorname{codim} \alpha_{1}, \operatorname{codim} \alpha_{2}, \operatorname{codim} \alpha_{3}\right)=(3,0,0)$ or $(2,1,0)$, the third term vanishes and the identity $\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}=\phi_{*}\left(\alpha_{1}\right) \cdot \phi_{*}\left(\alpha_{2}\right) \cdot \phi_{*}\left(\alpha_{3}\right)$ follows from Chow's moving lemma, which renders the intersections a same combination of 0-cycles in $X-C \stackrel{\phi}{\sim} X^{\prime}-C^{\prime}$.

For $\left(\operatorname{codim} \alpha_{1}, \operatorname{codim} \alpha_{2}, \operatorname{codim} \alpha_{3}\right)=(1,1,1)$, usineg Chow's moving lemma, one can assume that the two representing cycles for each of the following pairs meet properly

$$
\begin{gathered}
\left(\alpha_{1},[C]\right), \quad\left(\alpha_{2}, \alpha_{1}+[C]+\left[\alpha_{1} \cap C\right]\right), \\
\left(\alpha_{3}, \alpha_{1}+\alpha_{2}+[C]+\left[\alpha_{1} \cap \alpha_{2}\right]+\left[\alpha_{1} \cap C\right]+\left[\alpha_{2} \cap C\right]\right) .
\end{gathered}
$$

(For simplicity of notation, we identify $\alpha_{i}$ with their representing cycles in the discussion below.) This implies that $\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}$ is represented by a linear combination of 0 -cycles disjoint from $C$ and that $\phi_{*}\left(\alpha_{1}\right), \phi_{*}\left(\alpha_{2}\right), \phi_{*}\left(\alpha_{3}\right)$ meet properly with each other except along $C^{\prime}$, where an excess intersection of the three may occur. Under this choice of representing cycles, every two of $\phi_{*}(\alpha), \phi_{*}\left(\alpha_{2}\right)$, and $\phi_{*}\left(\alpha_{3}\right)$ meet properly along $C^{\prime}$ in $X^{\prime}$ if $C^{\prime}$ contains their intersection points. This implies that

$$
\begin{aligned}
& \phi_{*}\left(\alpha_{1}\right) \cdot \phi_{*}\left(\alpha_{2}\right) \cdot \phi_{*}\left(\alpha_{3}\right) \\
& \quad=\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}+\left(\text { contribution from the excess intersection along } C^{\prime}\right)
\end{aligned}
$$

for the triple intersecion. The intersection multiplicity $i\left(C^{\prime}, \phi_{*}\left(\alpha_{1}\right) \cdot \phi_{*}\left(\alpha_{2}\right) ; X^{\prime}\right)$ of $\phi_{*}\left(\alpha_{1}\right)$ and $\phi_{*}\left(\alpha_{2}\right)$ along $C^{\prime}$ in $X^{\prime}$ is given by $\left(-[C] \cdot \alpha_{1}\right)\left(-[C] \cdot \alpha_{2}\right)$. Since $\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{3}\right)=-[C] \cdot \alpha_{3}$, one concludes that

$$
\phi_{*}\left(\alpha_{1}\right) \cdot \phi_{*}\left(\alpha_{2}\right) \cdot \phi_{*}\left(\alpha_{3}\right)=\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}-\left([C] \cdot \alpha_{1}\right)\left([C] \cdot \alpha_{2}\right)\left([C] \cdot \alpha_{3}\right) .
$$

Finally, note that $\phi_{*}\left(q^{[C]} /\left(1-q^{[C]}\right)\right)=q^{\phi_{*}[C]} /\left(1-q^{\phi_{*}[C]}\right)=q^{-\left[C^{\prime}\right]} /\left(1-q^{-\left[C^{\prime}\right]}\right)$ is isomorphic to $-1-q^{\left[C^{\prime}\right]} /\left(1-q^{\left[C^{\prime}\right]}\right)$ by an analytic continuation. Combining all these and re-writing the triple intersection above as

$$
\begin{aligned}
& \phi_{*}\left(\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}\right) \\
& \quad=\phi_{*}\left(\alpha_{1}\right) \cdot \phi_{*}\left(\alpha_{2}\right) \cdot \phi_{*}\left(\alpha_{3}\right)-\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{1}\right)\right)\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{2}\right)\right)\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{3}\right)\right),
\end{aligned}
$$

one concludes that

$$
\begin{aligned}
& \phi_{*}\left(\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}+\left([C] \cdot \alpha_{1}\right)\left([C] \cdot \alpha_{2}\right)\left([C] \cdot \alpha_{3}\right) \frac{q^{[C]}}{1-q^{[C]}}\right) \\
& \quad \stackrel{\text { a.c. }}{=} \phi_{*}\left(\alpha_{1}\right) \cdot \phi_{*}\left(\alpha_{2}\right) \cdot \phi_{*}\left(\alpha_{3}\right)+\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{1}\right)\right)\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{2}\right)\right)\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{3}\right)\right) \frac{q^{\left[C^{\prime}\right]}}{1-q^{\left[C^{\prime}\right]}},
\end{aligned}
$$

where a.c. stands for "analytic continuation". The second line of this identity is exactly the summation of the first and the third part of the similar decomposition of the 3-point
function $\Psi^{X^{\prime}}\left(\phi_{*} \alpha_{1}, \phi_{*} \alpha_{2}, \phi_{*} \alpha_{3}\right)$. Recall Definition 3.0.1 of isomorphic power series. The whole discussions thus imply that:

Corollary 3.1.2 [3-point function under flop]. ([L-R: Corollary A.2].) The 3-point function $\Psi^{X^{\prime}}\left(\phi_{*}\left(\alpha_{1}\right), \phi_{*}\left(\alpha_{2}\right), \phi_{*}\left(\alpha_{3}\right)\right)$ of $X^{\prime}$ is isomorphic to $\phi_{*}\left(\Psi^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)\right)$.

Corollary 3.1.3 [Witten's wall-crossing formula]. Witten's wall-crossing formula [Wit5: Eq. (5.48)] of Yukawa couplings of the massless supermultiplets associated to $H^{1,1}(\cdot)$ in the 4-dimensional low-energy effective field theory from compactification of a superstring model can be completed to isomorphisms of 3-point functions of Gromov-Witten invariants for 3 -folds related by a standard flop in [L-R: Corollary A.2], cf. Corrollary 3.1.2. See Sec. A. 1 for more explanations on this stringy aspect of the problem.

### 3.2 Gromov-Witten invariants of 3 -folds under a small extremal transition.

In the diagram of a standard flop of 3 -folds at the beginning of Sec. 3.1, the singular variety $\underline{X}$ from contracting a curve $C \simeq \mathbb{P}^{1}$ in $X$ with normal sheaf $\mathcal{N}_{C / X} \simeq \mathcal{O}_{\mathbb{P}^{1}}(-1) \oplus \mathcal{O}_{\mathbb{P}^{1}}(-1)$ is a conifold. We will assume that $\underline{X}$ arises also from a degeneration $\pi^{\prime \prime}: W^{\prime \prime} \rightarrow \mathbb{A}^{1}$ with $W^{\prime \prime}$ smooth, $\pi^{\prime \prime}$ projective, and $W_{0}^{\prime \prime}=\underline{X}$ the fiber over $\mathbf{0} \in \mathbb{A}^{1}$. In this subsection, we explain how Gromov-Witten invariants of $X^{\prime \prime}:=W_{t_{0}}^{\prime \prime}$, for a fixed $t_{0} \neq \mathbf{0}$, are related to the Gromov-Witten invariants of $X$ in the algebro-geometric setting.

## Correspondences of algebraic classes.

Recall from Sec. 2 the semi-stable reduction $\pi^{s s}: W^{s s} \rightarrow \mathbb{A}^{1}$ of $\pi^{\prime \prime}$ from a degree- 2 base change with the fiber over $\mathbf{0}$ the gluing $W_{0}^{s s}=\widetilde{X} \cup_{E} Q$, where $Q$ is the smooth quadric hypersurface in $\mathbb{P}^{4}$ with $E \subset Q$ from a hyperplane section. The monodromy action of $\pi_{1}\left(\mathbb{A}^{1}-\mathbf{0}\right)$ is trivial on $H_{k}\left(X^{\prime \prime} ; \mathbb{Q}\right)$ except for $k=3$, which is not relevant in the current algebro-geometric setting. After the semi-stable reduction, the new monodromy action remains trivial on $H_{k}\left(W_{\sqrt{t_{0}}}^{s s} ; \mathbb{Q}\right)=H_{k}\left(X^{\prime \prime} ; \mathbb{Q}\right)$ for $k \neq 3$.

Given a class $\bar{\alpha} \in A_{*}\left(W^{\prime \prime}\right)$ reprented by a relative cycle (still denoted by $\bar{\alpha}$ ) of relative dimension $k \leq 2$, using Chow's moving lemma, one may assume that $\bar{\alpha}$ does not contain the conifold singularity in the fiber $W_{0}^{\prime \prime}=\underline{X}$. Then the lifting $\bar{\alpha}^{s s}$ of $\bar{\alpha}$ to $W^{s s}$ induces a class $\alpha:=\bar{\alpha}_{t_{0}} \in A_{k}\left(X^{\prime \prime}\right)$, and a class $\tilde{\alpha}:=\bar{\alpha}_{0} \in A_{k}(\tilde{X})$ by intersecting $\bar{\alpha}^{s s}$ with $W_{\sqrt{t_{0}}}^{s s}$ and $W_{0}^{s s}=\widetilde{X} \cup_{E} Q$ respectively. (The latter intersection is disjoint from $Q$ by our choice of the representative of $\bar{\alpha}$.) In this way, the subgroup of $H^{0}\left(R^{6-2 k} \pi_{*}^{\prime \prime} \mathbb{Q}_{W^{\prime \prime}}\right)$ that consists of elements representable by relative cycles on $W^{\prime \prime} / \mathbb{A}^{1}$ induces a correspondence

$$
\psi_{*}: A_{k}\left(X^{\prime \prime}\right) \vdash A_{k}(\tilde{X}) \quad \text { with } \quad \alpha \leadsto \tilde{\alpha}
$$

(Here, unlike in the smooth category, there exists no rational map $\psi: X^{\prime \prime} \xrightarrow{-\rightarrow} \widetilde{X}$. The lower $*$ is added only for a notation syncronization.) Define $\psi_{*}: A_{3}\left(X^{\prime \prime}\right) \vdash A_{3}(\widetilde{X})$ directly
by $\left[X^{\prime \prime}\right] \mapsto[\widetilde{X}]$. Let $A_{*}\left(X^{\prime \prime}\right)^{\circ}$ be the subgroup of $A_{*}\left(X^{\prime \prime}\right)$ that consists of elements whose image under $\psi_{*}$ is non-empty. Recall the blow-up $p_{1}: \widetilde{X} \rightarrow X$ from Sec. 3.1. Then one has a correspondence $\phi_{*}:=p_{1 *} \circ \psi_{*}: A_{*}\left(X^{\prime \prime}\right)^{\circ} \vdash A_{*}(X)$. (Again, there is no rational map $\phi$.) By construction, pairs of cycle classes in $A_{*}\left(X^{\prime \prime}\right) \times A_{*}(X)$ that are related by $\pi_{1 *} \circ \psi_{*}$ can be extended to pairs of relative cycle classes in $A_{*}\left(W^{s s} / \mathbb{A}^{1}\right) \times A_{*}\left(W / \mathbb{A}^{1}\right)$ with the common induced class on $\widetilde{X}$ disjoint from the other component in $W_{0}^{s s}$ or in $W_{0}$.

Recall the minimal lifting (curve class) $\tilde{\beta}^{0}$ in $H_{2}(\widetilde{X} ; \mathbb{Z})$ of a curve class $\beta \in H_{2}\left(X^{\prime \prime} ; \mathbb{Z}\right)$ in a conifold degeneration from Sec. 2.

Notation. Let $\alpha \in A_{*}\left(X^{\prime \prime}\right)^{\circ}$. We will adopt an abuse of notation that any element in the image set of the correspondence $\psi_{*}: A_{*}\left(X^{\prime \prime}\right)^{\circ} \vdash A_{*}(\widetilde{X})\left(\right.$ resp. $\left.\phi_{*}: A_{*}\left(X^{\prime \prime}\right)^{\circ} \vdash A_{*}(X)\right)$ will be denoted by $\psi_{*} \alpha$ (resp. $\phi_{*} \alpha$ ). Since there is no chance of confusion, the map on the set of curve classes in $H_{2}\left(X^{\prime \prime} ; \mathbb{Z}\right)$ to the set of curve classes in $H_{2}(X ; \mathbb{Z})$ by $\beta \mapsto p_{1 *} \tilde{\beta}^{0}$ will be denoted also by $\phi_{*}$.

## GW-invariants of $X^{\prime \prime}$ in terms of relative GW-invariants of $(\widetilde{X}, E)$.

Recall the moduli stack $\mathfrak{M}\left(\mathfrak{W}^{s s},(g, n ; \beta)\right)$ and the stabilized set $\Omega_{(g, n ; \beta)}$ of admissible triples in Sec. 2. Let $\left(\Gamma_{1}, \Gamma_{2}, I\right) \in \Omega_{(g, n ; \beta)}$ with $\beta_{i}:=b\left(\Gamma_{i}\right)$ and $\Gamma_{i}$ having $n_{i}$ legs and $r$ roots for $i=1,2$. Then, since $K_{Q}=-3 E$, the additivity relation of the (complex) dimension of virtual fundamental classes in Sec. 3.1 now reads:

$$
-\beta \cdot K_{X^{\prime \prime}}+n=\left(-\beta_{1} \cdot K_{\tilde{X}}+n_{1}+r-\beta_{1} \cdot E\right)+\left(2 \beta_{2} \cdot E+n_{2}+r\right)-2 r,
$$

where $\beta_{1} \neq 0$ unless $\beta=0, n=n_{1}+n_{2}$, and $\beta_{1} \cdot E=\beta_{2} \cdot E \geq r \geq 0$.
There are two cases.
Case $(a): v \operatorname{dim} \overline{\mathcal{M}}_{g, 0}\left(X^{\prime \prime}, \beta\right)=0$. In this case, one has the identity

$$
0=\left(-\beta_{1} \cdot K_{\tilde{X}}+r-\beta_{1} \cdot E\right)+\left(2 \beta_{2} \cdot E+r\right)-2 r,
$$

which implies that $\beta_{1} \cdot K_{\tilde{X}}=\beta_{1} \cdot E$. Thus,

$$
\operatorname{vdim} \mathfrak{M}\left(\widetilde{Y}, E ; \Gamma_{1}\right)=r-2 \beta_{1} \cdot E \quad \text { and } \quad \text { vdim } \mathfrak{M}\left(\widetilde{Y}, E ; \Gamma_{2}\right)=2 \beta_{2} \cdot E+r .
$$

This implies that, if $\beta_{2} \neq 0$, then $\operatorname{vdim} \mathfrak{M}\left(\widetilde{X}, E ; \Gamma_{1}\right)<0$. Consequently, the admissible triples in $\Omega_{(g, 0 ; \beta)}$ whose corresponding summand in the degeneration formula in Sec. 2 contributes must have $\beta_{2}=0$ and the degeneration formula in Sec. 2 reads

$$
\Psi_{(g, 0 ; \beta)}^{X^{\prime \prime}}=\sum_{\substack{l_{1,1}+l_{1,2}=\tilde{\beta}^{0} \cdot E \\ l_{1,1}, l_{1,2} \in \mathbb{Z}_{\geq 0}}} \Psi_{\left(g, 0 ; \tilde{\beta}^{0}+l_{1,1}[C]+l_{1,2}\left[C^{\prime}\right]\right)}^{(\tilde{X}, E)}
$$

where recall that $\tilde{\beta}^{0} \in H_{2}(\widetilde{X} ; \mathbb{Z})$ is the minimal lifting of $\beta$ in Sec. 2 and that $E=C \times C^{\prime}$ from the diagram for the standard flop in Sec. 3.1. For later use, we denote the (finite) set of $l_{1,1}$-values in the above summation by $I_{\beta}$.

Case $(b): \operatorname{vim} \overline{\mathcal{M}}_{g, 0}\left(X^{\prime \prime}, \beta\right)>0$. In this case, one considers the Gromov-Witten invariants $\Psi_{(g, n ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ with the summation of the codimension of $\alpha_{i}$ equal to $\operatorname{vdim} \overline{\mathcal{M}}_{g, n}\left(X^{\prime \prime}, \beta\right)$. Since $\Psi_{(g, n ; \beta)}^{X^{\prime \prime}}\left(\left[X^{\prime \prime}\right], \cdots\right)=0=\Psi_{(\ldots)}^{X}([X], \cdots)$ and our final goal is to relate $\Psi_{(g, n ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ to Gromov-Witten invariants of $X$, we will assume that $\alpha_{i} \in A_{*}\left(X^{\prime \prime}\right)^{\circ}$ and that $\alpha_{i} \neq\left[X^{\prime \prime}\right]$. Recall the construction of the classes $\psi_{*} \alpha_{i} \in A_{*}(\tilde{X})$. As cycles on $W_{0}^{s s}$, they are all disjoint from $Q$ by construction. This implies that the only admissible triples $\left(\Gamma_{1} \cdot \Gamma_{2}, I\right)$ in $\Omega_{(g, n ; \beta)}$ that contribute to the summation in the degeneration formula in Sec. 2 must have $\beta_{2}:=b\left(\Gamma_{2}\right)=0$ as well. Thus, we have a similar expression as in Case (a):

$$
\Psi_{(g, n ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \cdots, \alpha_{n}\right)=\sum_{\substack{l_{1,1}+l_{1,2}=\tilde{\beta}^{0} \cdot E \\ l_{1,1}, l_{1,2} \in \mathbb{Z}_{\geq 0}}} \Psi_{\left(g, n ; \tilde{\beta}^{0}+l_{1,1}[C]+l_{1,2}\left[C^{\prime}\right]\right)}^{\left(\tilde{X}, E \psi_{*} \alpha_{1}, \cdots, \psi_{*} \alpha_{n}\right) . . . ~ . ~}
$$

Again, we denote the (finite) set of $l_{1,1}$-values in the above summation by $I_{\beta}$.

## Gromov-Witten invariants under a small extremal transition.

We now want to convert the previous expression of $\Psi_{(g, n ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \cdots, \alpha_{n}\right)$ to one that is recognizable as a summation of Gromov-Witten invariants $\Psi_{(\ldots)}^{X}(\cdots)$ of $X$. Recall the blowup $p_{1}: \widetilde{X} \rightarrow X$. The curve class $\tilde{\beta}^{0}+l_{2,1}[C]+l_{2,2}\left[C^{\prime}\right]$ is the unique lifting on $\widetilde{X}$ of the curve class $\tilde{\beta}^{0}+l_{2,1}[C]$ on $X$ that satisfies the condition that $\left(\tilde{\beta}^{0}+l_{2,1}[C]+l_{2,2}\left[C^{\prime}\right]\right) \cdot E=0$. Observe that $\tilde{\beta}^{0}+l_{2,1}[C]$ can never be a pure multiple of $[C]$ unless $\beta=0$ and that $\psi_{*} \alpha_{i}=p_{1}^{-1}{ }_{*}\left(\phi_{*} \alpha_{i}\right)$ by construction. It follows thus from the discussion in Sec. 3.1 that

$$
\Psi_{\left(g, n ; \tilde{\beta}^{0}+l_{1,1}[C]+l_{1,2}\left[C^{\prime}\right]\right)}^{(\widetilde{X}, E}\left(\psi_{*} \alpha_{1}, \cdots, \psi_{*} \alpha_{n}\right)=\Psi_{\left(g, n ; \phi_{*}(\beta)+l_{1,1}[C]\right)}^{X}\left(\phi_{*} \alpha_{1}, \cdots, \phi_{*} \alpha_{n}\right) .
$$

This implies that:
Theorem 3.2.1 [GW-invariant under small extremal transition]. With notations from above, for $\beta \neq 0$,

$$
\Psi_{(g, n ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \cdots, \alpha_{n}\right)=\sum_{l \in I_{\beta}} \Psi_{\left(g, n ; \phi_{*}(\beta)+l[C]\right)}^{X}\left(\phi_{*} \alpha_{1}, \cdots, \phi_{*} \alpha_{n}\right) .
$$

If $X^{\prime \prime}$ is a Calabi-Yau 3-fold, then the above identity reduces to

$$
\Psi_{(g, 0 ; \beta)}^{X^{\prime \prime}}=\sum_{l \in I_{\beta}} \Psi_{\left(g, 0 ; \phi_{*}(\beta)+l[C]\right)}^{X} .
$$

Remark 3.2.2 [map on $\mathrm{H}_{2}$ ]. Topologically there is a surjective map (with the notation of $[\mathrm{L}-\mathrm{R}]) \varphi_{e}: H_{2}(X ; \mathbb{Z}) \rightarrow H_{2}\left(X^{\prime \prime} ; \mathbb{Z}\right)$ from the composition of canonical homomorphisms
$H_{2}(X ; \mathbb{Z}) \longrightarrow H_{2}(\underline{X} ; \mathbb{Z}) \xrightarrow{\sim} H_{2}\left(X^{\prime \prime} ; \mathbb{Z}\right)$. The kernel of $\varphi_{e}$ is generated by $[C]$. By construction, $\left(\varphi_{e} \circ \phi_{*}\right)(\beta)=\beta$.

For $\beta \neq 0$, the finite set of curve classes $\left\{\phi_{*}(\beta)+l[C] \mid l \in I_{\beta}\right\}$ only embeds in $\varphi_{e}^{-1}(\beta)$, and the latter can contain other curve classes as well. However, for any $\hat{\beta}$ in $\varphi_{e}^{-1}(\beta)-\left\{\phi_{*}(\beta)+l[C] \mid l \in I_{\beta}\right\}$, if $\hat{\beta}$ is not a curve class, then $\Psi_{(g, n ; \hat{\beta})}^{X}\left(\phi_{*} \alpha_{1}, \cdots, \phi_{*} \alpha_{n}\right)=0$ automatically; if $\hat{\beta}$ is a curve class, then note that there is no curve class $\tilde{\beta}$ in $H_{2}(\widetilde{X} ; \mathbb{Z})$ that satisfies both $p_{1 *}(\tilde{\beta})=\hat{\beta}$ and $\tilde{\beta} \cdot E=0$ since all such $\hat{\beta}$ are already contained in $\left\{\phi_{*}(\beta)+l[C] \mid l \in I_{\beta}\right\}$ by construction. The assumption that $\beta \neq 0$ implies also that such $\hat{\beta}$ cannot be a non-trivial multiple of $[C]$. It follows then from the discussion in Sec. 3.1 that all such $\Psi_{(g, n ; \hat{\beta})}^{X}\left(\phi_{*} \alpha_{1}, \cdots, \phi_{*} \alpha_{n}\right)$ must vanish as well since the degeneration $W / \mathbb{A}^{1}$ associated to blowing up $X$ along $C$ in Sec. 3.1 relates $\Psi_{(g, n ; \hat{\beta})}^{X}$ now to the relative GromovWitten invariant $\Psi_{\Gamma}^{(\widetilde{X}, E)}$ of $(\widetilde{X}, E)$ with $\Gamma \in \emptyset$ and the latter vanishes. Consequently, Theorem 3.2.1 above can be re-written simplier in terms of a superficial infinite sum:

Theorem 3.2.1' [GW-invariant under small extremal transition]. ([L-R: Theorem B and Corollary B.1].) With notations from above, for $\beta \neq 0$,

$$
\Psi_{(g, n ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \cdots, \alpha_{n}\right)=\sum_{\hat{\beta} \in \varphi_{e}^{-1}(\beta)} \Psi_{(g, n ; \hat{\beta})}^{X}\left(\phi_{*} \alpha_{1}, \cdots, \phi_{*} \alpha_{n}\right) .
$$

If $X^{\prime \prime}$ is a Calabi-Yau 3-fold, then the above identity reduces to

$$
\Psi_{(g, 0 ; \beta)}^{X^{\prime \prime}}=\sum_{\hat{\beta} \in \varphi_{e}^{-1}(\beta)} \Psi_{(g, 0 ; \hat{\beta})}^{X} .
$$

Consider now the 3 -point function for $X^{\prime \prime}$

$$
\Psi^{X^{\prime \prime}}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)=\left(\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}\right)_{0}+\sum_{\beta \in H_{2}\left(X^{\prime \prime} ; \mathbb{Z}\right)-\{0\}} \Psi_{(0,3 ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) q^{\beta} .
$$

By construction, $\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}=\phi_{*} \alpha_{1} \cdot \phi_{*} \alpha_{2} \cdot \phi_{*} \alpha_{3}$ since the relative cycles $\bar{\alpha}_{i} \in A_{*}\left(W^{\prime \prime} / \mathbb{A}^{1}\right)$ used to associate $\alpha_{i} \in A_{*}\left(X^{\prime \prime}\right)^{*}$ to $\phi_{*} \alpha_{i} \in A_{*}(X)$ are by definition flat over $\mathbb{A}^{1}$ and their lifting to $W^{s s} / \mathbb{A}^{1}$ are disjoint from the exceptional locus $E$ of $p_{1}: \widetilde{X} \rightarrow X$. Consequently, Theorem 3.2.1 ${ }^{\prime}$ and Remark 3.2.2 imply that

$$
\begin{aligned}
& \Psi^{X}\left(\phi_{*} \alpha_{1}, \phi_{*} \alpha_{2}, \phi_{*} \alpha_{3}\right) \\
& \quad=\left(\phi_{*} \alpha_{1} \cdot \phi_{*} \alpha_{2} \cdot \phi_{*} \alpha_{3}\right)_{0}+\sum_{\substack{\hat{\beta} \in H_{2}(X ; \mathbb{Z})-\{0\} \\
\hat{\beta} \neq m[C]}} \Psi_{(0,3 ; \hat{\beta})}^{X}\left(\phi_{*} \alpha_{1}, \phi_{*} \alpha_{2}, \phi_{*} \alpha_{3}\right) q^{\hat{\beta}} \\
& \quad \text { by noting that } \sum_{\hat{\beta}=m[C], m>0} \Psi_{\substack{(0,3 ; \hat{\beta})}}^{X}\left(\phi_{*} \alpha_{1}, \phi_{*} \alpha_{2}, \phi_{*} \alpha_{3}\right) q^{\hat{\beta}}=0 \text { since } C \cap \phi_{*} \alpha_{i}=\emptyset,
\end{aligned}
$$

$$
\begin{aligned}
&=\left(\phi_{*} \alpha_{1} \cdot \phi_{*} \alpha_{2} \cdot \phi_{*} \alpha_{3}\right)_{0}+\sum_{\beta \in H_{2}\left(X^{\prime \prime} ; \mathbb{Z}\right)-\{0\}} \sum_{\hat{\beta} \in \varphi_{e}^{-1}(\beta)} \Psi_{(0,3 ; \hat{\beta})}^{X}\left(\phi_{*} \alpha_{1}, \phi_{*} \alpha_{2}, \phi_{*} \alpha_{3}\right) q^{\hat{\beta}} \\
& q^{\hat{\beta}} \rightarrow q^{\varphi_{e}(\hat{\beta})}=q^{\beta} \\
&\left(\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}\right)_{0}+\sum_{\beta \in H_{2}\left(X^{\prime \prime} ; \mathbb{Z}\right)-\{0\}} \Psi_{(0,3 ; \beta)}^{X^{\prime \prime}}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) q^{\beta} \\
&=\Psi^{X^{\prime \prime}}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right) .
\end{aligned}
$$

In summary:
Corollary 3.2.3 [3-point function under small extremal transition]. ([L-R: Corollary B.2].) The 3-point function $\Psi^{X^{\prime \prime}}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ of $X^{\prime \prime}$ is equal to the 3-point function $\Psi^{X}\left(\phi_{*} \alpha_{1}, \phi_{*} \alpha_{2}, \phi_{*} \alpha_{3}\right)$ of $X$ with a change of variables $q^{\beta} \rightarrow q^{\varphi_{e}(\beta)}$ over $\beta \in H_{2}(X ; \mathbb{Z})$.

Remark 3.2.4 [Kähler 3-fold hierarchy vs. GW-hierarchy]. The discussions of this section can be summarized schematically by the following diagram:


## Appendix. The stringy and the symplectic aspect.

A very terse account of transformations of Gromov-Witten invariants from the stringy and the symplectic viewpoint is given in this appendix to complement the main text and to provide a slightly more (though never) complete picture of where the issue sits in all.

## A. 1 Transformation of GW-invariants from the stringy viewpoint.

The generating function of Gromov-Witten invariants gives a mathematical expression/ formulation for some correlation functions in superstring theory. The transformation of these correlation functions from one phase to another is an important issue in the theory. In this subsection, we discuss the concept of phase structures in superstring theory, focusing on those sharing an interface with birational geometry, and conclude with an explanation of Witten's wall-crossing formula in [Wit5] and its completion in [L-R].

## A glimpse of superstring theory.

String theory is a constantly fast-growing subject, containing a vast amount of contents with an ultimate goal to decipher God's code of creation and to unravel the mystery of the (physical) Universe. So far no-one can predict if the theory will finally settle to a final form, yet mathematicians have already witnessed its horrifying power of synthesizing drastically different branches of mathematics into an integrated entity via stringy dualities. Such ability and strength to bring together the microscopic world (particle physics), the macroscopic world (cosmology), and the man-made world (mathematics) make string theory very unique. A very conservative glimpse of it is given in DiAgram A-1-1, whose details are referred to $[\mathrm{G}-\mathrm{S}-\mathrm{W}]$, [Lü-T], [Polc1], and the literatures guided therein up to the end of year 1997. Surveys on more recent developments can be found in TASI Lectures from year 1996 on. $[\mathrm{Zw}]$ contains a lot of physicists' sense/intuitions/insights cleanly explained, which can be very helpful for mathematicians.

## Remark A.1.1. Two brief remarks follow:

(1) String $/ M / F$-theory. The five fundamental superstring models at 10 -dimensions are now known to be difference phases of a single $M$-theory, which also contains the 11-dimensional supergravity theory as one of its phases, [Wit6]. There is also a 12dimensional $F$-theory from geometrizing a complexified coupling constant in the low energy supergravity theory associated to the 10-dimensional IIB superstring theory, [Va2]. See [Polc1: Chapter 14] for a review of various dualities that connect all these theories/models. Compactifications of these theories bring $G_{2} 7$-manifolds and Spin (7) 8-manifolds into play as well. Transformations of correlations in superstring theory may be reduced to a reduction of transformations of correlations in these higher level theories.
(2) Though D-branes had already entered string theory in late 1980's ([D-L-P]), their very fundamental role in string theory began to be understood only in the mid 1990's ([Polc3]; see also [H-T] and [Wit6]). Since then, string theory has been revolutionized to such an extent that one has to re-ask What is string theory? What is space-time? and even What is a D-brane itself? Non-commutative geometry is expected to play a role here, cf. [Do2].

## Wilson's theory-space, phases, and dualities/correspondences.

A quantum field theory (QFT), as specified by its Lagrangian density, admits continuous deformations by, e.g. varying the coupling constants in the Lagrangian density. Wilson's theory-space, denoted by a general notation $\mathcal{S}_{\text {Wilson }}$ in this subsection, is meant to be the universal parameter/moduli space that encodes all such deformations of a field theory with a specified combinatorial type like number/type of fields involved (e.g. scalor theory, gauge theory with matters, gravity theory, nonlinear sigma model), dimension and symmetry of


Diagram A-1-1. The string world-sheet, the brane-world-volume, the total space-time, and the 4-dimensional effective field theory aspect of a string theory. Scattering amplitudes of fields and D-branes in space-time computed via string world-sheet methods and via space-time field theory method have to match order by order. Each of the four aspects itself has a Wilson's theory-space associated to it, containing all the phases of the field theory associated to that aspect of the string theory. Dualities can be realized either as a local isomorphisms or a coordinate change on the Wilson's theory space $\mathcal{S}_{\text {Wilson }}$ that induces an isomorphism on the universal family of Hilbert spaces of states, ring of operators, and correlation functions of these operators over $\mathcal{S}_{\text {Wilson }}$. In the weak string coupling regime, strings are light and branes are heavy and hence string are regarded as more fundamental. In the strong string coupling regime, branes can become light and strings become heavy and should be no longer treated as the most/unique fundamental object in the theory.
the field theory (e.g. $d=4, N=2$ super-Yang-Mills theory with gauge group $S U(n)$ ), types of couplings (e.g. $\phi^{4}$-theory, Yukawa theory), ..., etc. This is in the same spirit as the use of parameter/moduli spaces in mathematics for parameterizing e.g. varieties, Kähler manifolds, coherent sheavs, bundles, algebras, rings, ... etc. with a specification of combinatorial type like Hilbert polynomials, the number of generators, ..., etc.. With these said, however, quantum field theory, quantities defined therein and techniques used to draw conclusions remain full with surprises and mysteries for mathematicians (e.g. [AH-C-G] as a more recent example). While trying to spell out $\mathcal{S}_{\text {Wilson }}$ below, we have to admit as well our ignorance to $\mathcal{S}_{\text {Wilson }}$ and the universal family, in notation $\mathcal{U}_{\text {Wilson }}$, that goes with it (as in any moduli problem in mathematics). See e.g. [Ca], [P-S], [Poly], [Stra], [A-D-S], [G-V-W], [Polc2] for physicists' insights and [B-B] for a discussion on the mathematical structure in renormalization of QFT.

- $\mathcal{S}_{\text {Wilson }}$ as a universal parameter space. As a universal parameter space for QFT of a specified combinatorial type, a local coordinate system on the Wilson's theory-space $\mathcal{S}_{\text {Wilson }}$ should contain
(1) coupling constants of the field theories in the specified combinatorial types,
(2) a length/energy scale $\lambda$ that sets the effectiveness of the theory so that any effect from a smaller smaller scale (or larger energy) than the specified one is regarded as already integrated out,
(3) other natural parameters in the theory depending the contents, e.g. string tension $\alpha^{\prime}$ or the Planck's constant $\hbar$,
(4) all the additional parameters arising from renormalization/regularization schemes that are needed to render physical quantities of the theory finite.

Renormalization group flow (RG-flow) on $\mathcal{S}_{\text {Wilson }}$ takes a theory in one length scale $\lambda$ to another theory in a larger length scale $\lambda^{\prime}>\lambda$ by integrating out further the effect between $\lambda$ and $\lambda^{\prime}$.

- The universal object $\mathcal{U}_{\text {Wilson }}$ over $\mathcal{S}_{\text {Wilson }}$. A quantum field theory contains a long list of contents, e.g. the domain space on which fields are defined, the target spaces fields take value, bundles/sheaves whose sections give the fields, the Lagrangian density, symmetries, Hilbert space of states, observables/operators, correlation functions of fields/observables/operators, vacuum space/manifold $\cdots$. These pile together and form several types of universal objects over $\mathcal{S}_{\text {Wilson }}$. We will denote these universal objects collectively by $\mathcal{U}_{\text {Wilson }}$ over $\mathcal{S}_{\text {Wilson }}$.
- The topology and geometry on $\mathcal{S}_{\text {Wilson }}$. The use of coupling constants in Lagrangian densities and other parameters as the local coordinates on $\mathcal{S}_{\text {Wilson }}$ gives a topology on $\mathcal{S}_{\text {Wilson }}$. Infinitesimal deformations of a QFT $T$ can be realized as operators $O_{i}$, $i \in$ some index set $I$, in the theory $T$. The 2-point functions $\left\langle O_{i} O_{j}\right\rangle_{T}$ induces the (Zamolodchikov) metric on $\mathcal{S}_{\text {Wilson }}$. This makes $\mathcal{S}_{\text {Wilson }}$ a geometric object.
- Phase structure on $\mathcal{S}_{\text {Wilson }}$. In general, $\mathcal{S}_{\text {Wilson }}$ has a stratification determined by the behavior of the field theories it parameterizes. E.g. the 2-point (correlation) function can behavior asymptotically following a power law on on one region of $\mathcal{S}_{\text {Wilson }}$ while asymptotically exponentially on another region; or the vacuum space can be described by one deformation family of geometries/manifolds on one region of $\mathcal{S}_{\text {Wilson }}$ while by a different deformation family of geometries/manifolds on another region od $\mathcal{S}_{\text {Wilson }}$. This gives a phase structure on $\mathcal{S}_{\text {Wilson }}$ with each maximal stratum of the stratification called a phase of $\mathcal{S}_{\text {Wilson }}$ (or of the theory of the specified combinatorial type). Some nature of the quantum field theries parameterized by $\mathcal{S}_{\text {Wilson }}$ changes when passing from one phase to another.

Many mathematical moduli spaces/stacks with their universal object appear as a subfibration of $\mathcal{U}_{\text {Wilson }}$ over $\mathcal{S}_{\text {Wilson }}$. For example, in nonlinear sigm models (resp. compactifications of a superstring model) the moduli space of the target space (resp. compactifying space) (possibly with decorations like bundles/coherent-sheaves thereover or cycles therein) is naturally contained in the related $\mathcal{S}_{\text {Wilson }}$. And for gauge theories, the natural coordinates on $\mathcal{W}_{\text {Wilson }}$ can contain gauge instanton corrections, which involve integrations over the moduli space of connections or stable sheaves. Such links of $\mathcal{S}_{\text {Wilson }}$ to mathematical moduli problems turn many physical/stringy computations/statements to surprising mathematical conjectures, (cf. the discussion on geometric engineerings below).

Remark A.1.2 [reduced theory-space]. In the physicists' setting in describing Wilson's theory-space, a local chart thereon consists only of the coupling constants (the first in our four sets of parameters). Though physicists have developed techniques to use it fluently, mathematically this reduced description is less satisfying. The distinction is similar to (but more involved than) the distinction between a coarse moduli space versus the moduli stack in a moduli problem. However, in all our discussions below it is really this part that matters. Thus, we will resume physicists' conventional use/definition below; i.e. $\mathcal{S}_{\text {Wilson }}$ will be the parameter space of Largrangian densities of a fixed combinatorial type for the rest of the discussions. Depending on the level of the theory one explores - classical, quantum, or non-perturbative -, locally on $\mathcal{S}_{\text {Wilson }}$ these coupling coefficients as coordinates may be subject to corrections at each level. The length scale $\lambda$ at which the theory becmes effective comes in usually via an energy cutoff in the renormalization/regularization procedure. (It can come into play via other mechanism, e.g. dynamical generation, as well.)

Example A.1.3 [natural coordinates on $\mathcal{S}_{\text {Wilson }}$ and stringy duality]. Compactifications of different superstring models on different Calabi-Yau 3 -folds $X$ (possibly decorated with supersymmetric cycles) may give rise to isomorphic 4-dimensional effective supersymmetric field theories. (Such effective field theory will be denoted in Candelas-type notation by String $[X]$ in this subsection.) When this happens, each choice of the superstring models and the topological type of comapctifying spaces $X$ gives rise to a natural local coordinate system on the Wilson's theory-space $\mathcal{S}_{\text {Wilson }}$ of the related $d=4$ field theory. The local coordinate transformation between these different coordinate systems on $\mathcal{S}_{\text {Wilson }}$ is the much-explored "mirror map". Unfamiliar readers can use keyword search:
"Calabi-Yau", "mirror symmetry", "special geometry", and " $N=1$ or $N=2$ supersymmetry" to find more detailed explanations; see also [C-dlO-G-P], [Gre], [Polc1: Chapter 17 and Chapter 18]. We will give another example below when discussing a link between birational geometry and geometric engineering of quantum field theory.

## Phases in superstring theory and birational geometry.

Three themes in superstring theory that involve some key ingredients of Mori's program in birational geometry ( $[\mathrm{Kol}-\mathrm{M}]$ ) are listed below. (These had come before the language and techniques of derived categories, $t$-structures, and perverse sheaves came to play a role as well on both sides; see [As] and [Wa] for a review of these later related directions.)

- Phases of string world-sheet theory vs. transitions of Calabi-Yau manifolds. This is related to the theme of this note and we will come back to it. See also [Stro2] for a review from the aspect of $d=4$ supersymmetric effective field theories via compactifications of the superstring models on Calabi-Yau spaces.
- Special loci in the theory-space of $d=4$ effective SQFT vs. singularities of CalabiYau spaces: geometric engineering ([K-K-V]). Superstring theory contains also other dynamical extended objects, particularly D-branes. When a superstring model is compactified on a singular Calabi-Yau 3-space $X_{0}$ that arises from deforming either the complex or the Kähler structure of smooth Calabi-Yau manifolds $X$, D-branes that are wrapped around vanishing cycles or exceptional locus in $X$ associated to the complex or the Kähler degeneration (which contribute to massive spectrum of the $d=4$ effective theory String $[X])$ can give rise to extra degeneracy of the massless spectrum of the low-energy $d=4$ effective supersymmetric field theory String $\left[X_{0}\right]$, cf. [Stro1]. In particular, there can be enhanced gauge symmetry determined by the singular locus $X_{0 \text { sing }} \subset X_{0}$ when one moves in the theory-space of the $d=4$ SQFT from String $[X]$ to String $\left[X_{0}\right]$, e.g. [K-M-P]. Such phenomena are indeed required in establishing, e.g. IIA/Heterotic string duality and they go up also to compactifications of M- and F-theory. Focusing only on a neighborhood of $X_{0 \text { sing }} \subset X_{0}$ and its resolution gives rise to a corrrespondence (geometric engineering of SQFT):


In particular, this implies that there are three natural but different coordinates on a same region in the Wilosn's theory-space $\mathcal{S}_{\text {Wilson }}$ for the related $d=4$ supersymmetric gauge theory:
(1) from String $\left[X_{0}\right]$ : system of coordinates that contain string world-sheet instantons corrections;
(2) from local mirror to $X$ : system of coordinates that arise from periods of a (complex) curve $C$, the local mirror to $X$;
(3) as a $d=4$ super Yang-Mills theory in its own right: system of coordinates that contain gauge instanton correcions.

The related local coordinate transformations on $\mathcal{S}_{\text {Wilson }}$, phrased in mathematical language, say that some limit of generating function of Gromov-Witten invariants for the local Calabi-Yau 3-fold $X$ can be related to the generating function of Donaldson/Seiberg-Witten invariants for the $d=4$ gauge theory (on an appropriate compactification of $\mathbb{R}^{4}$ or $\mathbb{C}^{2}$ ) and that both can be reproduced by periods of curves; see $[\mathrm{N}-\mathrm{Y}]$ for some related review. On the other hand, a direct relation at the level of the moduli spaces involved (with the virtual fundamental classes thereon constructed) remains very technical to understand.

Incidentally, a singular $X_{0}$ can have different resolutions $X \leadsto X_{0}$. Thus these spacial loci can be where different phases/branches of the theory-space meet.

- Phases of D-brane world-volume theory vs. resolutions of quotient singularities. A D-brane carry charges for RR-fields in type II string theories. It is also where endpoints of open strings are sticked to. When a D3-brane sits in the singular locus of a quotient space-time $\mathbb{R}^{3+1} \times \mathbb{C}^{3} / \Gamma$, where $\Gamma$ is a finite subgroup in $U(3)$ and acts on $\mathbb{C}^{3}$ effectively and freely except at the origin, the induced field theory on the worldvolume of the D-brane is a $d=4$ SQFT with supermultiplets from dimensional reduction, orbifolding, and also excitations of open strings with one or both endpoints on the D-brane. The contents of this $d=4$ field theory can be encoded in a quiver diagram. Similar to the discussion in [Wit5], there are coupling constants in the theory that parameterizes the theory-space $\mathcal{S}_{\text {Wilson }}^{D-\text {-rane }}$ of such field theories on the D-brane. On different regions of $\mathcal{S}_{\text {Wilsone }}^{D \text {-brane }}$ the associated vacuum manifold of the theory can be different, giving rise to a phase structure on $\mathcal{S}_{\text {Wilson }}^{\text {D-brane }}$.

As a probe to the nature of its ambient space-time, before descended to a field theory expanded at a vacuum manifold, the field theory on the D-brane, regarded as a $d=4$ nonlinear sigma model with target the transverse space to the brane, "sees" a non-commutative ambient space with local coordinates transverse to the brane enhanced to Lie-algebra-valued depending on $\Gamma$. However, the vacuum manifolds obtained from solving minimal potential equation with flatness conditions from the superpotential recover the commutative nature of the transverse geometry to the brane yet they do not go back to the singular $\mathbb{C}^{3} / \Gamma$ that one starts with. Rather, these vacuum manifolds are various (partial) resolutions of $\mathbb{C}^{3} / \Gamma$. Thus, moving from one phase to another in $\mathcal{S}_{\text {Wilson }}^{\text {D-brane }}$ and via the descendant low energy field theory on the D-brane world-volume, the D-brane "sees" different ordinary (i.e. commutative) ambient space-times from the resolutions of $\mathbb{C}^{3} / \Gamma$, e.g. $[D-M],[D-G-M]$, and $[G-L-$

R]. In this way, the phase structure on the induced field theory on the D-brane world-volume is linked to resolutions of a quotient singularity. Figure A-1-2.
(It should be noted that there are other perspectives of phases of $D$-branes; see e.g. [H-I-V] and [Ma] in connection with McKay correspondence, mutations, and helixes; and e.g. $[\mathrm{A}-\mathrm{H}]$, $[\mathrm{H}-\mathrm{W}]$, and [Wit8] for phase structures of field theory from brane configurations.)


Figure A-1-2. The D-brane sitting along the singular locus perceives a noncommutative thickening of the singular space-time to begin with. At low energy, the space-time it perceives resumes an ordinary yet different space-time: a resolution of the original singular space-time. Each resolution corresponds to a geometric phase in the theory-space $\mathcal{S}_{\text {Wilson }}^{D-\text { brane }}$ of the QFT on the D-brane world-volume.

The world-sheet aspect of a superstring theory leads directly to the Gromov-Witten theory. We will now focus on this link. Some related physics background is highlighted in [Liu-L-Y: Appendix], which we will use directly.

## From superstrings to Gromov-Witten theory: A-model.

The path-integral quantization of superstrings in principle has the moduli spaces of holomorphic maps hidden in its integral. However, the technical issue of existence of global spinors on general string world-sheets and the fact that the path-integral involves more than such maps make these moduli spaces at best hidden. In [Wit4], Witten introduces the notion of topological twists. Associated to a Calabi-Yau manifold $X$ are two $d=2$, $N=(2,2)$ topological field theories: the A-model and the B-model for $X$. And the pathintegral of the A-model $A$-model $[X]$ for $X$ localizes on the space of holomorphic maps into $X$. This brings the moduli space of holomorphic maps of arbitrary genus manifestly out in superstring theory and links superstring theory directly to Gromov-Witten theory.

## Phase structure and the transformation of A-model correlation functions.

Though one's final goal is to understand the $d=4$ effective theory $\operatorname{String}[X]$ from superstring compactification on a Calabi-Yau 3 -space $X$, some quantities in $\operatorname{String}[X]$ can be computed via the $d=2$ world-sheet theory $A$-model $[X]$. In particular, one has the following diagram of local embeddings

where the horizontal arrow is the assignment to $X$ the 2-dimensional field theory $A$-model $[X]$ and the vertical arrow is the assignment to $X$ the 4-dimensional field theory $I I A[X]$ in Candelas' notation (one may use $I I B[X]$ as well). (The image of the horizontal arrow indeed lies in a small subspace of $\mathcal{S}_{\text {Wilson }}^{d=2, N=(2,2)}$ that parameterizes theories with also conformal symmetries.) Moving from one irreducible component to another irreducible component of $\mathcal{M}_{C Y^{3}}$ by deforming the complex and/or the Kähler structure of the 3 -space corresponds to moving from one phase to another phase in both $\mathcal{S}_{\text {Wilson }}^{d=2, N=(2,2)}$ and $\mathcal{S}_{\text {Wilson }}^{d=4, N=2}$. How correlation functions of the field theory transform when crossing the wall/boundary between different geometric phases are natural question to ask. For the phase transition corresponding to a standard flop of Calabi-Yau 3-fold, Witten conjectured a wall-crossing formula in [Wit5: Sec. 5.5]. (Some notations follow [Wit5] and are used only in this subsection. In particular, $E$ below is not our exceptional divisor $E$ in Sec. 3.1. See also the related discussions in $[\mathrm{K}-\mathrm{M}-\mathrm{P}]$.)

Consider a $d=2, N=(2,2)$ gauged linear sigma model (GLSM) with a fixed number of vector multiplets, chiral multiplets and appropriate choice of charge vectors so that the vacuum manifold in each geometric phase of the theory is a Calabi-Yau 3-fold. The descendant $d=2$ non-linear sigma models associated to the geometric phases of the GLSM lies in the image $A$-model $\left[\mathcal{M}_{C Y^{3}}\right]$; they can also be identified as a subspace of String $\left[\mathcal{M}_{C Y^{3}}\right]$ in $\mathcal{S}_{\text {Wilson }}^{d=4, N=2}$.
(1) Following [Wit5: Sec. 5.5], suppose that $X_{+}:=X$ and $X_{-}:=X^{\prime}$ are realized as the respective vacuum manifold in two adjacent geometric phases of the GLSM. On each such phase, there is a descendant non-linear sigma model with target the vacuum manifold $X^{ \pm}$. The (real) 2-dimensional space on which fields of the nonlinear sigma models are defined can be identified as the (Wick-rotated) string worldsheet. Let $\alpha_{1}=\alpha_{2}=\alpha_{3}=$ a divisor $E$ on $X_{+}$that intersects $C$ properly with $[C] \cdot E=+1$. They correspond to observables $O_{E}$ of the $d=2$ nonlinear sigma model with target $X_{+}$. Then under the topology change $X_{+} \rightarrow X_{-}$(which is our $\operatorname{map} \phi),\left[C^{\prime}\right] \cdot E=-1$. Let $\lambda_{+}\left(\right.$resp. $\left.\lambda_{-}\right)$be the contribution to the 3-point function of the non-linear sigma model $\left\langle O_{E} O_{E} O_{E}\right\rangle_{X_{+}}=\Psi^{X_{+}}(E, E, E)\left(\operatorname{resp} .\left\langle O_{E} O_{E} O_{E}\right\rangle_{X_{-}}=\right.$
$\Psi^{X_{-}}(E, E, E)$, which means $\Psi^{X_{-}}\left(\phi_{*} E, \phi_{*} E, \phi_{*} E\right)$ in Sec. 3.1) purely from the worldsheet instantons wrapped around $C$ (resp. $C^{\prime}$ ). Then, following [C-dlO-G-P] and [A-M], Witten deduced that

$$
\lambda_{+}-\lambda_{-}=-1 .
$$

[Wit5: Eq.(5.48)]
The "=" here is a direct formal manipulation on power series and can be interpreted as an analytic continuation from $\lambda_{+}+1$ to $\lambda_{-}$. Witten argued that it cannot be just $\lambda_{+}=\lambda_{-}$as one might naively expect and that the discrepancy -1 in the above equation has to do with the classical 3 -point function, namely the triple interseection product ( $=E \cdot E \cdot E$ in this example) of the observables (i.e. divisors on $X_{ \pm}$) chosen.. Taking together the classical 3 -point function plus the world-sheet instanton contribution, Witten concludes that the new stringy corrected 3-point function of the non-linear sigma model in the two phases of the gauged linear sigma model should be related by an analytic continuation when crossing the wall between the two phases.
(2) Compared with $[\mathrm{L}-\mathrm{R}]$ and Sec. 3.1, the $\lambda^{ \pm}$is the third term of the decomposition of the full 3 -point functions $\Psi^{X}\left(\alpha_{1}, \alpha_{2}, \alpha_{3}\right)$ and $\Psi^{X^{\prime}}\left(\phi_{*} \alpha_{1}, \phi_{*} \alpha_{2}, \phi_{*} \alpha_{3}\right)$. Witten's statement above is the identity

$$
\begin{aligned}
& \phi_{*}\left(\alpha_{1} \cdot \alpha_{2} \cdot \alpha_{3}+\left([C] \cdot \alpha_{1}\right)\left([C] \cdot \alpha_{2}\right)\left([C] \cdot \alpha_{3}\right) \frac{q^{[C]}}{1-q^{[C]}}\right) \\
& \quad \stackrel{\text { a.c. }}{=} \phi_{*}\left(\alpha_{1}\right) \cdot \phi_{*}\left(\alpha_{2}\right) \cdot \phi_{*}\left(\alpha_{3}\right)+\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{1}\right)\right)\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{2}\right)\right)\left(\left[C^{\prime}\right] \cdot \phi_{*}\left(\alpha_{3}\right)\right) \frac{q^{\left[C^{\prime}\right]}}{1-q^{\left[C^{\prime}\right]}}
\end{aligned}
$$

(with the necessary notation modification in $[\mathrm{L}-\mathrm{R}]$ to adjust to the push-pull direction of differential forms).
(3) The remaining contributation to the full 3-point function is the second term in the decomposition. From [L-R] and Sec. 3.1, it is intact; namely, it follows simply by applying $\phi_{*}$ without having to perform analytic continuations. In this aspect, the only difference for 3 -point functions when crossing the wall of geometric phases of a gauged linear sigma model does arise solely from the curve neighborhood of $C$ and $C^{\prime}$ that are involved in the topology change $X \rightarrow X^{\prime}$. It is in this way that A.-M. Li and Y. Ruan in [L-R] completed Witten's picture/wall-crossing-formula in [Wit5] mathematically.

## A. 2 Transformation of GW-invariants from the symplectic viewpoint.

In this appendix, we highlight the work of A.-M. Li and Y. Ruan [L-R] in the symplectic aspect. Besides for comparison to the algebraic main text, these techniques become important for new applications. See also [I-P1] and [I-P2] of E.-N. Ionel and T.H. Parker. (Slight modifications of notations are made so that they are linked to the main text directly.)

## Symplectic relative Gromov-Witten theory.

Let $\left(M, \partial M ; \omega_{0}\right)$ be a symplectic manifold-with-boundary equipped with a regular local Hamiltonian function $H: U_{0} \rightarrow(-\infty, 0] \subset \mathbb{R}$ defined on a neighborhood $U_{0}$ of $\partial M$ such that $H^{-1}(0)=\partial M$ and that the local Hamiltonian flow generated by the Hamiltonian vector field $X_{H}$ on $U_{0}$ renders $U_{0}$ a circle bundle over the associated quotient manifold $U_{0} / S^{1}$ with boundary $Z:=\partial M / S^{1}$. By construction, $Z$ is a symplectic reduction of $U_{0}$ and is equipped with a natural symplectic structure $\underline{\omega}$ induced from $\left(U_{0}, \omega_{0}\right)$. An infinite symplectic stretching is then appied to $\left(M, \omega_{0}\right)$ along $\partial M$. The resulting manifold-withend is topologically the peeled manifold $M^{\circ}:=M-\partial M$ with a collar of $\partial M$ pulled out to an infinite end diffeomorphic $[0,+\infty) \times \partial M . M^{\circ}$ is equipped with a symplectic structure $\omega$ from stretching $\omega_{0}$ along the end. One can choose an $\omega$-tamed almost complex structure $J$ on $M^{\circ}$ whose restriction to the infinite end is compatible with an $\underline{\omega}_{0}$-tamed almost structure on $Z$ and satisfies compatibility conditions with the $S^{1}$ Hamiltonian action on the end. These conditions imply that when the image of a $J$-holomorphic map $f$ enters the end, it has to go all the way through to infinity. Such occurrence corresponds to nonremovable singularities of $f$ on the domain Riemann surface. The pair $(J, \omega)$ defines also a Riemannian metric on $M^{\circ}$ and on $\mathbb{R}^{1} \times \partial M$ that is needed for defining various Sobolev spaces in the problem.

The energy for a $J$-holomorphic map $f$ from a Riemann surface $\Sigma$ (with possibly nodes and punctures) to $M^{\circ}$ is defined by $E(f):=\int_{\Sigma} f^{*} \omega$. For a $J$-holomorphic map $f$ to have finite energy, any small enough neighborhood of a non-removable singularity of $f$ on $\Sigma$ must be mapped to the end of $M^{\circ}$ in such a way that $f$ is asymptotically wrapping an orbit of the $S^{1}$-action on the end. Such maps can be identified with a relative map to $\left(\overline{M^{\circ}}, Z\right)$, where $\overline{M^{\circ}}$ is the closed manifold obtained from $M$ by collapsing the $S^{1}$-orbits in $\partial M$ via the quotient map $\partial M \rightarrow Z$. The structures $\omega$ and $J$ on $M^{\circ}$ descend naturally to the symplectic manifold-submanifold pair ( $\overline{M^{\circ}}, Z$ ).

Under deformation of $J$-holomorphic maps, non-vanishing positive energy can condense toward isolated points on the domain Riemann surface. This gives rise to the bubbling phenomena of $J$-holomorphic maps. For a target with infinite ends, there are two types of bubbles that can occur: (1) the ordinary ones that already occur in the study of absolute symplectic Gromov-Witten theory and (2) the ghost bubbles, which are a collection of $J$ holomorphic maps of finite enegry from $\mathbb{P}^{1}$ with punctures to $\mathbb{R}^{1} \times \partial M$. The latter has to satisfy some compatibility conditions for gluing the bubbles along ends at infinity. (This is the pre-deformability condition in the algebro-geometric setting.). Fix a topological type $\left(g, T_{m} ; \beta\right)$ of $J$-holomorphic relative maps, where $g$ is the arithmetic genus of domain Riemann surface (possibly with nodes), $T_{m}$ is an ordered set of $m$ non-negative integers labelling both the ordinary and the special marked points (corresponding to non-removable singularities of maps) on the Riemann surface and the asymptotic behavior of maps on the special marked points, and $\beta \in H_{2}\left(M^{\circ} ; \mathbb{Z}\right)$ is a curve class. Then adding in $J$ holomorphic maps on domains with bubbles gives rise to a compact Hausdorff moduli space $\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ of $J$-holomorphic stable relative maps of finite energy into $M^{\circ}$. (Such a map can be identified as a stable relative map of topological type ( $g, \cdot ; \beta$ ) to an
expanded relative pair $\left({\overline{M^{\circ}}}_{[l]}, Z_{[l]}\right)$ from the smooth variety-divisor pair $\left(\overline{M^{\circ}}, Z\right)$.)
In general, $\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ may have dimension greater than expected (either from the index of a natural differential operator or from the standard deformation-obstruction theory in the moduli problem) and is unsuitable to be directly used to define invariants of $(M, \partial M)$ constant under deformations. The virtual neighborhood construction in [Ru1], outlined below for the current case, can be employed to remedy this. (See [Si1] for a review and references on various alternative approaches in the symplectic category.)

Let $\overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ be the moduli space of $C^{\infty}$ finite-energy stable relative maps of topological type $\left(g, T_{m} ; \beta\right.$ ) without imposing the $J$-holomorphy condition: $d f+J \circ d f \circ j=$ 0 . Let $\mathcal{D}_{\left(g, T_{m} ; \beta\right)}$ be the set of weighted graphs with roots and legs that labels the following combinatorial data of stable maps: (i) the combinatorial/topological type of the domain Riemann surface with ordinary and special marked points, (ii) the index of the asymptotic wrapping (around the $S^{1}$-orbits in the end of $M^{\circ}$ or $\mathbb{R}^{1} \times \partial M$ ) associated to each special marked point, (iii) corresponding decomposition of curve classes $\beta=\sum_{i} \beta_{i}$ associated to the decomposition of domain Riemann surface in (i). Then $\mathcal{D}_{\left(g, T_{m} ; \beta\right)}$ is a finite set and both $\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ and $\overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ are naturally stratified with each statum labelled by a $\Gamma \in \mathcal{D}_{\left(g, T_{m} ; \beta\right)}$. Each stratum $\overline{\mathcal{B}}\left(M^{\circ}, \Gamma\right)$ of $\overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ is a Fréchet orbifold. There is an obstruction space fibration $\overline{\mathcal{F}} \rightarrow \overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ whose restriction to each stratum $\mathcal{F}_{\Gamma} \rightarrow \mathcal{B}_{\Gamma}$ is an orbi-bundle of infinite rank. The fiber of $\overline{\mathcal{F}}$ at $[f: \Sigma \rightarrow$ $\left.M^{0} \amalg\left(\mathbb{R}^{1} \times \partial M\right)\right]$ comes from $C^{\infty}$-sections of the sheaf $\Omega^{0,1} \otimes f^{*} T_{*}\left(M^{0} \amalg\left(\mathbb{R}^{1} \times \partial M\right)\right)$ on $\Sigma$ with finite appropriate Sobolev norm. The $J$-holomorphy condition can be realized as a section $\bar{\partial}_{J}: \overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right) \rightarrow \overline{\mathcal{F}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ with $\bar{\partial}_{J}^{-1}(0)=\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$. Let $\pi: \mathcal{C} \rightarrow \overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ be the universal Riemann surface over $\overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ and $f:$ $\mathcal{C} \rightarrow M^{0} \amalg\left(\mathbb{R}^{1} \times \partial M\right)$ be the universal $C^{\infty}$ stable maps. By passing to a multiple, one can assume that the induced symplectic 2 -form on $\overline{M^{\circ}}$, still denoted by $\omega$, is integral. Let $\mathcal{L}$ be a complex line bundle on $\overline{M^{\circ}}$ with a unitary connection such that $c_{1}(\mathcal{L})=[\omega]$. Let $\omega_{\mathcal{C} / \overline{\mathcal{B}}}$ be the relative dualizing sheaf of $\mathcal{C}$ over $\overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$. Identify finite-energy stable maps to $M^{\circ}$ with relative maps to $\left(\overline{M^{\circ}}, Z\right)$. Then, from the push-forward $R \pi_{*}\left(\omega_{\mathcal{C} / \overline{\mathcal{B}}} \otimes f^{*} \mathcal{L}\right)^{\otimes k}$ with $k$ large enough, one can construct a stratumwise complex orbi-bundle $\rho: \overline{\mathcal{E}} \rightarrow \mathcal{U}$ of finite constant rank over a neighborhood $\mathcal{U}$ of $\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ in $\overline{\mathcal{B}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ such that the section $\bar{\partial}_{J}$ of $\overline{\mathcal{F}}$ over $\mathcal{U}$ can be extended to a section $\mathcal{S}_{e}$ of $\rho^{*} \overline{\mathcal{F}}$ over $\overline{\mathcal{E}}$ with $D \mathcal{S}_{e}$ surjective. This implies the transversality of $\mathcal{S}_{e}$ with the zero-section of $\rho^{*} \overline{\mathcal{F}}$ over $\overline{\mathcal{E}}$. The data $\left(\overline{\mathcal{E}}, \mathcal{S}_{e}\right)$ is called a stabilization of the section $\bar{\partial}_{J}$ of $\overline{\mathcal{F}}$ along $\bar{\partial}_{J}^{-1}(0)$. Note that $\overline{\mathcal{E}}$ over $\mathcal{U}$ is oriented by construction.

The pair $\left(\overline{\mathcal{E}}, \mathcal{S}_{e}\right)$ defines a virtual neighborhood $(U, E, I)$ of $\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$, where

- $U:=\left(\mathcal{S}_{e}\right)^{-1}(0)$, which has dimension the rank of $\mathcal{E}$ plus the expected dimension of the moduli problem; $U$ has an induced stratification from that of $\overline{\mathcal{E}}$ labelled by $\mathcal{D}_{\left(g, T_{m} ; \beta\right)}$, this stratification has the property that each stratum $U_{\Gamma}$ is a smooth orbifold and that if $\mathcal{B}_{\Gamma_{1}} \subset \overline{\mathcal{B}_{\Gamma_{2}}}$ is a lower stratum, then $U_{\Gamma_{1}} \subset U_{\Gamma_{2}}$ is a submanifold of (real) codimension $\geq 2$;
- the stratumwise orbi-bundle map $\mathcal{E} \rightarrow \mathcal{U}$ induces a map $U \rightarrow \mathcal{U}$ and $E \rightarrow U$ is the
pull-back of $\mathcal{E}$ via this map; by construction $E$ is an oriented stratumwise orbi-bundle over $U$;
- $I$ is the tautological section of $E$ associated to the inclusion $U \subset \overline{\mathcal{E}}$; by construction $I^{-1}(0)=\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$.

The relative Gromov-Witten invariants can now be defined via the virtual neighborhood $(U, E, I)$ of $\overline{\mathcal{M}}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right)$ as follows.

Let $\Theta$ be the Thom class of the oriented stratumwise orbi-bundle $E$ over $U$. The ordinary and the special marked points on the domain Riemann surfaces define eveluation maps from compositions

$$
e v_{i}: U \longrightarrow \mathcal{M}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right) \longrightarrow M^{\circ}
$$

and

$$
e v_{j}: U \longrightarrow \mathcal{M}\left(M^{\circ},\left(g, T_{m} ; \beta\right)\right) \longrightarrow Z
$$

respectively. Let $\alpha_{i} \in H^{*}\left(M^{\circ}\right)$ and $\xi_{j} \in H^{*}(Z)$ represented by differential formas, then the (symplectic) relative Gromov-Witten invariants associated to these data is defined to be

$$
\Psi_{\left(g, T_{m} ; \beta\right)}^{(M, \partial M)}\left(\alpha_{1}, \cdots, \xi_{1}, \cdots\right)=\int_{U} I^{*} \Theta \wedge \prod_{i} e v_{i}^{*} \alpha_{i} \wedge \prod_{j} e v_{j}^{*} \xi_{j}
$$

The integral is defined stratum by stratum and only the strata of maximal dimension matter. Inequalities that bound appropriate Sobolev norms of the normal derivatives of gluing maps of ker $D \mathcal{S}_{e}$ from one stratum to a lower stratum imply that the integral on each stratum is finite. This shows that $\Psi_{\left(g, T_{m} ; \beta\right)}^{(M, \partial M)}\left(\alpha_{1}, \cdots, \xi_{1}, \cdots\right)$ is well-defined. These will be regarded also as the relative Gromov-Witten invariants for the symplectic pair $\left(\overline{M^{\circ}}, Z\right)$ and denoted by $\Psi_{\left(g, T_{m} ; \beta\right)}^{\left.\overline{M^{\circ}}, Z\right)}\left(\alpha_{1}, \cdots, \xi_{1}, \cdots\right)$.

Relative pairs of the form $\left(\overline{M^{0}}, Z\right)$ appear in symplectic cuts of a symplectic manifold and the relative Gromov-Witten invariants defined above form the basic blocks of a gluing formula of (symplectic) Gromov-Witten invariants, which we now turn to.

## Gluing formula and transformation of GW-invariants.

Let $M$ be a (closed) symplectic manifold with a regular local Hamiltonian function $H$ defined on an open subset $U_{0}$ of $M$ such that $H^{-1}(0)$ separates $M$ into a disjoint union $M-H^{-1}(0)=M^{+} \amalg M^{-}$and that the flow of the Hamiltonian vector field $X_{H}$ generates an $S^{1}$-aaction on $U$, which in particular renders $H^{-1}(0)$ a circle bundle $H^{-1}(0) \rightarrow Z$. Let $\bar{M}^{ \pm}$be the symplectic manifold obtained from $M^{ \pm} \cup H^{-1}(0)$ by collapsing the $S^{1}$-orbits in $H^{-1}(0)$ via $H^{-1}(0) \rightarrow Z$. Then $\left(\bar{M}^{+}, \bar{M}^{-}\right)$forms a symplectic cut of $M$ and there is a natural morphism $\pi: M \rightarrow \bar{M}^{+} \cup_{Z} \bar{M}^{-}$that is a symplectomorphism except along $H^{-1}(0)$, where $\pi$ sends $H^{-1}(0)$ onto $Z$, becoming the circle bundle map.

An infinite symplectic stretching can be applied to a small tubular neighborhood of $H^{-1}(0)$ in the domain of $H$. The resulting symplectic space with an infinite tube $M_{\infty}$ is a disjoint union of the infinite symplectic stretching of $M^{+}$and $M^{-}$respectively together with structure-preserving diffeomorphisms $\partial_{\infty} M^{+} \simeq \partial_{\infty} M^{-} \simeq H^{-1}(0)$ on the ideal boundary. The same construction for symplectic relative Gromov-Witten theory and invariants can be applied to construct a Gromov-Witten theory and invariants for $M_{\infty}$. Stokes' theorem then implies that Gromov-Witten invariants for $M$ and those for $M_{\infty}$ with matching curve classes and cohomology classes are equal. On the other hand, the GromovWitten invariants for $M_{\infty}$ can be readily expressed as a gluing of relative Gromov-Witten invariants of $\left(\bar{M}^{+}, Z\right)$ with relative Gromov-Witten invariants of $\left(\bar{M}^{-}, Z\right)$. Together, this gives a gluing formula for Gromov-Witten invariants of $M$ in terms of relative GromovWitten invariants of the pairs, $\left(\bar{M}^{+}, Z\right)$ and $\left(\bar{M}^{-}, Z\right)$, ([L-R: Theorem 5.7, Theorem 5.8]).

Given 3 -folds $M$ and $M^{\prime}$ related either by a flop or by a small extremal transition, one can associate symplectic cuts $\pi: M \rightarrow \bar{M}^{+} \cup_{Z} \bar{M}^{-}$and $\pi^{\prime}: M^{\prime} \rightarrow{\overline{M^{\prime}}}^{+} \cup_{Z^{\prime}}{\overline{M^{\prime}}}^{-}$ to $M$ and $M^{\prime}$ respectively in such a way that $\left(\bar{M}^{+}, Z\right)$ and $\left({\overline{M^{\prime}}}^{+}, Z^{\prime}\right)$ depends only on the curve neighborhood involved in the flop/small-extremal-transition while $\left(\bar{M}^{-}, Z\right)$ and $\left(\overline{M^{\prime}}, Z^{\prime}\right)$ are isomorphic. This enables one to compare Gromov-Witten invariants of both $M$ and $M^{\prime}$ to relative Gromov-Witten invariants of the symplectic pair $\left(\bar{M}^{-}, Z\right)$ and leads to transformation rules between Gromov-Witten invariants of $M$ and Gromov-Witten invariants of $M^{\prime},[$ L-R: Theorem A, Corollary A.2, Theorem B, Corollary B.2].

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