

# On exterior calculus and curvature in piecewise-flat manifolds

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Simplicial, piecewise-flat discretizations of manifolds provide a clear path towards curvature analysis on discrete geometries and for solutions of PDE's on manifolds of complex topologies. In this manuscript we review and expand on discrete exterior calculus methods using hybrid domains. We then analyze the geometric structure of curvature operators in a piecewise-flat lattice.

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## I. INTRODUCTION

Piecewise-flat (PF) simplicial manifolds are a crucial computational framework for systems with dynamic geometry or systems with complex topologies/geometries. In general relativity, PF manifolds are a cornerstone of the coordinate-free discretization introduced by Regge [1] often referred to as Regge calculus (RC). RC is often regarded as the backbone of the semi-classical or low-energy limit of, or even an effective theory [2] of, quantum gravity. It has also proven to be a useful computational tool in numerical relativity [3]. For numerical solutions to partial differential equations, PF manifolds are one clear method to discretize complex geometries over which the differential equations act [4] or in conformal transformations on 2 dimensional surfaces of arbitrary genus [5].

One may view RC as an approach to characterizing the intrinsic geometry of PF manifolds. In other regards the exterior calculus prescribes the calculus of fields on the curved background of smooth manifolds. Exterior calculus approaches to functions and fields on and the curvature tensors of the PF manifold have been of great use in preserving geometric notions on the discrete manifolds. In the canonical approach to RC, one can describe an exact action principle [6, 7], yet the discrete curvature tensors give only approximate expressions after smoothing over the discontinuities. The exterior calculus methods were also quite useful in developing a clearer understanding of the Einstein tensor in RC [8, 9]. In the numerical analysis for PDE's on PF manifolds, discrete exterior calculus (DEC) is used to preserve geometric symmetries of continuous systems on discrete manifolds [10] and for providing the bedrock on which to construct geometric flows on complex topologies. [5, 11] The latter has been extensively studied on 2D surfaces.[12–16]

In recent work we have extended and applied the methods of RC and DEC to derive a simplicial discretization of Hamilton's Ricci flow [17]. As we apply RC and discrete forms to dynamic, but not necessarily covariant, geometric flows on PF manifolds, it is necessary to develop a deeper understanding of the nature of curvature and exterior calculus in discrete geometries. In this manuscript we show how the curvature in PF manifolds gives rise to seemingly distinct notions of curvature tensors: (1) curvature with a single sectional curvature or (2) isotropic curvatures similar to those of Einstein spaces. Sec. IA and Sec. IB will review canonical RC and the standard approaches to discrete differential forms. Then in Sec. II we discuss the geometric principles behind hybrid cells as local measures and show how these hybrid volumes are core elements of a volume-based DEC. In Sec. III we discuss the representation of curvature operators over the hybrid measures and transformations between them.

### A. Canonical Regge Calculus

Suppose  $\mathcal{M}$  is a  $d$ -dimensional, smooth manifold endowed with a simplicial complex  $\mathcal{T}$ . A PF triangulation of  $\mathcal{M}$ ,  $\mathcal{T}_0$ , is a mapping from each  $d$ -simplex to a flat  $d$ -simplex in  $\mathbb{R}^d$  such that the proper lengths of the edges in the 1-skeleton of  $\mathcal{T}$  are preserved in  $\mathcal{T}_0$ . The simplicial manifold formed by  $\mathcal{T}_0$  is often called a Regge manifold or Regge skeleton. We now review the canonical approach to RC, as in [6, 18].

The interior geometry of any given simplex in  $\mathcal{T}_0$  is given by Euclidean or Minkowski geometry and represents a common tangent space for each of the vertexes of the simplex. Any such simplex has an induced metric that is uniquely determined by the proper squared-edge lengths of the simplex. The metric as a function of the edge lengths  $g_{\mu\nu}(\ell^2)$  gives the local, piecewise-constant approximation to the metric associated with  $\mathcal{M}$ . For two simplexes sharing a common boundary, the joint domain is isomorphic to  $\mathbb{R}^d$  and thus is intrinsically flat. While the two simplexes as viewed from an observer off the manifold may appear curved, i.e. with non-zero extrinsic curvature, a loop

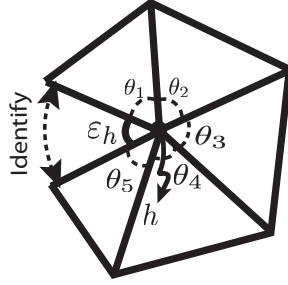


FIG. 1. 2-dimensional projection of a mapping of the neighborhood of a hinge  $h$  to  $\mathbb{R}^d$  with defect  $\varepsilon_h$ .

of parallel transport from simplex  $A$  to simplex  $B$  and back induces no change in orientation on a tangent vector in general position. Formally this is related to the requirement of the existence of a metric compatible connection in the PF manifold.

While there exists a flat connection across any  $(d-1)$ -boundary, curvature naturally arises when generating a map from a complete set of  $d$ -simplexes sharing a common  $(d-2)$ -simplex to  $\mathbb{R}^d$ . This is the first indication of curvature in the PF manifold. To effectively handle the discontinuity in mapping the neighborhood of a  $(d-2)$ -simplex, or codimension-2 hinge  $h$ , we can map the neighborhood of  $h$  to a subspace of  $\mathbb{R}^d$  and smoothly continue the mapping across the removed section of  $\mathbb{R}^d$  resulting from ‘breaking’ a  $(d-1)$ -simplex into two. By breaking a  $(d-1)$ -simplex into two to make the mapping to  $\mathbb{R}^d$ , a defect in the correspondence between the interior angles of the simplexes at  $h$  and  $2\pi$  is evident (Figure 1). We take the deviation of the interior angles of each simplex on  $h$  from an exact embedding in flat space to be the defect angle  $\varepsilon_h$  associated to  $h$  ;

$$\varepsilon_h = 2\pi - \sum_{i|h} \theta_i. \quad (1)$$

This defect angle is the measure of curvature associated with parallel transport of a vector around a loop that encircles  $h$ . Indeed, if we take any vector with components in the plane  $\hat{h}^*$  orthogonal to  $h$  and transport it around the boundary of any area  $\sigma^{\alpha\beta}$  (also with components in  $\hat{h}^*$ ), then the vector will have rotated by an amount equal to  $\varepsilon_h$ , and the rotation occurs in the plane of  $\hat{h}^*$ . Any loop  $\sigma^{\alpha\beta}$  with components in  $\hat{h}^*$  will generate such rotations, independent of the area enclosed. This has implications for the sectional curvature associated with  $h$ . The sectional curvature associated with the loop  $\sigma^{\alpha\beta}$  is given by

$$K_{\alpha\beta} = \frac{(\text{Angle of Rot'n})}{(\text{Area Enclosed})} = \frac{\varepsilon_h}{|\sigma^{\alpha\beta}|}. \quad (2)$$

Since the angle of rotation is independent of the area enclosed, we can take an infinitesimal area of rotation encircling  $h$  whose limit,  $|\sigma^{\alpha\beta}| \rightarrow 0$ , implying a singularity in the sectional curvature. The singularity is known as a conic singularity associated to the hinge  $h$ .

We can provide a representation for the sectional curvature for a loop of parallel transport, but can we also provide a clear expression for the Riemann curvature? From the continuum theory with infinitesimal rotations we can express the rotation of a vector  $A^\mu$  after transport along the boundary of an infinitesimal area by

$$\delta A^\nu = -R^\nu{}_{\mu\alpha\beta} A^\mu \mathbf{d}\sigma^{\alpha\beta}. \quad (3)$$

However, the curvature in PF manifolds is characterized by finite rotations. Friedberg and Lee [6] showed that a correspondence between the above notions of defect angle and a Riemann tensor can be approximately given by

$$R_{12\ 12}(h) \approx \varepsilon_h \delta(x_1) \delta(x_2) \approx -R_{12\ 21}(h) \quad (4a)$$

$$R_{1\ 1}(h) \approx \varepsilon_h \delta(x_1) \delta(x_2) \approx R_{2\ 2}(h) \quad (4b)$$

$$R(h) \approx 2\varepsilon_h \delta(x_1) \delta(x_2) \quad (4c)$$

$$\sqrt{|g|}R(h) = 2\varepsilon_h \delta(x_1) \delta(x_2), \quad (4d)$$

where Eq. (4d) is exact and the coordinates  $\{x_1, x_2\}$  lie in the plane of  $\hat{h}^*$ . In this approach, where one smooths out the discontinuities, one generates a limiting sequence of surfaces approximating the plane  $\hat{h}^*$  and takes the limit to the PF surface. This generates Dirac delta distributions such that the curvature is evaluated only on the hinge  $h$  and zero elsewhere. While the curvature tensors are only approximate in this sense, the integrand of the Einstein-Hilbert action,

$$I_{\text{E-H}} = \frac{c^4}{16\pi G} \int \sqrt{|g|} R d^d x$$

is exact. Thus we can take the standard action principle and have an exact expression for the lattice geometry (first locally),

$$\frac{c^4}{16\pi G} \int_{V_h} \sqrt{|g|} R d^d x = \frac{c^4}{8\pi G} \varepsilon_h A_h \quad (5)$$

where the integration is over a domain containing a single  $h$ . The global expression is obtained by summing over all hinges,

$$\frac{c^4}{16\pi G} \int \sqrt{|g|} R d^d x = \frac{c^4}{8\pi G} \sum_{h \in \mathcal{T}_0} \varepsilon_h A_h. \quad (6)$$

## B. Discrete Exterior Calculus

We now move away from the explicit geometry of PF manifolds to the properties of differential forms on simplicial manifolds. The framework we will follow below has stemmed from the work of Whitney [19] and, later, Bossavit [20]. Recent use of differential forms in PF manifolds corresponds to surface parameterization [11] or finite-element methods [4, 10]. The use of differential forms in the simplicial lattice has been independently used in RC (see for example [8, 21–23]). In the description below, we will follow some of the notation and conventions of [10].

The algebraic structure of the simplicial lattice is such that we have a natural representation of a discrete chain complex,

$$0 \xrightarrow{\tilde{\delta}} s^{(d)} \xrightarrow{\tilde{\delta}} s^{(d-1)} \xrightarrow{\tilde{\delta}} \dots \xrightarrow{\tilde{\delta}} s^{(1)} \xrightarrow{\tilde{\delta}} s^{(0)} \xrightarrow{\tilde{\delta}} 0, \quad (7)$$

and a discrete co-chain complex,

$$0 \xrightarrow{\tilde{d}} s^{(0)} \xrightarrow{\tilde{d}} s^{(1)} \xrightarrow{\tilde{d}} \dots \xrightarrow{\tilde{d}} s^{(d-1)} \xrightarrow{\tilde{d}} s^{(d)} \xrightarrow{\tilde{d}} 0. \quad (8)$$

Moreover, given a geometric dual lattice to the simplicial skeleton one can construct the dual chain and co-chain complexes.

Given a  $p$ -form  $\omega$  on the smooth manifold  $\mathcal{M}$ , we seek a representation of  $\omega$  on the PF manifold  $\mathcal{T}_0$ . The approach taken in DEC discretizations is to take the smooth image of  $\mathcal{T}_0$  on  $\mathcal{M}$ , i.e.  $\mathcal{T}$ , and evaluate  $\omega$  on the  $p$ -skeleton of  $\mathcal{T}$ . The simplicial approximation to  $\omega$  is obtained by integrating over individual elements of the  $p$ -skeleton of  $\mathcal{T}$  and associating that value with the image of the element in  $\mathcal{T}_0$ ;

$$\langle \omega | s^{(p)} \rangle := \omega_0(s^{(p)}) = \int_{s^{(p)}} \omega. \quad (9)$$

Here we use the notation that  $s^{(p)}$  denotes an element of the  $p$ -skeleton of  $\mathcal{T}$ . In general, we will use  $s^{(p)}$  to denote both elements of  $\mathcal{T}$  and  $\mathcal{T}_0$ . In the standard RC approach to discrete differential forms, the discrete forms may often be denoted by subscripts, e.g.  $\omega_\ell = \frac{1}{|\ell|} \langle \omega | \ell \rangle$  will generally denote a one-form  $\omega$  associated with an edge  $\ell$  in the simplicial lattice. Only when we are considering the initial discretization will  $s^{(p)}$  be in  $\mathcal{T}$ . Once the discrete differential form is assigned, all further manipulations take place within  $\mathcal{T}_0$ .

In addition to simplicial forms in  $\mathcal{T}_0$ , one can also discretize  $\omega$  on the dual lattice. Ambiguity arises here due to the multitude of ways of assigning a dual lattice. In general, there exist several natural geometric dual lattices, e.g. barycentric, circumcentric or incentric dual lattices, but there are also numerous non-intuitive dualities that can be constructed for an arbitrary simplicial lattice [24]. For our purposes and for clarity in later sections, we take the circumcentric dual as our prescribed dual lattice. The circumcentric dual of a  $k$ -simplex is given by [25]

$$\star s^{(k)} := \sum_{s^{(p)} \ni s^{(k)}: p > k} \text{sgn} [\mathcal{C}_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}_d], \quad (10)$$

where  $\mathcal{C}_p$  is the circumcenter of a  $s^{(p)}$  and  $\text{sgn}$  is used to ensure an orientation consistent with the encompassing  $s^{(d)}$ . The circumcentric dual has several nice properties that make it a natural choice: (1) a simplicial element and its circumcentric dual are (locally) orthogonal to one another, (2) the  $p$ -dimensional dual to a  $p$ -element is equidistant from each vertex on the  $p$ -element, and (3) in special cases the circumcentric dual lattice corresponds to the Voronoi lattice generated by the 0-skeleton of the simplicial lattice.

Once one has a dual lattice, discrete differential dual forms are obtained analogously to the simplicial forms. For a  $p$ -form  $\omega$  and a  $p$ -element  $\sigma^{(p)}$  of the dual lattice,  $\omega$  associated to  $\sigma^{(p)}$  is given by

$$\langle \omega | \sigma^{(p)} \rangle := \omega(\sigma^{(p)}) = \int_{\sigma^{(p)}} \omega. \quad (11)$$

This provides a way to directly discretize continuous forms on the dual lattice. Our next step is to show how to transform discrete forms on one lattice to forms on the other.

The discrete Hodge (or  $\star$ ) dual is an isomorphism between the simplicial  $k$ -forms and the dual  $(d-k)$ -forms. Given that the definitions of the dual/simplicial forms are integrated quantities over their respective lattice elements, we can assign an average scalar density to the lattice element by dividing by the volume of the lattice element. The Hodge dual isomorphism then says that these average scalar densities are equal for two dual elements;

$$\frac{1}{A_{s^{(k)}}} \langle \omega | s^{(k)} \rangle = \frac{1}{A_{\star s^{(k)}}} \langle \star \omega | \star s^{(k)} \rangle, \quad (12)$$

where  $A_{s^{(k)}} = |s^{(k)}|$  denotes the integrated measure, i.e. norm, of  $s^{(k)}$ . If we take the dual lattice to be the circumcentric dual, then the relation in Eq. (12) says that integration over a volume

spanned by  $s^{(k)}$  and  $\star s^{(k)}$  is preserved under the Hodge dual, i.e. the discrete manifestation of the self-adjointness of  $\star$  in the  $L^2$ -inner product.

It is useful to note here a property of the Hodge dual in the lattice that will be useful later on. As in the continuum, if one scales a differential  $k$ -form  $\omega$  by a scalar  $\alpha$ , then the dual  $\star(\alpha\omega)$  has the same orientation as  $\star\omega$ . In the simplicial lattice the Hodge dual acting on a subspace of a lattice element  $s^{(k)}$  returns back the lattice element  $\star s^{(k)}$ , possibly with a scalar coefficient. In particular, if one takes the dual of the portion of a triangle  $t$  closest to an edge  $\ell$ , then one simply obtains the dual element to  $t$  itself. The dual lattice element  $t^*$  orthogonal to  $t$  lies on the face of the polytope  $\ell^*$  and so entirely lies in the convex hull of  $\ell$  and  $\ell^*$ . Moreover it is orthogonal to  $t$  and therefore trivially orthogonal to the subspace of  $t$  closest to  $\ell$ .

Given a general dual lattice to the simplicial lattice and the Hodge dual, we can construct the complex

$$\begin{array}{ccccccccc} 0 & \xrightarrow{\tilde{d}} & s^{(0)} & \xrightarrow{\tilde{d}} & \dots & \xrightarrow{\tilde{d}} & s^{(d)} & \xrightarrow{\tilde{d}} & 0 \\ \downarrow \star & & \downarrow \star & & \downarrow \star & & \downarrow \star & & \downarrow \star \\ 0 & \xrightarrow{\tilde{\delta}} & \sigma^{(d)} & \xrightarrow{\tilde{\delta}} & \dots & \xrightarrow{\tilde{\delta}} & \sigma^{(0)} & \xrightarrow{\tilde{\delta}} & 0 \end{array} \quad (13)$$

containing the chain and co-chain complexes in the dual and simplicial lattices. In the above complex, the operators  $\tilde{d}$  and  $\tilde{\delta}$  define the co-boundary and boundary operators, respectively.

The next ingredient in the calculus of discrete differential forms is the exterior derivative  $d$  which maps  $k$ -forms to  $(k+1)$ -forms,

$$\langle d\omega|s^{(k+1)} \rangle = \langle \omega|\tilde{\delta}s^{(k+1)} \rangle = \sum_{s^{(k)} \in s^{(k+1)}} \langle \omega|s^{(k)} \rangle. \quad (14)$$

Here we have used the lattice boundary operator (via Stokes' Theorem) such that the action on a  $k$ -element of the simplicial  $k$ -skeleton returns its  $(k-1)$ -boundary. The discrete exterior derivative allows us to express  $d\omega$  in terms of the valuations of  $\omega$  on the boundary of a given  $s^{(k+1)}$ . By replacing the  $s^{(k)}(s^{(k+1)})$  with  $\sigma^{(k)}(\sigma^{(k+1)})$  we can translate the discrete exterior derivative from simplicial forms to dual forms.

$$\langle d\omega|\sigma^{(k+1)} \rangle = \langle \omega|\tilde{\delta}\sigma^{(k+1)} \rangle = \sum_{\sigma^{(k)} \in \sigma^{(k+1)}} \langle \omega|\sigma^{(k)} \rangle.$$

Similarly the exterior coderivative,  $\delta = \star d \star$ , on a  $(k-1)$ -form can be derived by using the adjoint relationship (in the continuum) between  $d$  and  $\delta$ ,

$$\langle \delta\omega|s^{(k-1)} \rangle = \langle \omega|\tilde{d}s^{(k-1)} \rangle = \sum_{s^{(k)} \ni s^{(k-1)}} \langle \omega|s^{(k)} \rangle. \quad (15)$$

Again, the same relationship holds on the dual forms

$$\langle \delta\omega|\sigma^{(k-1)} \rangle = \langle \omega|\tilde{d}\sigma^{(k-1)} \rangle = \sum_{\sigma^{(k)} \ni \sigma^{(k-1)}} \langle \omega|\sigma^{(k)} \rangle.$$

The co-derivative on discrete forms requires only our notion of the Hodge dual and the exterior derivative and so the above expressions come as a direct result of the adjoint relationships of these two operators in  $\langle \cdot | \cdot \rangle$ . We can, in fact, rebuild much of the algebra and calculus on exterior forms with these basic building blocks. The wedge product can also be reconstructed on the discrete forms as shown by Desbrun *et al.* [25]. We do not reproduce the wedge product here, but only mention

that this wedge product will, in general, be non-associative in the discrete scale. Associativity, however, is recovered in the continuum limit. One can construct an associative wedge product, but such a construction will fail to preserve the anti-commutativity property. It is generally a feature of discrete physics with finite angles that certain continuum symmetries fail at the discrete level only to be regained in the infinitesimal edge-length limit.

## II. THE LOCAL STRUCTURE OF DISCRETE FORMS

We have described thus far the previous approaches to analysis of curvature and differential forms on a  $\mathcal{T}_0$ . What was evident when we examined the discrete forms and the relationship between forms on the dual lattices was that the integrated measure of a lattice differential form was preserved under the Hodge dual. Given this preservation, we can interpret the discrete forms as measures over a volume spanned by a lattice element and its dual, i.e. the convex hull of the vertexes of a lattice element and the vertexes of the dual element. This suggests a volume-based approach to discrete forms. Since many of the operations applied to discrete forms rely on transforming one discrete form on a  $k$ -skeleton to a discrete form on a  $p$ -skeleton (where we may have  $p \neq k$ ), we will be required to transform objects from one domain to a non-coinciding (but possibly overlapping) domain of integration. We now turn to the local properties of discrete forms and their inherent domains of support. This will allow us to reconstruct transformations between lattice elements in terms of hybrid simplicial-dual volumes, henceforth called hybrid cells or hybrid volumes.

### A. A Menagerie of Hybrid Cells

A hybrid volume is a domain that is at once the local measure of a lattice element  $s^{(k)}$  and the orthogonal subspace dual to  $s^{(k)}$ . For standard tensor analysis it is sometimes convenient to take the  $d$ -simplexes as the local domains over which a function or tensor is piecewise evaluated. This is due to the ability to define an unambiguous tangent space to any  $d$ -simplex in  $\mathcal{T}_0$ . However, an arbitrary  $k$ -form will, in general, be contained in multiple  $d$ -simplexes and thus requires junction conditions to hold across the multiple coordinate charts assigned to the  $d$ -simplexes sharing a given  $k$ -simplex. We will show how hybrid cells can be viewed as natural domains for differential forms in  $\mathcal{T}_0$  such that there are local orthogonal frames containing the carrier of the geometric content of the discrete form. The lattice element hybrid cells are atomized via local domains that contain the minimal, non-trivial amount of information about the lattice  $k$ -skeleton. We build these hybrid cells from local constructions and discuss how they are representative of the measure of the discrete forms.

We first construct domains that are shared by a set of  $k$ -simplexes, one  $k$ -simplex for each  $k = 0, \dots, d$ . These shared domains become the “atoms” of our geometry in the sense that these are the simplest meaningful  $d$ -volumes in a PF manifold. Of course one could always subdivide these irreducible domains in some arbitrary way, even to go so far as to define infinitesimal domains. However, doing so yields no further discrete information about the lattice or the differential forms on the lattice.

**Definition 1.** *An irreducible hybrid cell,  $V_{s^{(0)}s^{(1)}\dots s^{(d)}}$ , in a PF  $d$ -manifold  $\mathcal{T}_0$  is the  $d$ -simplex*

$$\text{sgn}_{0\dots d}([\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_d]) = \frac{1}{d!} \epsilon_{i_1 \dots i_d} e^{i_1} \wedge \dots \wedge e^{i_d},$$



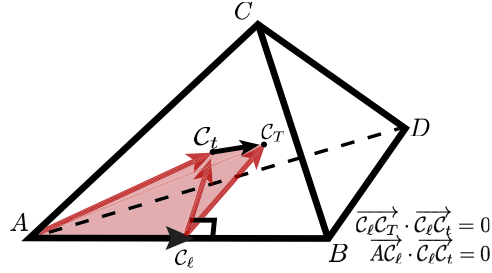


FIG. 2. An irreducible domain in three dimensions. The domain (red, shaded) is that domain on which a 0-form on  $A$ , a 1-form on  $[AB] = \ell$ , a 2-form on  $[ABC] = t$ , and a 3-form on  $[ABCD] = T$  are mutually defined. The black edges bounding the irreducible domain are edges that lie in either the simplicial or dual 1-skeleton of  $\mathcal{T}_0$ . The points  $C_\ell$ ,  $C_t$ , and  $C_T$  label the circumcenters of  $\ell$ ,  $t$ , and  $T$  respectively. Moreover, there is a natural orthogonal basis in this domain given by the edges  $\overrightarrow{C_v C_\ell}$ ,  $\overrightarrow{C_\ell C_t}$ ,  $\overrightarrow{C_t C_T}$ . This irreducible domain is foundation of the DEC as applied to the geometry of PF manifolds.

given that  $s^{(k)} \in s^{(k+1)}$  for every  $k < d$ ,  $\epsilon_{i_1 i_2 \dots i_d}$  is the orientation of  $s^{(d)}$ , and the vectors  $e_i$  are the vectors emanating from  $C_0$ . When a circumcenter lies outside  $\mathcal{T}_0$ , we only take the domain of  $V_{s^{(0)} s^{(1)} \dots s^{(d)}}$  that lies in  $\mathcal{T}_0$ .

Here a  $k$ -simplex, or convex hull of  $k + 1$  points, is denoted  $[a_0, a_1, \dots, a_k]$  and  $C_k$  is the circumcenter of the simplicial element  $s^{(k)}$ . The factor  $\text{sgn}_{0\dots d}$  ensures the volume is consistent with the induced orientation from  $s^{(d)}$ , i.e. if the orientation of  $[C_0, C_1, \dots, C_d]$  is opposite to that induced in  $s^{(d)}$  then the orientation is flipped. This is matched by the totally-antisymmetric tensor  $\epsilon_{i_1 i_2 \dots i_d}$  whose orientation is induced by  $s^{(d)}$ . In cases where the circumcenter of an element lies outside the element, the volume gives negative contribution to sums over the irreducible hybrid cell. The 2-form still contains a vector with negative 1-dimensional orientation even after being made compatible with the containing volume's orientation. In general, the circumcenter lying outside the simplicial element leads to an over-counting of volume. This careful accounting of orientation ensures the conservation of total volume, e.g. over the simplicial element  $s^{(d)}$ .

It is clear from the definition that the 0-skeleton of an irreducible hybrid domain consists of the circumcenters of each of the  $k$ -simplexes that share the domain. In the 0-skeleton only the vertexes  $C_0 = s^{(0)}$  and  $C_d = \sigma^{(0)}$  are vertexes that are also members of the simplicial or dual skeletons. Similarly, the 1-skeleton of  $V_{s^{(0)} s^{(1)} \dots s^{(d)}}$  consists of 6 vectors, two of which are subspaces of either the simplicial or dual 1-skeleton of  $\mathcal{T}_0$ . Figure 2 shows an irreducible hybrid cell in three dimensions and highlights the members of the 1-skeleton that lie in either of the 1-skeletons of  $\mathcal{T}_0$ .

In each irreducible hybrid cell we can form an orthogonal (or orthonormal) basis from the set of vectors  $\{m_i \mid m_i = \overrightarrow{C_i C_{i+1}}\}$ . Moreover, since each of these cells are subspaces of  $\mathbb{R}^d$ , we have the volume element given by  $\tilde{V} = \frac{1}{d!} \epsilon_{i_1 \dots i_d} m^{i_1} \wedge \dots \wedge m^{i_d}$ . In this basis of differential forms, we carry the information about the discrete 1-forms  $\ell \propto m_0$  and  $\lambda \propto m_{d-1}$ . All other edges in the 0-skeleton of this cell are virtual in the sense that these 1-forms only become basis elements of higher-dimensional  $k$ -forms while not carrying direct information about the discretization of 1-forms on the lattice. Similarly, the irreducible hybrid cells contain discretization content for only one simplicial and one dual  $k$ -form, for all  $k$ . For simplicial  $k$ -forms, this is clear from the definition. For dual  $k$ -forms it is evident from the restriction of the cell to exactly one  $(d - k)$ -simplicial element. The rest of the  $\binom{d+1}{k+1}$   $k$ -forms in the irreducible hybrid cell carry the information about the graded algebra in the simplex  $s^{(d)}$ , but only indirectly given the flat interior of the simplex.



The irreducible hybrid domains give the smallest domain of support for any given  $k$ -form in the dual lattices of  $\mathcal{T}_0$ . From these irreducible domains we wish to reconstitute local “natural volumes” associated to each lattice element of  $\mathcal{T}_0$ . The irreducible hybrid cells do tile the PF manifold  $\mathcal{T}_0$  but will not generally provide a disjoint cover of  $\mathcal{T}_0$ . Only when the triangulation is well-centered (i.e. when the circumcenter of each simplicial element is contained in the simplicial element) does the set of irreducible hybrid cells form a disjoint cover. We can now state a result with regard to the measure of the hybrid cell.

**Theorem 1.** *The convex hull (interior to  $\mathcal{T}_0$ ) of a simplicial element and its dual,  $CH(s^{(k)}, \star s^{(k)}) \cap \mathcal{T}_0$ , defines the domain of support for a discrete form on  $s^{(k)}$  ( $\star s^{(k)}$ ) which is given by the set theoretic union of the irreducible hybrid cells (contained in  $\mathcal{T}_0$ ). The volume measure of discrete forms on  $s^{(k)}$  ( $s^{(k)}$ ) is given by  $V_{s^{(k)}} = \frac{1}{(d)} |s^{(k)}| | \star s^{(k)}|$ .*

*Proof.* For each irreducible hybrid cell containing a given  $s^{(k)}$ ,  $V_{\dots s^{(k)} \dots}$ , any  $k$ -form  $\omega$  will generally have a non-zero evaluation over  $s^{(k)}$  and hence have a non-zero component in each irreducible cell containing  $s^{(k)}$ . We can check that the set union of  $V_{\dots s^{(k)} \dots}$  is convex by examining the convex sum of extremal points on the  $V_{\dots s^{(k)} \dots}$ . First we examine the sum over  $s^{(p)}$  for  $p < k$ , which gives

$$\bigcup_{s^{(0)}, \dots, s^{(k-1)}} [\mathcal{C}_0, \dots, \mathcal{C}_{k-1}, \mathcal{C}_k, \dots, \mathcal{C}_d] \cap \mathcal{T}_0 = \underbrace{[v_0, v_1, \dots, v_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}_d]}_{s^{(k)}} \cap \mathcal{T}_0,$$

which is clearly convex. Consider first the convex sum of  $\mathcal{C}_p$  and  $\mathcal{C}'_p$  for two  $s^{(p)}$ s ( $p > k$ ) for which the irreducible hybrid cell contain  $s^{(k)}$  and  $s^{(p)}$  is non-zero. We can examine the two irreducible hybrid cells who only differ by  $s^{(p)}$ . There exists a boundary between the two cells  $[v_0, \dots, v_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}_p, \dots, \mathcal{C}_d]$  and  $[v_0, \dots, v_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}'_p, \dots, \mathcal{C}_d]$ . If  $p = d$  then the boundary is a subspace of the common  $s^{(d-1)}$ . In this case, the convex sum of the two  $\mathcal{C}_d$  and  $\mathcal{C}'_d$  is the dual edge  $\lambda$ , which is a straight-line entirely contained on the combined domain (when the two  $d$ -simplexes are mapped onto  $\mathbb{R}^d$ ). If  $p < d$  then we know from the circumcentric duality that the convex sum

$$\overrightarrow{\mathcal{C}_p \mathcal{C}'_p} = \rho \mathcal{C}_p + (1 - \rho) \mathcal{C}'_p \quad \rho \in [0, 1]$$

forms angles  $\angle \overrightarrow{\mathcal{C}_{p-1} \mathcal{C}_p}, \overrightarrow{\mathcal{C}_p \mathcal{C}'_p} < \frac{\pi}{2}$  and  $\angle \overrightarrow{\mathcal{C}_p \mathcal{C}'_p}, \overrightarrow{\mathcal{C}'_p \mathcal{C}_p} < \frac{\pi}{2}$  and hence lies within both the irreducible hybrid cells. We therefore conclude that the set union of these irreducible cells for a given lattice element is  $CH(s^{(k)}, \star s^{(k)})$ .

However, this entire domain may not contribute to the final oriented sum over the  $V_{\dots s^{(k)} \dots}$ . Two irreducible hybrid cells covering the same domain with opposite orientations will give zero contribution from the overlapping volume. When summing over the  $s^{(p)}$  for  $p < k$ , we simply get back a simplex

$$\sum_{p < k} [\mathcal{C}_0, \dots, \mathcal{C}_k, \dots, \mathcal{C}_d] = \underbrace{[v_0, \dots, v_k, \mathcal{C}_{k+1}, \dots, \mathcal{C}_d]}_{s^{(k)}}$$

Meanwhile summing over the  $s^{(p)}$  for  $p > k$  returns

$$\sum_{p > k} [\mathcal{C}_0, \dots, \mathcal{C}_k, \dots, \mathcal{C}_d] = [\mathcal{C}_0, \dots, \mathcal{C}_k, \{\sigma^{(d-k)}\}].$$

Using these two results, the full sum gives a bipyramid with base  $\sigma^{(d-k)}$  and  $k$ -dimensional altitude given by  $s^{(k)}$  whose volume is  $V_k = \frac{1}{(d)} |s^{(k)}| | \star s^{(k)}|$ .  $\square$

The hybrid domains associated to lattice elements of  $\mathcal{T}_0$  define the local measure for the discrete forms on the lattice. This domain defined by the measure may not actually encompass the lattice element, especially when the lattice is non-Pittway (when the element and its dual have empty intersection). However, the domain of support is generally more expansive and necessarily contains the lattice element. It is particularly insightful to notice that for a simplicial element and its dual the domains of support and local measures for a discrete form  $\omega$  and its dual  $\star\omega$  coincide. These lattice-element hybrid measures and the convex domain of support then become fundamental to the discretization and algebra of discrete forms. Henceforth, when we think of the hybrid cell, we will take this to mean the local lattice measure and not the domain of compact support, since it is only the former that is necessary for explicit computations. Examples of the menagerie of the local measure hybrid cells in 3 dimensions is shown in Figure 3.


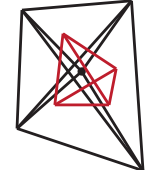




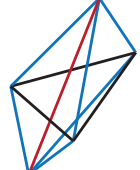

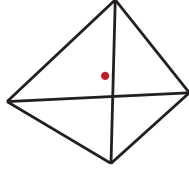
	Simplicial	Dual	Hybrid Cell
$\binom{0}{3}$	$\bullet v$	 $v^*$	
$\binom{1}{2}$	 $l$	 $l^*$	
$\binom{2}{1}$	 $\lambda^*$	 $\lambda$	
$\binom{3}{0}$	 $\nu^*$	$\bullet \nu$	

FIG. 3. In three dimensions there are four distinct classes of hybrid cells in  $\mathcal{T}_0$ . We show in this figure representative depictions of these four classes. The labeling  $\binom{k}{d-k}$  labels the dimension of a  $k$  simplicial element  $s^{(k)}$  and its dual. The hybrid cell is heuristically constructed by connecting the vertexes of  $s^{(k)}$  with the vertexes of the dual cell  $\star s^{(k)}$ . In the cases of  $\binom{0}{d}$  and  $\binom{d}{0}$ , the hybrid cell is just equal to  $s^{(d)}$  or  $\sigma^{(d)}$ , respectively.

We have built up the lattice element hybrid cells from irreducible domains. We now take a step in the reverse direction to examine hybrid cells common to multiple lattice elements, not just a given lattice element. Such constructions are very useful when we are required to relate discrete forms on a  $k$ -skeleton to discrete forms from a  $p$ -skeleton ( $p \neq k$ ). For example, the exterior derivative  $d\omega$  is a map from  $k$ -forms to  $(k+1)$ -forms and requires a relationship to be formed

between hybrid cells that overlap but do not coincide. The integration over the hybrid domain to  $s^{(k+1)}$  picks up contributions from each of the  $s^{(k)} \in s^{(k+1)}$ , but they do not contribute equally. Rather the “democratic” allotment of domains by the hybrid cells for each  $s^{(k)}$  defines a domain of support for each of the  $s^{(k)}$ . The integration over the hybrid domain for the  $s^{(k+1)}$  then acts as a restriction on the integration for the  $s^{(k)}$  and we find a domain common to  $s^{(k)}$  and  $s^{(k+1)}$  for each term in the discrete  $d\omega$ .

**Corollary 1.** *The hybrid cell common to two simplicial elements  $s$  and  $s'$  (with  $\dim(s) < \dim(s')$ ) is given by sum of oriented volumes*

$$V_{ss'} = \sum_{s^{(k)} \neq s, s'} V_{s^{(0)} \dots s^{(k)} \dots s^{(d)}} + \sum_{s^{(k)}|_s s^{(m)}|_{s'}} V_{s^{(0)} \dots s^{(k)} \dots s^{(d)} \cap V_{s^{(0)} \dots s^{(m)} \dots s^{(d)}}. \quad (18a)$$

When the simplicial lattice is well-centered (i.e.  $c(s^{(d)}) \in s^{(d)}$ ,  $\forall s^{(d)}$ ), then the hybrid cell common to a set of simplicial elements  $\mathcal{S} = s_1^{(i_1)}, s_2^{(i_2)}, \dots, s_n^{(i_n)}$  is given by the sum;

$$V_{1, \dots, n} = \sum_{s^{(k)} \notin \mathcal{S}} V_{s^{(0)} \dots s^{(i_1)} \dots s^{(i_n)} \dots s^{(d)}}. \quad (18b)$$

*Proof.* The hybrid domain common to any two simplicial elements is obtained by the intersection of the hybrid cells for each element  $s$  and  $s'$ . This reduces the problem to pairwise set intersections over irreducible hybrid domains. For an irreducible hybrid cell containing both  $s$  and  $s'$ , the intersection returns the full irreducible cell. Any irreducible hybrid cell for  $s$  that is not also a hybrid cell for  $s'$  will yield the subspace common to both irreducible cells, but with an orientation opposite of the simplicial complex. These contribute with negative volume and remove subspaces not in any shared  $s^{(d)} \ni s, s'$ . Hence, the hybrid cell common to  $s$  and  $s'$  is reduced to a sum over irreducible hybrid cells plus negative volume terms obtained by intersections of non-shared irreducible cells. The extension to  $n$  simplicial elements is straightforward, though quickly becomes cumbersome. In the case of well-centered simplicial complexes, all irreducible hybrid cells are disjoint and trivially factorize any given  $s^{(d)}$ . Hence, the second summation Eq. (18a) vanishes.  $\square$

The above definitions have only focused on hybrid cells for simplicial lattice elements. As we noted before for the hybrid cell for a given lattice element, the hybrid domain for  $s^{(k)}$  is coincident with the hybrid domain for  $\star s^{(k)}$  as a result of the self-adjointness of the Hodge dual in the  $L^2$ -inner product. Generalization from simplicial elements for the reduced hybrid cells or domains common to multiple lattice elements is, therefore, trivial by simply taking the dual of an element in the dual lattice.

## B. Solder Forms and Moment Arms

We consider a frame bundle on our manifold. In each simplex we have a fibre that is a copy of the tangent space on the base manifold, which we call the space of values. The tangent space of the base manifold is the horizontal section while the fibre or space of values is the vertical section. On  $\mathcal{T}_0$  we assign a tangent space to any  $d$ -simplex or any pair of neighboring  $d$ -simplexes. For each of these tangent spaces, we have a copy in the space of values.

The PF manifold has a hard-wired simplicial skeleton and from that we construct a dual skeleton, entirely determined by its rigid predecessor lattice. We can think of these two lattices as

complements of one another much the way we consider the space of vectors and one-forms as complements. Moreover, since the dual lattice is decomposed into subspaces of  $d$ -simplexes, we can define the dual lattice entirely in terms of the vectors in the tangent spaces. Thus, determining transformations between the simplicial and dual lattices is tantamount to determining an appropriate solder form in the  $d$ -simplexes, or more appropriately in the irreducible hybrid cells. A solder form, the unit vector-valued one-form, is the identity map from the tangent space—with one-form basis  $e^a$ —to the space of values—with vector basis  $\tilde{e}_a$ ;

$$d\mathcal{P} = \tilde{e}_a e^a. \quad (19)$$

When acting on a vector  $d\mathcal{P}$  transforms a vector in the tangent space to a vector in the space of values. The summation ensures that we retain the basis expansion but in the new vector space. We can now construct a representation of the solder form in  $\mathcal{T}_0$  that allows for transformations between the simplicial and dual lattices.

In the discrete manifold, the solder form is an object that carries information about the relationship between the space of values (vertical section) and the tangent space (horizontal section). In conjunction with the Hodge dual, the solder form acts as a transform between  $k$ -forms,  $\omega$ , and  $p$ -forms ( $p \leq d - k$ ) orthogonal to  $\omega$ . Through wedge products we can form a  $(d - p)$ -dimensional space orthogonal to a desired  $p$ -form that contains our  $k$ -form. The Hodge dual and a summation over discrete  $k$ -forms that are able to construct such a space then provide a transformation between a set of  $k$ -forms and the desired  $p$ -form. Another route is to take the Hodge dual of  $k$ -forms orthogonal to a desired  $p$ -form and contract over the directions orthogonal to the  $p$ -form. Together these paths form two routes towards a notion of the double-dual of a vector-space valued differential (discrete) form.

We start the construction in an irreducible hybrid cell. In this irreducible cell we have exactly one  $\ell$  and  $\lambda$  that are representative edges of the simplicial and dual 1-skeletons, respectively. As discussed above, this domain has volume form given by

$$V_{s^{(0)}, \ell, \dots, \lambda, s^{(d)}} = \frac{1}{d(d-1)} (\ell \wedge M_{\ell\lambda} \wedge \lambda), \quad (20)$$

where the  $M_{\ell\lambda} = \frac{1}{(d-2)!} m_1 \wedge \dots \wedge m_{d-2}$  is the  $(d - 2)$ -dimensional subspace orthogonal to both  $\ell$  and  $\lambda$ , which we call the moment arm from  $\ell$  to  $\lambda$ . Both  $\ell$  and  $\lambda$  are understood to consist of only the segment of  $\ell$  and  $\lambda$  within a given irreducible hybrid cell. Taking the set  $\{\ell, \lambda, \{m_i\}_{i=1}^{d-2}\}$  as an orthonormal basis (where  $\{m_i\}$  are the one-forms that span  $M_{\ell\lambda}$ ), we have the relationship between the lattice forms and their induced vectors;

$$\ell(\ell) = 1, \quad \ell(\lambda) = 0, \quad \ell(m_i) = 0 \quad (21a)$$

$$\lambda(\lambda) = 1, \quad \lambda(\ell) = 0, \quad \lambda(m_i) = 0 \quad (21b)$$

$$m_i(m_i) = 1, \quad m_i(\ell) = 0, \quad m_i(\lambda) = 0, \quad (21c)$$

where  $M_{\ell\lambda}$  represents any 1-form or vector in the subspace defined by  $M_{\ell\lambda}$ . We can then identify individual maps from each basis form to an identified basis element in the vertical section (space of values). In general, we make no real distinction between the lattice element as a scalar-valued differential form or as a vector-space valued differential form. In the context of this manuscript, the lattice elements always take the meaning of a vector-valued differential form, or as a map from the tangent space to the space of values. The solder form is thus

$$d\mathcal{P}_0 = \ell + \lambda(s^{(d)}) + \sum_{m_1}^{m_{(d-2)}} m_i(s^{(d)}). \quad (22)$$

The combination of the solder form and the Hodge dual provide us with practical tools for transforming between the two lattices. We first examine the use of the solder form in such a transformation that is well-known in general relativity. In E. Cartan's [26] approach to general relativity, the Einstein tensor need not be defined with relation to the Ricci curvature and Ricci scalar, but as the dual to the moment of rotation;

$$\begin{aligned}
\mathbf{G} &\equiv \star (d\mathcal{P} \wedge \mathcal{R}) \\
&= \star \left( \frac{1}{4} \tilde{e}_\mu \delta^\mu_\nu e^\nu \wedge \tilde{e}_\sigma \wedge \tilde{e}_\tau R^{\sigma\tau}{}_{\alpha\beta} e^\alpha \wedge e^\beta \right) \\
&= \frac{1}{4} \tilde{e}_\xi \tilde{e}^\xi{}_{\mu\sigma\tau} R^{\sigma\tau}{}_{\alpha\beta} e^\mu \wedge e^\alpha \wedge e^\beta \\
&= \tilde{e}_\xi \underbrace{\frac{1}{4} \tilde{e}^\xi{}_{\mu\sigma\tau} R^{\sigma\tau}{}_{\alpha\beta} \epsilon_\zeta^{\mu\alpha\beta}}_{G^\xi{}_\zeta = (\star\mathcal{R}\star)^\xi{}_\zeta} d\Sigma^\zeta, \tag{23}
\end{aligned}$$

where we have specifically worked in dimension 4. Here the solder form plays the crucial role of defining a subspace orthogonal to both the 3-volume for the moment of rotation and imposing the trace on the 3-form that results from the wedge product. In higher dimensions, we keep appending onto the Riemann curvature  $(d-3)$  solder forms prior to taking the Hodge dual to obtain the moment of rotation  $(d-1)$ -form. In this light, the solder forms take on the interpretation as moment arms between the original basis elements and elements in the orthogonal subspace. This map provides a way to identify mappings between the dual lattices not provided by the Hodge dual alone.

The solder form is an instrumental tool in the transformation of vector-space valued differential  $k$ -forms to differential  $p$ -forms. Of particular interest to us now is the transformation of 1-forms  $\omega \in \Lambda^{(1)}$  to 1-forms in the dual space  $\omega' \in \Lambda^{*(1)}$ , which is the raising (or lowering) operation on differential forms. We take a lattice 1-form  $\ell$  and construct a map to a dual one-form  $\lambda$ . This is equivalent to mapping the 1-form  $\ell$  to the dual space of 1-forms and asking for the components along  $\lambda$ .

To make such a map, we first make use of a discrete analog of a continuum property relating the Hodge dual, the wedge product  $\wedge$  and the inner derivative  $\iota$ .

**Theorem 2.** *Let  $\mathcal{T}_0$  be a PF simplicial manifold and choose a simplicial  $k$ -element  $s^{(k)} \in \mathcal{T}_0$ , a one-form  $m_k$  that extends from  $\mathcal{C}_k$  to  $\mathcal{C}_{k+1}$  of a  $s^{(k+1)} \ni s^{(k)}$ , and a circumcentric Hodge dual operator,  $\star$ , on  $\mathcal{T}_0$ . Then there exists a commutative diagram*

$$\begin{array}{ccc}
\Lambda^{(k)} & \xrightarrow{\wedge m_i} & \Lambda^{(k+1)} \\
\downarrow \star & & \downarrow \star \\
\Lambda^{*(d-k)} & \xrightarrow{(d-1)\iota m_i} & \Lambda^{*(d-k-1)}, \tag{24}
\end{array}$$

using the wedge product with  $m_k$  and the inner derivative (contraction),  $\iota_{m_k}$ , over the vector,  $\vec{m}_k$ , corresponding to  $m_k$ .

*Proof.* We first prove this for the case  $k=1$ . Using the definition of the circumcentric Hodge dual and its property that any subspace of a  $p$ -simplex,  $s^{(p)}$ , gives back  $\star s^{(p)}$  yields

$$\star (m_1 \wedge \ell) = \sum_{s^{(p)} \ni \ell, t: p > 2} \text{sgn} [C_2, C_3, \dots, C_d] = \sum_{m_i: i > 3} \frac{1}{(d-2)!} m_2 \wedge \dots \wedge m_{d-1}, \tag{25a}$$

where the summations are over the  $s^{(k)}$  (for  $k > 3$ ) containing  $\ell$ . Meanwhile, taking the Hodge dual first then the inner derivative results in

$$\begin{aligned} \iota_{m_1}(\star\ell) &= \iota_{m_1} \left( \sum_{m_j: j>1} \frac{1}{(d-1)!} m_1 \wedge \cdots \wedge m_d \right) \\ &= \sum_{m_j: j>2} \frac{1}{(d-1)!} m_2 \wedge \cdots \wedge m_d \end{aligned} \quad (25b)$$

where we use the normalization property  $m_i(\overrightarrow{m_j}) = \delta_{ij}$  that picks out a given  $s^{(2)}$  from the summation. Multiplying by  $(d-1)$  yields the desired result.

The case for arbitrary  $k$  follows in a straightforward manner.  $\square$

This theorem shows that there exist two paths to map from an arbitrary  $k$ -form on the simplicial lattice to a  $(d-k-1)$ -form on the dual lattice. These two paths stem from continuum descriptions of the double-dual transformations of tensors (as was used to construct the Einstein tensor). Since discrete forms are mapped as coefficients on the lattice elements, we need only know how the space of forms on the lattice elements get mapped to one another. The scalar coefficient gets carried through with no change.

As a result of Theorem 2, we can show a subsequent commutative diagram for maps from  $\Lambda^{(1)}$  to  $\Lambda^{*(1)}$ . We will drop combinatoric factors from the commutative diagrams for simplicity.

**Corollary 2.** *On  $\mathcal{T}_0$  there exist maps from the simplicial 1-skeleton to the dual 1-skeleton such that*

$$\begin{array}{ccc} \Lambda^{(1)} & \xrightarrow{\wedge^{(d-2)}} & \Lambda^{(d-1)} \\ \downarrow \star & & \downarrow \star \\ \Lambda^{*(d-1)} & \xrightarrow{\iota^{(d-2)}} & \Lambda^{*(1)}. \end{array} \quad (26)$$

*Proof.* The proof follows by induction and successive applications of Theorem 2 as in the diagram below:

$$\begin{array}{ccccccc} \Lambda^{(1)} & \xrightarrow{\wedge_{m_1}} & \Lambda^{(2)} & \xrightarrow{\wedge_{m_2}} & \Lambda^{(3)} & & \\ \star \downarrow & & \downarrow \star & & \downarrow \star & & \\ \Lambda^{*(d-1)} & \xrightarrow{\iota_{m_1}} & \Lambda^{*(d-2)} & \xrightarrow{\iota_{m_2}} & \Lambda^{*(d-3)}. & & \end{array} \quad (27)$$

Iteratively applying the above diagram, we finally construct the desired result.  $\square$

The two paths in Eq. (26) define our two notions of the double-dual of a vector-space valued differential form. The Down-right path acts as the trace of the double dual to obtain a vector oriented in a given direction, while the Right-down path builds a  $(d-1)$  subspace orthogonal to the desired vector before taking the dual to obtain the desired result. The Right-down path corresponds to the first line of Eq. (23) while the Down-right path corresponds to the last line of Eq. (23).

Taking the inner-derivative of the hybrid volume form  $V_{\ell\lambda}$  with a desired  $\lambda$  gives

$$d\iota_\lambda V_{\ell\lambda} = \frac{1}{(d-1)} \ell \wedge M_{\ell\lambda}, \quad (28)$$

which defines the orthogonal subspace to  $\lambda$ . We find the moment arm by taking the inner-derivative again,

$$d(d-1)\iota_{\ell}\iota_{\lambda}V_{\ell\lambda} = M_{\ell\lambda}. \quad (29)$$

The inner-derivatives are used to find the components of the moment arm that maps  $\ell$  to  $\lambda$  (or the reverse). Choosing another  $\ell$  or another  $\lambda$  changes the moment arm. Using our above commutative diagrams, we then have

$$\begin{array}{ccc} \Lambda^{(1)} & \xrightarrow{\wedge^{M_{\ell\lambda}}} & \Lambda^{(d-1)} \\ \downarrow \star & & \downarrow \star \\ \Lambda^{*(d-1)} & \xrightarrow{\iota^{M_{\ell\lambda}}} & \Lambda^{*(1)}. \end{array} \quad (30)$$

Given that these moment arms play the same role as the solder forms in mapping  $k$ -forms to  $p$ -forms, we will often refer to the moment arms as generalized solder forms. Just as the Einstein tensor can be expressed as

$$\mathbf{G} = \star \left( \underbrace{d\mathcal{P} \wedge \dots \wedge d\mathcal{P}}_{d-3\text{-times}} \wedge \mathcal{R} \right) = \text{Tr}(\star\mathcal{R}\star),$$

using the solder forms to map the 2-form to a 1-form, the moment arms  $M_{\ell\lambda}$  allow us to define a map from the simplicial 1-skeleton to the dual 1-skeleton on the dual lattice. In the more general case (Figure 4) of mapping a simplicial  $k$ -form  $s$  to a dual  $p$ -form  $\sigma$  ( $p < d - k$ ), we define the moment arm  $M_{s\sigma}$

$$M_{s\sigma} = \binom{d}{k} \binom{d-k}{p} \iota_s \iota_{\sigma} V_{s\sigma} = \frac{d!}{k!(d-k-p)!p!} \iota_s \iota_{\sigma} V_{s\sigma}. \quad (31)$$

This is the  $\binom{k}{p}$ -solder form between the simplicial and dual lattices. Of course, this solder form only makes sense when  $\sigma$  and  $s$  have overlapping domains, i.e. when  $s \in \star\sigma$  or  $\sigma \in \star s$ . Moreover, it is a natural consequence that the solder form between a lattice element and its dual is given by the point of intersection, and so we have a consistent framework to map between the dual and simplicial lattice elements.

We can also construct solder forms between simplicial lattice elements. Given the duality between  $\sigma$  and a simplicial element  $s^{(d-p)}$ , Eq. (31) also defines a moment arm between two simplicial elements. If we have two simplicial elements,  $s \in \{s^{(k)}\}$  and  $s' \in \{s^{(p)}\}$  then the moment arm or solder form between  $s$  and  $s'$  is given by

$$M_{ss'} = \binom{d}{k} \binom{d-k}{d-p} \iota_s \iota_{\star s'} V_{\star s, s'}. \quad (32)$$

A similar result holds for solder forms between two dual lattice elements. This provides a geometric foundation for measures of discrete forms. The irreducible, reduced and standard hybrid cells form a topological framework on which we form local measures for lattice elements, while the moment arms/solder forms allow us freedom to transform from one lattice to another. This has yet to tell us anything about the structure of discrete differential forms in this context, and so we now shift our focus to the exterior calculus using the hybrid domains.



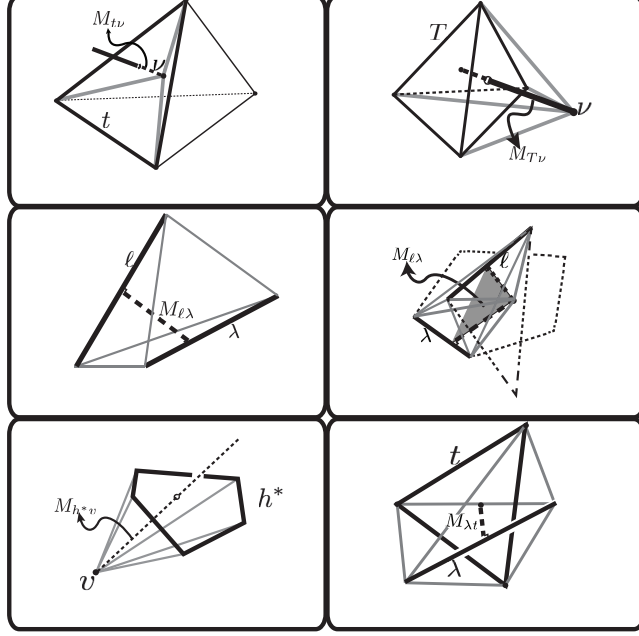


FIG. 4. Solder Forms are shown in three (*left*) and four (*right*) dimensions. When the dimensionality of the space spanned by two elements  $s^{(k)}$  and  $\sigma^{(p)}$  is  $(d-1)$ , a solder form between a  $s^{(k)}$  and a  $\sigma^{(p)}$  is given by the vector from  $\mathcal{C}_{s^{(k)}}$  to  $\mathcal{C}_{\sigma^{(p)}}$ . In more general cases, the generalized solder form between two elements is given by the  $(d-p-k)$  subspace orthogonal to both  $s^{(k)}$  and  $\sigma^{(p)}$ .

### C. The Algebraic Structure of Forms on Hybrid Cells

A common property of exterior calculus on discrete manifolds is the distributional nature of the discrete forms. In the canonical approach to DEC, we have discussed how the differential forms are obtained by integration of the continuous differential form over the simplicial element in  $\mathcal{T}$  corresponding to a simplicial element in  $\mathcal{T}_0$ . These discrete differential forms take on values when evaluated on the simplicial or dual skeletons.

In applying a DEC formalism to curvature operators on the PF manifolds, we want to utilize the known properties of curvature in RC and use as our guide the Regge action principle. It is known from the canonical, continuum analysis of PF curvature that any loop of parallel transport in a plane orthogonal to a hinge will non-trivially transform tangent vectors carried around the loop. This occurs even when the loop of parallel transport does not intersect the orthogonal dual polygon to a hinge (independent of how one defines the dual lattice). As a result, the curvature remains non-zero as long as the loop of parallel transport contains a non-trivial projection onto a surface parallel to the dual polygon. Therefore, the discrete measure of the curvature in the hybrid domain of a hinge is given by

$$R \rightarrow \int_{V_h} R dV_{\text{proper}} = \frac{1}{\binom{d}{2}} \int_h \int_{h^*} (\mathbf{Riem} \cdot dh^*) dh = 2\varepsilon_h A_h, \quad (33)$$

where the combinatoric normalization comes from the decomposition of the volume. We then notice that this is the total curvature across the hybrid cell. Yet it has little direct indication of the local tensorial content of the Riemann tensor. From a DEC perspective, we wish to find a projection of  $\mathbf{Riem}$  into  $\mathcal{T}_0$  that preserves this total curvature.

Since the curvature in PF manifolds is entirely projected onto the polygonal dual  $h^*$  to  $h$  we then require that this result be constant over  $h$  in order to recover Eq. (33). This is a direct

indication that the discrete form defines a constant field over  $h$ . If we take the Hodge dual on the discrete forms, then we also have,

$$\int_h \left[ \int_{h^*} \mathbf{Riem} \cdot dh^* \right] dh = 2\varepsilon_h A_h = \int_{h^*} \left[ \int_h \star \mathbf{Riem} \cdot dh \right].$$

The scalar coefficient is thus constant over the hybrid domain, and we now define an approach based on the standard DEC that explicitly assigns the discrete forms over the hybrid measures.

In an explicit volume-based DEC, the base discretization is done as before by projection of the differential  $\omega$  onto a simplex or dual polytope in  $\mathcal{T}$  that corresponds to a simplex or dual polytope in  $\mathcal{T}_0$ ,

$$\omega_{s^{(k)}} = \frac{\langle \omega | s^{(k)} \rangle}{|s^{(k)}|}.$$

Our discrete forms then become the geometric objects  $\omega_{s^{(k)}} s^{(k)}$  where  $s^{(k)}$ 's are the  $k$ -forms in the hybrid cells for  $s^{(k)}$ . The  $s^{(k)}$  does not merely represent the lattice element, but the family of surfaces in  $V_{s^{(k)}}$  parallel to  $s^{(k)}$ , each such surface mapped back to  $s^{(k)}$  when viewed from outside  $V_{s^{(k)}}$ , i.e. when coarse-grained to smooth over the internal structure. As a comparison to the continuum, we have the relationship

$$\left( \omega, s^{(k)} \right) = \int_V \omega_{s^{(k)}} dV,$$

using our previous definition of  $\omega_{s^{(k)}}$ . Then, using the projection of  $\omega$  onto a lattice element,  $\langle \omega | s^{(k)} \rangle = \int \omega_{s^{(k)}} s^{(k)}$  we have

$$\left( \omega, s^{(k)} \right) = \frac{1}{\binom{d}{k}} \int_{\star s^{(k)}} \langle \omega | s^{(k)} \rangle d(\star s^{(k)}). \quad (34)$$

We then take the coefficient  $\omega_{s^{(k)}}$  as a scalar function defined over the dual polytope  $\star s^{(k)}$  but with components only lying in the surfaces parallel to  $s^{(k)}$ . The discrete measure of a  $k$ -form  $\omega$  is

$$\left( \omega, s^{(k)} \right) := \frac{1}{\binom{d}{k}} \int_{V_{s^{(k)}}} \omega_{s^{(k)}} s^{(k)} \wedge \star s^{(k)}. \quad (35)$$

Further, this volume measure has the standard property that the Hodge dual preserves the coefficient and so we have

$$\left( \star \omega, \star s^{(k)} \right) = \left( \omega, \star(\star s^{(k)}) \right) = \left( \omega, s^{(k)} \right). \quad (36)$$

The definitions of the exterior derivative and co-derivative follow from Stokes' theorem applied to the  $L^2$ -inner product;

$$\begin{aligned} \left( d\omega, s^{(k+1)} \right) &= \frac{1}{\binom{d}{k+1}} \int_{V_{s^{(k+1)}}} \langle d\omega | s^{(k+1)} \rangle ds^{(k+1)} d(\star s^{(k+1)}) \\ &= \frac{1}{\binom{d}{k+1}} \frac{1}{k+1} \int_{V_{s^{(k+1)}}} \sum_{s^{(k)}} \langle \omega | s^{(k)} \rangle ds^{(k)} dM_{s^{(k)} s^{(k+1)}} d\star s^{(k+1)} = \left( \omega, \tilde{\delta} s^{(k+1)} \right) |_{s^{(k)}}. \end{aligned} \quad (37)$$

where we have decomposed  $s^{(k+1)}$  into  $s^{(k)}$ 's and the moment arms  $M_{s^{(k)}s^{(k+1)}}$ . Moreover, the integral is maintained over the measure corresponding to  $s^{(k+1)}$  and so the total volume integral is over the volume  $V_{s^{(k)}s^{(k+1)}}$ . After integrating and combining appropriate terms, we then have

$$\left(d\omega, s^{(k+1)}\right) = \frac{1}{\binom{d}{k+1}} \frac{1}{(k+1)} \sum_{s^{(k)} \in s^{(k+1)}} \omega_{s^{(k)}} |s^{(k)}| |m_{s^{(k)}s^{(k+1)}}| \star s^{(k+1)} = \sum_{s^{(k)}} \omega_{s^{(k)}} V_{s^{(k)}s^{(k+1)}} \quad (38)$$

Using similar properties of the co-derivative in the local inner-product, we obtain the measure of the discrete exterior co-derivative;

$$\left(\delta\omega, s^{(k-1)}\right) = \sum_{s^{(k)} \ni s^{(k-1)}} \omega_{s^{(k)}} V_{s^{(k)}s^{(k-1)}}. \quad (39)$$

Some key points and assumptions are useful to highlight before directly using this formalism. A core assumption for the DEC approach to vector-space valued differential forms is that we only explicitly discretize the base manifold. In this sense, we only discretize the tangent space components and examine the constraints this puts on components in the space of values. This is crucial for understanding the role of vector-space valued forms in the lattice. For a basis of the discrete forms, we expand a  $k$ -chain in terms of the lattice elements, while the basis in the space of values is given by the unit  $p$ -forms with magnitude given by the coefficients defined in the discretization. Secondly, while discretization is done on a given lattice element, the  $k$ -form field is extended as a constant field domain of support for the  $k$ -element and measured on the hybrid cell. Under a standard assumption of the assignment of a field value to the lattice element, the discrete form would be distributionally valued on the lattice element. However, our measure requires the valuation to be constant across the orthogonal subspace to the lattice element. This is essential to properly recover local integral measures as averages over a given finite domain.

We have provided a scheme based on standard DEC where the measures of  $k$ -forms are given by integral  $d$ -measures instead of local  $k$ -measures on  $k$ -elements. This has the advantage of simplifying our evaluation of the discrete forms and providing a consistent framework for the understanding of the Hilbert action. However, we have only specified how one takes differential forms and discretizes them, but not how the inherent properties of the lattice affect the coefficients in the discretization. To examine the DEC approach to curvature in PF manifolds, we must examine properties of local curvature operators in addition to the integral measures provided by the Hilbert action. In the next section we analyze the standard curvature operators and the manifold geometry from the joint perspective of DEC and RC.

### III. CURVATURE FORMS IN PIECEWISE-FLAT MANIFOLDS

Having laid out the foundations for analyzing curvature we can now provide measures of local components for the standard curvature operators. In particular, we examine the nature of curvature in the hybrid domain to codimension 2 hinges, simplicial edges, and simplicial vertexes. We have already discussed the role of the Hilbert action in measuring the curvature. We now specify how this can be obtained and used to derive local curvature operators.

The Riemann tensor is defined as a  $\binom{1}{1}$ -tensor valued 2-form, or (by the raising operation in the space of values) as a bivector-valued 2-form;

$$\mathbf{Riem} = \frac{1}{2} \tilde{e}_\mu \wedge \tilde{e}^\nu R^\mu{}_{\nu\sigma\tau} e^\sigma \wedge e^\tau = \frac{1}{4} \tilde{e}_\mu \wedge \tilde{e}_\nu \underbrace{R^{\mu\nu}{}_{\sigma\tau}}_{g^{\nu\alpha} R^\mu{}_{\alpha\sigma\tau}} e^\sigma \wedge e^\tau. \quad (40)$$

The Riemann tensor thus takes as an argument a bivector from the base manifold to return a rotation operator or rotation bivector in the space of values, i.e. for a loop of parallel transport enclosing a given area an arbitrary vector  $\mathbf{A} = \tilde{e}_\mu A^\mu$  will be transformed as  $\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \delta\mathbf{A}$  with

$$\delta A^\mu = -R^\mu{}_{\nu\sigma\tau} A^\nu \Sigma^{\sigma\tau}.$$

In the discretization, the magnitude of  $\delta\mathbf{A}$  will be determined by the geometry, while the basis bivectors in the space of values will remain a set of orthonormal bivectors. We can then set the basis of the tangent space as the integrated measures of the lattice elements.

Similarly the Ricci curvature tensor is a vector-valued 1-form which is given by inserting an inverse solder form (a 1-form valued vector) into the bivector-valued curvature 2-form;

$$\mathbf{Rc} = \iota_{d\mathcal{P}-1} \mathbf{Riem} = \mathbf{Riem}(\tilde{e}^\mu e_\mu) = \tilde{e}_\nu R^{\mu\nu}{}_{\mu\tau} e^\tau. \quad (41)$$

The inverse solder form induces a trace on the Riemann curvature tensor. Inserting the inverse solder form into the Ricci curvature yields the scalar curvature,

$$R = \iota_{d\mathcal{P}-1} \iota_{d\mathcal{P}-1} \mathbf{Riem} = \mathbf{Riem}(\tilde{e}^\mu e_\mu, \tilde{e}^\nu e_\nu) = \mathbf{Rc}(\tilde{e}^\nu e_\nu) = R^{\mu\nu}{}_{\mu\nu}. \quad (42)$$

In this section we will illustrate some properties of the representation of curvature via local operators in the PF manifold using the inherent discrete structure. Our general strategy is to formalize the notion of parallel transport and curvature interior to the hybrid cells at the scale lengths shorter than the local scale of the discretization. We then generate a representation of the Riemann curvature operator from the scale of the discretization, i.e. a zeroth order coarse-graining over the interior structure of the hybrid cells. This process is then used to reexamine the Ricci tensor from [27].

### A. The Riemann Tensor, Locally Einsteinian Structure, and the Conic Singularity

Given the PF manifold  $\mathcal{T}_0$ , the domain of support associated to a codimension 2 hinge,  $h$ , is the convex hull (interior to  $\mathcal{T}_0$ ) of the hinge and a polygonal loop of parallel transport. Meanwhile the meaningful domain is the measure which is given by the oriented sum of irreducible hybrid cells. In an irreducible domain, there exists an orthogonal basis of vectors  $\{m_i\}$  which are defined the same as before. Since the subspace of the hinge  $h$  is isomorphic to  $\mathbb{R}^{d-2}$ , we can choose a complete set of the  $\{m_i\}$  for  $0 \leq i \leq d-3$  from any irreducible hybrid cell containing  $h$ . The last two vectors of any irreducible hybrid cell containing  $h$  then span  $h^*$ . We take these two vectors as  $M_{h\lambda}$  and  $\lambda$ . In the full measure of  $V_h$ , we have a set of  $\{M_{h\lambda}, \lambda\}$  for each  $\lambda \in \tilde{\delta}h^*$  such that  $h^* = \frac{1}{2} \sum_{\lambda \in \tilde{\delta}h^*} \lambda \wedge M_{h\lambda}$  (with a change in orientation as necessary to ensure consistent orientation across the domain). We instead focus on a basis of bivectors (and 2-forms) that leads to the representation of  $\mathbf{Riem}$  as a  $\binom{d}{2} \times \binom{d}{2}$  matrix. We then ask ‘‘How do we assign coefficients to the discretized Riemann tensor in this basis?’’

On the interior of the hybrid cell  $V_h$  the tangential components of the metric on the boundary between two neighboring tangent spaces are constant while there is generally a discontinuous change in the normal components of the metric across the boundary, when viewed in the basis of one of the  $d$ -simplexes [6, 28]. As we encircle a hinge with a loop of parallel transport, we notice that the components of the metric tangential to  $h$  are always constant across this domain (the surfaces parallel to  $h$  are always flat and remain parallel to  $h$ ). Hence, the components of any vector in the space spanned by  $h$  will be unaffected by parallel transport around  $h$ . In our vector basis, the  $\{m_i\}$  for  $0 \leq i \leq d-3$  will be unaffected by parallel transport. However, the  $M_{h\lambda}$ s, which is

tangential to a given  $s^{(d-1)}$ , will generally have components normal to any other  $s^{(d-1)} \in V_h$  and so will experience a rotation when transported around a loop encircling  $h$ . The amount of this rotation is always given by the deficit,  $\varepsilon_h$ . Moreover, this is true regardless of the size of the loop and depends only on whether the loop has a non-trivial projection into the plane orthogonal to  $h$ , i.e.  $h^*$ . This indicates that the basis we have chosen is anomalous and that for a more robust analysis we must take care in our choice of basis.

**Theorem 3.** *The Riemann tensor, **Riem**, on codimension 2 hinges of  $\mathcal{T}_0$  are rank 1 tensors with an eigen-decomposition*

$$\mathbf{Riem} = \hat{h}^* \binom{d}{2} \frac{\varepsilon_h}{A_h^*} h^*.$$

*Proof.* We assign a basis  $\{\sigma^i\}$  such that for each  $\sigma^i$  we have

$$\sigma^i \cdot h^* = 2g^{\mu\nu} g^{\alpha\beta} \sigma_{(\mu\alpha}^i h_{\nu\beta}^* \neq 0 \quad (43a)$$

where  $\sigma_{(\mu\alpha}^i = \frac{1}{2} [\sigma_{\mu\alpha}^i + \sigma_{\alpha\mu}^i]$ . Since each  $\sigma^i$  has a non-zero projection onto  $h^*$ , we have

$$\mathbf{Riem}(\sigma_i) = \hat{h}^* \varepsilon_h. \quad (43b)$$

Therefore the Riemann tensor associates to each basis bivector  $\sigma_i$  a rotation bivector oriented along  $h^*$  with magnitude of rotation equal to  $\varepsilon_h$ . Here we have inserted an oriented area in **Riem** to obtain

$$\mathbf{Riem}(\sigma_i) = \hat{h}^* \varepsilon_h. \quad (43c)$$

This allows us to assign a matrix representation to the Riemann curvature tensor,

$$\mathbf{Riem} \doteq \begin{pmatrix} \varepsilon_h (\hat{h}^* \cdot \sigma^1) & \varepsilon_h (\hat{h}^* \cdot \sigma^1) & \varepsilon_h (\hat{h}^* \cdot \sigma^1) & \cdots \\ \varepsilon_h (\hat{h}^* \cdot \sigma^2) & \varepsilon_h (\hat{h}^* \cdot \sigma^2) & \varepsilon_h (\hat{h}^* \cdot \sigma^2) & \cdots \\ \varepsilon_h (\hat{h}^* \cdot \sigma^3) & \varepsilon_h (\hat{h}^* \cdot \sigma^3) & \varepsilon_h (\hat{h}^* \cdot \sigma^3) & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad (43d)$$

where each row has identical elements and the sum of each column gives  $\varepsilon_h \hat{h}^*$ . Since a Riemann tensor is a symmetric tensor across the basis of 2-forms, we require that the matrix representation adhere to the symmetry. The asymmetry that appears in Eq. (43d) is due to our asymmetric use of the conic singularity. To account for this, we further require that the inner product  $\hat{h} \cdot \sigma^i$  be normalized in both the space of values and tangent space. This is equivalent to requiring that any loop of parallel transport is treated as a 2-surface in the space with constant sectional curvature equal to the sectional curvature along  $h^*$ . Hence any loop with non-trivial projection onto  $h^*$  gets maximally projected onto the dual polygon  $h^*$  as in Figure 5 and the Riemann tensor acts only on this projected loop.

The Riemann tensor becomes a matrix with uniform entries, i.e. a rank 1 matrix with eigenvalue given by the sum over a row or column. Since there are  $\binom{d}{2}$  basis 2-forms, we have a Riemann tensor with a single component

$$\mathbf{Riem} = \hat{h}^* \frac{d(d-1)}{2} \varepsilon_h \hat{h}^* \doteq \frac{d(d-1)}{2} \begin{pmatrix} \varepsilon_h & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (43e)$$

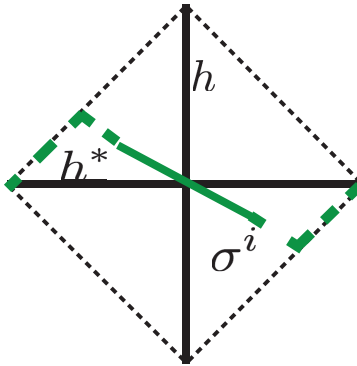


FIG. 5. The conic singularity has the peculiar property that the Riemann curvature tensor operates on any area of parallel transport (with non-zero projection onto  $h^*$ ) as if the area were projected identically onto  $h^*$ . While we take the area of this projection to be that of  $h^*$ , this is not a necessity of the PF manifold. All that is required is that each area only be acted upon by the sectional curvature of the plane  $h^*$ . Taking the area to be equal to  $A_{h^*} = |h^*|$  is a normalization choice that is natural given the dual operation defined and the inherent orthogonality of the dual lattices, see for example [17, 29].

If we then put the basis of the horizontal section as the full lattice elements  $h^*$  and the orthogonal 2-forms, then we have

$$\mathbf{Riem} = \hat{h}^* \frac{d(d-1)}{2} \frac{\varepsilon_h}{A_{h^*}} h^*. \quad (43f)$$

□

In the eigenvalue analysis, we have assigned a projection operation on the hybrid cell that treats any non-trivial loop of parallel transport as a parallel transport around  $h^*$ . This tells us that the Riemann tensor in this domain has the special property that there is exactly one non-zero sectional curvature in the plane  $h^*$ , and a trivial flat subspace of bivectors orthogonal to  $h^*$ .

While this is now a local measure of the Riemann tensor as obtained from an arbitrary sampling of the space and evaluation of its eigenspace decomposition, it does not carry with it the flavor of the Riemann tensor as one would see from an evaluation on a single loop of parallel transport. This comes from the imposition of the conic singularity that all bivectors (with non-zero projection on  $h^*$ ) have assigned to them a equal sectional curvatures. Therefore, a basis in general position (all basis elements satisfying Eq. (43a)) acts as though the space were an Einstein space. The eigenvalue is the local measure over the hybrid cell and is the value through which we perform analysis in the DEC.

From this eigen-decomposition of the Riemann tensor we can regain a local representation by normalizing the non-trivial eigenvalue by the combinatoric factor counting the basis elements of a basis in general position. In our case there are  $\binom{d}{2}$  2-forms and the normalization factor is given by  $\binom{d}{2}^{-1} = 1/\binom{d}{2}$ . This allows us to back-track from the eigenvalue ( as a measure of all surfaces in the domain) to the local evaluation of the Riemann tensor as a measure of the 2-surfaces in the domain of support;

$$\mathbf{Riem} = \hat{h}^* \frac{\varepsilon_h}{A_{h^*}} h^*.$$

This normalization process is a final step in the description of the discretized tensors on the PF manifold. It should be noted that any and all calculus is done on the local measures of the

differential forms and tensors, i.e. the tensors integrated over the local volume. The normalized, local tensors are convenient representations that are made possible by the locally simple structure of the hybrid cells.

Further, we can assign a Ricci tensor and Ricci scalar in this hybrid domain. If we span  $h^*$  by the local choice of  $\{M_{h\lambda}, \lambda\}$ , then we have

$$\bar{R}^{M_{h\lambda}\lambda}_{M_{h\lambda}\lambda} = -\bar{R}^{\lambda M_{h\lambda}}_{M_{h\lambda}\lambda} = -\bar{R}^{M_{h\lambda}\lambda}_{\lambda M_{h\lambda}} = \bar{R}^{\lambda M_{h\lambda}}_{\lambda M_{h\lambda}} = \binom{d}{2} \frac{\varepsilon_h}{A_{h^*}}, \quad (44a)$$

which leads to Ricci tensor components

$$\bar{R}^{M_{h\lambda}}_{M_{h\lambda}} = \bar{R}^\lambda_\lambda = \binom{d}{2} \frac{\varepsilon_h}{A_{h^*}}, \quad (44b)$$

and scalar curvature

$$\bar{R} = 2 \binom{d}{2} \frac{\varepsilon_h}{A_{h^*}}, \quad (44c)$$

where the barred notation indicates the unnormalized representation of the eigenvalues. Normalizing these curvature tensors given their differential forms character requires one to normalize by a factor of  $\binom{d}{2}$  for the Riemann tensor,  $\binom{d}{1}$  for the Ricci tensor and  $\binom{d}{0}$  for the Ricci scalar. Hence the normalized curvature tensors are

$$R^{M_{h\lambda}\lambda}_{M_{h\lambda}\lambda} = -R^{\lambda M_{h\lambda}}_{M_{h\lambda}\lambda} = -R^{M_{h\lambda}\lambda}_{\lambda M_{h\lambda}} = R^{\lambda M_{h\lambda}}_{\lambda M_{h\lambda}} = \frac{\varepsilon_h}{A_{h^*}} \quad (45a)$$

$$R^{M_{h\lambda}}_{M_{h\lambda}} = R^\lambda_\lambda = \frac{d-1}{2} \frac{\varepsilon_h}{A_{h^*}} \quad (45b)$$

$$R = 2 \binom{d}{2} \frac{\varepsilon_h}{A_{h^*}}. \quad (45c)$$

These are the oriented and normalized versions of the Riemann, Ricci and scalar curvatures within a hybrid domain  $V_h$ . We often only need the unoriented measure of the curvature forms and so summing over the orientations introduces a factor of 2 into Eqs. (44a), (44b), (45a) and (45b). We now have an accounting of the measures of curvature on the hybrid domain on a hinge that is analogous to that obtained by Friedberg and Lee [6]. On a qualitative scale, we have the same form of the Riemann, Ricci and scalar curvatures obtained in [6]. The quantitative distinction comes from our use of the hybrid volume as a domain of support and our treatment of the Dirac distribution on  $h$  as spread out over the entire domain  $V_h$ , instead of distributionally valued only on  $h$ .

## B. The Ricci tensor and its double dual

The Riemann tensor had a natural association to the dual polygons  $h^*$  to the codimension 2 hinges  $h$  given that its differential form properties are that of a bivector-valued 2-form. Hence one need only project two indices of the Riemann tensor onto the discretization. In evaluating the Riemann tensor in the simplicial discretization we also were able to express representations of the Ricci tensor and scalar curvature in this domain. We now shift attention to the PF representation



of the Ricci tensor. Since the Ricci tensor's natural representation is that of a vector-valued one-form, its direct discretization is on the 1-skeletons of the dual and simplicial lattices. In [27] we derived representations of the Ricci tensor in both the dual and simplicial 1-skeletons as weighted averages of the Riemann curvatures. We now try to elucidate the properties of these derivations and draw comparisons with an updated understanding of the curvature.

The Ricci tensor is given by the bivector-valued 2-form curvature operator acting on the inverse solder form  $d\mathcal{P}^{-1} = \tilde{e}^\mu e_\mu$ , inducing a trace on the Riemann tensor. As a vector-valued 1-form, the Ricci tensor is directly discretized on 1-forms orthogonal to the hinges. Since **Riem** only has components in the planes  $h^*$ , **Rc** only takes components along  $\lambda$  or  $M_{h\lambda}$ . The components along  $M_{h\lambda}$  trivially give only one component from **Riem** and are contained in  $V_h$ . Calculating a Ricci tensor in the direction of  $M_{\lambda h}$  becomes a sum over directions orthogonal to  $M_{\lambda h}$ , only one of which gives a non-zero contribution. Moreover, the  $M_{h\lambda}$  are virtual-being members of neither the simplicial or dual lattices— carry no inherent discrete differential forms. This is simply a statement about the non-independence of the curvature directed along  $M_{h\lambda}$  and the Riemann curvature associated to  $h$ . At the same time, in a small domain surrounding any given  $\lambda$ , there are  $d$  distinct holonomies with independent curvature operators. Each of these distinct curvature tensors have Ricci curvature components oriented along  $\lambda$ . Therefore, there exist non-trivial representation of **Rcs**, and distinct from the **Riem** of the hinges, on the  $\lambda$ 's of the dual lattice. In [27] we sought to ensure that the integrated measure of curvature associated  $h^*$  and  $\lambda$  was preserved over the domain common to these two elements. This is equivalent to the continuum requirement that

$$\text{tr}(\mathbf{Rc}, \mathbf{e}^a)_\Omega = \text{tr}(\mathbf{Riem}, \mathbf{e}^a \wedge \mathbf{e}^b)_\Omega$$

over some common domain  $\Omega$ . The measure of curvature one obtains from integration over the domain is given by the one non-trivial eigenvalue of the curvature in that domain, and hence the unnormalized measure.

To trace the Riemann curvature in the domain of  $\lambda$  is to sum over the polygonal loops  $h^* \ni \lambda$  given that the individual domains of overlap between  $\lambda$  and each  $h^*$  satisfies

$$\bar{R}_\lambda V_{\lambda h^*} = \bar{R}_{h^*} V_{h^* \lambda}. \quad (46)$$

Doing so gives a scalar measure of the Ricci curvature on  $\lambda$

$$\bar{R}_\lambda = \frac{\sum_{h^*} R_{h^*} V_{h^* \lambda}}{V_\lambda}. \quad (47)$$

Given that this is an integrated measure, it samples all orientations of the loops of parallel transport and one naturally picks up the unnormalized, integrated form of the Riemann curvature tensor and an overall factor  $d(d-1)$ ,

$$\bar{R}_\lambda = d(d-1) \left\langle \frac{\varepsilon_h}{A_{h^*}} \right\rangle_\lambda, \quad (48)$$

where we have used the volume-weighted average

$$\left\langle \frac{\varepsilon_h}{A_{h^*}} \right\rangle_\lambda = \frac{\sum_{h^* \ni \lambda} A_h V_{h\lambda}}{V_\lambda}.$$

This is an association of a scalar quantity to a 1-form on the lattice. We can again normalize by the dimension of the space of 1-forms to obtain

$$\bar{R}_\lambda = (d-1) \left\langle \frac{\varepsilon_h}{A_{h^*}} \right\rangle_\lambda. \quad (49)$$

We can further associate to this measure a directionality. As we have already mentioned, each of the curvature tensors contributing to  $R_\lambda$  ( $\bar{R}_\lambda$ ) is already oriented along  $\lambda$  in each subdomain  $V_{\lambda h}$  since the measure of  $\mathbf{Rc}$  on those domains is obtained by contraction of  $\mathbf{Riem}$  with  $M_{\lambda h}$ . Therefore, the measures  $\bar{R}_\lambda$  and  $R_\lambda$  can be considered as coefficients on the one-form oriented along  $\lambda$ . This assigns both directionality and magnitude to  $\mathbf{Rc}$  on a given  $\lambda$ .

It is useful to note at this point that we have associated a scalar quantity to  $\mathbf{Rc}$  on  $\lambda$  and assigned to it a directionality. However, the scalar coefficient is dependent on the domain of integration. While we associate to  $\lambda$  a component of  $\mathbf{Rc}$  in the direction of  $\lambda$ , this object is not of the same class as the  $\mathbf{Riem}$  on the hinges. Whereas the curvature operators on the hinges are constant over the codimension 2 hinges and treated as constant over the domains  $V_h$  (whenever a subdomain also encompasses the hinge itself), the Ricci curvatures are composed explicitly in terms of components of curvature operators whose valuations only make sense within the subdomains  $V_{\lambda h}$ . It is therefore notable that the scalar coefficient is only an appropriate measure whenever a domain of interest encompasses the entire  $V_\lambda$ . If a domain of interest only intersects a portion of  $V_\lambda$ , then one must suitably restrict the measures in the definition of  $R_\lambda$  ( $\bar{R}_\lambda$ ). This will come into play as we now seek representations of  $\mathbf{Rc}$  on the simplicial lattice.

The natural discretization of  $\mathbf{Rc}$  is on the dual lattice; however, we have shown in [27] that a representation on the simplicial edges is also possible by requiring that domains of overlap between  $V_\lambda$  and  $V_\ell$  give rise to the same measure of integrated curvature. We now want to show that is related to the discrete version of the 1-form double-dual of a vector-space valued 1-form.

In each irreducible hybrid cell common to both  $\lambda$  and  $\ell$ , we can form a basis from the two vectors  $\ell$  and  $\lambda$  as well as the set of  $d - 2$  vectors spanning the subspace  $M_{\ell\lambda}$ ,  $\{m_i \mid 1 \leq i \leq d - 2\}$ . The transformation from the dual lattice to the simplicial lattice is done using the orthogonal subspace  $M_{\ell\lambda}$  applied through the commutative diagram from Eq. (26). If we take the discrete form along  $\lambda$ , then we have  $(\bar{R}_\lambda)\lambda$  as a vector-valued 1-form with  $\lambda$  as the trivial map from  $\lambda$  in the horizontal section to  $\lambda$  in the vertical section. Taking the dual of this assigns  $\bar{R}_\lambda$  to  $\lambda^*$ . If a given  $\ell$  is not contained in  $\lambda^*$  there is no contribution of  $\bar{R}_\lambda$  to  $\bar{R}_\ell$  and likewise in the reverse. So we only consider when  $\lambda \in \ell^*$  and  $\ell \in \lambda^*$ . Moreover,  $\bar{R}_\lambda$  is constructed from the Riemann tensors evaluated on  $h^*$ s and so we further expand the  $\bar{R}_\lambda$ 's such that the dependence on  $\bar{R}_{h^*}$  is explicit. If we then take the double dual and inner-derivative over the inverse solder forms on  $M_{\ell\lambda}$ , we obtain

$$\bar{R}_\ell V_{\ell\lambda} = (\iota_{M_{\ell\lambda}}(\star\bar{R}_\lambda), \ell)_{V_{\ell\lambda}} = (\star\bar{R}_\lambda, \ell \wedge M_{\ell\lambda})_{V_{\ell\lambda}} = (\bar{R}_\lambda, \star(\ell \wedge M_{\ell\lambda}))_{V_{\ell\lambda}} = \bar{R}_\lambda V_{\ell\lambda}. \quad (50)$$

This object is a measure of the contribution from those curvature tensors with support in the defined domain. The above integrated curvatures are entirely connected to their domains of integration and the restriction of the domains induces a restriction of the integration on their definitions. We then view Eq. (50) as statement of dependence of the simplicial lattice Ricci curvature on the restriction of the dual skeleton Ricci curvature. As the latter depends on multiple curvature tensors in multiple domains, the restriction ensures that only those curvature operators with values in the specified domain contribute the final result. Summing over all  $\lambda$  that are incident to  $\ell$ , i.e. all  $\lambda \in \ell^*$ , gives the final measure of the unnormalized Ricci tensor on  $\ell$ ;

$$\bar{R}_\ell = \frac{\sum_{\lambda \in \ell^*} \bar{R}_\lambda V_{\lambda\ell}}{V_\ell} = \frac{\sum_{h \ni \ell} \bar{R}_h V_{h\ell}}{V_\ell} = d(d-1) \frac{\langle \varepsilon_h \rangle_\ell}{\langle A_{h^*} \rangle_\ell} \quad (51)$$

where

$$\langle B_h \rangle_\ell := \frac{\sum_{h \ni \ell} B_h A_{h\ell}}{\sum_{h \ni \ell} A_{h\ell}},$$

and we have used the relation

$$V_\ell = \sum_{h \ni \ell} \frac{1}{\binom{d}{2}} A_{h\ell} A_{h^*}.$$

It should be clear from Eq. (50) that the object being assigned to an  $\ell$  is not the component of  $\mathbf{Rc}$  along  $\ell$ , but rather a measure of  $\mathbf{Rc}$  in the orthogonal complement to  $\ell$ . Summing over all  $\lambda$ 's in  $\ell^*$  provides a complete measure of the  $\mathbf{Rc}$  in that orthogonal complement to  $\ell$ .

After normalizing by the combinatorial factor  $\binom{d}{1}$  we get the normalized Ricci tensor on the simplicial edge  $\ell$ ,

$$R_\ell = (d-1) \frac{\langle \varepsilon_h \rangle_\ell}{\langle A_{h^*} \rangle_\ell}. \quad (52)$$

This demonstrates that we can formalize the transformation between the dual and simplicial lattices via the trace of the double dual. Moreover it provides a geometrically clear picture of the transformation via the moment arm or generalized solder form between  $\ell$  and  $\lambda$ .

The double-dual and its traces provide a direct path for the raising and lowering operators common in general relativity and differential geometry. These were applied in [27] as a method to obtain a dualized view of the Ricci tensor on the simplicial lattice. What is a particularly important lesson to be drawn from this is that the Ricci tensor associated with any  $\ell$  is not a measure of the Ricci tensor in the direction of  $\ell$  but an average of Ricci tensors in the  $(d-1)$ -subspace orthogonal to  $\ell$ . This was a crucial understanding in [17] since one must ensure that metric components in Hamilton's Ricci flow [30] change in proportion to those components of the Ricci tensor. If instead one were to simply take metric components along  $\ell$ , Hamilton's Ricci flow would not be recovered. Rather one would obtain an orthogonal flow to that of Hamilton's.

We have done this explicitly for the Ricci curvature and similar results hold for the scalar curvature. In particular, taking the trace of the Ricci curvature, or the double trace of the Riemann, we assign a scalar curvature to a vertex,  $\nu$ , of the dual lattice;

$$R_\nu = \frac{\sum_{h^* \ni \nu} \bar{R}_{h^*} V_{h^* \nu}}{V_\nu}. \quad (53)$$

Since this is the scalar curvature, there is no distinction between the normalized and unnormalized curvatures and hence we drop the bars at the very beginning. In [31] we showed that scalar curvature on a vertex  $v$  of the simplicial lattice is given by

$$R_v = d(d-1) \frac{\langle \varepsilon_h \rangle_v}{\langle A_{h^*} \rangle_v}. \quad (54)$$

We have thus described an intrinsic geometric derivation of the curvature operators and curvature scalar without explicit reference a limiting smooth sequence of surfaces.

#### IV. DISCUSSION AND CONCLUSIONS

We have shown in this manuscript a revised formulation of DEC that is based on the volume measures of differential forms on local domains of compact support. This formulation makes explicit the assignment of a discrete form to a family of surfaces in a volume local to a lattice element. The characterization of a discrete form to the family of surfaces provides a transparent view of the operations on discrete forms, such as the exterior (co)-derivative. This makes the DEC approach more directly amenable to use in RC.

We built the volume-based DEC from the irreducible domains of the lattice—the *monads* of space—that form the most basic structures of the PF manifold. These irreducible domains are domains of supports for arbitrary  $k$ -forms in any given tangent space. From these irreducible cells we identified generalized solder forms and local solder forms to allow for transformations between

the simplicial and dual lattices. Moreover, since solder forms provide a unit map from the tangent space (horizontal section) to the space of values (vertical section), a discretization of the solder form allows one to work within a framework that covers both scalar-valued and vector-space valued differential forms.

We have also shown how curvature operators can be given explicit constructions as bivector-valued two-forms in the lattice by examining the conic singularities around the hinges  $h$  of  $\mathcal{T}_0$ . It was found that the PF Riemann curvature operators have an eigenspectrum with only one non-trivial eigenspace, that aligned along the plane orthogonal to the codimension 2 hinges. Noticing that the **Riem** operator takes a form analogous to a space with constant sectional curvature for bases in general position, we define local, normalized curvature operators that can then be compared to local continuum quantities. This provides flexibility to insist that not only should the average integrated curvature compare to the integrated curvature over a finite domain, but that the local curvature operators compare to sectional curvatures in the continuum. It was then shown how the Ricci curvature can be formulated on the dual lattice and how the simplicial representation is viewed not as the components of **Rc** along a simplicial edge  $\ell$  but the components of the double dual of **Rc**.

This general framework and the volume-based DEC is a purely discrete foundation for the analysis of the geometry of PF manifolds. The continuum is only used at the level of the discretization and for the locally flat behavior of irreducible hybrid cells. We can therefore characterize this approach as a stratified view of discrete geometry with three distinct regimes: (1) the scale smaller than the local discretization where one admits ignorance of the internal structure and must assume some approximate behavior (e.g. flatness, constant curvature, etc), (2) the discrete scale where the geometric properties are based on the PF structure, and (3) a coarse-grained scale that regains the continuum behavior of the manifold. We have shown how to treat the discrete scale by inferring behavior from the connectivity of irreducible hybrid cells within a given domain.

These results have been applied in [17] to a simplicial discretization of Hamilton's Ricci flow and in earlier stages in [32]. These results utilize the foundations laid in RC to open up DEC to a variety of geometric objects characterizable as vector-space valued differential forms.

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