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Absence of Zero Energy States in Reduced SU(N) 3d Supersymmetric Yang Mills Theory

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Abstract

For the SU(N) invariant supersymmetric matrix model related to membranes in 4 space-time dimensions we argue that $\langle\Psi,\chi\rangle = 0$ for the previously obtained solution of $Q\chi = 0, Q^\dagger\Psi = 0$.

In a series of 3 short papers [1] it was recently shown how to obtain, for a certain class of supersymmetric matrix models, solutions of $Q\psi = 0$ resp. $Q^\dagger\psi = 0$. The models of interest [2] are SU(N) gauge-invariant and can be formulated with either 2, 3, 5 or 9 times $(N^2-1)\cdot 2$ bosonic degrees of freedom. This letter mainly concerns the first case, $d = 2$ (corresponding to membranes in 4 space-time dimensions [3], and to 2+1 dimensional (Susy) Yang-Mills theory with spatially constant fields [3]), while our method can also be applied to the other cases. For different approaches to the problem see [4],[5],[6],[7].
The supercharges of the model are given by

$$\begin{aligned} Q &= iq_a\lambda_a + 2\partial_a\frac{\partial}{\partial\lambda_a} =: M_a\lambda_a + D_a\partial_{\lambda_a} \\ Q^\dagger &= -iq_a\frac{\partial}{\partial\lambda_a} - 2\overline{\partial}_a\lambda_a =: M_a^\dagger\partial_{\lambda_a} + D_a^\dagger\lambda_a \end{aligned} \tag{1}$$

where $\partial_a = \frac{\partial}{\partial z_a}$, $z_a \in \mathbb{C}, a = 1\cdots N^2-1, q_a := \frac{i}{2}f_{abc}z_b\overline{z}_c$ (f_{abc} being totally antisymmetric, real, structure constants of SU(N)) and λ_a ($\frac{\partial}{\partial\lambda_a}$) being fermionic creation (annihilation) operators satisfying $\{\lambda_a, \frac{\partial}{\partial\lambda_b}\} = \delta_{ab}$, $\{\lambda_a,\lambda_b\} = 0 = \{\frac{\partial}{\partial\lambda_a}, \frac{\partial}{\partial\lambda_b}\}$. In $\mathcal{H}_+$, the Hilbert-space of SU(N)-invariant square-integrable wavefunctions

$$\Psi = \psi + \frac{1}{2}\psi_{ab}\lambda_a\lambda_b + \cdots + \frac{1}{\Lambda!}\psi_{a_1\cdots a_\Lambda}\lambda_{a_1}\cdots\lambda_{a_\Lambda} , \tag{2}$$

$\Lambda :=\ N^2-1$ (even) the general solution of

$$Q^\dagger\Psi = 0 , \ Q\chi = 0 \tag{3}$$

was shown [1] to be of the form

$$\begin{aligned} \Psi &= (\mathbf{1} - A)^{-1}\ \Psi^{(h)} \\ \chi &= (\mathbf{1} - B)^{-1}\ \chi^{[h]} \end{aligned} \tag{4}$$

with

$$\begin{aligned} A := (I^\dagger\cdot\lambda)(D^\dagger\cdot\lambda)\ ,\ B &= (I\cdot\partial_\lambda)(D\cdot\partial_\lambda) \\ I_a : &= i\frac{q_a}{q^2} \end{aligned} \tag{5}$$

and

$$(M^\dagger\cdot\partial_\lambda)\ \Psi^{(h)} = 0\ ,\ (M\cdot\lambda)\ \chi^{[h]} = 0\ . \tag{6}$$

As

$$\Psi = \Psi^{(h)} + A\Psi\ ,\ \chi = \chi^{[h]} + B\chi\ , \tag{7}$$

and (from (3))

$$\begin{aligned} A\Psi &= -\ (I^\dagger\cdot\lambda)\ (M^\dagger\partial_\lambda)\ \Psi = \frac{q_aq_b}{q^2}\lambda_a\partial_{\lambda_b}\Psi\ \in\ \mathcal{H}_+ \\ B\chi &= -\ (I\cdot\partial_\lambda)\ (M\cdot\lambda)\ \chi = \chi - \frac{q_aq_b}{q^2}\lambda_a\partial_{\lambda_b}\chi\ \in\ \mathcal{H}_+ \end{aligned} \tag{8}$$

one can see that $\Psi^{(h)}$, resp. $\chi^{[h]}$, have to be elements of $\mathcal{H}_+$. The scalar product of any two solutions of (3) is therefore

$$\begin{aligned}\langle\Psi,\chi\rangle &= \langle(\mathbf{I}-A)^{-1}\Psi^{(h)}\ ,\ (\mathbf{I}-B)^{-1}\chi^{[h]}\rangle \\ &= \langle(\mathbf{I}-B^{\dagger})^{-1}(\mathbf{I}-A)^{-1}\ \Psi^{(h)}\ ,\ \chi^{[h]}\rangle \\ &= \langle(\mathbf{I}-C)^{-1}\Psi^{(h)}\ ,\ \chi^{[h]}\rangle \end{aligned} \tag{9}$$

with

$$C := A + B^{\dagger} = \{I^{\dagger}\cdot\lambda, D^{\dagger}\cdot\lambda\}\ . \tag{10}$$

As $H_M := \{M\cdot\lambda, M^{\dagger}\cdot\partial_\lambda\} = q^2 > 0$, (6) implies

$$\Psi^{(h)} = (M^{\dagger}\cdot\partial_\lambda)\Psi_-^{(h)}, \chi^{[h]} = (M\cdot\lambda)\chi_-^{[h]} \tag{11}$$

for some $\Psi_-^{(h)}, \chi_-^{[h]}$.
Furthermore, C commutes with $M^{\dagger}\cdot\partial_\lambda$ (between SU(N) invariant states), as

$$[M^{\dagger}\cdot\partial_\lambda\ ,\ \{I^{\dagger}\cdot\lambda, D^{\dagger}\cdot\lambda\}] = -[I^{\dagger}\cdot\lambda, \{D^{\dagger}\cdot\lambda, M^{\dagger}\cdot\partial_\lambda\}] - [D^{\dagger}\cdot\lambda, \{M^{\dagger}\cdot\partial_\lambda, I^{\dagger}\cdot\lambda\}] \tag{12}$$

and $\{D^{\dagger}\cdot\lambda, M^{\dagger}\cdot\partial_\lambda\} = -iz_aJ_a$,

$$J_a := -if_{abc}(z_b\partial_c + \bar{z}_b\bar{\partial}_c + \lambda_b\partial_{\lambda_c})\ . \tag{13}$$

One therefore has

$$\langle\Psi,\chi\rangle = \langle(M^{\dagger}\cdot\partial_\lambda)\ (1-C)^{-1}\Psi_-^{(h)}\ ,\ (M\cdot\lambda)\ \chi_-^{[h]}\rangle = 0, \tag{14}$$

showing that $Q^{\dagger}\Psi = 0 = Q\Psi$ implies $\Psi \equiv 0$ (in $\mathcal{H}_+$). The same holds in $\mathcal{H}_-$ (as the extra-conditions $D^{\dagger}\cdot\lambda\psi^{(h)}_{a_1\cdots a_{\Lambda-1}}\lambda_{a_1}\cdots\lambda_{a_{\Lambda-1}} = 0$, $(D\cdot\partial_\lambda)\ \chi_a^{[h]}\lambda_a = 0$, are automatically satisfied for SU(N)-invariant states, due to (11)).
Let us close with a remark on d=9: in order to prove the existence of a zero-energy state for the supersymmetric matrix model related to membranes in 11 space-time dimensions it is sufficient to show that for one particular solution of $Q^{\dagger}\Psi = 0$, and one particular solution of $Q\chi = 0$, one has $\langle\Psi,\chi\rangle \neq 0$.
Note added: Due to the singularity at $q = 0$ the above argument is not yet complete.

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