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Azumaya structure on D-branes and deformations and resolutions of a conifold revisited: Klebanov-Strassler-Witten vs. Polchinski-Grothendieck

Chien-Hao Liu and Shing-Tung Yau

Abstract

In this sequel to [L-Y1], [L-L-S-Y], and [L-Y2] (respectively arXiv:0709.1515 [math.AG], arXiv:0809.2121 [math.AG], and arXiv:0901.0342 [math.AG]), we study a D-brane probe on a conifold from the viewpoint of the Azumaya structure on D-branes and toric geometry. The details of how deformations and resolutions of the standard toric conifold Y can be obtained via morphisms from Azumaya points are given. This should be compared with the quantum-field-theoretic/D-brany picture of deformations and resolutions of a conifold via a D-brane probe sitting at the conifold singularity in the work of Klebanov and Witten [K-W] (arXiv:hep-th/9807080) and Klebanov and Strasser [K-S] (arXiv:hep-th/0007191). A comparison with resolutions via noncommutative desingularizations is given in the end.

Key words: D-brane, Azumaya structure, Polchinski-Grothendieck Ansatz, Azumaya point, conifold.

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In memory of a young string theorist Ti-Ming Chiang,
whose path I crossed accidentally and so briefly.[†]

[†] *From C.-H.L.* During the years I was attending Prof. Candelas's group meetings, I learned more about Calabi-Yau manifolds and mirror symmetry and got very fascinated by the works from Brian Greene's group. Because of this, I felt particularly lucky knowing later that I was going to meet one of his students, Ti-Ming, - a young string theorist with a PhD from Cornell at his very early 20's - and perhaps to cooperate with him. Unfortunately that anticipated cooperation never happened. Ti-Ming had become unwell just before I resettled. Except the visits to him at the hospital and some chats when he showed up in the office, I didn't really get the opportunity to interact with him intellectually. Further afterwards I was informed of Ti-Ming's passing away. Like a shooting star he reveals his shining so briefly and then disappears. The current work is the last piece of Part 1 of the D-brane project. It is grouped with the earlier D(1), D(2), D(3) under the hidden collective title: "Azumaya structure on D-branes and its tests". Here we address in particular a conifold from the viewpoint of a D-brane probe with an Azumaya structure. This is a theme Ti-Ming may have felt interested in as well, should he still work on string theory, since conifolds have play a role in understanding the duality web of Calabi-Yau threefolds - a theme Ti-Ming once worked on - and D-brane resolution of singularities is a theme Brian Greene's group once pursued vigorously. We thus dedicate this work to the memory of Ti-Ming.

0. Introduction and outline.

Conifolds, i.e. Calabi-Yau threefolds with ordinary double-points, have been playing special roles at various stages of string theory.[1] In this sequel to [L-Y1], [L-L-S-Y], and [L-Y2], we study a D-brane probe on a conifold from the viewpoint of Azumaya structure on D-branes and toric geometry. This should be compared with the quantum-field-theoretic/D-brany picture of deformations and resolutions of a conifold in the work of Klebanov and Strasser [K-S] and Klebanov and Witten [K-W].

Effective-space-time-filling D3-brane at a conifold singularity.

In [K-W], Klebanov and Witten studied the $d = 4$, $N = 1$ superconformal field theory (SCFT)[2] on the D3-brane world-volume X ($\simeq \mathbb{R}^4$ topologically) that is embedded in the product space-time $\mathbb{M}^{3+1} \times Y$ as $\mathbb{M}^{3+1} \times \{\mathbf{0}\}$,[3] and its supergravity dual - a compactification of $d = 10$, type-IIB supergravity theory on $\mathrm{AdS}^5 \times (S^3 \times S^2)$ - along the line of the AdS/CFT correspondence of Maldacena [Ma]. Here $\mathbb{M}^{3+1}$ is the $d = 3 + 1$ Minkowski space-time, Y is the conifold $\{z_1z_2 - z_3z_4 = 0\} \subset \mathbb{C}^4$ (with coordinates (z_1, z_2, z_3, z_4)), $\mathbf{0}$ is the conifold singularity on Y, and AdS^5 is the $d = 4 + 1$ anti-de Sitter space-time.

In the simplest case when there is a single D3-brane sitting at the conifold point of Y, the *classical moduli space* of the *supersymmetric vacua* of the associated $U(1)$ super-Yang-Mills theory coupled with matter on the D3-brane world-volume comes from the *D-term* of the *vector multiplet* and the coefficient $\zeta \in \mathbb{R}$ of the *Fayet-Iliopoulos term* in the Lagrangian.[4] By varying ζ, one realizes the two small resolutions, Y_+ and Y_-, of Y as the classical moduli space Y_ζ of the above $d = 4$ SCFT.[5] A flop $X_+ \dashrightarrow Y_-$ happens when Y_ζ crosses over $\zeta = 0$.

To describe the physics for N-many parallel D3-branes sitting at the conifold singularity, Klebanov and Witten proposed to enlarge the gauge group for the super-Yang-Mills theory on the common world-volume of the stacked D3-brane to $U(N) \times U(N)$ (rather than the naive $U(N)$) *and* introduce a *superpotential* W for the chiral multiplets. The classical moduli space of the theory comes from a system with equations of the type above (i.e. *D-term equations*) and equations from the superpotential term W (i.e. *F-term equations*). In particular, the N-fold symmetric product $Sym^n Y$ of Y can be realized as the classical moduli space of the $d = 4$ SCFT on the D3-brane world-volume with $\zeta = 0$.

In [K-S], Klebanov-Strassler studied further $d = 4$, $N = 1$ supersymmetric quantum field theory (SQFT) on the D3-brane world-volume that arises from a D3-brane configuration with both N-many above *full*/free D3-branes and M-many new *fractional*/trapped D3-branes[6] sitting

[1]Readers are referred to, for example, [C-dlO] (1989); [Stro], [G-M-S], [C-G-G-K] (1995); [G-V] (1998); [Be], [C-F-I-K-V] (2001) and references therein to get a glimpse of conifolds in string theory around the decade 1990s.

[2]There will be a few standard physicists' conventional notations in this highlight of the relevant part of [K-W] and [K-S]: N that counts the *number of supersymmetries* (susy) via the multiple number of minimal susy numbers in each space-time dimension vs. N that appears in the *gauge group* $U(N)$ or $SU(N)$ vs. N that counts the *multiplicity* of stacked D-branes.

[3]In string-theorist's terminology, the D3-brane is "*sitting at the conifold singularity*". We will also adopt this phrasing for convenience. Note that in such a setting, the internal part is a D0-brane on the conifold Y. The latter is what we will study in this work.

[4]ζ is part of the parameters to give local coordinates of the *Wilson's theory-space* in the problem; cf. [L-Y2: Introduction] for brief words. See also [W-B] for the standard SUSY jargon.

[5]See also [Wi] and [D-M] for details of such a construction.

[6]See [G-K] and references therein for the detail of such *fractional D-branes*.

at the conifold singularity $\mathbf{0}$ of Y. For infrared physics, the theory now has the gauge group $SU(N+M) \times SU(N)$. It follows from the work of Affleck, Dine, and Seiberg [A-D-S][7] that an *additional term* to the previous superpotential W is now *dynamically generated*. This deforms the classical moduli space of SUSY vacua of the $d = 4$ SQFT on the D3-brane world-volume. In the simplest case when $N = M = 1$, this enforces a deformation of the classical moduli space from a conifold to a deformed conifold Y' ($\simeq T^*S^3$ topologically). Cf. FIGURE 0-1.

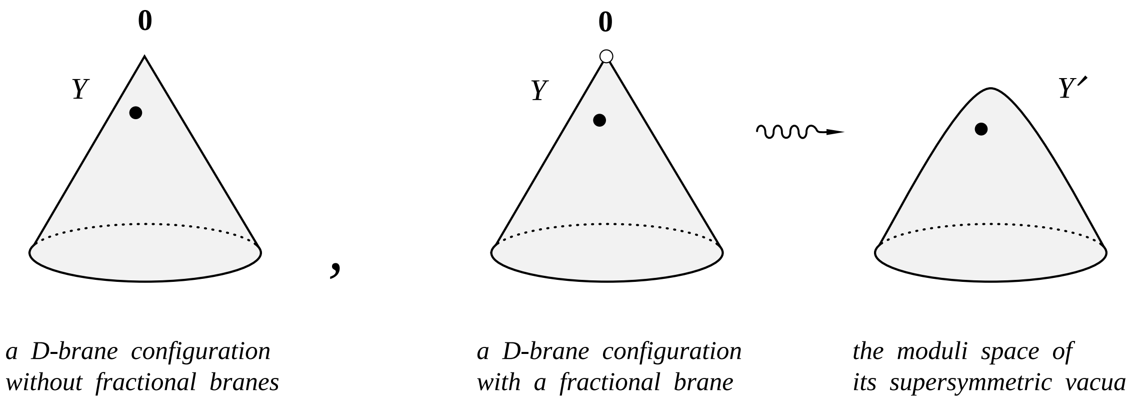


FIGURE 0-1. (Cf. [Stra: Figures 25, 26, 27].) When a fractional/trapped D3-brane sits at the conifold singularity $\mathbf{0} \in Y$, the full/free D3-brane "sees" a smooth deformed conifold Y' ($\simeq T^*S^3$ topologically) as its classical vacua manifold. I.e., in very low energy for this situation the free D3-brane "feels" as if it lives on Y' instead of Y! In the figure, a full D3-brane is indicated by $\bullet$ while a fractional D3-brane by $\circ$.

While giving only a highlight of key points in [K-W] and [K-S] that are most relevant to us, we should remark that, in addition to further quantum-field-theoretical issues on the gauge theory side, there is also a gravity side of the story that was studied in [K-W] and [K-S].[8]

Azumaya structure on D-branes and its tests.

In D(1) [L-Y1], D(2) [L-L-S-Y], D(3) [L-Y2] and the current work D(4), we illuminate the Azumaya geometry as a key feature of the geometry on D-brane world-volumes in the algebro-geometric category. These four together center around the very remark of Polchinski:

> ([Po: vol. 1, Sec. 8.7, p. 272]) "*For the collective coordinate X^μ, however, the meaning is mysterious: the collective coordinates for the embedding of n D-branes in space-time are now enlarged to $n \times n$ matrices. This 'noncommutative geometry' has proven to play a key role in the dynamics of D-branes, and there are conjectures that it is an important hint about the nature of space-time.*",

[7] See also [Arg: Chapter 3] and [Te: Chapter 9].

[8] See [A-G-M-O-O] and [Stra] for a review with more emphasis on respectively the gravity and the gauge theory side in the correspondence; e.g. [G-K], [K-N] for developments between [K-W] and [K-S]; and e.g. [D-K-S] for a more recent study.

which was taken as a guiding question as to what a D-brane is in this project, cf. [L-Y1: Sec. 2.2]. D(2), D(3), and the current D(4) are meant to give more explanations of the highlight [L-Y1: Sec. 4.5]. In this consecutive series of four, we learned that :

Lesson 0.1 [Azumaya structure on D-branes]. *This "enhancement to $n \times n$ matrices" Polchinski alluded to says even more fundamentally the nature of D-branes themselves, i.e. the Azumaya structure thereupon. This structure gives them the power to detect the nature of space-time. We also learned that Azumaya structures on D-branes and morphisms therefrom can be used to reproduce/explain several stringy/brany phenomena of stringy or quantum-field-theoretical origin that are very surprising/mysterious at a first mathematical glance.*

This is a basic test to ourselves to believe that Azumaya structures play a special role in understanding/desccribing D-branes in string theory. Having said this, we should however mention that D-brane remains a very complicated object and the Azumaya structure addressed here is only a part of it. Further issues are investigated in separate works.

Convention. Standard notations, terminology, operations, facts in (1) physics aspects of strings and D-branes; (2) algebraic geometry; (3) toric geometry can be found respectively in (1) [Po], [Jo]; (2) [Ha]; (3) [Fu].

- *Noncommutative algebraic geometry* is a very technical topic. For the current work, [Art] of Artin, [K-R] of Kontsevich and Rosenberg, and [leB1] of Le Bruyn are particularly relevant. See [L-Y1: References] for more references.

Outline.

1 D-branes in an affine noncommutative space.

We recall definitions and notions in [L-Y1] that are needed for the current work. Readers are referred to ibidem for more details and references. See also [L-L-S-Y] and [L-Y2] for further explanations and examples.

Affine noncommutative spaces and their morphisms.

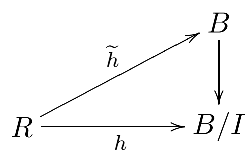
An *affine noncommutative space* over $\mathbb{C}$ is meant to be a "space" $Space\,R$ that is associated to an associative unital $\mathbb{C}$-algebra R. In general, it can be tricky to truly realize $Space\,R$ as a set of points with a topology in a natural/functorial way. However,"geometric" notions can still be pursued - despite not knowing what $Space\,R$ really is - via imposing the fundamental geometry/algebra ansatz:

- [*geometry = algebra*] The correspondence $R \leftrightarrow Space\,R$ gives a contravariant equivalence between the category $\mathcal{A}lg_{\mathbb{C}}$ of associative unital $\mathbb{C}$-algebras and the category $\mathcal{A}ffine\mathcal{S}pace_{\mathbb{C}}$ of "affine noncommutative spaces" over $\mathbb{C}$.

For example,

Definition 1.1. [smooth affine noncommutative space]. ([C-Q: Sec. 3], [K-R: Sec. 1.1.4].) An affine noncommutative space $Space\,R$ over $\mathbb{C}$ is said to be *smooth* if the associative unital $\mathbb{C}$-algebra R is finitely generated and satisfies the following property:

- (*lifting property for nilpotent extensions*) for any $\mathbb{C}$-algebra S, two-sided nilpotent ideal $I \subset R$ (i.e. $I = BIB$ and $I^n = 0$ for $n >> 0$), and $\mathbb{C}$-algebra homomorphism $h : R \to B/I$, there exists an $\mathbb{C}$-algebra homomorphism $\widetilde{h} : R \to S$ such that the diagram



commutes. Here $B \to B/I$ is the quotient map.

The following two classes of smooth affine noncommutative spaces are used in this work.

Example 1.2. [noncommutative affine space]. ([K-R: Sec. 2: Example (E1)].) The *noncommutative affine n-space $N\mathbb{A}^n := Space\,(\mathbb{C}\langle\, \xi_1\,, \cdots, \xi_n \,\rangle)$ over* $\mathbb{C}$ is smooth. Here $\mathbb{C}\langle\, \xi_1\,, \cdots, \xi_n \,\rangle$ is the associative unital $\mathbb{C}$-algebra freely generated by the elements in the set $\{\, \xi_1\,, \cdots, \xi_n \,\}$.

Example 1.3. [Azumaya-type noncommutative space]. ([C-Q: Sec. 5 and Proposition 6.2], [K-R: Sec. 1.2, Examples (E2) and (C4)].) Let $M_r(R)$ be the $\mathbb{C}$-algebra of $r \times r$-matrices with entries in a commutative regular $\mathbb{C}$-algebra R. Then the *Azumaya-type noncommutative space* $Space\,M_r(R)$ is smooth (over $\mathbb{C}$). Furthermore, it is also smooth over $Spec\,R$.

As a consequence of the Geometry/Algebra Ansatz, a *morphism* $\varphi : X = Space\,R \to Y = Space\,S$ is defined contravariantly to be a $\mathbb{C}$-algebra homomorphism $\varphi^\sharp : S \to R$. The *image*, denoted $Im\,\varphi$ or $\varphi(X)$, of X under φ is defined to be $Space\,(S/Ker\,\varphi^\sharp)$. The latter is canonically included in Y via the morphism $\iota : \varphi(X) \hookrightarrow Y$ defined by the $\mathbb{C}$-algebra quotient-homomorphism $\iota^\sharp : S \to S/Ker\,\varphi^\sharp$. This extends what is done in Grothendieck's theory of (commutative) schemes. The benefit of thinking a morphism between affine noncommutative spaces this way is actually *two* folds:

(1) *As a functor of point*: The space $X = \mathit{Space}\,R$ defines a functor

$$\begin{array}{rcccc} h_X & : & \mathcal{A}\!f\!f\!ine\mathcal{S}pace_{\mathbb{C}} & \longrightarrow & \mathcal{S}et^{\circ} \\ & & Y & \longmapsto & \mathit{Mor}\,(Y, X)\,; \end{array}$$

i.e. a functor

$$\begin{array}{rcccc} h_R & : & \mathcal{A}lg_{\mathbb{C}} & \longrightarrow & \mathcal{S}et \\ & & S & \longmapsto & \mathit{Hom}\,(R, S)\,. \end{array}$$

Here $\mathcal{S}et$ is the category of sets, $\mathcal{S}et^{\circ}$ its opposite category, and $\mathit{Hom}\,(R,S)$ is the set of $\mathbb{C}$-algebra-homomorphisms.

(2) *As a probe*: $X = \mathit{Space}\,R$ defines another functor

$$\begin{array}{rcccc} g_X & : & \mathcal{A}\!f\!f\!ine\mathcal{S}pace_{\mathbb{C}} & \longrightarrow & \mathcal{S}et \\ & & Y & \longmapsto & \mathit{Mor}\,(X, Y)\,; \end{array}$$

i.e. a functor

$$\begin{array}{rcccc} g_R & : & \mathcal{A}lg_{\mathbb{C}} & \longrightarrow & \mathcal{S}et^{\circ} \\ & & S & \longmapsto & \mathit{Hom}\,(S, R)\,. \end{array}$$

Aspect (1) is by now standard in algebraic geometry. It allows one to define the various *local geometric properties* of a "space" via algebra-homomorphisms; for example, Definition 1.1. It suggests one to think of X as a sheaf over $\mathcal{A}\!f\!f\!ine\mathcal{S}pace_{\mathbb{C}}$. Thus, after the notion of *coverings* and *gluings* is selected, it allows one to extend the notion of a noncommutative space to that of a "*noncommutative stack*". Aspect (2) is especially akin to our thought on D-branes. It says, in particular, that the geometry of $X = \mathit{Space}\,R$ can be revealed through an $\mathbb{C}$-subalgebra of R.

Example 1.4. [**Azumaya point**]**.** Consider the *Azumaya point of rank* r: $\mathit{Space}\,M_r(\mathbb{C})$. Its only two-sided prime ideal is (0), the zero ideal. Thus, naively, one would expect $\mathit{Space}\,M_r(\mathbb{C})$ to behave like a point with an Artin $\mathbb{C}$-algebra as its function ring. *However*, for example, from the $\mathbb{C}$-algebra monomorphism $\times^r\mathbb{C} \hookrightarrow M_r(\mathbb{C})$ with image the diagonal matrices in $M_r(\mathbb{C})$, one sees that $\mathit{Space}\,M_r(\mathbb{C})$ - which is topologically a one-point set if one adopts its interpretation as $\mathit{Spec}\,M_r(\mathbb{C})$ - can dominate $\amalg_r \mathit{Spec}\,\mathbb{C}$ - which is topologically a disjoint union of r-many points -. Furthermore, consider, for example, the morphism $\varphi : \mathit{Space}\,M_r(\mathbb{C}) \to \mathbb{A}^1 = \mathit{Spec}\,\mathbb{C}[z]$ defined by $\varphi^{\sharp} : \mathbb{C}[z] \to M_r(\mathbb{C})$ with $\varphi^{\sharp}(z) = m$ that is diagonalizable with r distinct eigenvalues $\lambda_1, \cdots, \lambda_r$. Then $\mathit{Im}\,\varphi$ is a collection of r-many $\mathbb{C}$-points on $\mathbb{A}^1$, located at $z = \lambda_1, \cdots, \lambda_r$ respectively. In other words, the Azumaya noncommutativity cloud $M_r(\mathbb{C})$ over the seemingly one-point space $\mathit{Space}\,M_r(\mathbb{C})$ can really "split and condense" to a collection of concrete geometric points! Cf. FIGURE 1-1. See [L-Y1: Sec. 4.1] for more examples. Such phenomenon generalizes to Azumaya schemes; in particular, see [L-L-S-Y] for the case of Azumaya curves.

Definition 1.5. [surrogate associated to morphism]. Given $X = \mathit{Space}\,R$, let $R' \hookrightarrow R$ be a $\mathbb{C}$-subalgebra of R. Then, the space $X' := \mathit{Space}\,R'$ is called a *surrogate* of X. By definition, there is a built-in dominant morphism $X \to X'$, defined by the inclusion $R' \hookrightarrow R$. Given a morphism $\varphi : \mathit{Space}\,R \to \mathit{Space}\,S$ defined by $\varphi^{\sharp} : S \to R$, then $\mathit{Space}\,R_{\varphi}$, where R_{φ} is the image $\varphi^{\sharp}(S)$ of S in R, is called the *surrogate of* X *associated to* φ.

As Example 1.4 illustrates, commutative surrogates may be used to manifest/reveal the hidden geometry of a noncommutative space.

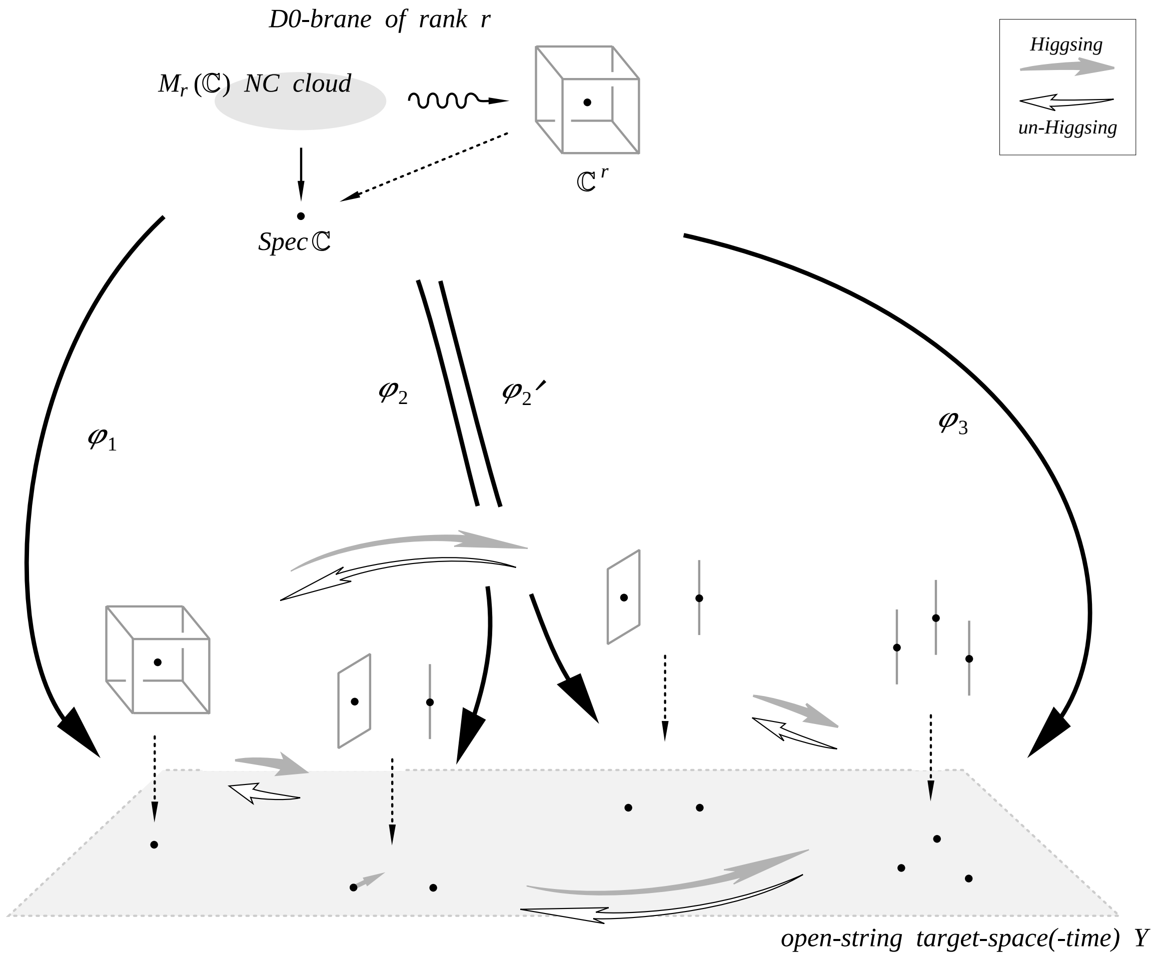


FIGURE 1-1. ([L-L-S-Y: FIGURE 2-1-1].) Despite that $\mathit{Space}\, M_r(\mathbb{C})$ may look only one-point-like, under morphisms the Azumaya "noncommutative cloud" $M_r(\mathbb{C})$ over $\mathit{Space}\, M_r(\mathbb{C})$ can "split and condense" to various schemes with a rich geometry. The latter schemes can even have more than one component. The Higgsing/un-Higgsing behavior of the Chan-Paton module of D0-branes on Y occurs due to the fact that when a morphism $\varphi : \mathit{Space}\, M_r(\mathbb{C}) \to Y$ deforms, the corresponding push-forward $\varphi_*\mathbb{C}^r$ of the fundamental module $\mathbb{C}^r$ on $\mathit{Space}\, M_r(\mathbb{C})$ can also change/deform. These features generalize to morphisms from Azumaya schemes to Y. Here, a module over a scheme is indicated by a dotted arrow ┈┈➢ .

Definition 1.6. [push-forward of module]. Given a morphism $\varphi : X = Space\, R \to Y = Space\, S$, defined by $\varphi^\sharp : S \to R$, and a (left) R-module M, the *push-forward* of M from X to Y under φ, in notation $\varphi_* M$ or ${}_S M$ when φ is understood, is defined to be M as a (left) S-module via $\varphi^\sharp$. Since $Ker\,\varphi^\sharp \cdot M = 0$, we say that the S-module $\varphi_* M$ on Y is *supported* on $\varphi(X) \subset Y$.

In particular, any R-module M on $X = Space\, R$ has a push-forward on any surrogate of X.

D-branes in an affine noncommutative space à la Polchinski-Grothendieck Ansatz.

A *D-brane* is geometrically a locus in space-time that serves as the boundary condition for open strings.[9] Through this, open strings dictate also the fields and their dynamics on D-branes. In particular, when a collection of D-branes are stacked together, the fields on the D-brane that govern the deformation of the brane are enhanced to matrix-valued, cf. Polchinski in [Po: vol. I, Sec. 8.7]. This open-string-induced phenomenon on D-branes, when re-read from Grothendieck's contravariant equivalence between the category of geometries and the category of algebras, says that D-brane world-volume carries an Azumaya-type noncommutative structure. I.e.

- *Polchinski-Grothendieck Ansatz*: D-brane has a geometry that is generically locally associated to algebras of the form $M_r(R_0)$, where R_0 is an $\mathbb{R}$-algebra.

See [L-Y1: Sec. 2.2] for detailed explanations.

For this work, we will be restricting ourselves to affine situations in noncommutative algebraic geometry with R_0 a commutative Noetherian $\mathbb{C}$-algebra. Thus:

Definition 1.7. [affine D-brane in affine target]. A *D-brane* (or *D-brane world-volume*) in an affine noncommutative space $Y = Space\, S$ is a triple that consists of

- a $\mathbb{C}$-algebra R that is isomorphic to $M_r(R_0)$ for an R_0,
- a (left) generically simple R-module M, which has rank r as an R_0-module,
- a morphism $\varphi : Space\, R \to Y$, defined by a $\mathbb{C}$-algebra-homomorphism $\varphi^\sharp : S \to R$.

We will write $\varphi : (Space\, R, M) \to Y$ for simplicity of notations. $\varphi(X) = Im\,\varphi$ is called the *image-brane* on Y. M is called the *fundamental module* on $Space\, R$ and the push-forward $\varphi_* M$ is called the *Chan-Paton module* on the image-brane $\varphi(X)$.

Definition/Example 1.8. [D0-brane as morphism from Azumaya point with fundamental module]. A *D0-brane of length r* on an affine noncommutative space $Y = Space\, S$ is given by a morphism $\varphi : (Space\, End(V), V) \to Y$, where $V \simeq \mathbb{C}^r$. In other words, a D0-brane on Y is given by

- a finite-dimensional $\mathbb{C}$-vector space V and a $\mathbb{C}$-algebra-homomorphism: $\varphi^\sharp : S \to End(V)$.

[9] This is how one would think of a D-brane to begin with. Later development of string theory enlarges this picture considerably. See [L-Y1: References] to get a glimpse.

This is precisely a realization of a finite-dimensional $\mathbb{C}$-vector space V as an S-module.[10] A *morphism* from $\varphi_1 : (\mathit{Space}\,\mathit{End}\,(V_1), V_1) \to Y$ to $\varphi_2 : (\mathit{Space}\,\mathit{End}\,(V_2), V_2) \to Y$ is a $\mathbb{C}$-vector-space isomorphism $h : V_2 \stackrel{\sim}{\to} V_1$ such that the following diagram commutes

$$
\begin{array}{ccc}
\mathit{End}\,(V_1) & \xleftarrow{\ \varphi_1^\sharp\ } & S \\
{\scriptstyle h}\uparrow & \swarrow{\scriptstyle \varphi_2^\sharp} & \\
\mathit{End}\,(V_2) & &
\end{array} \quad .
$$

Here, the h-induced isomorphism $\mathit{End}\,(V_2) \stackrel{\sim}{\to} \mathit{End}\,(V_1)$ is also denoted by h. In other words, a morphism between φ_1 and φ_2 is an isomorphism of the corresponding V_1 and V_2 as S-modules.

It follows from the above definition/example that the moduli stack $\mathfrak{M}_r^{\mathrm{D0}}(Y)$ of D0-branes of length r on $Y = \mathit{Space}\,S$ has an atlas given by the *representation scheme* $\mathit{Rep}\,(S, M_r(\mathbb{C}))$ that parameterizes all $\mathbb{C}$-algebra-homomorphisms $S \to M_r(\mathbb{C})$. The latter commutative scheme serves also as the moduli space of morphisms $\mathit{Space}\,M_r(\mathbb{C}) \to Y$ with $M_r(\mathbb{C})$ treated as fixed. From [K-R] and [leB1], one expects that noncommutative geometric structures/properties of $Y = \mathit{Space}\,S$ are reflected in properties/structures of the discrete family of commutative schemes $\mathit{Rep}\,(S, M_r(\mathbb{C}))\,,\ r \in \mathbb{Z}_{>0}$. This anticipation from noncommutative algebraic geometry rings hand in hand with the stringy philosophy to use D-branes as a probe to the nature of space-time!

2 Deformations of a conifold via an Azumaya probe.

Using a toric setup for a conifold that is meant to match Klebanov-Witten [K-W], we discuss how an Azumaya probe "sees" deformations of the conifold in a way that resembles Klebanov-Strassler [K-S].

A toric setup for the standard local conifold.

The standard local conifold $Y = \mathit{Spec}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4))$ can be given an affine toric variety description as follows. Let $N = \oplus_{i=1}^4 \mathbb{Z}e_i$ be the rank 4 lattice and Δ be the fan in N that consists of the single non-strongly convex polyhedral cone $\sigma = \oplus_{i=1}^6 \mathbb{R}_{\ge 0} v_i$ in $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$, where

$$
\begin{array}{llll}
v_1 = e_1\,, & v_2 = e_2\,, & v_3 = e_3\,, & v_4 = -e_1 + e_2 + e_3\,, \\
v_5 = e_1 - e_2 - e_3 + e_4\,, & v_6 = -v_5 = -e_1 + e_2 + e_3 - e_4\,. & &
\end{array}
$$

Let $M = \mathit{Hom}\,(N, \mathbb{Z})$ be the dual lattice of N, with the dual basis $\{e_1^*, e_2^*, e_3^*, e_4^*\}$. Then, the dual cone $\sigma^\vee$ of σ is given by $\mathit{Span}_{\mathbb{R}_{\ge 0}}\{e_1^* + e_2^*, e_3^* + e_4^*, e_1^* + e_3^*, e_2^* + e_4^*\} \subset M_{\mathbb{R}}$. This determines a commutative semigroup

$$
S_\sigma = \sigma^\vee \cap M = \mathit{Span}_{\mathbb{Z}_{\ge 0}}\{e_1^* + e_2^*, e_3^* + e_4^*, e_1^* + e_3^*, e_2^* + e_4^*\}
$$

[10] Thus, a D0-brane on $\mathit{Space}\,S$ is precisely an S-module that is of finite dimension as a $\mathbb{C}$-vector space. Such a direct realization of a D-brane as a module on a target-space is a special feature for D0-branes. For high dimensional D-branes, such modules on the target-space give only a subclass of D-branes that describe solitonic branes in space-time.

with generators $e_1^*+e_2^*$, $e_3^*+e_4^*$, $e_1^*+e_3^*$, $e_2^*+e_4^*$. The corresponding group-algebra

$$\mathbb{C}[S_\sigma] \;=\; \mathbb{C}[\,\xi_1\xi_2\,,\,\xi_3\xi_4\,,\,\xi_1\xi_3\,,\,\xi_2\xi_4\,] \;\subset\; \mathbb{C}[\xi_1,\xi_2,\xi_3,\xi_4]\,,$$

where $\xi_i=\exp(e_i^*)$, $i=1,\,2,\,3,\,4$, defines then the conifold

$$Y \;=\; U_\sigma \;=\; \mathit{Spec}\,(\mathbb{C}[S_\sigma]) \;=\; \mathit{Spec}\,(\mathbb{C}[z_1,z_2,z_3,z_4]/(z_1z_2-z_3z_4))\,,$$

where

$$z_1 \;=\; \xi_1\xi_2\,,\quad z_2 \;=\; \xi_3\xi_4\,,\quad z_3 \;=\; \xi_1\xi_3\,,\quad z_4 \;=\; \xi_2\xi_4\,.$$

Note that built into this construction is the morphism

$$\mathbb{A}^4_{[\xi_1,\xi_2,\xi_3,\xi_4]} \;:=\; \mathit{Spec}\,(\mathbb{C}[\xi_1,\xi_2,\xi_3,\xi_4]) \;\longrightarrow\; Y \;\hookrightarrow\; \mathbb{A}^4_{[z_1,z_2,z_3,z_4]} \;:=\; \mathit{Spec}\,(\mathbb{C}[z_1,z_2,z_3,z_4])\,,$$

where the first morphism is surjective.

An Azumaya probe to a noncommutative space and its commutative descent.

Guided by [K-W] and [K-S], where ξ_i's here play the role of scalar component of chiral superfields involved in ibidem, consider the noncommutative space

$$\begin{aligned}\Xi \;&:=\; \mathit{Space}\,(R_\Xi)\\ &:=\; \mathit{Space}\left(\frac{\mathbb{C}\langle\,\xi_1,\xi_2,\xi_3,\xi_4\,\rangle}{([\xi_1\xi_3,\xi_2\xi_4]\,,\,[\xi_1\xi_3,\xi_1\xi_4]\,,\,[\xi_1\xi_3,\xi_2\xi_3]\,,\,[\xi_2\xi_4,\xi_1\xi_4]\,,\,[\xi_2\xi_4,\xi_2\xi_3]\,,\,[\xi_1\xi_4,\xi_2\xi_3])}\right)\,,\end{aligned}$$

where $\mathbb{C}\langle\xi_1,\xi_2,\xi_3,\xi_4\rangle$ is the associative unital $\mathbb{C}$-algebra generated by $\{\xi_1,\xi_2,\xi_3,\xi_4\}$, $(\cdots)$ in the denominator is the two-sided ideal generated by $\cdots$, and $[\,\bullet\,,\,\bullet'\,]$ is the commutator. Here, $\mathit{Space}\,(\,\bullet\,)$ is the would-be space associated to the ring $\bullet$. We do not need its detail as all we need are morphisms between spaces which can be contravariantly expressed as ring-homomorphisms. By construction, the scheme-morphism $\mathbb{A}^4_{[\xi_1,\xi_2,\xi_3,\xi_4]}\to\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$, whose image is Y, extends to a morphism

$$\pi^\Xi \;:\; \Xi \;\longrightarrow\; \mathbb{A}^4_{[z_1,z_2,z_3,z_4]}\,,$$

whose image is now the whole $\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$. The underlying ring-homomorphism is given by

$$\begin{array}{cccccc}\pi^{\Xi,\sharp} & : & \mathbb{C}[z_1,z_2,z_3,z_4] & \longrightarrow & R_\Xi & \\ & & z_1 & \longmapsto & \xi_1\xi_3 & \\ & & z_2 & \longmapsto & \xi_2\xi_4 & \\ & & z_3 & \longmapsto & \xi_1\xi_4 & \\ & & z_4 & \longmapsto & \xi_2\xi_3 & .\end{array}$$

Consider a D0-brane moving on the conifold Y via the chiral superfields. In terms of Polchinski-Grothendieck Ansatz, this is realized by the descent of morphisms $\widetilde{\varphi}:\mathit{Space}\,M_1(\mathbb{C})=\mathit{Spec}\,\mathbb{C}\to\Xi$ to $\varphi:\mathit{Space}\,M_1(\mathbb{C})=\mathit{Spec}\,\mathbb{C}\to Y$ by the specification of ring-homomorphisms

$$\widetilde{\varphi}^\sharp \;:\; \xi_1 \;\longmapsto\; a_1\,;\quad \xi_2 \;\longmapsto\; a_2\,;\quad \xi_3 \;\longmapsto\; b_1\,;\quad \xi_4 \;\longmapsto\; b_2\,.$$

The corresponding

$$\varphi^\sharp \;:\; z_1 \;\longmapsto\; a_1b_1\,;\quad z_2 \;\longmapsto\; a_2b_2\,;\quad z_3 \;\longmapsto\; a_1b_2\,;\quad z_4 \;\longmapsto\; a_2b_1$$

gives a morphism $\varphi:\mathit{Spec}\,\mathbb{C}\to Y$, i.e. a $\mathbb{C}$-point on the conifold Y.

Deformations of the conifold via an Azumaya probe: descent of noncommutative superficially-infinitesimal deformations.

We now consider what happens if we add a D0-brane to the conifold point of Y. This D0-brane together with the D0-brane probe is the image of a morphism from the Azumaya point $\mathit{Space}\, M_2(\mathbb{C})$ to Y. Thus we should consider morphisms $\widetilde{\varphi} : \mathit{Space}\, M_2(\mathbb{C}) \to \Xi$ of noncommutative spaces and their descent φ on related commutative spaces.

Definition 2.1. [superficially infinitesimal deformation]. Given finitely-presented associative unital rings, $R = \langle r_1, \ldots, r_m \rangle / \sim$ and S, and a ring-homomorphism $h : R \to S$. A *superficially infinitesimal deformation* of h *with respect to the generators* $\{r_1, \ldots, r_m\}$ *of* R is a ring-homomorphism $h_\varepsilon : R \to S$ such that $h_\varepsilon(r_i) = h(r_i) + \varepsilon_i$ with $\varepsilon_i^2 = 0$, for $i = 1, \ldots, m$.

Remark 2.2. [*commutative* S]. Note that when S is commutative, a superficially infinitesimal deformation of $h_\varepsilon : R \to S$ is an infinitesimal deformation of h in the sense that $h_\varepsilon(r) = h(r) + \varepsilon_r$ with $(\varepsilon_r)^2 = 0$, for all $r \in R$. This is no longer true for general noncommutative S.

To begin, consider the diagram of morphisms of spaces

$$\begin{array}{ccccc}
\mathit{Space}\, M_2(\mathbb{C}) & \xrightarrow{\quad\widetilde{\varphi}\quad} & & & \Xi = \mathit{Space}\, R_\Xi \\
\Vert & & & & \big\downarrow{\scriptstyle \pi^\Xi} \\
\mathit{Space}\, M_2(\mathbb{C}) & \xrightarrow{\quad\varphi\quad} & Y & \hookrightarrow & \mathbb{A}^4
\end{array}$$

given by ring-homomorphisms

$$\begin{array}{ccccc}
M_2(\mathbb{C}) & \xleftarrow{\qquad\qquad\widetilde{\varphi}^\sharp\qquad\qquad} & & & R_\Xi \\
\Vert & & & & \big\uparrow{\scriptstyle \pi^{\Xi,\sharp}} \\
M_2(\mathbb{C}) & \xleftarrow{\quad\varphi^\sharp\quad} & \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) & \twoheadleftarrow & \mathbb{C}[z_1, z_2, z_3, z_4]
\end{array}$$

with

$$A_1\,;\;\; A_2\,;\;\; B_1\,;\;\; B_2 \xleftarrow{\qquad\qquad\widetilde{\varphi}^\sharp\qquad\qquad} \xi_1\,;\;\; \xi_2\,;\;\; \xi_3\,;\;\; \xi_4$$

$$\begin{array}{ccccc}
& & & & \xi_1\xi_3\,;\;\; \xi_2\xi_4\,;\;\; \xi_1\xi_4\,;\;\; \xi_2\xi_3 \\
& & & & \big\uparrow{\scriptstyle \pi^{\Xi,\sharp}} \\
A_1B_1\,;\;\; A_2B_2\,;\;\; A_1B_2\,;\;\; A_2B_1 & \xleftarrow{\;\varphi^\sharp\;} & \overline{z_1}\,;\;\; \overline{z_2}\,;\;\; \overline{z_3}\,;\;\; \overline{z_4} & \longleftarrow & z_1\,;\;\; z_2\,;\;\; z_3\,;\;\; z_4
\end{array}$$

where

$$A_1 \;=\; \begin{bmatrix} a_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad A_2 \;=\; \begin{bmatrix} a_2 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_1 \;=\; \begin{bmatrix} b_1 & 0 \\ 0 & 0 \end{bmatrix}, \qquad B_2 \;=\; \begin{bmatrix} b_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

The image D-brane $\varphi(\mathit{Space}\, M_2(\mathbb{C}))$ is supported on a subscheme Z of Y associated to the ideal

$$\mathit{Ker}\,\varphi \;=\; \left\{ \begin{array}{ll} (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}) \cap (\overline{z_1} - a_1 b_1, \overline{z_2} - a_2 b_2, \overline{z_3} - a_1 b_2, \overline{z_4} - a_2 b_1) & \\ \qquad\qquad\qquad\qquad \text{if the tuple } (a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1) \neq (0,0,0,0)\,, & \\ (\overline{z_1}, \overline{z_2}, \overline{z_3}, \overline{z_4}) \quad \text{if the tuple } (a_1 b_1, a_2 b_2, a_1 b_2, a_2 b_1) = (0,0,0,0)\,. & \end{array} \right.$$

The former corresponds to two simple non-coincident D0-branes, each with Chan-Paton module $\mathbb{C}$, on the conifold Y with one of them sitting at the conifold point $\mathbf{0}$ and the other sitting at the $\mathbb{C}$-point with the coordinate tuple $(a_1b_1,\, a_2b_2,\, a_1b_2,\, a_2b_1)$ while the latter corresponds to coincident D0-branes at $\mathbf{0}$ with the Chan-Paton module enhanced to $\mathbb{C}^2$ at $\mathbf{0}$. In both situations, the support Z of the D-brane is reduced. This is the transverse-to-the-effective-space-time part of the D3-brane setting in [K-W] and [K-S].

Consider now a superficially infinitesimal deformation of $\widetilde{\varphi}$ given by:

$$\begin{array}{ccc}
\mathit{Space}\,M_2(\mathbb{C}) & \xrightarrow{\widetilde{\varphi}_{(\delta_1,\delta_2,\eta_1,\eta_2)}} & \Xi = \mathit{Space}\,R_\Xi \\
M_2(\mathbb{C}) & \xleftarrow{\widetilde{\varphi}^\sharp_{(\delta_1,\delta_2,\eta_1,\eta_2)}} & R_\Xi \\
A_1\,;\;\; A_2\,;\;\; B_1\,;\;\; B_2 & \longleftarrow\!\shortmid & \xi_1\,;\;\xi_2\,;\;\xi_3\,;\;\;\xi_4
\end{array}$$

where

$$A_1 \;=\; \begin{bmatrix} a_1 & \delta_1 \\ 0 & 0 \end{bmatrix}, \qquad A_2 \;=\; \begin{bmatrix} a_2 & \delta_2 \\ 0 & 0 \end{bmatrix}, \qquad B_1 \;=\; \begin{bmatrix} b_1 & 0 \\ \eta_1 & 0 \end{bmatrix}, \qquad B_2 \;=\; \begin{bmatrix} b_2 & 0 \\ \eta_2 & 0 \end{bmatrix}.$$

Should $\mathit{Space}\,M_2(\mathbb{C})$ be a commutative space, this would give only an infinitesimal deformation of φ. However, $\mathit{Space}\,M_2(\mathbb{C})$ is not a commutative space and, hence, the naive anticipation above could fail. Indeed, the descent $\varphi_{(\delta_1,\delta_2,\eta_1,\eta_2)}$ of $\widetilde{\varphi}_{(\delta_1,\delta_2,\eta_1,\eta_2)}$ is given by

$$\begin{array}{ccc}
\mathit{Space}\,M_2(\mathbb{C}) & \xrightarrow{\varphi_{(\delta_1,\delta_2,\eta_1,\eta_2)}} & \mathbb{A}^4 \\
M_2(\mathbb{C}) & \xleftarrow{\varphi^\sharp_{(\delta_1,\delta_2,\eta_1,\eta_2)}} & \mathbb{C}[z_1,z_2,z_3,z_4] \\
A_1B_1\,;\;\; A_2B_2\,;\;\; A_1B_2\,;\;\; A_2B_1 & \longleftarrow\!\shortmid & z_1\,;\;\; z_2\,;\;\; z_3\,;\;\; z_4\,,
\end{array}$$

i.e.

$$\begin{bmatrix} a_1b_1+\delta_1\eta_1 & 0 \\ 0 & 0 \end{bmatrix};\; \begin{bmatrix} a_2b_2+\delta_2\eta_2 & 0 \\ 0 & 0 \end{bmatrix};\; \begin{bmatrix} a_1b_2+\delta_1\eta_2 & 0 \\ 0 & 0 \end{bmatrix};\; \begin{bmatrix} a_2b_1+\delta_2\eta_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\longleftarrow\!\shortmid \qquad z_1\,;\;\; z_2\,;\;\; z_3\,;\;\; z_4\,.$$

The image $Z := \varphi_{(\delta_1,\delta_2,\eta_1,\eta_2)}\,(\mathit{Space}\,M_2(\mathbb{C}))$ of the Azumaya point $\mathit{Space}\,M_2(\mathbb{C})$ under $\varphi_{(\delta_1,\delta_2,\eta_1,\eta_2)}$ remains a 0-dimensional reduced scheme, consisting of either two $\mathbb{C}$-points - with one of them at $\mathbf{0}$ - or $\mathbf{0}$ alone. However,

$$z_1z_2 - z_3z_4 \;=\; \begin{vmatrix} z_1 & z_3 \\ z_4 & z_2 \end{vmatrix} \;=\; \begin{vmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{vmatrix} \cdot \begin{vmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{vmatrix}$$

vanishes if and only if either $\begin{vmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{vmatrix}$ or $\begin{vmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{vmatrix}$ is 0. In other words, while the image $\varphi_{(\delta_1,\delta_2,\eta_1,\eta_2)}\,(\mathit{Space}\,M_2(\mathbb{C}))$ still contains the conifold-point $\mathbf{0}$ in Y, as a whole it may longer lie completely even in any infinitesimal neighborhood of the conifold Y in $\mathbb{A}^4$. I.e.:

Lemma 2.3. [deformation from descent of superficially infinitesimal deformation]. *The descent $\varphi_{(\delta_1,\delta_2,\eta_1,\eta_2)}$ of a superficially infinitesimal deformation of $\widetilde{\varphi}$ can truly deform φ. Thus, an appropriate choice of a subspace of the space of morphisms $\widetilde{\varphi}_{(\,\bullet\,)} : \mathit{Space}\,M_2(\mathbb{C}) \to \Xi$ can descend to give a space of morphisms $\varphi_{(\,\bullet\,)} : \mathit{Space}\,M_2(\mathbb{C}) \to \mathbb{A}^4$ that is parameterized by a deformed conifold Y'.*

This realizes a deformed conifold as a moduli space of morphisms from an Azumaya point and is the reason why the Azumaya probe can see a deformation of the conifold Y from the viewpoint of Polchinski-Grothendieck Ansatz. FIGURE 2-1.

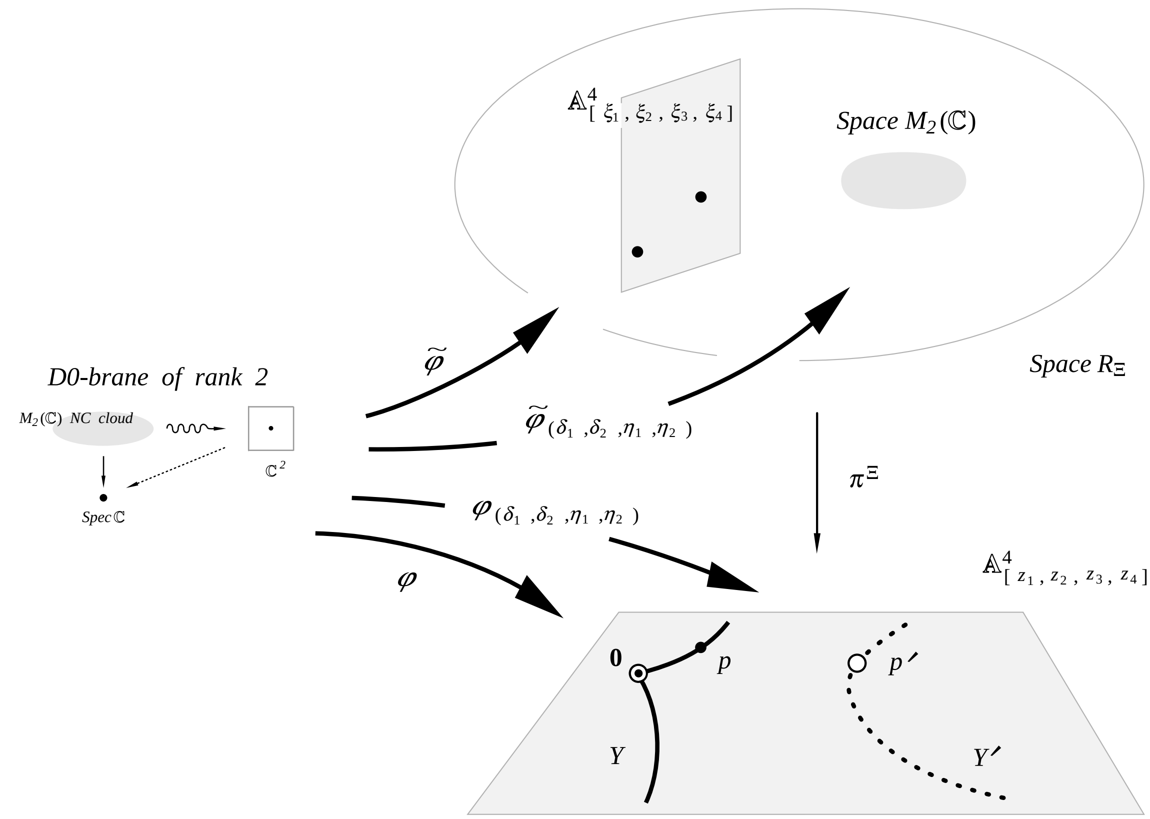


FIGURE 2-1. A generic superficially infinitesimal deformation $\widetilde{\varphi}_{(\delta_1,\delta_2,\eta_1,\eta_2)}$ of $\widetilde{\varphi}$ has a noncommutative image $\simeq$ *Space* $M_2(\mathbb{C})$. It then descends to $\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$ and becomes a pair of $\mathbb{C}$-points on $\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$. One of the points is the conifold singularity $\mathbf{0} = V(z_1, z_2, z_3, z_4) \in Y$ and the other is the point $p' = V(\, z_1 - a_1b_1 - \delta_1\eta_1\, ,\, z_2 - a_2b_2 - \delta_2\eta_2\, ,\, z_3 - a_1b_2 - \delta_1\eta_2\, ,\, z_4 - a_2b_1 - \delta_2\eta_1\,)$ *off* Y (generically). Through such deformations, any $\mathbb{C}$-point on $\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$ can be reached. Thus, one can realizes a deformation Y' of Y in $\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$ by a subvariety in $Rep\,(R_\Xi, M_2(\mathbb{C}))$. This is the Azumaya-geometry origin of the phenomenon in Klebanov-Strassler [K-S] that a trapped D-brane sitting on the conifold singularity may give rise to a deformation of the moduli space of SQFT on the D3-brane probe, turning a conifold to a deformed conifold. Our D0-brane here corresponds to the internal part of the effective-space-time-filling D3-brane world-volume of [K-S].

Remark 2.4. [*generalization*]. This phenomenon can be generalized beyond a conifold. In particular, recall that an A_n-singularity on a complex surface is also a toric singularity. Similar mechanism/discussion can be applied to deform a transverse A_n-singularity via morphisms from an Azumaya probe.

Deformations of the conifold via an Azumaya probe: details.

We now give an explicit construction that realizes Lemma 2.3. For convenience[11], we will take $\mathit{Space}\,M_2(\mathbb{C})$ as fixed, and is equipped with the defining fundamental (left) $M_2(\mathbb{C})$-module $\mathbb{C}^2$. Then, the space $\mathit{Mor}^a(\mathit{Space}\,M_2(\mathbb{C}),\Xi)$ of admissible morphisms of the form $\widetilde{\varphi}_{(\bullet)}$ in the previous theme is naturally realized as a subscheme $\mathit{Rep}^a(R_\Xi, M_2(\mathbb{C}))$ of the representation scheme $\mathit{Rep}\,(R_\Xi, M_2(\mathbb{C}))$ that parameterizes elements in $\mathit{Mor}_{\mathbb{C}\text{-}\mathit{Alg}}(R_\Xi, M_2(\mathbb{C}))$. From the previous discussion,

$$\begin{aligned}\mathit{Rep}^a(R_\Xi, M_2(\mathbb{C})) \;&=\; \mathit{Spec}\,\mathbb{C}[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]\\ &=:\; \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]} \;=\; \mathbb{A}^4_{[a_1,a_2,\delta_1,\delta_2]}\times_{\mathbb{C}}\mathbb{A}^4_{[b_1,b_2,\eta_1,\eta_2]}\,.\end{aligned}$$

Consider also the space $\mathit{Mor}^a(\mathit{Space}\,M_2(\mathbb{C}),\mathbb{A}^4)$ of morphisms from Azumaya point to $\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$ with the associated $\mathbb{C}$-algebra-homomorphism of the form

$$z_1\longmapsto\left[\begin{array}{cc}c_1&0\\0&0\end{array}\right],\;\; z_2\longmapsto\left[\begin{array}{cc}c_2&0\\0&0\end{array}\right],\;\; z_3\longmapsto\left[\begin{array}{cc}c_3&0\\0&0\end{array}\right],\;\; z_4\longmapsto\left[\begin{array}{cc}c_4&0\\0&0\end{array}\right].$$

Denote the associated representation scheme by

$$\mathit{Rep}^a(\mathbb{C}[z_1,z_2,z_3,z_4], M_2(\mathbb{C}))\,,\quad \text{which is}\quad \mathit{Spec}\,\mathbb{C}[c_1,c_2,c_3,c_4] \;=:\; \mathbb{A}^4_{[c_1,c_2,c_3,c_4]}\,.$$

The $\mathbb{C}$-algebra homomorphism $\pi^{\Xi,\sharp}:\mathbb{C}[z_1,z_2,z_3,z_4]\to R_\Xi$ induces a morphism of representation schemes

$$\pi_{\mathit{Rep}}\;:\;\mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}\;\longrightarrow\;\mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$$

with $\pi^\sharp_{\mathit{Rep}}$ given in a matrix form by

$$\pi^\sharp_{\mathit{Rep}}\;:\;\left[\begin{array}{cc}c_1&c_3\\c_4&c_2\end{array}\right]\longmapsto\left[\begin{array}{cc}a_1&\delta_1\\a_2&\delta_2\end{array}\right]\cdot\left[\begin{array}{cc}b_1&b_2\\\eta_1&\eta_2\end{array}\right].$$

Lemma 2.5. [enough superficially infinitesimally deformed morphisms].

$$\pi_{\mathit{Rep}}\;:\;\mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}\;\longrightarrow\;\mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$$

is surjective.

There are three homeomorphism classes of fibers of π_{Rep} over a closed point of $\mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$, depending on the rank of $\left[\begin{array}{cc}c_1&c_3\\c_4&c_2\end{array}\right]$.

Lemma 2.6. [topological type of fibers of π_{Rep}]. *Let $C^3_{[c_1,c_2,c_3,c_4]}$ be the subvariety of $\mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$ associated to the ideal $(c_1c_2-c_3c_4)$. Similarly, for $C^3_{[a_1,a_2,\delta_1,\delta_2]}$ and $C^3_{[b_1,b_2,\eta_1,\eta_2]}$. Then:*

(0) Over $\mathbf{0}$, the fiber is given by $\mathbb{A}^4_{[a_1,a_2,\delta_1,\delta_2]}\cup\mathbb{A}^4_{[b_1,b_1,\eta_1,\eta_2]}\cup\Pi^5$, where Π^5 is a 5-dimensional irreducible affine scheme meeting $\mathbb{A}^4_{[a_1,a_2,\delta_1,\delta_2]}\cup\mathbb{A}^4_{[b_1,b_2,\eta_1,\eta_2]}$ along $C^3_{[a_1,a_2\,\delta_1,\delta_2]}\cup C^3_{[b_1,b_2\,\eta_1,\eta_2]}$.

[11] If $\mathit{Space}\,M_2(\mathbb{C})$ is not fixed, then one studies Artin stacks that parameterizes morphisms in question from $\mathit{Space}\,M_2(\mathbb{C})$ to $\mathit{Space}\,R_\Xi$, the conifold Y, and $\mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$ respectively. The discussion given here is then on an atlas of the stack in question.

(1) Over a closed point of $C^3_{[c_1,c_2,c_3,c_4]} - \{\mathbf{0}\}$*, the fiber is the union* $\Pi^4_1 \cup \Pi^4_2$ *of two irreducible 4-dimensional affine scheme meeting at a deformed conifold.*

(2) Over a closed point of $\mathbb{A}^4_{[c_1,c_2,c_3,c_4]} - C^3_{[c_1,c_2,c_3,c_4]}$*, the fiber is isomorphic to* $\mathbb{A}^4_{[a_1,a_2\,\delta_1,\delta_2]} - C^3_{[a_1,a_2\,\delta_1,\delta_2]} \simeq \mathbb{A}^4_{[b_1,b_2\,\eta_1,\eta_2]} - C^3_{[b_1,b_2\,\eta_1,\eta_2]}$.

The lemma follows from a straightforward computation.[12] Note that the fundamental group as an analytic space is given by

$$\begin{aligned}\pi_1(\mathbb{A}^4_{[c_1,c_2,c_3,c_4]} - C^3_{[c_1,c_2,c_3,c_4]}) &\simeq \pi_1(\mathbb{A}^4_{[a_1,a_2,\delta_1,\delta_2]} - C^3_{[a_1,a_2,\delta_1,\delta_2]}) \\ &\simeq \pi_1(\mathbb{A}^4_{[b_1,b_2,\eta_1,\eta_2]} - C^3_{[b_1,b_2,\eta_1,\eta_2]}) \;\simeq\; \mathbb{Z}\end{aligned}$$

and that the smooth bundle-morphism

$$\pi_{Rep} \;:\; \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]} - \pi_{Rep}^{-1}(C^3_{[c_1,c_2,c_3,c_4]}) \longrightarrow \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} - C^3_{[c_1,c_2,c_3,c_4]}$$

exhibits a monodromy behavior which resembles that of a Dehn twist.

The map $\pi_{Rep} : \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]} \to \mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$ admits sections, i.e. morphism $s : \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} \to \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}$ such that $\pi_{Rep} \circ s =$ the identity map on $\mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$.

Example 2.7. [section of π_{Rep}]. Let $t \in GL_2(\mathbb{C})$, then a simple family of sections of π_{Rep}

$$s_t \;:\; \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} \longrightarrow \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}$$

is given compactly in a matrix expression by (with t also in its defining 2×2-matrix form)

$$s_t^\sharp \;:\; \left(\begin{bmatrix} a_1 & \delta_1 \\ a_2 & \delta_2 \end{bmatrix}, \begin{bmatrix} b_1 & b_2 \\ \eta_1 & \eta_2 \end{bmatrix} \right) \longmapsto \left(\begin{bmatrix} c_1 & c_3 \\ c_4 & c_2 \end{bmatrix} \cdot t^{-1}, t \right).$$

Through any section $s : \mathbb{A}^4_{[c_1,c_2,c_3,c_4]} \to \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]}$, one can realize $Y' \amalg \{\mathbf{0}\}$, where Y' is a deformation of the conifold Y in $\mathbb{A}^4 = \mathbb{A}^4_{[z_1,z_2,z_3,z_4]}$ and $\mathbf{0}$ is the singular point on Y, as the descent of a family of superficially infinitesimal deformations of morphisms from Azumaya point to the noncommutative space Ξ. In string theory words,

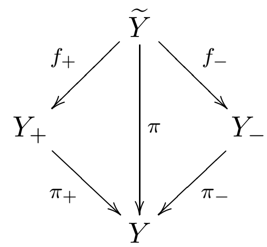
- *deformations of a conifold via a D-brane probe are realized by turning on D-branes at the singularity appropriately; the conifold is deformed and becomes smooth while leaving the trapped D-branes at the singularity behind.*

Cf. FIGURE 2-1.

[12] It is very instructive to think of the fibration $\pi_{Rep} : \mathbb{A}^8_{[a_1,a_2,\delta_1,\delta_2,b_1,b_2,\eta_1,\eta_2]} \to \mathbb{A}^4_{[c_1,c_2,c_3,c_4]}$ as defining a one-matrix-parameter family of "matrix nodal curves" in the sense of noncommutative geometry.

3 Resolutions of a conifold via an Azumaya probe.

In this section, we consider resolutions of the conifold $Y = \mathit{Spec}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4))$ from the viewpoint of an Azumaya probe. Recall the following diagram of resolutions of Y from blow-ups of Y:



,

where

- $\pi : \widetilde{Y} = \mathit{Bl}_{V(I)} Y = \mathit{Proj}\,(\oplus_{i=0}^{\infty} I^i) \rightarrow Y$ with $I = (z_1, z_2, z_3, z_4)$,
- $\pi_+ : Y_+ = \mathit{Bl}_{V(I_+)} Y = \mathit{Proj}\,(\oplus_{i=0}^{\infty} I_+^i) \rightarrow Y$ with $I_+ = (z_1, z_3)$, and
- $\pi_- : Y_- = \mathit{Bl}_{V(I_-)} Y = \mathit{Proj}\,(\oplus_{i=0}^{\infty} I_-^i) \rightarrow Y$ with $I_- = (z_1, z_4)$

are blow-ups of Y along the specified subschemes $V(\,\bullet\,)$ associated respectively to the ideals I, I_+, and I_- of $\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$ as given. Here, we set $I^0_{(\pm)} = \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4)$. Let $\mathbf{0} = V(z_1, z_2, z_3, z_4)$ be the singular point of Y. Then the exceptional locus in each case is given respectively by $\pi^{-1}(\mathbf{0}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\pi_+^{-1}(\mathbf{0}) \simeq \mathbb{P}^1$, and $\pi_-^{-1}(\mathbf{0}) \simeq \mathbb{P}^1$; Y_+ and Y_- as schemes/Y are related by a flop; and the restriction of birational morphisms $f_\pm : \widetilde{Y} \rightarrow Y_\pm$ to $\pi^{-1}(\mathbf{0})$ corresponds to the projections of $\mathbb{P}^1 \times \mathbb{P}^1$ to each of its two factors.

D-brane probe resolutions of a conifold via the Azumaya structure.

An atlas for the stack of morphisms from $\mathit{Space}\, M_2(\mathbb{C})$ to Y is given by the representation scheme $\mathit{Rep}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$ with the $PGL_2(\mathbb{C})$-action induced from the $GL_2(\mathbb{C})$-action on the fundamental module $\mathbb{C}^2$. For convenience, we will also call this a $GL_2(\mathbb{C})$-action on $\mathit{Rep}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$. Let

$$W \;=\; \mathit{Rep}^{\,\mathit{singleton}}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$$

be the subscheme of $\mathit{Rep}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$ that parameterizes D0-branes $\varphi : (\mathit{Spec}\,\mathbb{C}, M_2(\mathbb{C}), \mathbb{C}^2) \rightarrow Y$ with $(\mathit{Im}\,\varphi)_{\mathrm{red}}$ a single $\mathbb{C}$-point on Y. Explicitly, W is the image scheme of

$$GL_2(\mathbb{C}) \times W_{ut} \;\longrightarrow\; \mathit{Rep}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$$

where

$$\begin{aligned} W_{ut} \;&=\; \left\{ \rho : \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4) \rightarrow M_2(\mathbb{C}) \;\middle|\; \rho(z_i) \text{ is of the form } \begin{bmatrix} a_i & \varepsilon_i \\ 0 & a_i \end{bmatrix} \right\} \\ &\subset\; \mathit{Rep}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C})) \end{aligned}$$

and the morphism $\longrightarrow$ is from the restriction of the $GL_2(\mathbb{C})$-group on $\mathit{Rep}\,(\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1 z_2 - z_3 z_4), M_2(\mathbb{C}))$. Using this notation, as a scheme,

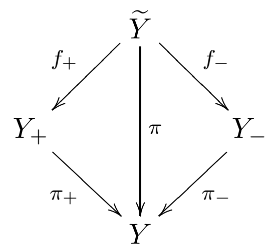
$$\begin{aligned} W_{ut} \;&=\; \mathit{Spec}\,(\mathbb{C}[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]/(\, a_1 a_2 - a_3 a_4\,,\; a_2\varepsilon_1 + a_1\varepsilon_2 - a_4\varepsilon_3 - a_3\varepsilon_4\,)) \\ &\subset\; \mathit{Spec}\,(\mathbb{C}[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]) \quad =: \quad \mathbb{A}^8_{[a_1, a_2, a_3, a_4, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4]} \,. \end{aligned}$$

Imposing the trivial $GL_2(\mathbb{C})$-action on Y, then by construction, there is a natural $GL_2(\mathbb{C})$-equivariant morphism

$$\pi^W \;:\; W \longrightarrow Y$$

defined by $\pi^{W,\sharp}(z_i) = \frac{1}{2}\, Tr\,\rho(z_i) = a_i$ in the above notation. This is the morphism that sends a $\varphi : (Spec\,\mathbb{C}, M_2(\mathbb{C}), \mathbb{C}^2) \to Y$ under study to $(Im\,\varphi)_{\mathrm{red}} \in Y$.

Lemma 3.1. [Azumaya probe to conifold singularity]. *There exists $GL_2(\mathbb{C})$-invariant subschemes $\widetilde{Y}'$, Y'_+, and Y'_- of W such that their geometric quotient $\widetilde{Y}'/GL_2(\mathbb{C})$, $Y'_+/GL_2(\mathbb{C})$, $Y'_-/GL_2(\mathbb{C})$ under the $GL_2(\mathbb{C})$-action exist and are isomorphic to $\widetilde{Y}$, Y_+, and Y_- respectively. Furthermore, under these isomorphisms, the restriction of $\pi^W : W \to Y$ to $\widetilde{Y}'$, Y'_+, and Y'_- descends to morphisms from the quotient spaces $\widetilde{Y}'/GL_2(\mathbb{C})$, $Y'_+/GL_2(\mathbb{C})$, $Y'_-/GL_2(\mathbb{C})$ to Y that realize the resolution diagram*



of Y at the beginning of this section.

It is in the sense of the above lemma we say that

- *an Azumaya point of rank ≥ 2 and hence a D-brane probe of multiplicity ≥ 2 can "see" all the three different resolutions of the conifold singularity.*

It should also be noted that Lemma 3.1 is a special case of a more general statement that reflects the fact that the stack of morphisms from Azumaya points to a (general, possibly singular, Noetherian) scheme Y is a generalization of the notion of jet-schemes of Y. Cf. [L-Y2: Figure 0-1, caption].

An explicit construction of $\widetilde{Y}'$, Y'_+, and Y'_-.

An explicit construction of $\widetilde{Y}'$, Y'_+, and Y'_-, and hence the proof of Lemma 3.1, follows from a lifting-to-W of an affine atlas of $Proj\,(\oplus_{i=0}^{\infty} I^i_{(\pm)})$.

To construct $\widetilde{Y}'$, recall that $I = (z_1, z_2, z_3, z_4)$. An affine atlas of $\widetilde{Y}$ is given by the collection

$$U^{(z_i)} = Spec\,((\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0) \;\simeq\; \left\{ \begin{array}{ll} Spec\,(\mathbb{C}[z_1, u_2, u_3, u_4]/(u_2 - u_3 u_4)) \simeq \mathbb{A}^3_{[z_1, u_3, u_4]} & \text{for } i = 1\,; \\ Spec\,(\mathbb{C}[u_1, z_2, u_3, u_4]/(u_1 - u_3 u_4)) \simeq \mathbb{A}^3_{[z_2, u_3, u_4]} & \text{for } i = 2\,; \\ Spec\,(\mathbb{C}[u_1, u_2, z_3, u_4]/(u_1 u_2 - u_4)) \simeq \mathbb{A}^3_{[u_1, u_2, z_3]} & \text{for } i = 3\,; \\ Spec\,(\mathbb{C}[u_1, u_2, u_3, z_4]/(u_1 u_2 - u_3)) \simeq \mathbb{A}^3_{[u_1, u_2, z_4]} & \text{for } i = 4\,. \end{array} \right.$$

Here, $z_i \in I$ has grade 1 and $(\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0$ is the grade-0 component of the graded algebra $(\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]$. Each $U^{(z_i)}$ is equipped with a built-in morphism $\pi^{(i)} : U^{(z_i)} \to Y$ in such a way that, when all four are put together, they glue to give the resolution $\pi : \widetilde{Y} \to Y$.

Consider the lifting $\{\pi^{(i)\,\prime} : U^{(z_i)} \to W \,|\, i = 1,2,3,4\}$ of the atlas $\{\pi^{(i)} : U^{(z_i)} \to Y \,|\, i = 1,2,3,4\}$ of $\widetilde{Y}$ that is given by the lifting $\{\pi^{(i)\,\prime} : U^{(z_i)} \to W_{ut} \subset W \,|\, i = 1,2,3,4\}$ defined by

$$\begin{array}{lclcll}
\pi^{(1)\,\prime,\sharp} & : & a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_1\,,\, z_1u_2\,,\, z_1u_3\,,\, z_1u_4\,,\, 1\,,\, u_2\,,\, u_3\,,\, u_4 & \text{respectively}\,,\\
\pi^{(2)\,\prime,\sharp} & : & a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_2u_1\,,\, z_2\,,\, z_2u_3\,,\, z_2u_4\,,\, u_1\,,\, 1\,,\, u_3\,,\, u_4 & \text{respectively}\,,\\
\pi^{(3)\,\prime,\sharp} & : & a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_3u_1\,,\, z_3u_2\,,\, z_3\,,\, z_3u_4\,,\, u_1\,,\, u_2\,,\, 1\,,\, u_4 & \text{respectively}\,,\\
\pi^{(4)\,\prime,\sharp} & : & a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_4u_1\,,\, z_4u_2\,,\, z_4u_3\,,\, z_4\,,\, u_1\,,\, u_2\,,\, u_3\,,\, 1 & \text{respectively}\,.
\end{array}$$

$\pi^{(i)\,\prime}$, $i = 1, 2, 3, 4$, are now embeddings into W with the property that for any geometric point $p \in U^{(z_i)} \times_{\widetilde{Y}} U^{(z_j)}$, $\pi^{(i)\,\prime}(p)$ and $\pi^{(j)\,\prime}(p)$ lies in the same $GL_2(\mathbb{C})$-orbit in W. In other words, up to the pointwise $GL_2(\mathbb{C})$-action, they are gluable. Let $\widetilde{Y}'$ be the image scheme of the morphism

$$GL_2(\mathbb{C}) \times (U^{(z_1)} \amalg U^{(z_2)} \amalg U^{(z_3)} \amalg U^{(z_4)}) \longrightarrow W$$

via $\pi^{(1)\,\prime} \amalg \pi^{(2)\,\prime} \amalg \pi^{(3)\,\prime} \amalg \pi^{(4)\,\prime}$ and the $GL_2(\mathbb{C})$-action on W. Then it follows that the geometric quotient $\widetilde{Y}'/GL_2(\mathbb{C})$ exists and is equipped with a built-in isomorphism $\widetilde{Y}'/GL_2(\mathbb{C}) \xrightarrow{\sim} \widetilde{Y}$, as schemes over Y, through the defining embeddings $U^{(z_i)} \hookrightarrow \widetilde{Y}$, $i = 1, 2, 3, 4$.

For Y'_+, recall that $I_+ = (z_1, z_3)$. An affine atlas of Y_+ is given by the collection

$$U_+^{(z_i)} = \mathit{Spec}\,((\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0) \simeq \left\{ \begin{array}{ll} \mathit{Spec}\,(\mathbb{C}[z_1, z_2, u_3, z_4]/(z_2 - z_4u_3)) \simeq \mathbb{A}^3_{[z_1,u_3,z_4]} & \text{for } i = 1\,;\\ \mathit{Spec}\,(\mathbb{C}[u_1, z_2, z_3, z_4]/(z_2u_1 - z_4)) \simeq \mathbb{A}^3_{[u_1,z_2,z_3]} & \text{for } i = 3\,. \end{array}\right.$$

Each $U_+^{(z_i)}$ is equipped with a built-in morphism $\pi_+^{(i)} : U_+^{(z_i)} \to Y$ in such a way that, when both are put together, they glue to give the resolution $\pi_+ : Y_+ \to Y$.

Consider the lifting $\{\pi_+^{(i)\,\prime} : U_+^{(z_i)} \to W \,|\, i = 1,3\}$ of the atlas $\{\pi_+^{(i)} : U_+^{(z_i)} \to Y \,|\, i = 1,3\}$ of Y_+ that is given by the lifting $\{\pi_+^{(i)\,\prime} : U_+^{(z_i)} \to W_{ut} \subset W \,|\, i = 1,3\}$ defined by

$$\begin{array}{lclcll}
\pi_+^{(1)\,\prime,\sharp} & : & a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_1\,,\, z_4u_3\,,\, z_1u_3\,,\, z_4\,,\, 1\,,\, 0\,,\, u_3\,,\, 0 & \text{respectively}\,,\\
\pi_+^{(3)\,\prime,\sharp} & : & a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_3u_1\,,\, z_2\,,\, z_3\,,\, z_2u_1\,,\, u_1\,,\, 0\,,\, 1\,,\, 0 & \text{respectively}\,.
\end{array}$$

The pair, $\pi_+^{(1)\,\prime}$ and $\pi_+^{(3)\,\prime}$, are now embeddings into W that, as in the case of $\widetilde{Y}$, are gluable up to the pointwise $GL_2(\mathbb{C})$-action. Same construction as in the case of $\widetilde{Y}$ gives then a $GL_2(\mathbb{C})$-invariant subscheme Y'_+ of W whose geometric quotient $Y'_+/GL_2(\mathbb{C})$ exists and is equipped with a built-in isomorphism $Y'_+/GL_2(\mathbb{C}) \xrightarrow{\sim} Y_+$ as schemes over Y.

For Y'_-, recall that $I_- = (z_1, z_4)$. The construction is identical to that in the case of Y_+ after relabelling. An affine atlas of Y_- is given by the collection

$$U_-^{(z_i)} = \mathit{Spec}\,((\oplus_{j=0}^{\infty} I^j)[z_i^{-1}]_0) \simeq \left\{ \begin{array}{ll} \mathit{Spec}\,(\mathbb{C}[z_1, z_2, z_3, u_4]/(z_2 - z_3u_4)) \simeq \mathbb{A}^3_{[z_1,z_3,u_4]} & \text{for } i = 1\,;\\ \mathit{Spec}\,(\mathbb{C}[u_1, z_2, z_3, z_4]/(z_2u_1 - z_3)) \simeq \mathbb{A}^3_{[u_1,z_2,z_4]} & \text{for } i = 4\,. \end{array}\right.$$

Each $U_-^{(z_i)}$ is equipped with a built-in morphism $\pi_-^{(i)} : U_-^{(z_i)} \to Y$ in such a way that, when both are put together, they glue to give the resolution $\pi_- : Y_- \to Y$.

Consider the lifting $\{\pi_-^{(i)\,\prime} : U_-^{(z_i)} \to W \,|\, i = 1,4\}$ of the atlas $\{\pi_-^{(i)} : U_-^{(z_i)} \to Y \,|\, i = 1,4\}$ of Y_- that is given by the lifting $\{\pi_-^{(i)\,\prime} : U_-^{(z_i)} \to W_{ut} \subset W \,|\, i = 1,4\}$ defined by

$$\begin{array}{llll} \pi_-^{(1)\,\prime,\sharp} & : \; a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_1\,,\, z_3u_4\,,\, z_3\,,\, z_1u_4\,,\, 1\,,\, 0\,,\, 0\,,\, u_4 \quad \text{respectively}\,, \\ \pi_-^{(4)\,\prime,\sharp} & : \; a_1\,,\, a_2\,,\, a_3\,,\, a_4\,,\, \varepsilon_1\,,\, \varepsilon_2\,,\, \varepsilon_3\,,\, \varepsilon_4 & \longmapsto & z_4u_1\,,\, z_2\,,\, z_2u_1\,,\, z_4\,,\, u_1\,,\, 0\,,\, 0\,,\, 1 \quad \text{respectively}\,. \end{array}$$

The pair, $\pi_-^{(1)\,\prime}$ and $\pi_-^{(4)\,\prime}$, are now embeddings into W that are gluable up to the pointwise $GL_2(\mathbb{C})$-action. Same construction as in the case of $\widetilde{Y}$ gives then a $GL_2(\mathbb{C})$-invariant subscheme $Y_-^\prime$ of W whose geometric quotient $Y_-^\prime/GL_2(\mathbb{C})$ exists and is equipped with a built-in isomorphism $Y_-^\prime/GL_2(\mathbb{C}) \stackrel{\sim}{\rightarrow} Y_-$ as schemes over Y.

This concludes the explicit construction.

Remark 3.2. [*lifting to jet-scheme*]. Note that there is a one-to-one correspondence between $GL_2(\mathbb{C})$-orbits in W and isomorphism classes of 0-dimensional torsion sheaves of length 2 on the conifold Y (i.e. the push-forward Chan-Paton sheaves on Y under associated morphisms from the Azumaya point $\mathit{Space}\, M_2(\mathbb{C})$ with the fundamental module $\mathbb{C}^2$) with connected support. Under this correspondence, the various special liftings-to-W in the construction above:

$$(\pi^{(1)\,\prime}\,,\, \pi^{(2)\,\prime}\,,\, \pi^{(3)\,\prime}\,,\, \pi^{(4)\,\prime})\,, \quad (\pi_+^{(1)\,\prime}\,,\, \pi_+^{(3)\,\prime})\,, \quad (\pi_-^{(1)\,\prime}\,,\, \pi_-^{(4)\,\prime})\,,$$

and the gluing property, up to the pointwise $GL_2(\mathbb{C})$-action, in each tuple follow from the underlying lifting property to the related jet-schemes, which is the total space of the tangent sheaf $\mathcal{T}_Y$ of Y in our case.

A comparison with resolutions via noncommutative desingularizations.

Consider the *conifold algebra* defined by[13]

$$\Lambda_c \;:=\; \frac{\mathbb{C}\langle\, \xi_1\,,\, \xi_2\,,\, \xi_3\,\rangle}{(\, \xi_1^2\xi_2 - \xi_2\xi_1^2\,,\, \xi_1\xi_2^2 - \xi_2^2\xi_1\,,\, \xi_1\xi_3 + \xi_3\xi_1\,,\, \xi_2\xi_3 + \xi_3\xi_2\,,\, \xi_3^2 - 1\,)}\,,$$

where the numerator is the associative unital $\mathbb{C}$-algebra generated by $\{\xi_1,\, \xi_2,\, \xi_3\}$ and the denominator is the two-sided ideal generated by the elements of $\mathbb{C}\langle \xi_1\, \xi_2,\, \xi_3\rangle$ as indicated.

Lemma 3.3. [center of Λ_c]. ([leB-S: Lemma 5.4].) *The* $\mathbb{C}$*-algebra monomorphism*

$$\begin{array}{lccc} \tau^\sharp \; : & \mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4) & \longrightarrow & \Lambda_c \\ & z_1 & \longmapsto & \xi_1^2 \\ & z_2 & \longmapsto & \xi_2^2 \\ & z_3 & \longmapsto & \frac{1}{2}(\xi_1\xi_2 + \xi_2\xi_1) \,+\, \frac{1}{2}(\xi_1\xi_2 - \xi_2\xi_1)\xi_3 \\ & z_4 & \longmapsto & \frac{1}{2}(\xi_1\xi_2 + \xi_2\xi_1) \,-\, \frac{1}{2}(\xi_1\xi_2 - \xi_2\xi_1)\xi_3 \end{array}$$

realizes $\mathbb{C}[z_1, z_2, z_3, z_4]/(z_1z_2 - z_3z_4)$ *as the center of* Λ_c*.*

Proposition 3.4. [representation variety of Λ_c]. ([leB-S: Proposition 5.7].) *The representation variety* $\mathit{Rep}\,(\Lambda_c, M_2(\mathbb{C}))$ *is a smooth affine variety with three disjoint irreducible components. Two of these components are a point. The third* $\mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))$ *has dimension* 6*.*

[13] The highlight here follows [leB-S] with some change of notations for consistency and mild rephrasings to link ibidem directly with us.

This implies[14] that Λ_c is a *smooth order* over $\mathbb{C}[z_1,z_2,z_3,z_4]/(z_1z_2 - z_3z_4)$ and, if one defines $\mathit{Spec}\,\Lambda_c$ to be the set of two-sided prime ideals of Λ_c with the Zariski topology, then the natural morphism

$$\mathit{Spec}\,\Lambda_c \;\longrightarrow\; \mathit{Spec}\,(\mathbb{C}[z_1,z_2,z_3,z_4]/(z_1z_2 - z_3z_4))$$

by intersecting a two-sided prime ideal of Λ_c with the center of Λ_c gives a *smooth noncommutative desingularization* of Y. ([leB-S: Proposition 5.7].)

Up to the conjugation by an element in $GL_2(\mathbb{C})$, a $\mathbb{C}$-algebra homomorphism $\rho:\Lambda_c \to M_2(\mathbb{C})$ can be put into one the following three forms: (In (1) and (2) below, 0 and Id are respectively the zero matrix and the identity matrix in $M_2(\mathbb{C})$.)

(1) $\rho(\xi_1)=0\,,\ \rho(\xi_2)=0\,,\ \rho(\xi_3)=\ \ Id;$

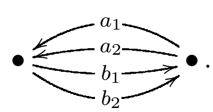
(2) $\rho(\xi_1)=0\,,\ \rho(\xi_2)=0\,,\ \rho(\xi_3)=-Id;$

(3)
$$\rho(\xi_1)\;=\;\begin{bmatrix} 0 & a_1\\ b_1 & 0\end{bmatrix},\quad \rho(\xi_2)\;=\;\begin{bmatrix} 0 & a_2\\ b_2 & 0\end{bmatrix},\quad \rho(\xi_3)\;=\;\begin{bmatrix} 1 & 0\\ 0 & -1\end{bmatrix}.$$

Form (1) and Form (2) correspond to the two point-components in $\mathit{Rep}\,(\Lambda_c, M_2(\mathbb{C}))$ and Form (3) corresponds to elements in $\mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))$. On the subvariety $\mathbb{A}^4_{[a_1,b_1,a_2,b_2]}$ of $\mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))$ that parameterizes ρ of the form (3), the $GL_2(\mathbb{C})$-action on $\mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))$ reduces to the $\mathbb{C}^*\times\mathbb{C}^*$-action

$$(a_1,b_1,a_2,b_2)\xrightarrow{\qquad (t_1,t_2)\qquad}(\,t_1t_2^{-1}a_1\,,\ t_1^{-1}t_2b_1\,,\ t_1t_2^{-1}a_2\,,\ t_1^{-1}t_2b_2\,)\,,$$

where $(t_1,t_2)\in\mathbb{C}^*\times\mathbb{C}^*$. The pair $(\rho(\xi_1),\rho(\xi_2))$ in Form (3) realizes this $\mathbb{A}^4_{[a_1,b_1,a_2,b_2]}$ as the representation variety of the quiver



Impose the trivial $GL_2(\mathbb{C})$-action on Y, then note that there is a natural $GL_2(\mathbb{C})$-equivariant morphism from $\mathit{Rep}\,(\Lambda_c, M_2(\mathbb{C}))$ to Y, as the composition

$$\mathbb{C}[z_1,z_2,z_3,z_4]/(z_1z_2 - z_3z_4)\ \stackrel{\tau^\sharp}{\longrightarrow}\ \Lambda_c\ \stackrel{\rho}{\longrightarrow}\ M_2(\mathbb{C})$$

has the form

$$z_i\;\longmapsto\;0\,,\quad i\;=\;1\,,\,2\,,\,3\,,\,4\,,$$

for ρ conjugate to Form (1) or Form (2);

$$z_1\;\longmapsto\;a_1b_1\,Id\,,\quad z_2\;\longmapsto\;a_2b_2\,Id\,,\quad z_3\;\longmapsto\;a_1b_2\,Id\,,\quad z_4\;\longmapsto\;a_2b_1\,Id$$

for ρ conjugate to Form (3).[15] One can now follow the setting of [Ki] to define the stable structures for the $GL_2(\mathbb{C})$-action on $\mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))$. There are two different choices, θ_+ and

[14]Readers are referred to [leB1] for a general study of the several notions involved in this paragraph. We do not need their details here.

[15]Note that when restricted to $\mathbb{A}^4_{[a_1,b_1,a_2,b_2]}\subset \mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))$, this is the morphism $\mathbb{A}^4_{[\xi_1,\xi_2,\xi_3,\xi_4]}\to Y$ in Sec. 2 after the substitution: a_1 (here) $\to \xi_1$ (there), $a_2\to\xi_2$, $b_1\to\xi_3$, $b_2\to\xi_4$.

θ_-, of such structures in the current case. The corresponding stable locus on the quiver variety $\mathbb{A}^4_{[a_1,b_1,a_2,b_2]}$ is given respectively by

$$\mathbb{A}^{4,\theta_+}_{[a_1,b_1,a_2,b_2]} \;=\; \mathbb{A}^4_{[a_1,b_1,a_2,b_2]} - V(b_1,b_2) \quad \text{and} \quad \mathbb{A}^{4,\theta_-}_{[a_1,b_1,a_2,b_2]} \;=\; \mathbb{A}^4_{[a_1,b_1,a_2,b_2]} - V(a_1,a_2)\,,$$

where $V(a_1,a_2)$ (resp. $V(b_1,b_2)$) is the subvariety of $\mathbb{A}^4_{[a_1,b_1,a_2,b_2]}$ associated to the ideal (a_1,a_2) (resp. (b_1,b_2)). The corresponding GIT quotients

$$\begin{array}{ccc} \mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))/\!/^{\theta_+}GL_2(\mathbb{C}) & & \mathit{Rep}^{\,0}(\Lambda_c, M_2(\mathbb{C}))/\!/^{\theta_-}GL_2(\mathbb{C}) \\ & {\scriptstyle \pi^{\theta_+}}\searrow \quad \swarrow{\scriptstyle \pi^{\theta_-}} & \\ & Y & \end{array}$$

recover

$$\begin{array}{ccc} Y_+ & & Y_- \\ & {\scriptstyle \pi_+}\searrow \quad \swarrow{\scriptstyle \pi_-} & \\ & Y & \end{array}$$

at the beginning of the section. See [leB-S], [leB2] for the mathematical detail and [Be], [B-L], [K-W] for the SQFT/stringy origin.

From the viewpoint of the Polchinski-Grothendieck Ansatz, *both* the Azumaya-type noncommutative structure on D-branes and a noncommutative structure over Y described by $\mathit{Space}\,\Lambda_c$ come into play in the above setting. As indicated by the explicit expression for $\rho\circ\tau^\sharp$ above, any morphism $\tilde{\varphi}:\mathit{Space}\,M_2(\mathbb{C})\rightarrow\mathit{Space}\,\Lambda_c$ has the property:

- The composition

$$\mathit{Space}\,M_2(\mathbb{C}) \xrightarrow{\;\tilde{\varphi}\;} \mathit{Space}\,\Lambda_c \xrightarrow{\;\tau\;} Y$$

  is a morphism $\varphi := \tilde{\varphi}\circ\tau$ from the Azumaya point $pt^{Az} = \mathit{Space}\,M_2(\mathbb{C})$ to Y with the associated surrogate $pt_\varphi \simeq \mathit{Spec}\,\mathbb{C}$.

Thus, the new ingredient of target-space noncommutativity comes into play as another key role toward resolutions of Y in the above setting while the generalized-jet-resolution-of-singularity picture in our earlier discussion disappears.

Remark 3.5. [*world-volume noncommutativity vs. target-space(-time) noncommutativity*]. Such a "trading" between a noncommutativity target and morphisms from Azumaya schemes to a commutative target suggests a partial duality between D-brane world-volume noncommutativity and target space(-time) noncommutativity.

FIGURE 3-1.

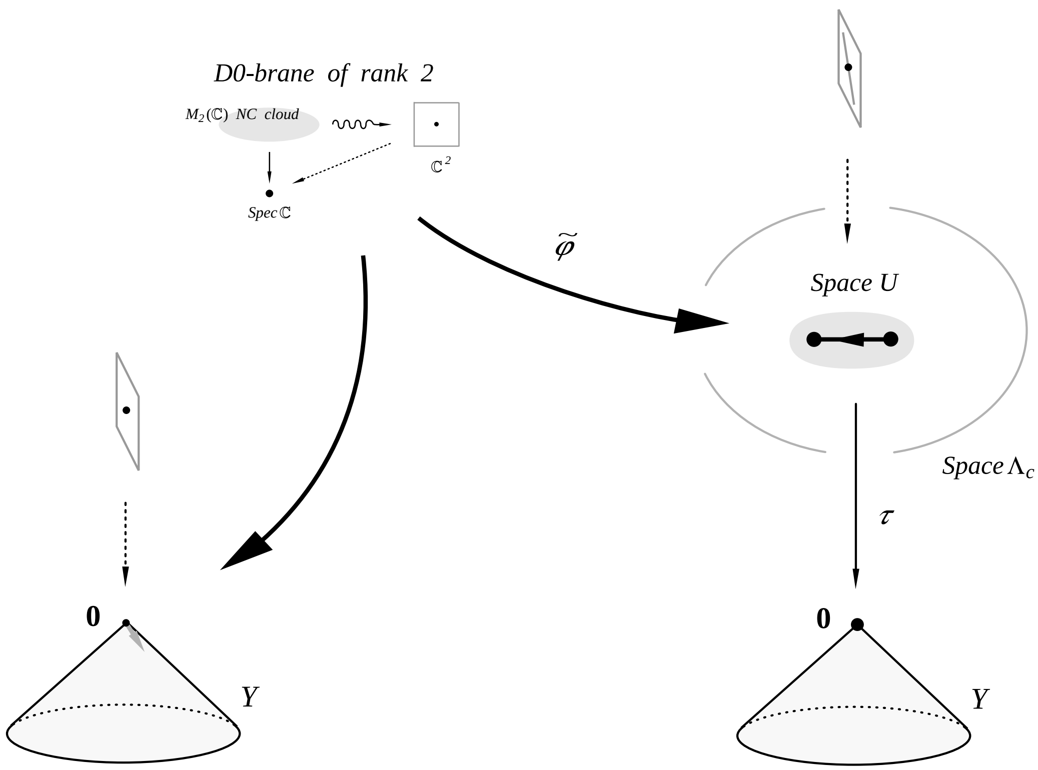


FIGURE 3-1. Trading of morphisms from $Space\,M_2(\mathbb{C})$ directly to the conifold Y with those to the noncommutative space $Space\,\Lambda_c$ over Y. Note that for generic $\rho \in Rep\,(\Lambda_c, M_2(\mathbb{C}))$ such that $\rho \circ \tau^\sharp = 0$, $\rho(\Lambda_c)$ is similar to the $\mathbb{C}$-subalgebra U of upper triangular matrices in $M_2(\mathbb{C})$. The noncommutative point $Space\,U$ is also smooth, with $Spec\,U$ consisting of two $\mathbb{C}$-points connected by a directed nilpotent bond. It is thus represented by a quiver $\bullet \longrightarrow \bullet$ in the figure. Furthermore, let $\widetilde{\varphi} : Space\,M_2(\mathbb{C}) \to Space\,\Lambda_c$ be the corresponding morphism. Then $\widetilde{\varphi}$ determines also a flag in the Chan-Paton module $\widetilde{\varphi}_*\mathbb{C}^2$ on the image D0-brane $Im\,\widetilde{\varphi}$. On the other hand, over a generic $p \neq \mathbf{0}$ on Y, the generic image of a $\widetilde{\varphi}'$ that maps to p after the composition with τ will be simply $Space\,M_2(\mathbb{C})$.

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