

**Further studies on
the notion of differentiable maps from Azumaya/matrix manifolds
I. The smooth case**

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Abstract

In this follow-up of our earlier two works D(11.1) (arXiv:1406.0929 [math.DG]) and D(11.2) (arXiv:1412.0771 [hep-th]) in the D-project, we study further the notion of a ‘differentiable map from an Azumaya/matrix manifold to a real manifold’. A conjecture is made that the notion of differentiable maps from Azumaya/matrix manifolds as defined in D(11.1) is equivalent to one defined through the contravariant ring-homomorphisms alone. A proof of this conjecture for the smooth (i.e. C^∞) case is given in this note. Thus, at least in the smooth case, our setting for D-branes in the realm of differential geometry is completely parallel to that in the realm of algebraic geometry, cf. arXiv:0709.1515 [math.AG] and arXiv:0809.2121 [math.AG]. A related conjecture on such maps to \mathbb{R}^n , as a C^k -manifold, and its proof in the C^∞ case is also given. As a by-product, a conjecture on a division lemma in the finitely differentiable case that generalizes the division lemma in the smooth case from Malgrange is given in the end, as well as other comments on the conjectures in the general C^k case. We remark that there are similar conjectures in general and theorems in the smooth case for the fermionic/super generalization of the notion.

Key words: D-brane; Azumaya manifold, matrix manifold, smooth map, ring-homomorphism, spectral subscheme; germ of differentiable functions, Malgrange Division Theorem, division lemma

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*Chien-Hao Liu dedicates this note to the memory of
Beina Yao (1981–2015).*

(From C.H.L.) Playing the flute, as well as other woodwind instruments, is really meant to sing through the piece of metal (or wood). This is why when Beina's singing came across my world by accident in late 2014, it immediately attracted and touched me. Her voice has a rich and solid timbre with the flexibility to cover a wide range usually only a mechanical musical instrument can reach without losing its elegance, cleanness, and fullness. Despite years of rigorous conservatory training in Chinese traditional folk music, her voice remains so natural and her singing looks so effortless. (An over-trained voice tends to become unhuman, which I never liked.) But she won't be that unique if this is all she got. What's truly inspiring and respectable is her brave twist of her path in music to follow her own heart and avoid a safe but otherwise mediocre life, her courage to battle with her cancer, and her noble and philanthropic legacy in the end of her life, donating what her body could offer to enable other people to see the world again. At this moment of another return-to-zero/origin in both the project and my life, her example rings particularly deep in my heart. Last but not the least, my respect and consolation to her parents, who reminded me of a musical family that influenced me forever. The major part of the current note was typed with the company of her songs.

0. Introduction and outline

The notion of differentiable maps from an Azumaya/matrix manifold (with a fundamental module) to a real manifold was developed in [L-Y2] (D(11.1)) as a natural mathematical language to describe D-branes as a dynamical/fundamental object in string theory. (See [Liu] for a review.) Its fermionic/super generalization was given in [L-Y3] (D(11.2)). At the first sight and in comparison with the setting in [L-Y1] (D(1)), the mathematical setting for D-branes in the realm of differential geometry look more involved/constrained. The core notion behind is the notion of ‘ C^k -admissible ring-homomorphisms’ from the C^k -function ring of a C^k -manifold to the Azumaya/matrix-function ring of an Azumaya/matrix C^k -manifold; ([L-Y2: Definition 5.1.2] (D(11.1), reviewed in Definition 1.2, Sec. 1 of the current note). The setup of this notion meant to deal with the technical issue that a noncommutative ring cannot be made a C^k -ring and was guided by the principle that any good notion of a ‘map’ should be accompanied by a corresponding natural notion of the ‘graph’ of the map.

In this follow-up of [L-Y2] (D(11.1)) and [L-Y3] (D(11.2)), we examine further the notion of a ‘differentiable map from an Azumaya/matrix manifold to a real manifold’. A conjecture is made that the notion of differentiable maps from Azumaya/matrix manifolds as defined in [L-Y2] (D(11.1)) is equivalent to one defined through the contravariant ring-homomorphisms alone. A proof of this conjecture for the smooth (i.e. C^∞) case is given in this note. Thus, at least in the smooth case, our setting for D-branes in the realm of differential geometry is completely parallel to that in the realm of algebraic geometry, cf. [L-Y1] (D(1)) and [L-L-S-Y] (D(2)). A related conjecture on such maps to \mathbb{R}^n , as a C^k -manifold, and its proof in the C^∞ case is also given. As a by-product, a conjecture on a division lemma in the finitely differentiable case that generalizes the division lemma in the smooth case from Malgrange is given in the end, as well as other comments on the conjectures in the general C^k case.

We remark that there are similar conjectures in general and theorems in the smooth case for the fermionic/super generalization of the notion.

Convention. References for standard notations, terminology, operations and facts in (1) algebraic geometry; (2) synthetic geometry, C^∞ -algebraic geometry; (3) D-branes can be found respectively in (1) [Ha]; (2) [Du1], [Du2], [Jo], [Ko], [M-R]; (3) [Po].

- For clarity, the *real line* as a real 1-dimensional manifold is denoted by \mathbb{R}^1 , while the *field of real numbers* is denoted by \mathbb{R} . Similarly, the *complex line* as a complex 1-dimensional manifold is denoted by \mathbb{C}^1 , while the *field of complex numbers* is denoted by \mathbb{C} .
- The inclusion ‘ $\mathbb{R} \hookrightarrow \mathbb{C}$ ’ is referred to the *field extension of \mathbb{R} to \mathbb{C}* by adding $\sqrt{-1}$, unless otherwise noted.
- The *complexification* of an \mathbb{R} -module M is denoted by $M^{\mathbb{C}}$ ($:= M \otimes_{\mathbb{R}} \mathbb{C}$).
- The *real n -dimensional vector spaces* $\mathbb{R}^{\oplus n}$ vs. the *real n -manifold* \mathbb{R}^n ; similarly, the *complex r -dimensional vector space* $\mathbb{C}^{\oplus r}$ vs. the *complex r -fold* \mathbb{C}^r .
- All C^k -manifolds, $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, are paracompact, Hausdorff, admitting a (locally finite) partition of unity, and embeddable into some \mathbb{R}^N as closed C^k -submanifolds. We adopt the *index convention for tensors* from differential geometry. In particular, the tuple coordinate functions on an n -manifold is denoted by, for example, (y^1, \dots, y^n) . However, no up-low index summation convention is used.

- ‘*differentiable*’ = k -times differentiable (i.e. C^k) for some $k \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$; ‘*smooth*’ = C^∞ ; C^0 = *continuous* by standard convention.
- $\text{Spec } R$ ($:= \{\text{prime ideals of } R\}$) of a commutative Noetherian ring R in algebraic geometry vs. $\text{Spec } R$ of a C^k -ring R ($:= \text{Spec }^{\mathbb{R}} R := \{C^k\text{-ring homomorphisms } R \rightarrow \mathbb{R}\}$).
- *morphism* between schemes in algebraic geometry vs. C^k -*map* between C^k -manifolds or C^k -schemes in differential topology and geometry or C^k -algebraic geometry.
- *matrix* m vs. manifold of *dimension* m .

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1 Conjectures on the notion of C^k -maps from Azumaya/matrix C^k -manifolds

Let

- X, Y be C^k -manifolds, $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, with their C^k -function ring denoted $C^k(X)$ and $C^k(Y)$ respectively; (and their structure sheaf \mathcal{O}_X and \mathcal{O}_Y respectively);
- E be a complex C^k vector bundle on X of rank r ; and $C^k(\text{End}_{\mathbb{C}}(E))$ be its endomorphism algebra.

Note that if an \mathbb{R} -subalgebra $S \subset C^k(\text{End}_{\mathbb{C}}(E))$ admits a C^k -ring structure, that structure is unique up to C^k -ring isomorphisms.

Definition 1.1. [weakly C^k -admissible ring-homomorphism]. A ring-homomorphism

$$\varphi^{\sharp} : C^k(Y) \longrightarrow C^k(\text{End}_{\mathbb{C}}(E))$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$ is said to be *weakly C^k -admissible* if the \mathbb{R} -subalgebra $\text{Im} \varphi^{\sharp}$ of $C^k(\text{End}_{\mathbb{C}}(E))$ admits a C^k -ring structure with respect to which φ^{\sharp} is a C^k -ring-homomorphism.

Definition 1.2. [C^k -admissible ring-homomorphism].

(Cf. [L-Y2: Definition 5.1.2] (D(11.1)).) A ring-homomorphism

$$C^k(\text{End}_{\mathbb{C}}(E)) \longleftarrow \varphi^{\sharp} \longrightarrow C^k(Y)$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$ is said to be *C^k -admissible* if it extends to the following commutative diagram of ring-homomorphisms

$$\begin{array}{ccc} C^k(\text{End}_{\mathbb{C}}(E)) & \xleftarrow{\varphi^{\sharp}} & C^k(Y) \\ \uparrow & \swarrow \tilde{\varphi}^{\sharp} & \downarrow \text{pr}_Y^{\sharp} \\ C^k(X) & \xrightarrow{\text{pr}_X^{\sharp}} & C^k(X \times Y) \end{array}$$

such that

- (1) $\tilde{\varphi}^{\sharp} : C^k(X \times Y) \rightarrow \text{Im}(\tilde{\varphi}^{\sharp})$ is a C^k -normal quotient (i.e. the C^k -ring structure on $C^k(X \times Y)$ descends to a C^k -ring structure on $\text{Im}(\tilde{\varphi}^{\sharp})$),
- (2) replacing $C^k(\text{End}_{\mathbb{C}}(E))$ with

$$A_{\varphi} = C^k(X) \langle \text{Im}(\varphi^{\sharp}) \rangle := \text{Im}(\tilde{\varphi}^{\sharp}) \subset C^k(\text{End}_{\mathbb{C}}(E))$$

with the C^k -ring structure induced from that of $C^k(X \times Y)$ by Condition (1), then

$$\begin{array}{ccc} A_{\varphi} & \xleftarrow{\varphi^{\sharp}} & C^k(Y) \\ \uparrow & \swarrow \tilde{\varphi}^{\sharp} & \downarrow \text{pr}_Y^{\sharp} \\ C^k(X) & \xrightarrow{\text{pr}_X^{\sharp}} & C^k(X \times Y) \end{array}$$

is a commutative diagram of C^k -ring homomorphisms.

Clearly, for a correspondence $\varphi^\sharp : C^k(Y) \rightarrow C^k(\text{End}_{\mathbb{C}}(E))$,

$$\begin{aligned} & \varphi^\sharp \text{ is a } C^k\text{-admissible ring-homomorphism over } \mathbb{R} \hookrightarrow \mathbb{C} \\ \implies & \varphi^\sharp \text{ is a weakly } C^k\text{-admissible ring-homomorphism over } \mathbb{R} \hookrightarrow \mathbb{C} \\ \implies & \varphi^\sharp \text{ is a ring-homomorphism over } \mathbb{R} \hookrightarrow \mathbb{C}. \end{aligned}$$

And φ^\sharp in Definition 1.2 is what we employed in [L-Y2] to define the notion of a C^k -map

$$\varphi : (X^{\text{Az}}, E) \longrightarrow Y,$$

following the spirit of Grothendieck's setting for modern (commutative) Algebraic Geometry. It resolves the issue that $C^k(\text{End}_{\mathbb{C}}(E))$, $r \geq 2$, can never be made a C^k -ring and at the same time makes the notion of the 'graph' of φ , a $\mathcal{O}_{X \times Y}^{\mathbb{C}}$ -module $\hat{\mathcal{E}}_\varphi$, naturally built into the definition of the differentiable map φ .

Conjecture 1.3. [C^k -map vs. ring-homomorphism]. *Let X and Y be C^k -manifolds and E be a complex C^k vector bundle of rank r on X . Given a correspondence*

$$\varphi^\sharp : C^k(Y) \longrightarrow C^k(\text{End}_{\mathbb{C}}(E)).$$

Then, the following three statements are equivalent:

- (1) φ^\sharp is a ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$.
- (2) φ^\sharp is a weakly C^k -admissible ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$.
- (3) φ^\sharp is a C^k -admissible ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$.

Thus, if justified, any of the φ^\sharp in Statements (1), (2), or (3) above can be used to define the notion of a C^k -map $\varphi : (X^{\text{Az}}, E) \rightarrow Y$; cf. [L-Y2: Sec. 5] (D(11.1)). The resulting notions are the same/equivalent.

Next, we recall the notion of 'nilpotency' in three situations and then bring forth the second conjecture of the current note.

Definition 1.4. [nilpotency]. We define the notion of *nilpotency* in three situations.

- (1) Let $a \in R$ be a nilpotent element of a ring (commutative or not). We say a has nilpotency $\leq l \in \mathbb{Z}_{\geq 1}$ if $a^l = 0$. The minimal such l is called the *nilpotency* of a .
- (2) Let R be a ring. We say that R has nilpotency $\leq l$ if $a^l = 0$ for all nilpotent elements of R . The minimal such l is called the *nilpotency* of R .
- (3) Let $m \in M_{r \times r}(\mathbb{C})$ be an $r \times r$ -matrix with entries in \mathbb{C} . We say that m has nilpotency $\leq l$ if each elementary Jordan block of m

$$\left[\begin{array}{cccc} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & \\ & & & \lambda \end{array} \right]_{l' \times l'},$$

with all entries not on the diagonal nor on the first upper off-diagonal being equal to zero, has the size $l' \leq l$. The minimal such l is called the *nilpotency* of m .

Conjecture 1.5. [C^k -map to \mathbb{R}^n]. Let X be a C^k -manifold and E be a complex C^k vector bundle of rank r on X . Let (y^1, \dots, y^n) be a global coordinate system on \mathbb{R}^n , as a C^k -manifold, and

$$\eta : y^i \mapsto m_i \in C^k(\text{End}_{\mathbb{C}}(E)), \quad i = 1, \dots, n,$$

be an assignment such that

- (1) $m_i m_j = m_j m_i$, for all i, j ;
- (2) for every $p \in X$, the eigenvalues of the restriction $m_i(p) \in \text{End}_{\mathbb{C}}(E|_p) \simeq M_{r \times r}(\mathbb{C})$ are all real;
- (3) for every $p \in X$, the nilpotency of $m_i(p) \leq k + 1$.

Then, η extends to a unique C^k -admissible ring-homomorphism

$$\varphi_{\eta}^{\sharp} : C^k(\mathbb{R}^n) \longrightarrow C^k(\text{End}_{\mathbb{C}}(E))$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$ and, hence, defines a C^k -map $\varphi_{\eta} : (X^{\text{Az}}, E) \rightarrow Y$.

Note that the set of conditions (1), (2), and (3) are necessary for η to be extendable to a ring-homomorphism $C^k(Y) \rightarrow C^k(\text{End}_{\mathbb{C}}(E))$; cf. [L-Y2: Sec. 3] (D(11.1)). This conjecture rings with the fact that any C^k -map $f : Z \rightarrow \mathbb{R}^k$ from a C^k -manifold Z to \mathbb{R}^n is specified by the component maps $f_i := pr_i \circ f : Z \rightarrow \mathbb{R}$, where $pr_i : \mathbb{R}^n \rightarrow \mathbb{R}$ is the projection map to the i -th coordinate of \mathbb{R}^n , $1 \leq i \leq n$.

Similarly, there are also the fermionic/super version of these conjectures, which would give, in particular, equivalent notions of C^k -maps from an Azumaya/matrix super C^k -manifold to a C^k -manifold or super C^k -manifold to that defined in [L-Y3: Sec. 4] (D(11.2)).

The goal of the current note is to prove Conjecture 1.3 and Conjecture 1.5 in the case $k = \infty$. Their fermionic/super version hold similarly. Before that, let us take a look at the preliminary case when X is a point.

2 Preliminaries: When X is a point

For general $k \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, we examine and prove Conjecture 1.3 and Conjecture 1.5 for the special case that X is a point. For simplicity of notation, some of the explicit expressions in the discussion are meant for k being finite; for example, Taylor polynomials at a point or infinitesimal neighborhoods around a point. They can be readily converted to the case $k = \infty$ (by restricting the Taylor polynomial to degree r or the nilpotency of the infinitesimal neighborhood to r).

The canonical C^k -ring structure on a commutative finite-dimensional \mathbb{R} -algebra

Lemma 2.1. [canonical C^k -ring structure on finite-dimensional \mathbb{R} -algebra]. Let A be a commutative finite-dimensional \mathbb{R} -algebra of nilpotency $\leq l$. Then, for all $k \geq l - 1$, A admits a canonical C^k -ring structure that is compatible with the underlying ring structure of A .

Proof. Since A factorizes into a product

$$A = A_1 \times \cdots \times A_s$$

of Weil algebras (i.e. commutative finite-dimensional \mathbb{R} -algebra with a unique maximal ideal) that is unique up to a permutation of the factors and a product of C^k -rings admits a canonical C^k -ring structure from the factors, without loss of generality we may assume that A is a Weil algebra.

In this case, there is a built-in \mathbb{R} -algebra quotient

$$\alpha : A \longrightarrow \mathbb{R}$$

whose kernel is the maximal ideal \mathfrak{m} of A . Together with the built-in inclusion $\mathbb{R} \hookrightarrow A$ for any \mathbb{R} -algebra, one has a sequence of \mathbb{R} -algebra homomorphisms

$$\mathbb{R} \hookrightarrow A \xrightarrow{\alpha} \mathbb{R}$$

with the composition the identity homomorphism on \mathbb{R} . This gives a canonical splitting

$$A = \mathbb{R} \oplus \mathfrak{m}$$

as \mathbb{R} -vector spaces, with \mathfrak{m} identical with the nil-ideal of A of nilpotency $\leq l$.

Let $h \in C^k(\mathbb{R}^n)$ for any $n \in \mathbb{Z}_{\geq 1}$. Then, for any $a_1, \dots, a_n \in A$, define $h(a_1, \dots, a_n)$ by setting

$$h(a_1, \dots, a_n) = \sum_{s=0}^k \frac{1}{s!} \sum_{d_1 + \dots + d_n = s} \partial_1^{d_1} \cdots \partial_n^{d_n} h(b_1, \dots, b_n) c_1^{d_1} \cdots c_n^{d_n},$$

where

- $a_i = b_i + c_i$, $i = 1, \dots, n$, is the decomposition of a_i according to $A = \mathbb{R} \oplus \mathfrak{m}$;
- $\partial_1^{d_1} \cdots \partial_n^{d_n} h$ is the partial derivative of h with respect to the first variable d_1 -times, the second variable d_2 -times, ..., and the n -th variable d_n -times.

Notice that $(b_1, \dots, b_n) \in \mathbb{R}^n$. Thus, while

$$\partial_1^{d_1} \cdots \partial_n^{d_n} h, \quad d_1, \dots, d_n \in \mathbb{Z}_{\geq 0}, \quad d_1 + \dots + d_n = s,$$

in general lie only in $C^{k-s}(\mathbb{R}^n)$, the evaluation $\partial_1^{d_1} \cdots \partial_n^{d_n} h(b_1, \dots, b_n)$ remains well-defined, as is required for $h \in C^k(\mathbb{R}^n)$.

This defines the canonical C^k -ring structure on A . Clearly, it is compatible with the underlying ring structure of A . □

The following two lemmas follow by construction. They are indications that the canonical C^k -ring structure on a finite-dimensional \mathbb{R} -algebra, when exists, is functorial/natural.

Lemma 2.2. [\mathbb{R} -algebra homomorphism vs. C^k -ring homomorphism, I]. *Let A and B be commutative finite-dimensional \mathbb{R} -algebras with both of nilpotency $\leq k + 1$. Then*

$$\text{Hom}_{\mathbb{R}\text{-Algebras}}(B, A) = \text{Hom}_{C^k\text{-Rings}}(B, A)$$

with respect to the canonical C^k -ring structure on A and B respectively.

Lemma 2.3. [\mathbb{R} -algebra homomorphism vs. C^k -ring homomorphism, II]. *Let A be a commutative finite-dimensional \mathbb{R} -algebra of nilpotency $\leq k + 1$ and Y be a C^k -manifold. Then*

$$\text{Hom}_{\mathbb{R}\text{-Algebras}}(C^k(Y), A) = \text{Hom}_{C^k\text{-Rings}}(C^k(Y), A)$$

with respect to the canonical C^k -ring structure on A .

Validity of Conjecture 1.3 when X is a point

The following lemma follows from [L-Y2: Sec. 3] (D(11.1)):

Lemma 2.4. [ring-homomorphism to matrix algebra]. *Given a C^k -manifold Y , let*

$$\varphi^\sharp : C^k(Y) \longrightarrow M_{r \times r}(\mathbb{C})$$

be a ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$. Then

- (1) *For all $f \in C^k(Y)$, $\varphi^\sharp(f) \in M_{r \times r}(\mathbb{C})$ has only real eigenvalues.*
- (2) *φ^\sharp factors through a finite-Taylor-expansion map at a finite set $\{q_1, \dots, q_s\}$ of Y*

$$\begin{array}{ccc} C^k(Y) & \xrightarrow{\varphi^\sharp} & M_{r \times r}(\mathbb{C}) \\ \oplus_{j=1}^s T_{q_j}^{(k)} \downarrow & \nearrow \varphi^\sharp & \\ \oplus_{j=1}^s \frac{\mathbb{R}[y_j^1, \dots, y_j^n]}{(y_j^1, \dots, y_j^n)^{k+1}} & & \end{array},$$

where

- (y_j^1, \dots, y_j^n) is a local coordinate system in a neighborhood of $q_i \in Y$ with the coordinates of q_j all 0,
- $T_{q_j}^{(k)}$ is the map ‘taking Taylor polynomial (of elements in $C^k(Y)$) at q_j with respect to (y_j^1, \dots, y_j^n) up to and including degree k ’, and
- φ^\sharp is an (algebraic) ring-homomorphism over $\mathbb{R} \subset \mathbb{C}$.

- (3) *Nilpotency of $\varphi^\sharp(C^k(Y))$ is bounded by $\min\{k + 1, r\}$.*

See FIGURE 2-1 for the contravariant geometry behind; cf. [L-Y2: FIGURE 3-4-1] (D(11.1)).

It follows from Lemma 2.1 that both $\oplus_{j=1}^s \frac{\mathbb{R}[y_j^1, \dots, y_j^n]}{(y_j^1, \dots, y_j^n)^{k+1}}$ and $A_\varphi := \text{Im}(\varphi^\sharp) \subset M_{r \times r}(\mathbb{C})$ admit a canonical C^k -ring structure that is compatible with their underlying ring structure. In terms of this, both ring-homomorphisms $\oplus_{j=1}^s T_{q_j}^{(k)}$ and φ^\sharp , regarded now as a ring-morphism to A_φ , are also C^k -ring homomorphisms. Thus,

Corollary 2.5. [φ^\sharp C^k -admissible]. *As a ring-homomorphism to A_φ , $\varphi^\sharp : C^k(Y) \rightarrow A_\varphi$ is a C^k -ring homomorphism. Thus, as a ring-homomorphism to $M_{r \times r}(\mathbb{C})$, $\varphi^\sharp : C^k(Y) \rightarrow M_{r \times r}(\mathbb{C})$ is C^k -admissible.*

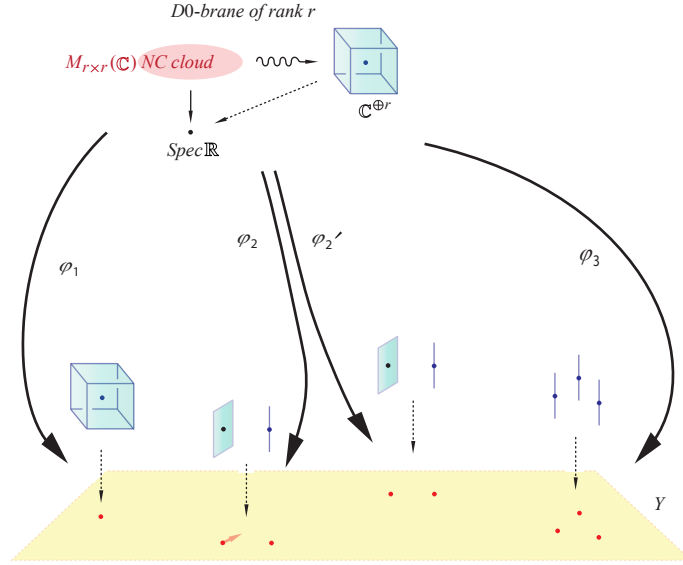


FIGURE 2-1. Four examples of C^k maps $\varphi : (p^{Az}, \mathbb{C}^{\oplus r}) \rightarrow Y$ from an Azumaya/matrix point with a fundamental module to a C^k -manifold Y are illustrated. The nilpotency of the image scheme $Im\varphi$ in Y is bounded by $\min\{k + 1, r\}$. In the figure, the push-forward of the fundamental module in each example is also indicated.

This proves Conjecture 1.3 when X is a point.

Validity of Conjecture 1.5 when X is a point

Given \mathbb{R}^n , as a C^k -manifold, with coordinate $(y^1 \cdots, y^n)$ and an assignment

$$\eta : y^i \mapsto m_i \in M_{r \times r}(\mathbb{C}), \quad i = 1, \dots, n,$$

that satisfies

- (1) $m_i m_j = m_j m_i$, for all i, j ;
- (2) the eigenvalues of m_i are all real;
- (3) the nilpotency of $m_i \leq k + 1$, for all i .

Then Properties (1) and (2) together imply that the collection $\{m_1, \dots, m_n\}$ of matrices are simultaneously triangularizable

$$m_i \sim \begin{bmatrix} \lambda_1^i & * & * \\ & \ddots & * \\ 0 & & \lambda_r^i \end{bmatrix}_{r \times r},$$

with $\lambda_j^i \in \mathbb{R}$, $1 \leq i \leq n$, $1 \leq j \leq r$. Let

$$q_j = (\lambda_j^1, \dots, \lambda_j^n) \in \mathbb{R}^n.$$

Then, after removing repetitive copies and relabelling, the finite set of points q_1, \dots, q_s , for some $s \leq r$, in \mathbb{R}^n is an invariant of the commuting tuple (m_1, \dots, m_n) of matrices in $M_{r \times r}(\mathbb{C})$. Furthermore, the fundamental representation $\mathbb{C}^{\oplus n}$ of $M_{r \times r}(\mathbb{C})$ decomposes into the direct sum

$$\mathbb{C}^{\oplus n} = V_1 \oplus \dots \oplus V_s$$

of s -many common invariant subspaces of m_1, \dots, m_n such that the Jordan form of

$$m_i|_{V_j} : V_j \longrightarrow V_j$$

has diagonal entries all equal to λ_j^i . After a change of basis of $\mathbb{C}^{\oplus n}$ and for simplicity of notation, we may assume that the decomposition is given by

$$\mathbb{C}^{\oplus n} = \mathbb{C}^{\oplus r_1} \oplus \dots \oplus \mathbb{C}^{\oplus r_s},$$

with the j -th summand $\mathbb{C}^{\oplus r_j}$ being associated to $(\lambda_j^1, \dots, \lambda_j^n)$. In terms of this decomposition, one can re-express η as an assignment

$$\eta : y^i \longmapsto (m_{i,1}, \dots, m_{i,s}) \in M_{r_1 \times r_1}(\mathbb{C}) \times \dots \times M_{r_s \times r_s}(\mathbb{C}) \subset M_{r \times r}(\mathbb{C}).$$

In this expression, it is immediate that η extends to a ring-homomorphism

$$\varphi_\eta^\# : C^k(\mathbb{R}^n) \longrightarrow M_{r \times r}(\mathbb{C})$$

from the composition of ring-homomorphisms

$$\begin{array}{ccc} C^k(\mathbb{R}^n) & & M_{r_1 \times r_1} \times \dots \times M_{r_s \times r_s}(\mathbb{C}) \subset M_{r \times r}(\mathbb{C}) \\ \oplus_{j=1}^s T_{q_j}^{(k)} \downarrow & \nearrow & \\ \oplus_{j=1}^s \frac{\mathbb{R}[y^1 - \lambda_j^1, \dots, y^n - \lambda_j^n]}{(y^1 - \lambda_j^1, \dots, y^n - \lambda_j^n)^{k+1}} & \xrightarrow{\varphi^\# = (\varphi_1^\#, \dots, \varphi_s^\#)} & \end{array},$$

where

- $T_{q_j}^{(k)}$ is the map ‘taking Taylor polynomial (of elements in $C^k(\mathbb{R}^n)$) at q_j with respect to coordinate (y^1, \dots, y^n) up to and including degree k , and

$$\varphi_j^\# : \frac{\mathbb{R}[y^1 - \lambda_j^1, \dots, y^n - \lambda_j^n]}{(y^1 - \lambda_j^1, \dots, y^n - \lambda_j^n)^{k+1}} \longrightarrow M_{r_j \times r_j}(\mathbb{C}),$$

is the \mathbb{R} -algebra homomorphism generated by sending $y^i \mapsto m_{i,j}$, $i = 1, \dots, n$.

Equip $\frac{\mathbb{R}[y^1 - \lambda_j^1, \dots, y^n - \lambda_j^n]}{(y^1 - \lambda_j^1, \dots, y^n - \lambda_j^n)^{k+1}}$ and $Im(\varphi_j^\#)$ with the canonical C^k -ring structure. Then all of $T_{q_j}^{(k)}$ and $\varphi_j^\#$, $j = 1, \dots, s$, become C^k -ring homomorphisms. Let $A_{\varphi_\eta} := Im(\varphi_\eta^\#)$ be equipped with the canonical C^k -ring structure. It follows then that the ring-homomorphism

$$\varphi_\eta^\# : C^k(\mathbb{R}^n) \longrightarrow A_{\varphi_\eta}$$

is also a C^k -ring homomorphism. This proves Conjecture 1.5 when X is a point.

3 Proof of Conjectures in the C^∞ case

Conjecture 1.3 and Conjecture 1.5 in the C^∞ case are examined in this section for general X .

3.1 Proof of Conjecture 1.3 in the C^∞ case

We prove in this subsection Conjecture 1.3 in the C^∞ case:

Theorem 3.1.1. [C^∞ -map vs. ring-homomorphism]. *Let X and Y be C^∞ -manifolds and E be a complex C^∞ vector bundle of rank r on X . Given a correspondence*

$$\varphi^\# : C^\infty(Y) \longrightarrow C^\infty(\text{End}_{\mathbb{C}}(E)).$$

Then, the following three statements are equivalent:

- (1) $\varphi^\#$ is a ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$.
- (2) $\varphi^\#$ is a weakly C^∞ -admissible ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$.
- (3) $\varphi^\#$ is a C^∞ -admissible ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$.

Proof. Since Statement(3) \Rightarrow Statement (2) \Rightarrow Statement (1), one only needs to show that Statement (1) \Rightarrow Statement (3).

Step (a) : The only natural candidate extension

Let $\varphi^\# : C^\infty(Y) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ be a ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$. Consider the \mathbb{C} -algebra $C^{-\infty}(\text{End}_{\mathbb{C}}(E))$ of sections of the endomorphism bundle $\text{End}_{\mathbb{C}}(E) \rightarrow X$ as a map between sets. Then $\varphi^\#$ extends canonically to a ring-homomorphism

$$\begin{array}{ccccc} C^{-\infty}(\text{End}_{\mathbb{C}}(E)) & \longleftarrow & C^\infty(\text{End}_{\mathbb{C}}(E)) & \xleftarrow{\varphi^\#} & C^\infty(Y) \\ & & & & \downarrow \\ & & & & C^\infty(X \times Y) \\ & \swarrow \tilde{\varphi}^\# & & & \end{array}$$

over $\mathbb{C} \hookrightarrow \mathbb{R}$, where both inclusions in the diagram are tautological, as follows:

- Associated to each $f \in C^\infty(X \times Y)$ is the subset

$$X^{f;1} := \{(p, f|_{\{p\} \times Y}) : p \in X\} \subset X \times C^\infty(Y).$$

- The map $\text{Id}_X \times \varphi^\# : X \times C^\infty(Y) \rightarrow X \times C^\infty(\text{End}_{\mathbb{C}}(E))$ sends $X^{f;1}$ to the subset

$$X^{f;2} = \{(p, \varphi^\#(f|_{\{p\} \times Y})) : p \in X\} \subset X \times C^\infty(\text{End}_{\mathbb{C}}(E)).$$

- Which produces a section of $\text{End}_{\mathbb{C}}(E) \rightarrow X$ as a map between sets:

$$s_f = \{(p, (\varphi^\#(f|_{\{p\} \times Y}))|_p) : p \in X\} \in C^{-\infty}(\text{End}_{\mathbb{C}}(E)).$$

- $\tilde{\varphi}^\# : C^\infty(X \times Y) \rightarrow C^{-\infty}(\text{End}_{\mathbb{C}}(E))$ is now defined by $f \mapsto s_f$.

By construction, $\tilde{\varphi}^\#$ is a ring-homomorphism over $\mathbb{R} \hookrightarrow \mathbb{C}$ and it makes the following diagram of ring-homomorphisms commute:

$$\begin{array}{ccccc} C^{-\infty}(\text{End}_{\mathbb{C}}(E)) & \longleftarrow & C^\infty(\text{End}_{\mathbb{C}}(E)) & \xleftarrow{\varphi^\#} & C^\infty(Y) \\ & & \uparrow & & \downarrow pr_Y^\# \\ & & C^\infty(X) & \xrightarrow{pr_X^\#} & C^\infty(X \times Y), \\ & \swarrow \tilde{\varphi}^\# & & & \end{array}$$

where $pr_X : X \times Y \rightarrow X$ and $pr_Y : X \times Y \rightarrow Y$ are the projection maps and $C^\infty(X) \hookrightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$ follows from the inclusion of the center $C^\infty(X)^{\mathbb{C}}$ of $C^\infty(\text{End}_{\mathbb{C}}(E))$. (Cf. The construction in [L-Y2: Sec. 5.1, theme ‘A generalization to ring-homomorphisms to Azumaya/matrix algebras’] (D(11.1)).) Notice that $\tilde{\varphi}^\sharp$ is the only extension of φ^\sharp to $C^\infty(X \times Y)$ that satisfies the above commutative diagram and the natural condition that

$$\tilde{\varphi}^\sharp|_{p \times Y} = \varphi^\sharp|_p : C^\infty(Y) \longrightarrow \text{End}_{\mathbb{C}}(E|_p),$$

for all $p \in X$.

Step (b) : From the aspect of germs over X

To understand whether $\tilde{\varphi}^\sharp$ takes its values in $C^\infty(\text{End}_{\mathbb{C}}(E))$, one needs to know how $\tilde{\varphi}^\sharp|_{p \times Y} : C^\infty(Y) \rightarrow \text{End}_{\mathbb{C}}(E|_p)$ varies as p varies along X . This leads us to studying the germs-over- X aspect of $\tilde{\varphi}^\sharp$, which we now proceed.

Definition 3.1.1.1. [**spectral locus/subscheme φ^\sharp**]. Let $I_\varphi \subset C^\infty(X \times Y)$ be the ideal of $C^\infty(X \times Y)$ generated by the set

$$\{ \det(f \cdot Id_{r \times r} - \varphi^\sharp(f)) \mid f \in C^\infty(Y) \}$$

of elements in $C^\infty(X \times Y)$, where $Id_{r \times r}$ is the $r \times r$ identity matrix. I_φ defines a C^∞ -subscheme Σ_φ of $X \times Y$, called interchangeably the *spectral locus* or the *spectral subscheme* of φ^\sharp in $X \times Y$.

Notice that while the local matrix presentation of $\varphi^\sharp(f)$ depends on the local trivialization of E chosen, the determinant $\det(f \cdot Id_{r \times r} - \varphi^\sharp(f))$ does not and, hence, is well-defined.

Some properties of Σ_φ that follow immediately from the defining ideal I_φ are listed below:

- Σ_φ is finite over X in the sense that, for all $p \in X$, the preimage $pr_X^{-1}(p)$ of the morphism $pr_X : \Sigma_\varphi \rightarrow X$ from the restriction of the projection map $X \times Y \rightarrow Y$ are all 0-dimensional C^∞ -scheme with the function-ring given by a (commutative) finite-dimensional \mathbb{R} -algebra.
- A comparison with the study of ring-homomorphisms from $C^\infty(\mathbb{R}^n)$ to $M_{r \times r}(\mathbb{C})$ in [L-Y2: Sec. 3.2] (D(11.1)) implies that
 - $\tilde{\varphi}^\sharp(I_\varphi) = 0$.
 - for all $f \in C^\infty(X \times Y)$, $\tilde{\varphi}^\sharp(f) \in C^{-\infty}(\text{End}_{\mathbb{C}}(E))$ depends only on the restriction of f on the C^∞ -subscheme $\Sigma_\varphi \subset X \times Y$.

We emphasize that, being a C^∞ -scheme defined by an ideal of $C^\infty(X \times Y)$, the spectral locus Σ_φ of φ is more than just a closed subset of $X \times Y$; cf. FIGURE 3-1-1-1.

Recall the morphism $pr_X : \Sigma_\varphi \rightarrow X$. Let $p \in X$. Then since $pr_X^{-1}(p)$ is 0-dimensional, there exists an open neighborhood U of p such that $pr_X^{-1}(U)$ is contained in an open subset $U \times V$ of $X \times Y$, where V is an open subset of Y that is diffeomorphic to \mathbb{R}^n with $n = \dim Y$. Under the diffeomorphism $V \simeq \mathbb{R}^n$, let (y^1, \dots, y^n) be coordinates on V and let

$$(pr_X^{-1}(p))_{red} = \{q_1, \dots, q_s\}$$

be the set of closed points in $pr_X^{-1}(p) \subset \Sigma_\varphi$. (For notation, $q_j = (p; q_j^1, \dots, q_j^n) \in U \times V$ in the coordinate system (y^1, \dots, y^n) on V .) Consider the auxiliary C^∞ -subscheme

$$\Sigma_{(y^1, \dots, y^n)} \subset U \times V$$

defined by the ideal

$$I_{(y^1, \dots, y^n)} := (g_1, \dots, g_n) \subset C^\infty(U \times V), \quad \text{where } g_i := \det(y^i \cdot Id_{r \times r} - \varphi^\sharp(y^i)).$$

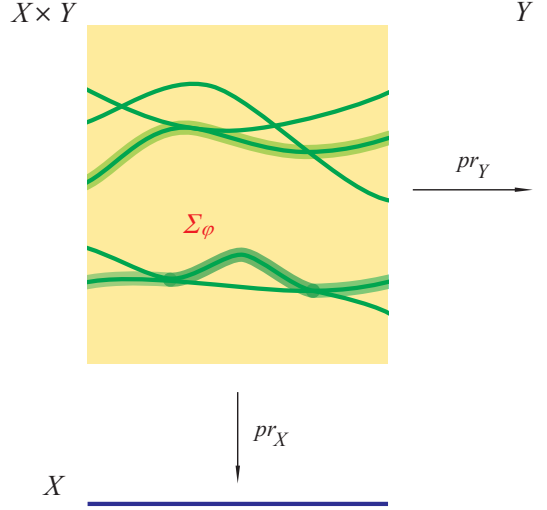


FIGURE 3-1-1-1. The spectral subscheme Σ_φ (in green color, with the green shade indicating the nilpotent structure/cloud on Σ_φ) in $X \times Y$ associated to a ring-homomorphism $\varphi^\sharp : C^\infty(Y) \rightarrow C^\infty(\text{End}_{\mathbb{C}}(E))$. More than just a point-set with topology, it is a C^∞ -scheme that is finite over X .

Then,

$$\Sigma_\varphi \cap U \times V \subset \Sigma_{(y^1, \dots, y^n)}$$

and, again, one has

$$\tilde{\varphi}^\sharp(I_{(y^1, \dots, y^n)}) = 0.$$

Now let

$$d_{i,1}, \dots, d_{i,s}$$

be the regularity of g_i along the y^i -coordinate direction at q_1, \dots, q_s respectively (cf. [Br: 6.1 Definition]). I.e.

$$g_i(q_j) = \partial_i g_i(q_j) = \dots = \partial_i^{d_{i,j}-1} g_i(q_j) = 0 \quad \text{while} \quad \partial_i^{d_{i,j}} g_i(q_j) \neq 0.$$

Here, $\partial_i := \partial/\partial y^i$. Then, it follows from the Malgrange Division Theorem ([Mal]; see also [Br], [Mat1], [Mat2], [Ni]) that

the germ of $f \in C^\infty(X \times Y)$ at q_j admits a normal form

$$f = f_0^{(q_j)} + f_1^{(q_j)}$$

with

$$f_0^{(q_j)} \in C^\infty(U)[y^1, \dots, y^n] \quad \text{of } (y^1, \dots, y^n)\text{-degree } \leq (d_{1,j} - 1, \dots, d_{n,j} - 1)$$

$$\text{and} \quad f_1^{(q_j)} \in I_{(y^1, \dots, y^n)}.$$

After shrinking the neighborhood U of $p \in X$ further, if necessary, and capping $f_0^{(q_j)}$ (still denoted by $f_0^{(q_j)}$) by a smooth cutoff function with support a disjoint union of small enough coordinate balls around q_j , $j = 1, \dots, s$,

$$\tilde{\varphi}^\sharp(f)|_U = \tilde{\varphi}^\sharp(\sum_{j=1}^s f_0^{(q_j)}) \in C^\infty(X)[\varphi^\sharp(y_1), \dots, \varphi^\sharp(y^n)] \subset C^\infty(\text{End}_{\mathbb{C}}(E|_U))$$

since

- $\tilde{\varphi}^\sharp(h) = \varphi^\sharp(h)$ for all $h \in C^\infty(Y)$, and particularly for y^i , $i = 1 \dots, n$;
- $\tilde{\varphi}^\sharp(h) = h \cdot Id_E$ for all $h \in C^\infty(X)$, where Id_E is the identity map on E .

Since smoothness is a local (indeed, infinitely infinitesimal) property, smoothness of $\tilde{\varphi}^\sharp(f)$ for all $f \in C^\infty(X \times Y)$ follows. This shows that $Im(\tilde{\varphi}^\sharp) \subset C^\infty(End_{\mathbb{C}}(E))$.

Step (c) : Conclusion

Let $A_\varphi := Im(\tilde{\varphi}^\sharp)$, which is identical to the $C^\infty(X)$ -subalgebra $C^\infty\langle Im(\varphi^\sharp) \rangle$ of $C^\infty(End_{\mathbb{C}}(E))$ generated by $C^\infty(X)$ and $Im(\varphi^\sharp)$ in $C^\infty(End_{\mathbb{C}}(E))$. Then, in the C^∞ case, as a consequence of the Hadamard's Lemma, the C^∞ -ring structure on $C^\infty(X \times Y)$ always descends, via $\tilde{\varphi}^\sharp$, to a C^∞ -ring structure on A_φ that is compatible with the underlying ring-structure of A_φ . In this way, one obtains a commutative diagram

$$\begin{array}{ccc}
 A_\varphi & \xleftarrow{\varphi^\sharp} & C^\infty(Y) \\
 \uparrow \text{J} & \swarrow \tilde{\varphi}^\sharp & \downarrow pr_Y^\sharp \\
 C^\infty(X) & \xrightarrow{pr_X^\sharp} & C^\infty(X \times Y)
 \end{array}$$

of C^∞ -ring homomorphisms. This shows that φ^\sharp is C^∞ -admissible and proves the theorem. \square

3.2 Proof of Conjecture 1.5 in the C^∞ case

We prove in this subsection Conjecture 1.5 in the C^∞ case:

Theorem 3.2.1. [C^∞ -map to \mathbb{R}^n]. *Let X be a C^∞ -manifold and E be a complex C^∞ vector bundle of rank r on X . Let (y^1, \dots, y^n) be a global coordinate system on \mathbb{R}^n , as a C^∞ -manifold, and*

$$\eta : y^i \mapsto m_i \in C^\infty(End_{\mathbb{C}}(E)), \quad i = 1, \dots, n,$$

be an assignment such that

- (1) $m_i m_j = m_j m_i$, for all i, j ;
- (2) for every $p \in X$, the eigenvalues of the restriction $m_i(p) \in End_{\mathbb{C}}(E|_p) \simeq M_{r \times r}(\mathbb{C})$ are all real.

Then, η extends to a unique C^∞ -admissible ring-homomorphism

$$\varphi_\eta^\sharp : C^\infty(\mathbb{R}^n) \longrightarrow C^\infty(End_{\mathbb{C}}(E))$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$ and, hence, defines a C^∞ -map $\varphi_\eta : (X^{Az}, E) \rightarrow \mathbb{R}^n$.

Notice that Condition (3): for every $p \in X$, the nilpotency of $m_i(p) \leq k + 1$ in the statement of Conjecture 1.5 is automatically satisfied in the C^∞ case.

Proof. Given η in the statement of the theorem, it follows from Sec. 2 that for all $p \in X$, the assignment from restriction

$$\eta_p : y^i \mapsto m_i(p) \in \text{End}_{\mathbb{C}}(E|_p), \quad i = 1, \dots, n,$$

extends uniquely to a ring-homomorphism

$$\varphi_{\eta_p}^{\#} : C^{\infty}(\mathbb{R}^n) \longrightarrow \text{End}_{\mathbb{C}}(E|_p)$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$ that is C^{∞} -admissible over p . As p varies, η extends uniquely to a ring-homomorphism

$$\varphi_{\eta}^{\#} : C^{\infty}(\mathbb{R}^n) \longrightarrow C^{-\infty}(\text{End}_{\mathbb{C}}(E))$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$. The same construction as Step (a) in the proof of Theorem 3.1.1 extends $\varphi_{\eta}^{\#}$ further and uniquely to a ring-homomorphism

$$\tilde{\varphi}_{\eta}^{\#} : C^{\infty}(X \times \mathbb{R}^n) \longrightarrow C^{-\infty}(\text{End}_{\mathbb{C}}(E))$$

over $\mathbb{R} \hookrightarrow \mathbb{C}$ that fits into the following commutative diagram

$$\begin{array}{ccc}
 & & \varphi_{\eta}^{\#} \\
 & \longleftarrow & \longleftarrow \\
 C^{-\infty}(\text{End}_{\mathbb{C}}(E)) & \longleftarrow & C^{\infty}(\mathbb{R}^n) \\
 & \uparrow & \downarrow \text{pr}_{\mathbb{R}^n}^{\#} \\
 & C^{\infty}(X) & \xrightarrow{\text{pr}_X^{\#}} C^{\infty}(X \times \mathbb{R}^n) \\
 & \uparrow \tilde{\varphi}_{\eta}^{\#} & \\
 & & \varphi_{\eta}^{\#}
 \end{array}$$

of ring-homomorphisms while satisfying the condition that

$$\tilde{\varphi}_{\eta}^{\#}|_{p \times Y} = \varphi_{\eta_p}^{\#} : C^{\infty}(\mathbb{R}^n) \longrightarrow \text{End}_{\mathbb{C}}(E|_p),$$

for all $p \in X$.

The same argument as Step (b) in the proof of Theorem 3.1.1 implies that indeed $\tilde{\varphi}_{\eta}^{\#}$ takes values in $C^{\infty}(\text{End}_{\mathbb{C}}(E))$. Thus, so does $\varphi_{\eta}^{\#}$. As in Step (c) there, one thus has the following commutative diagram

$$\begin{array}{ccc}
 A_{\varphi_{\eta}} & \longleftarrow & C^{\infty}(\mathbb{R}^n) \\
 & \uparrow & \downarrow \text{pr}_{\mathbb{R}^n}^{\#} \\
 C^{\infty}(X) & \xrightarrow{\text{pr}_X^{\#}} & C^{\infty}(X \times \mathbb{R}^n) \\
 & \uparrow \tilde{\varphi}_{\eta}^{\#} & \\
 & & \varphi_{\eta}^{\#}
 \end{array}$$

of C^{∞} -ring homomorphisms, where $A_{\varphi_{\eta}} := \text{Im}(\tilde{\varphi}_{\eta}^{\#}) \subset C^{\infty}(\text{End}_{\mathbb{C}}(E))$. This shows that $\varphi_{\eta}^{\#}$ is C^{∞} -admissible and proves the theorem. \square

4 Remarks on the general C^k case

Even if not conceptually, technically the finitely differentiable case seems to be more difficult than the smooth (i.e. infinitely differentiable) case. Some remarks are collected here as a guide to verify Conjecture 1.3 and Conjecture 1.5 in full (or to correct them, taking the statements as the reference starting point and see how things could break).

Reflections on C^∞ - vs. general C^k -algebraic geometry, and the proof

(1) From the construction of the canonical C^k -ring structure on a commutative finite-dimensional \mathbb{R} -algebra in Sec. 2, one learns that while C^∞ -algebraic geometry is self-contained (in the sense that only elements in $\cup_{l=0}^\infty C^\infty(\mathbb{R}^l)$ are involved), C^k -algebraic geometry with k finite may not (in the sense that elements in $\cup_{k' < k} \cup_{l=0}^\infty C^{k'}(\mathbb{R}^l)$ that come from partial derivatives of elements in $\cup_{l=0}^\infty C^\infty(\mathbb{R}^l)$ are involved as well when the C^k -scheme considered is not reduced).

(2) The proof of Conjecture 1.5 in the smooth case (Theorem 3.2.1) by first constructing φ_η^\sharp and $\tilde{\varphi}_\eta^\sharp$ with values in $C^{-\infty}(End_{\mathbb{C}}(E))$ and then proving that they actually take values in $C^\infty(End_{\mathbb{C}}(E))$ reminds one of wall-crossing phenomena in string theory¹ in which some quantities (e.g. soliton numbers; here, canonical-form-rendering automorphisms/frames) jump cell by cell in order that a related geometric quantity (e.g. flat sections from solutions to a differential system; here, endomorphisms of a complex vector bundle) can be kept continuous (here even differentiable). A simple example serves to illuminate this:

Example 4.1. [wall-crossing of frames vs. smoothness of endomorphism]. Let X^{Az} be the Azumaya/matrix smooth line $(\mathbb{R}^{1,Az}, E)$, where E is a complex vector bundle of rank 2 on $X = \mathbb{R}^1$ with coordinate x , and $Y = \mathbb{R}^1$ be the smooth real line with coordinate y . For convenience, we assume that E is trivialized. Consider the assignment

$$\eta : y \mapsto m = \begin{bmatrix} x & 1 \\ 0 & -x \end{bmatrix} \in C^\infty(End_{\mathbb{C}}(E)).$$

To extend η to a C^∞ -admissible ring-homomorphism

$$\varphi_\eta^\sharp ; C^\infty(Y) \longrightarrow C^\infty(End_{\mathbb{C}}(E)),$$

consider the following chamber structure on X and Jordan-form-of- m -rendering frames (e_1, e_2) of E over each chamber:

$$\begin{aligned} \text{for } x \neq 0, \quad (e_1, e_2) &= \begin{bmatrix} 1 & 1 \\ 0 & 2x \end{bmatrix} =: S_1; \\ \text{for } x = 0, \quad (e_1, e_2) &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =: S_0. \end{aligned}$$

Let $f \in C^\infty(Y)$. Then, for $x \neq 0$, one has

$$m|_{x \neq 0} = S_1 \cdot \begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \cdot S_1^{-1};$$

¹Readers are referred to [C-V] (1993) when such notion started to surface in string theory before becoming a major study, and to keyword search for more recent fast and vast development from various stringy and/or mathematical aspects, including counting solitonic D-brane systems.

thus, over each of the two chambers $\{x > 0\}$ and $\{x < 0\}$,

$$f(m|_{x \neq 0}) = S_1 \cdot f \left(\begin{bmatrix} x & 0 \\ 0 & -x \end{bmatrix} \right) \cdot S_1^{-1} = S_1 \cdot \begin{bmatrix} f(x) & 0 \\ 0 & f(-x) \end{bmatrix} \cdot S_1^{-1} = \begin{bmatrix} f(x) & \frac{f(x)-f(-x)}{2x} \\ 0 & f(-x) \end{bmatrix}.$$

While at $x = 0$,

$$m(0) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and

$$f(m(0)) = f \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \right) = \begin{bmatrix} f(0) & f'(0) \\ 0 & f(0) \end{bmatrix},$$

where $f'(0) = \frac{df}{dy}(0)$. Notice that for $f \in C^\infty(\mathbb{R}^1)$, $(f(x) - f(-x))/(2x)$ is smooth at 0 (an immediate consequence of the Malgrange Division Theorem again) and hence at all x , with its value at 0 equal to $f'(0)$. Thus, while the Jordan form J_m of m , Jordan-form-of- m -rendering S , and, hence, all the factors in the product $S f(J_m) S^{-1}$ are discontinuous at $x = 0$, the product $S f(J_m) S^{-1}$, which gives $f(m)$, remains continuous, even smooth, over all X .

A conjecture on a division lemma in the finitely differentiable case

In the proof of Theorem 3.1.1 and Theorem 3.2.1, though differentiability of $\tilde{\varphi}^\sharp(f)$ or $\tilde{\varphi}_\eta^\sharp(f) \in C^{-\infty}(\text{End}_{\mathbb{C}}(E))$ at a point $p \in X$ is an issue that involves only an infinitesimal neighborhood of $p \in X$, technically it looks very difficult to prove it without employing consequences from the Malgrange Division Theorem, which is a theorem at the level of germs on a small neighborhood of $p \in X$. (Cf. Readers may try to prove directly that $(f(x) - f(-x))/(2x)$ for $f \in C^\infty(\mathbb{R}^1)$ in Example 4.1 is smooth at $x = 0$ without employing the Malgrange Division Theorem or its similar construction or argument.) While such theorem looks more than we need, if it is indispensable, then one would expect a version of it in the finitely differentiable case would prove both Conjecture 1.3 and Conjecture 1.5 since all other part of the proof of Theorem 3.1.1 and Theorem 3.2.1 works also for finite k . The following conjecture is guided by the Taylor expansion of a C^k -function in the normal direction to a codimension-1 C^k subscheme:

Conjecture 4.2. [generalized division lemma in finitely differentiable case]. *Let*

- 0 be the origin of \mathbb{R}^{m+1} (and same notation also for the origin of \mathbb{R}^m , if necessary),
- y be the $(m+1)$ -th coordinate of $\mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}^1$,
- $h \in C^k(\mathbb{R}^{m+1})$ such that

$$h(0) = \partial_y h(0) = \dots = \partial_y^{s-1} h(0) = 0 \quad \text{while} \quad \partial_y^s h(0) \neq 0$$

for some $s \leq k$.

Denote by $C^k(\mathbb{R}^{m+1})_{(0)}$ the germs of C^k -functions on \mathbb{R}^{m+1} at 0; and similarly for $C^s(\mathbb{R}^m)_{(0)}$. Then, for all $f \in C^k(\mathbb{R}^{m+1})_{(0)}$, there exists $g \in C^k(\mathbb{R}^{m+1})_{(0)}$ (or some sensible subset of $\cup_{k'=0}^k C^{k'}(\mathbb{R}^{m+1})_{(0)}$) and $a_i \in C^{k-i}(\mathbb{R}^m)_{(0)}$, $i = 1, \dots, s$, such that

$$f = gh + \sum_{i=1}^s a_i y^{s-i}$$

in $C^k(\mathbb{R}^{m+1})_{(0)}$.

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