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On a borderline class of non-positively curved compact Kähler manifolds

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0 Introduction and statement of results

Let M be a compact complex manifold. Denote by $\mathscr{F}(M)$ the space of all Kähler metrics on M with non-positive holomorphic bisectional curvature. Since the summation of two such metrics still has the same curvature property, $\mathscr{F}(M)$ forms a convex subset in $\mathscr{C}(M)$, the linear span of the space of all Kähler metrics on M.

Definition. M is said to be *semi-rigidly non-positively curved*, or simply *semi-rigid*, if $\mathscr{F}(M)$ is not empty, and its linear span in $\mathscr{C}(M)$ is finite dimensional.

It is not hard to see that for a finite unbranched cover $\pi\colon M \to N$, M is semi-rigid if and only if N is so; and for a product manifold $M = M_1 \times M_2$, M is semi-rigid if and only if both M_1 and M_2 are so.

Apparently, if there is a metric g on M which has strictly negative holomorphic bisectional curvature at a point $x \in M$, then any small perturbation of g near x is also in $\mathscr{F}(M)$, so M cannot be semi-rigid. In other words, semi-rigidity is likely to occur only when the cotangent bundle T^*_M is semi-ample but not ample in certain strong way, which could give lots of flat directions for the curvature of any g in $\mathscr{F}(M)$. This would tie the elements of $\mathscr{F}(M)$ together.

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For example, any complex torus $M=\mathbf{T}^n$ is semi-rigid, since each $g\in\mathscr{F}(M)$ must be flat, and the flat metrics on M are contained in a finite dimensional linear space.

But certainly it is more interesting to consider the case when M is of general type. The first result in this direction was obtained by Mok in [M]. Let us state the following special case of his main theorem:

Theorem (Mok) *Let $M=D/\Gamma$ be a smooth compact quotient of an irreducible bounded symmetric domain of rank bigger than one. Then $\mathscr{F}(M)=\mathbf{R}^+\{g_0\}$, i.e., the non-positively curved metric on M is unique up to a constant multiple.*

In this paper, we study a class of general type manifolds which are semi-rigid (but not rigid in general). Our main results can be stated as the following.

Theorem A. *Let (M,g) be a n-dimensional compact Kähler manifold with:*

(1) *$n\geq 2$, and $(c_1^2-c_2)\cdot[\omega_h]^{n-2}=0$ for a Kähler metric h;*

(2) *$g\in\mathscr{F}(M)$, i.e., the holomorphic bisectional curvature is non-positive;*

(3) *$\{x\in M:\mathrm{Ric}_g^n(x)\neq 0\}$ is dense in M, and $\{x\in M:\mathrm{Ric}_g^{n-1}(x)\neq 0\}$ is a Zariski open subset in M.*

Then there exists an isometric holomorphic immersion f: $(\tilde{M},\tilde{g})\to(\mathbf{C}^{n+1},g_0)$ from the universal covering space of (M,g) into the complex euclidean space, and for each $\gamma\in\pi_1(M)$, there is a rigid motion ϕ_γ in $\mathbf{C}^{n+1}$ such that $f\circ\gamma=\phi_\gamma\circ f$.

Theorem B. *Let (M,g) be as in Theorem A. Then for any $h\in\mathscr{F}(M)$, (M,h) also satisfies the condition (3); and for any two isometric holomorphic immersions $f:(\tilde{M},\tilde{g})\to(\mathbf{C}^{n+1},g_0)$ and $f^{(h)}$: $(\tilde{M},\tilde{h})\to(\mathbf{C}^{n+1},g_0)$, there always exists an affine transformation ϕ in $\mathbf{C}^{n+1}$ such that $f^{(h)}=\phi\circ f$.*

In particular, M is semi-rigid.

Note that by the beautiful theorem of Fulton and Lazarsfeld [F-L], the semi-ampleness of the cotangent bundle T_M^* gives a bunch of inequalities on the Chern classes, where $c_1^2-c_2\geqq 0$ is among the first a few ones; and all these inequalities become sharp when T_M^* is ample. Therefore, the condition (1) in Theorem A says in a strong tone that T_M^* is not ample. The third condition guarantees that M is of general type, and it is satisfied by any smooth theta divisor in a complex torus $\mathbf{T}^{n+1}$.

From the differential-geometric point of view, Theorem A says that locally (M,g) looks like a piece of hypersurface in the complex euclidean space; while Theorem B implies that for any two metrics in $\mathscr{F}(M)$, their holomorphic bisectional curvature tensore are conformal to each other.

1 Decomposition of the holomorphic bisectional curvature

Through out this section, let us assume that (M,g) is a fixed Kähler manifold satisfying the conditions in Theorem A.

Under any tangent frame $\{e_1,\ldots,e_n\}$, let θ, Θ and g be the $n\times n$ matrices of the connection, curvature, and the metric itself. Let $\Omega=\Theta\cdot g$ be the matrix of holomorphic bisectional curvature. Also write $e={}^t(e_1,\ldots,e_n)$, $\varphi={}^t(\varphi_1,\ldots,\varphi_n)$ as column vectors, where $\{\varphi_1,\ldots,\varphi_n\}$ is the coframe dual to e. Let U_g denote the dense open subset $\{x\in M:\mathrm{Ric}_g^n(x)\neq 0\}=\{x\in M:\mathrm{Ric}_g(x)<0\}$.

Proposition 1 *For any $x \in M$ and any $v \in T_x M$, $(\Omega_{v\bar{v}})^2 = 0$.*

Proof. Let $\pi: P = \mathbf{P}(T_M) \to M$ be the projectified tangent bundle and L the dual of the tautological line bundle on P. For any $(x, [v]) \in P$, let $(z_1, \ldots, z_n)$ be a holomorphic coordinate contered at x, and $\{e_1, \ldots, e_n\}$ a holomorphic tangent frame near x which is normal at x and $[e_1(0)] = [v]$. Then $(z, t) = (z_1, \ldots, z_n; t_2, \ldots, t_n)$ gives the holomorphic coordinates of the point $\left(z, \left[\sum_{i,j=1}^{n} t_i \bar{t}_j g_{i\bar{j}}\right]\right)$ near $(x, [v])$. Here $t_1 \equiv 1$.

Let $\hat{g}$ be the metric on L induced by g, and ω be the Kähler form of h in (1) of Theorem A, then at $(x, [v])$,

$$\begin{aligned} c_1(L, \hat{g})|_{(x,[v])} &= \partial\bar{\partial} \log\left(\sum_{i,j=1}^{n} t_i \bar{t}_j g_{i\bar{j}}(z)\right)\Bigg|_{(0,0)} \\ &= -\frac{\Omega_{v\bar{v}}}{g_{v\bar{v}}} + dt_2 \wedge d\bar{t}_2 + \ldots + dt_n \wedge d\bar{t}_n \\ &\cdot (c_1(L, \hat{g}))^{n+1} \wedge \pi^*(\omega^{n-2})|_{(x,[v])} \\ &= c\left(-\frac{\Omega_{v\bar{v}}}{g_{v\bar{v}}}\right)^2 \wedge \pi^*(\omega^{n-2}) \wedge (dt_2 \wedge d\bar{t}_2 + \ldots + dt_n \wedge d\bar{t}_n)^{n+1} \geqq 0. \end{aligned}$$

Since $c_1^2 - c_2 \cdot [\omega]^{n-2} = 0$ and $L^n - L^{n-1} \cdot \pi^* C_1 + \ldots + \pi^* c_n = 0$ as cohomology classes, we get the pointwise identity:

$$(\Omega_{v\bar{v}})^2 \wedge (\omega)^{n-2} = 0$$

hence $(\Omega_{v\bar{v}})^2 = 0$ as $\Omega_{v\bar{v}} \leqq 0$. QED

Lemma 1 *For any i, j, k in $\{1, 2, \ldots, n\}$:*

$$\Omega_{i\bar{i}} \wedge \Omega_{i\bar{j}} = \Omega_{j\bar{j}} \wedge \Omega_{i\bar{j}} = 0$$

$$(\Omega_{i\bar{j}})^2 = \Omega_{i\bar{i}} \wedge \Omega_{j\bar{k}} + \Omega_{j\bar{k}} \wedge \Omega_{j\bar{i}} = 0.$$

Proof. Let $v = e_i + t e_j$. Then $\Omega_{v\bar{v}} = \Omega_{i\bar{i}} + \bar{t}\Omega_{i\bar{j}} + t\Omega_{j\bar{i}} + t\bar{t}\Omega_{i\bar{i}}$. Since $(\Omega_{v\bar{v}})^2 = 0$ for arbitrary t, one gets $\Omega_{i\bar{i}} \wedge \Omega_{i\bar{j}} = \Omega_{j\bar{j}} \wedge (\Omega_{i\bar{j}})^2 = 0$. Similarly, let $v = e_i + t e_j + s e_k$ and consider the $t\bar{s}$ terms in $(\Omega_{v\bar{v}})^2 = 0$, one gets $\Omega_{i\bar{i}} \wedge \Omega_{j\bar{k}} + \Omega_{i\bar{k}} \wedge \Omega_{j\bar{i}} = 0$. QED

Proposition 2 *For any $x \in U_g$, and any tangent frame e near x, there exist (1,0)-forms $\psi_1, \ldots, \psi_n$ in a neighbourhood of x, such that*

$$\Omega = -\psi \wedge \psi^*$$

where $\psi = {}^t(\psi_1, \ldots, \psi_n)$, and $\psi^* = {}^t\bar{\psi}$.

Proof. Without loss of generality, we may assume that e is an unitary frame. For each i between 1 and n, write $\Omega_{i\bar{i}} = {}^t\varphi A \bar{\varphi}$. Then $\operatorname{rank}(A) \leqq 1$ by Proposition 1, and $A(x) \neq 0$ as $x \in U_g$, hence $\operatorname{rank}(A) = 1$ in a small neighbourhood of x. Since A is Hermitian and semi-negative definite, we get $\Omega_{i\bar{i}} = -\psi_i \wedge \bar{\psi}_i$ for some (1,0)-form ψ_i. By $\operatorname{Ric}_x < 0$, ψ forms a coframe near x.

For any $i<j$, write $\Omega_{i\bar{j}}=\sum_{k,l=1}^{n} c_{k\bar{l}}\psi_k \wedge \bar{\psi}_l$. Then by Lemma 1, $\Omega_{i\bar{j}}=c_{i\bar{j}}\psi_i \wedge \bar{\psi}_j + c_{j\bar{i}}\psi_j \wedge \bar{\psi}_i$, and $c_{i\bar{j}}\cdot c_{j\bar{i}}=0$, $|c_{i\bar{j}}|^2+|c_{j\bar{i}}|^2=1$. By the first Bianchi identity ${}^t\varphi \wedge \Theta = 0$, here $\Omega=\Theta$ as e is unitary, we know that $c_{j\bar{i}}$ must vanish. Therefore, $\Omega_{i\bar{j}} = c_{i\bar{j}}\psi_i \wedge \bar{\psi}_j$, $c_{i\bar{i}}=1$, and $|c_{i\bar{j}}|^2=1$.

Let $C=(c_{i\bar{j}})$. Then C is a nowhere zero Hermitian matrix. By the last equality in Lemma 1, $\operatorname{rank}(C)\leqq 1$, hence $C=b\cdot b^*$ for a column vector b. Replace ψ_i by $b_i\psi_i$, we get the desired decomposition of Ω. QED

Proposition 3 *For any $x\in U_g$, and any tangent frame e near x, let ψ be a coframe near x satisfying $\Omega=-\psi \wedge \psi^*$ as in Proposition 2. Then there exists a 1-form λ near x such that $\bar{\lambda}=-\lambda$, $d\psi=\theta\wedge\psi-\lambda\wedge\psi$ and $d\lambda=-\mathrm{Ric}_g$ (θ is the connection matrix under e).*

Proof. Again we may assume that e is unitary. Since $d\varphi=-{}^t\theta\wedge\varphi$, and ψ forms a coframe, one can write $d\psi=\theta\wedge\psi+\xi\wedge\psi$ for some $n\times n$ matrix of 1-forms ξ. Plug it into the second Bianchi identity $d\Theta=\theta\wedge\Theta-\Theta\wedge\theta$, and $\Theta=\Omega=-\psi\wedge\psi^*$, one gets:

$$\xi\wedge\psi\wedge\psi^*+\psi\wedge\psi^*\wedge\xi^*=0.$$

Its (2, 1)-parts gives:

$$\xi^{(1,0)}\wedge\psi\wedge\psi^*+\psi\wedge\psi^*\wedge\xi^{(0,1)*}=0.$$

This implies that

$$\xi^{(0,1)*}=\alpha I$$
$$\xi^{(1,0)}\wedge\psi=-\alpha\wedge\psi.$$

Therefore

$$\xi\wedge\psi=-(\alpha-\bar{\alpha})\wedge\psi.$$

Let $\lambda=\alpha-\bar{\alpha}$, then

$$\bar{\lambda}=-\lambda; \qquad d\psi=\theta\wedge\psi-\lambda\wedge\psi.$$

Differentiate the last equality, one gets $d\lambda=-\mathrm{Ric}_g$. QED

2 Curvature decomposition in the degenerate case

Let V_g be the Zariski open set $\{x\in M: \mathrm{Ric}_g^{n-1}(x)\neq 0\}$. In this section, we shall consider the decomposition of Ω in V_g, since it will be needed later in the proof of Theorem A.

Let us fix a point $x\in V_g\setminus U_g$. Choose an unitary frame e with the dual frame φ such that

$$-\mathrm{Ric}_g=\lambda_1\varphi_1\wedge\bar{\varphi}_1+\ldots+\lambda_n\varphi_n\wedge\bar{\varphi}_n$$

where $\lambda_1\geqq\ldots\geqq\lambda_{n-1}>\lambda_n\geqq 0$ in a neighbourhood V of x. Write $U=V\cap U_g$, then $\lambda_n>0$ in U and $=0$ along $V\setminus U$.

Since $\Omega_{i\bar{i}}\leqq 0$, and $\mathrm{tr}_\omega\Omega_{i\bar{i}}(x)=\mathrm{Ric}(e_i,\bar{e}_i)|_x=-\lambda_i(x)$, hence $\Omega_{i\bar{i}}(x)\neq 0$ for $1\leqq i\leqq n-1$ and $\Omega_{n\bar{n}}(x)=0$.

Therefore, there exist (1,0)-forms $\psi_1,\ldots,\psi_{n-1}$ in V such that $\Omega_{i\bar{i}}=-\psi_i\wedge\bar{\psi}_i$ for each $i\leqq n-1$, and $\psi_1\wedge\ldots\wedge\psi_{n-1}\neq 0$ in V.

Write $\psi_i = \sum_{j=1}^{n-1} a_{ij}\varphi_j + b_i\varphi_n$, and $A=(a_{ij})$. Then $\Omega_{n\bar n}(x)=0$ gives ${}^t A\bar A(x) = \mathrm{diag}(\lambda_1(x),\dots,\lambda_{n-1}(x))>0$, hence $\det A(x)\neq 0$. Thus by shrinking V if necessary, we have $\psi_1\wedge\dots\wedge\psi_{n-1}\wedge\varphi_n\neq 0$ in V.

For any $y\in U$, Proposition 2 gives that $\Omega=-\psi'\wedge{}^t\bar\psi'$ for some coframe $\psi'=B\varphi$ near y. Then for $1\leqq i\leqq n-1$, $\psi'_i=\alpha_i\psi_i$ near y for some $|\alpha_i|=1$. By the first Bianchi identity: ${}^t\varphi\wedge\Theta=0$, hence ${}^tB=B$.

Write:

$$B=\begin{pmatrix} H & b\\ {}^tb & c\end{pmatrix}.$$

Since

$$\mathrm{Ric}_g=\mathrm{tr}(\Omega)=-{}^t\psi'\wedge\bar\psi'=-{}^t\varphi(B\bar B)\bar\varphi$$

we have

$$H\bar H+b\,{}^t\bar b=\mathrm{diag}(\lambda_1,\dots,\lambda_{n-1})$$
$$H\bar b+b\bar c=0$$
$${}^tb\bar b+c\bar c=\lambda_n$$

therefore

$$\sum_{i=1}^{n-1}\lambda_i|b_i|^2=\lambda_n\sum_{i=1}^{n-1}|b_i|^2.$$

This together with the fact that $\lambda_1\geqq\dots\geqq\lambda_{n-1}>\lambda_n\geqq 0$ implies that near y:

$$b=0;\qquad \Omega_{n\bar n}=-c\bar c\varphi_n\wedge\bar\varphi_n.$$

Now if we write $\Omega_{n\bar n}=-{}^t\varphi E\bar\varphi$ in V, where

$$E=\begin{pmatrix} F & h\\ {}^th & a\end{pmatrix}\geqq 0.$$

Then $h=0$ in U, hence in V. Since $\mathrm{rank}(E)\leqq 1$, while in U, $a=|c|^2>0$, therefore $F=0$ in U, hence in V. Namely we have

$$\Omega_{n\bar n}=-\psi_n\wedge\bar\psi_n;\qquad \psi_n=\tau\varphi_n$$

in the whole neighbourhood V.

Use the denseness of U and $\{\psi_1,\dots,\psi_{n-1},\varphi_n\}$ as the coframe, a little modification of the proofs of Propositions 2 and 3 gives the following:

Proposition 4 *For any $x\in V_g$ and any frame e near x, there exist $(1,0)$-forms $\psi_1,\dots,\psi_n$ and 1-form λ in a neighbourhood of with $\bar\lambda=-\lambda$ such that*

$$\Omega=-\psi\wedge{}^t\bar\psi;\qquad d\psi=\theta\wedge\psi-\lambda\wedge 4;\qquad d\lambda=-\mathrm{Ric}_g.$$

3 Proof of Theorem A

First let us recall the fundamental theorem for complex hypersurfaces.

Suppose (X, g) is a n-dimensional Kähler manifold. Let $\{e_1, \ldots, e_n\}$ be a tangent frame, with $\{\varphi_1, \ldots, \varphi_n\}$ its dual frame. Still denote by θ, Ω the matrix of connection and holomorphic bisectional curvature.

For a covariant 2-tensor A of type $(2,0)$, write $A = \sum_{i,j=1}^{n} a_{ij} \varphi_i \otimes \varphi_j$. Let $\psi_i = \sum_{j=1}^{n} a_{ij} \varphi_j$. Then A can be written as ${}^t\varphi \otimes \psi$ and A is symmetric if and only if ${}^t\varphi \wedge \psi = 0$, where $\varphi = {}^t(\varphi_1, \ldots, \varphi_n)$, $\psi = {}^t(\psi_1, \ldots, \psi_n)$ are the column vectors. Let us call ψ the associated (column of) $(1,0)$-forms of A under the frame e.

Definition. A covariant 2-tensor A is called a *second fundamental tensor*, if A is symmetric, of type $(2,0)$, and there exists a 1-form λ with $\bar{\lambda} = -\lambda$, such that the associated $(1,0)$-forms ψ of A satisfies:

$$\Omega = -\psi \wedge {}^t\bar{\psi}; \quad d\psi = \theta \wedge \psi - \lambda \wedge \psi; \quad d\lambda = -\mathrm{Ric}_g.$$

Note that λ (once exists) is uniquely determined by A (or ψ) while the above conditions are independent of the choices of the frames. Now we can state a weak version of the fundamental theorem for complex hypersurfaces as the following:

Theorem. *Let (X, g) be a n-dimensional Kähler manifold with* Ric_g *being negative definite in a dense open subset. Assume that for any $x \in X$, there exists a second fundamental tensor near x. Then*:

(1) *For any $x \in X$, there exists a neighborhood U of x and an isometric holomorphic immersion from U into $\mathbf{C}^{n+1}$, the $(n+1)$-dimensional complex euclidean space; and such map is unique up to a rigid motion in $\mathbf{C}^{n+1}$.*

(2) *If furthermore X is simply-connected, then there is a global isometric holomorphic immersion f: $X \to \mathbf{C}^{n+1}$, which is also unique up to a rigid motion. (In particular, for any isometry γ on X, there exists a rigid motion ϕ_γ in $\mathbf{C}^{n+1}$ such that $f \circ \gamma = \phi_\gamma \circ f$.)*

The proof is standard. Its key point is that the following system of linear equations has local existence and uniquencess for any initial conditions:

$$\nabla \xi - \rho A = 0$$
$$d\rho - \rho\lambda + g(\xi, \bar{A}) = 0$$

where ρ is a function, ξ is a $(1,0)$-form, and $g(\xi, \bar{A})$ denote the 1-form $\sum_{i,j=1}^{n} \xi_i g^{i\bar{j}} \bar{\psi}_j$ under a frame e $\left(\text{with } \varphi \text{ its dual frame and } \xi = \sum_{i=1}^{n} \xi_i \varphi_i\right)$.

For any point $x \in X$ and any unitary coframe $\{\sigma_1, \ldots, \sigma_n\}$ at x, let (ξ^ν, ρ^ν) be the unique solution near x of the above system together with the initial condition

$$(\xi^\nu(x), \rho^\nu(x)) = (\sigma_\nu, 0); \quad \nu = 1, \ldots, n$$
$$(\xi^\nu(x), \rho^\nu(x)) = (0, 1); \quad \nu = n+1$$

then the symmetry property ${}^t\varphi \wedge \psi = 0$ will imply that ξ^ν's are exact forms. Write $\xi^\nu = df^\nu$, then $f = (f^1, \ldots, f^{n+1})$ is an isometric holomorphic immersion into $\mathbf{C}^{n+1}$ (with $(\rho^1, \ldots, \rho^{n+1})$ an unit normal vector field along its image).

When Ric_g is negatively definite in a dense subset, A (or equivalently, ψ) is determined by the curvature Ω up to a multiple function τ with $|\tau| = 1$. While when A is replaced by τA, λ will be changed to $\lambda - \tau^{-1} d\tau$. In this case the ξ-part of the solution remains the same, that is to say, different second fundamental tensors on (X, g) will change the isometric immersions from X into $\mathbf{C}^{n-1}$ only by the composition of a rigid motion.

Now let us turn to the proof of Theorem A, but first we need the following:

Lemma 2 *Suppose that (M, g) satisfies the conditions in Theorem* A. *Then M is projective, with canonical line bundle K_M ample, and the analytic subset $M \setminus V_g = \{x \in M\colon \mathrm{Ric}_g^{n-1}(x) = 0\}$ is of codimension at least two.*

Proof. Since $U_g = \{x \in M\colon \mathrm{Ric}_g^n \neq 0\}$ is not empty, so $K_M^n = (-c_1)^n > 0$. By the Riemann-Roch Theorem,

$$\chi(m K_M) = \frac{K_M^n}{n!} m^n + O(m^{n-1}).$$

On the other hand, since $\mathrm{Ric} \leqq 0$ and < 0 in a non-empty set, a generalized version of the Kodaira vanishing theorem (see Theorem 2.27 in [S-S], for example) says that $h^q(m K_M) = 0$ for any $q > 0$ and $m \geqq 2$. So M is of general type, hence projective as it is Kählerian.

By a result of Kawamata [K], if the canonical line bundle K_M of a general type projective manifold M is not ample, the M will contain a rational curve. So in our case, K_M must be ample.

Now if $\{x \in M\colon \mathrm{Ric}_g^{n-1} = 0\}$ contains a divisor D, then $K_M^{n-1} \cdot D = 0$ which contradicts the ampleness of K_M, therefore this set is of codimension at least two. QED

Proof of Theorem A. Let $\pi\colon (\tilde{M}, \tilde{g}) \to (M, g)$ be the universal covering space. Then $\pi^{-1} V_g$ is still simply connected as its complement is of codimension at least two. Now by Proposition 4 and the fundamental theorem for complex hypersurfaces we get an isometric holomorphic immersion $f\colon (\pi^{-1} V_g, \tilde{g}) \to (\mathbf{C}^{n+1}, g_0)$, which is unique up to a rigid motion in $\mathbf{C}^{n+1}$. The Hartogs' extension theorem then gives us the map that we want. QED

Remark. One can replace the condition (3) in Theorem A by the following:

(3′) M is of general type, and g is real analytic.

In this case U_g is not empty (otherwise $K_M^n = 0$, so the numerical Kodaira dimension of M is less than n). The above argument shows that any simply-connected open subset in U_g can be isometrically immersed into $(\mathbf{C}^{n+1}, g_0)$. By the result of Calabi [C] one has the global isometric immersion $f\colon (\tilde{M}, \tilde{g}) \to (M, g)$.

4 The conformal relations of holomorphic bisectional curvature

Throughout this section, let us assume that (M, g) satisfies the conditions in Theorem A, and $h \in \mathscr{F}(M)$ is another metric on M with non-positive holomorphic bisectional curvature. Let $S=\nabla^h-\nabla^g$, and $s=\theta(h)-\theta(g)$ be the matrix of S under a frame e.

Lemma 3 *In U_g, s can be locally written as $s=\psi\,{}^t\alpha$, where ψ is as in Proposition 2 and $\alpha={}^t(\alpha_1,\ldots,\alpha_n)$ is a column vector of smooth functions.*

Proof. Since $h+g$ also belongs to $\mathscr{F}(M)$, by Proposition 1, we have $(\Omega_{v\bar v}(h+g))^2=(\Omega_{v\bar v}(h))^2=0$. Let $\eta=\Omega(g)+\Omega(h)-\Omega(h+g)$, then it is easy to verify that:

$$\eta=sg(h+g)^{-1}h\,{}^t\bar s$$

under any frame e. This implies $\eta\geqq 0$, therefore the vanishing of the square of

$$(-\Omega_{v\bar v}(h+g))=(-\Omega_{v\bar v}g))+(-\Omega_{v\bar v}(h))+\eta_{v\bar v}$$

gives:

$$\Omega_{v\bar v}(g)\wedge\eta_{v\bar v}=\Omega_{v\bar v}(g)\wedge\Omega_{v\bar v}(h)=0$$

for any v.

For any $x\in U_g$, let e be a g-unitary frame near x such that the matrix of h under e (also denote by h) is diagonal:

$$h=\operatorname{diag}(a_1,\ldots,a_n).$$

By Proposition 2, there exist ψ such that $\Omega(g)=-\psi\wedge{}^t\bar\psi$. Write $v=\sum_{i=1}^n v_ie_i$, then we have

$$\Omega_{v\bar v}=-\psi_v\wedge\bar\psi_v;\qquad \psi_v=\sum_{i=1}^n v_i\psi_i$$

$$\eta_{v\bar v}=\sum_{k=1}^n\frac{a_k}{1+a_k}\left(\sum_{j=1}^n v_js_{jk}\right)\wedge\left(\overline{\sum_{j=1}^n v_js_{jk}}\right)$$

therefore $\psi_v\wedge\sum_{j=1}^n v_js_{jk}=0$ for each k. This implies $s_{ij}=\psi_i\alpha_j$ for each i,j. QED

Next we prove the following:

Proposition 5 *There is a positive constant c such that:*

$$\Omega(h)\cdot\det(h)=c\cdot\Omega(g)\cdot\det(g).$$

Proof. We need only to prove this identity in the dense open subset U_g. For any direction v at a point $x\in U_g$, by the fact $(\Omega_{v\bar v}(h))^2=\Omega_{v\bar v}(h)\wedge\Omega_{v\bar v}(g)=0$ and $\Omega_{v\bar v}(g)=-\psi_v\wedge\bar\psi_v$, we know that

$$\Omega_{v\bar v}(h)=\rho_v\Omega_{v\bar v}(g)$$

where ρ_v depends (continuously) on the direction v. Since both g and h are Kählerian, we get:

$$(\rho_v-\rho_u)R_{v\bar vu\bar u}(g)=0$$

for any two tangent directions u, v at x. As $\mathrm{Ric}_g(x)<0$, $R_{v\bar vu\bar u}(g)\neq 0$ for generic u and v. Hence $\rho=\rho(x)$ is independent of the directions.

Write $\Omega(h)\cdot\det(h)=f(x)\cdot\Omega(g)\cdot\det(g)$. f is a non-negative smooth function in U_g which is independent of the choices of the frames. We want to show that $df\equiv 0$.

Fix a point $x\in U_g$, and let e be a holomorphic tangent frame near x which is g-normal at x, i.e., $g(x)=I$, $\theta_g(x)=0$. Write $\xi=hg^{-1}$, $\beta=\dfrac{\det(h)}{\det(g)}$, then:

$$\Theta(h)\cdot\xi\cdot\beta=f\cdot\Theta(g)$$

(recall that $\Theta_g\cdot g=\Omega_g$). Differentiate the above equality and take the trace, we have:

$$\begin{aligned}\partial f\wedge \mathrm{Ric}_g &= \mathrm{tr}\,\partial(\Theta_h\xi\beta)\\ &=\mathrm{tr}\,\partial(\Theta_h\xi)\beta+\mathrm{tr}\,(\Theta_h\xi)\wedge\partial\beta\\ &=\mathrm{tr}\,(s\wedge\Theta_h\xi)\beta+\mathrm{tr}\,(\Theta_h\xi)\wedge\beta\,\mathrm{tr}\,(s)\\ &=f\{\mathrm{tr}\,(s)\wedge\mathrm{tr}\,(\Omega_g)+\mathrm{tr}\,(s\wedge\Omega_g)\}\\ &=-f\{\mathrm{tr}\,(\psi\,{}^t\alpha)\wedge\mathrm{tr}\,(\psi\wedge{}^t\bar\psi)+\mathrm{tr}\,(\psi\,{}^t\alpha\wedge\psi\wedge{}^t\bar\psi)\}\\ &=0.\end{aligned}$$

Therefore $\partial f=0$, hence $df=0$ in U_g as f is real-valued. This implies that f is constant in each connected component of U_g. Since (M,h) cannot be flat, and V_g is connected, it is not hard to see that these constants must be equal. QED

Corollary 1 *If (M,g) satisfies the conditions in Theorem* A, *then any $h\in\mathscr{F}(M)$ also satisfies the same conditions, and $U_h=U_g$; $V_h=V_g$.*

5 Proof of Theorem B

Again let (M,g) be a Kähler manifold satisfying the conditions in Theorem A, and $h\in\mathscr{F}(M)$ be another Kähler metric on M with non-positive holomorphic bisectional curvature. By Corollary 1, (M,h) also satisfies those conditions, so Theorem A gives isometric holomorphic immersions:

$$f:(\tilde M,\tilde g)\to(\mathbf{C}^{n+1},g_0);\qquad f^{(h)}:(\tilde M,\tilde h)\to(\mathbf{C}^{n+1},g_0)$$

from the universal covering spaces into the complex euclidean space. In this section we are going to show that there exists an affine transformation ϕ in $\mathbf{C}^{n+1}$ such that $f^{(h)}=\phi\circ f$.

First of all let us recall that in the fundamental theorem for complex hypersurfaces, a second fundamental tensor A for (M,g) with associated 1-form λ gives a linear system $(*_g)$:

$$\begin{aligned}&\nabla\xi-\rho A=0\\ &d\rho-\rho\lambda+g(\xi,\bar A)=0\end{aligned}$$

and if $\{(\xi^\nu,\rho^\nu);\ \nu=1,2,\dots,n+1)\}$ is a basis of solutions of $(*_g)$ under suitable initial conditions, then ξ^ν's are all exact: $\xi^\nu=df^\nu$, and $f=(f^1,\dots,f^n)$ gives the local isometric immersion. For (M,h), one has parallel situations and the corresponding system $(*_h)$. So for our purpose it would be sufficient to show that the ξ-part solutions of $(*_g)$ and $(*_h)$ are the same. In order to do this, first let us fix some notations.

Locally near a point $x \in U_g = U_h$, let ψ, λ be as in Proposition 4 for (M, g), and ψ', λ' for (M, h). Then ψ, ψ' both give local coframes near x. By Proposition 5 we have

$$\psi' = e^{a+ib}\psi; \qquad \Omega_h = e^{2a}\Omega_g$$

where both a and b are real. Also be Lemma 3, we can write $s \equiv \theta_h - \theta_g = \psi\, {}^t\alpha$.

Lemma 4 *Under the above notations,*

$$\lambda' = \lambda - i\,db + \partial a - \bar{\partial} a; \qquad 2\partial a = -{}^t\psi\,\alpha.$$

Proof. Plug $\psi' = e^{a+bi}\psi$ and $d\psi = \theta \wedge \psi - \lambda \wedge \psi$ into $d\psi' = \theta' \wedge \psi' - \lambda' \wedge \psi'$, we get:

$$(\lambda' - \lambda - s + da + i\,db) \wedge \psi' = 0.$$

Since ψ' is a coframe, the $(0,1)$-part of the braces must vanish, while s is of type $(1,0)$, hence

$$\lambda' = \lambda - i\,db + \partial a - \bar{\partial} a.$$

Next, since $e^{2a} = c\dfrac{\det g}{\det h}$, so (when e is a holomorphic frame):

$$2\partial a = \partial \log \det g - \partial \log \det h = -\operatorname{tr} s = -{}^t\psi\,\alpha.$$

Note that the two very end terms are independent of the choice of the frame, therefore this equality holds for any frame e. QED

Lemma 5 *For any* $(1,0)$*-form* $\xi = \sum_{i=1}^{n} \xi_i \varphi_i$, *write* $\sigma(\xi) = \sum_{i=1}^{n} \alpha_i \xi_i$. *Then*:

$$\nabla' \xi - \nabla \xi = -\sigma(\xi) \cdot A.$$

Proof. By $s = \psi \wedge {}^t\alpha$ and $A = {}^t\varphi \otimes \psi = {}^t\psi \otimes \varphi$, a straight calculation shows that:

$$\nabla' \xi - \nabla \xi = S(\xi) = \sum_{i=1}^{n} \xi_i S(\varphi_i) = \sum_{i=1}^{n} \xi_i(-s_{ji} \otimes \varphi_j)$$

$$= -\sum_{i,j=1}^{n} \xi_i \psi_j \alpha_i \otimes \varphi_j = -\sigma(\xi) \cdot A. \quad \text{QED}$$

Proposition 6 *If* (ξ, ρ) *is a solution of* $(*_g)$, *let* $\rho' = e^{-a-ib}(\rho - \sigma(\xi))$. *Then* (ξ, ρ') *is a solution for* $(*_h)$.

Proof. Consider the system $(*_h)$:

$$\nabla' \xi - \rho' A' = 0$$
$$d\rho' - \rho' \lambda' + h(\xi, \bar{A}') = 0.$$

Since (ξ, ρ) satisfies $(*_g)$ and $\rho' = e^{-a-ib}(\rho - \sigma(\xi))$, the first equation of $(*_h)$ is immediate by Lemma 5:

$$\nabla' \xi = \nabla \xi - \sigma(\xi) A = \rho A - \sigma(\xi) A = \rho' A'.$$

For the second equation, let B denote the value of its right hand side multiply by e^{a+ib}. Then by Lemma 4 we have:

$$\begin{aligned} B &= e^{a+ib}\{d\rho' - \rho'\lambda' + h(\xi, \bar{A}')\} \\ &= d(\rho - \sigma(\xi)) - (\rho - \sigma(\xi))(da + idb + \lambda - idb + \partial a - \bar{\partial} a) + e^{2a} h(\xi, \bar{A}) \\ &= d(\rho - \sigma(\xi)) - (\rho - \sigma(\xi))(\lambda - {}^t\psi\alpha) + e^{2a} h(\xi, \bar{A}) \\ &= (d\rho - \rho\lambda) - d\sigma(\xi) + \sigma(\xi)\lambda + (\rho - \sigma(\xi))^t\psi\alpha + e^{2a} h(\xi, \bar{A}) \\ &= e^{2a} h(\xi, \bar{A}) - g(\xi, \bar{A}) - d\sigma(\xi) + \sigma(\xi)\lambda + (\rho - \sigma(\xi))^t\psi\alpha. \end{aligned}$$

Write $\xi = {}^t(\xi_i, \ldots, \xi_n)$ also as the column vector of its coefficients under the coframe φ, then as it satisfies $(*_g)$, one has: $\nabla({}^t\xi\varphi) = d\,{}^t\xi \otimes \varphi - {}^t\xi\,{}^t\theta \otimes \varphi = \rho A = \rho\,{}^t\psi \otimes \varphi$. Hence $d\,{}^t\xi\,{}^t\theta + \rho\,{}^t\psi$ and:

$$d\sigma(\xi) = d({}^t\xi\alpha) = {}^t\xi\,{}^t\theta\alpha + {}^t\xi\, d\alpha + \rho\,{}^t\psi\alpha$$

while

$$e^{2a} h(\xi, \bar{A}) - g(\xi, \bar{A}) = {}^t\xi(e^{2a}\,{}^th^{-1} - {}^tg^{-1})\bar{\psi}$$

therefore

$$B = {}^t\xi\{(e^{2a}\,{}^th^{-1} - {}^tg^{-1})\bar{\psi} - {}^t\theta\alpha - d\alpha + \alpha\lambda - \alpha\,{}^t\psi\alpha\}.$$

Let us denote the $\{\ \}$ term by η, which is a column vector of 1-forms. In order to show that η vanishes, it suffices to check it under a special frame. For any point x, let e be a holomorphic frame which is g-normal at x, i.e., $g(x) = I$, $dg(x) = 0$. So at x, $\theta = 0$, $\theta' = s = \psi\,{}^t\alpha$, and:

$$ds = d(\theta' - \theta) = \Theta' + \theta' \wedge \theta' - \Theta = \Omega' h^{-1} - \Omega g^{-1} + s \wedge s.$$

By Proposition 5, and take trace, we have that at x:

$$d\,{}^ts = (e^{2a}\,{}^th^{-1} - {}^tg^{-1})\bar{\psi} \wedge {}^t\psi - {}^ts \wedge \alpha\,{}^t\psi.$$

On the other hand,

$$d\,{}^ts = d(\alpha\,{}^t\psi) = d\alpha \wedge {}^t\psi - \alpha\lambda \wedge {}^t\psi$$

therefore at the point x,

$$\{(e^{2a}\,{}^th^{-1} - {}^tg^{-1})\bar{\psi} - {}^ts\alpha - d\alpha + \alpha\lambda\} \wedge {}^t\psi = 0$$

hence $\eta(x) = 0$. Since x is arbitrary, we proved that $\eta \equiv 0$, so $B = 0$ as we wanted. QED

Proof of Theorem B. Let (M, g) and $h \in \mathscr{F}(M)$ be as in Theorem B. By Corollary 1, (M, h) also satisfies the same conditions. Hence there are isometric holomorphic immersions:

$$f: (\tilde{M}, \tilde{g}) \to (\mathbf{C}^{n+1}, g_0)$$
$$f^{(h)}: (\tilde{M}, \tilde{h}) \to (\mathbf{C}^{n+1}, g_0)$$

from the universal covering spaces into the complex euclidean space. By Proposition 6, we know that locally f and $f^{(h)}$ differ only by an affine transformation in $\mathbf{C}^{n+1}$, that is, there is an open covering $\{\mathcal{U}_\alpha\}$ of M' and affine transformations ϕ_α in $\mathbf{C}^{n+1}$ with $f^{(h)}|_{\mathcal{U}_\alpha}=\phi_\alpha \circ f|_{\mathcal{U}_\alpha}$ for each α. For any α, β with $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$, one has $\phi_\alpha \circ \phi_\beta = \mathrm{id}$ on $f(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$. Since the fixed point set of any non-trivial affine transformation is an proper affine subspace (or empty), while $f(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)$ can not be flat as $\mathrm{Ric}_g < 0$ in a dense subset. Therefore $\phi_\alpha = \phi_\beta$ whenever $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \neq \phi$, hence all the ϕ_α's are the same, and Theorem B is proved. QED

In the following, let us give an alternative and shorter proof of Theorem B, by using the conformal relation Proposition 5 and Theorem A (for both g and h). This proof is suggested to us by the referee of this paper:

Proposition 6′ *Suppose f, f' are two holomorphic embeddings from a small piece of complex n-manifold X into the complex Euclidean space $(\mathbf{C}^{n+1}, g_0)$. Let A, A' be any second fundamental tensor corresponds to f and f', respectively. Suppose $\sigma: X \to X$ is a biholomorphism such that $A(u,v)=0$ if and only if $A'(\sigma_* u, \sigma_* v)=0$. If A or A' is of rank $\geqq 2$ somewhere, then σ is induced by an affine transformation on $\mathbf{C}^{n+1}$.*

Proof. Without loss of generality, let us assume that σ is the identity map, and locally the two hypersurfaces are given by graphs $z_{n+1}=F(z_1,\ldots,z_n)$ and $z_{n+1}=G(z_1,\ldots,z_n)$, respectively. Under the coordinate $\{z_1,\ldots,z_n\}$, a straightforward calculation shows that the bisectional curvature tensors are $\Omega = (1+|F_1|^2+\ldots+|f_n|^2)^{-1}\,{}^t(\partial F_1,\ldots,\partial F_n)\wedge(\overline{\partial F_1},\ldots,\overline{\partial F_n})$ and similarly for Ω'. Now the condition in Proposition 6′ says that A and A' are proportional to each other, therefore Ω and Ω' are also proportional. Hence there is a holomorphic function b on X such that $G_{ij}=b \cdot F_{ij}$ for any $1 \leqq i,j \leqq n$. From this one gets $b_k F_{ij} - b_j F_{ik} = 0$ for any i,j,k. It follows that $b_k \equiv 0$ for all k, since the rank of the $n \times n$ matrix (F_{ij}) is $\geqq 2$ by the assumption. Therefore b is a constant. So by the relation $G_{ij}=b \cdot F_{ij}$ we know that f and f' (or rather $f' \circ \sigma$) differ by an affine transformation of $\mathbf{C}^{n+1}$. QED

6 An example

In this final section, let us consider the following class of manifolds which satisfies the conditions of Theorem A.

Assume that M is a n-dimensional compact complex manifold of general type such that there is a holomorphic immersion $\gamma: M \to \mathbf{T}^{n+1}$ from M into a complex $(n+1)$-torus.

Lemma 6 *For any flat metric g_0 on $\mathbf{T}^{n+1}$, let $g=\gamma^*(g_0)$. Then (M,g) satisfies the conditions in Theorem* A.

Proof. Certainly (M,g) is a compact Kähler manifold with non-positive holomorphic bisectional curvature, and the short exact sequence:

$$0 \to T_M \to \gamma^* T_{\mathbf{T}^{n+1}} = \mathcal{O}_M^{n+1} \to \mathcal{N} \to 0$$

implies that $\mathcal{N} \cong K_M$, and $c_1^2(M) - c_2(M) = 0$.

Let $D_k=\{x\in M: \mathrm{Ric}_g^k(x)=0\}$. Then $V_g=M\backslash D_{n-1}\supseteq U_g=M\backslash D_n$. Since M does not contain any rational curve, by [K], $K_M^n>0$, hence $D_n\neq M$, or $U_g\neq\phi$. Now it suffices to show that each D_k is an analytic subset in M. In order to see this, locally let $(z,\ldots,z_{n+1})$ be natural coordinates of $\mathbf{T}^{n+1}$ such that a small piece (say, U) of $\gamma(M)$ is defined by a holomorphic function $z_{n+1}=f(z_1,\ldots,z_n)$. Then it is a straight calculation to verify that $D_k\cap U=\{x\in U: \mathrm{rank}(F)<k\}$, where F is the $n\times n$ matrix with entries $F_{ij}=\dfrac{\partial^2 f}{\partial z_i\partial z_j}$ which are holomorphic functions. QED

Corollary 2 *Suppose that $\gamma: M\to\mathbf{T}^{n+1}$ is a holomorphic immersion from a n-dimensional general type manifold into a complex $(n+1)$-torus. Then $\mathscr{F}(M)$ consists of the pull-backs of the flat metrics on $\mathbf{T}^{n+1}$.*
In particular, M is semi-rigid.

Proof. Let $g=\gamma^*(g_0)$ be the pull-back of a flat metric on $\mathbf{T}^{n+1}$, and suppose that h is an arbitrary metric in $\mathscr{F}(M)$. Let $f:(M',g')\to(\mathbf{C}^{n+1},g_0)$ be the lift of γ to the universal covering spaces. By Lemma 6 and Theorems A and B, there is also an isometric holomorphic immersion

$$f^{(h)}:(\tilde{M},\tilde{h})\to(\mathbf{C}^{n+1},g_0)$$

for h, and an affine transformation ϕ in $\mathbf{C}^{n+1}$ such that $f^{(h)}=\phi\circ f$. Furthermore, for any deck transformation $\sigma\in\pi_1(M)$, there are rigid motions ϕ_σ, ϕ'_σ in $\mathbf{C}^{n+1}$ such that

$$f\circ\sigma=\phi_\sigma\circ f;\qquad f^{(h)}\circ\sigma=\phi'_\sigma\circ f^{(h)}.$$

Combine these two identities we get $\phi'_\sigma=\phi\circ\phi_\sigma\circ\phi^{-1}$ for each $\sigma\in\pi_1(M)$.

Since f is the lift of γ to the universal covers, all ϕ_σ's are translations, hence the conjugations ϕ'_σ are also translations, which preserve any flat metric on $\mathbf{C}^{n+1}$, hence $h=\gamma^*(h_0)$ for a flat metric h_0 on $\mathbf{T}^{n+1}$. QED

Remark. In Theorem A, if one assume that $\pi_1(M)$ contains an abelian subgroup of finite index, then it is easy to show that there exists an holomorphic isometric immersion from a finite cover of M into an abelian variety $\mathbf{T}^{n+1}$. However, we do not know if this should be the case in general.

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