Math. Ann. (2017) 369:1271–1282
DOI 10.1007/s00208-016-1486-y

Mathematische Annalen

Nonexistence for complete Kähler–Einstein metrics on some noncompact manifolds

Peng Gao[1] · Shing-Tung Yau[1] · Wubin Zhou[2]

Received: 31 March 2016 / Revised: 21 September 2016 / Published online: 15 October 2016

Abstract Let M be a compact Kähler manifold and N be a subvariety with codimension greater than or equal to 2. We show that there are no complete Kähler–Einstein metrics on $M - N$. As an application, let E be an exceptional divisor of M. Then $M - E$ cannot admit any complete Kähler–Einstein metric if blow-down of E is a complex variety with only canonical or terminal singularities. A similar result is shown for pairs.

1 Introduction and main theorem

A basic question in Kähler geometry is how to find on each Kähler manifold a canonical metric such as Kähler–Einstein metric, constant scalar curvature Kähler metric, or even extremal metric. When the Kähler manifold is compact with negative or zero first Chern class, the question has been solved by the senior author's celebrated work on Calabi's conjecture [21]. Whereas the first Chern class is positive, there is an obstruction called Futaki invariant for the existence of Kähler–Einstein metric.

For the noncompact case, the majority of work focuses on open Käher manifolds or quasi-projective manifolds. A complex manifold M is open (resp. quasi-projective) if there is a compact Kähler (resp. projective) manifold $\bar{M}$ with an effective divisor D

✉ Wubin Zhou
wbzhou@tongji.edu.cn

Peng Gao
penggao@math.harvard.edu

Shing-Tung Yau
yau@math.harvard.edu

1 Department of Mathematics, Harvard University, Cambridge, MA 02138, USA

2 School of Mathematical Sciences, Tongji University, Shanghai 200092, China

such that M is biholomorphic to $\bar{M} - D$. The second author raised the question concerning existence of complete Kähler–Einstein metrics on quasi-projective varieties in [22], where the results of [3] were announced for the case with constant negative scalar curvature. Also, the second author announced in [22] the existence of complete Ricci-flat Kähler metrics on the complement of an anticanonical divisor, using methods following his earlier work in [19,21].

Cheng and Yau [3] constructed complete negative Kähler–Einstein metrics on $\bar{M} - D$ when D is a normal crossing divisor and $K_M + D$ positive. In [15,16] the second author showed that there exists a complete Ricci flat Kähler metric on quasi-projective M, if D is a neat and almost ample smooth divisor on M; or $\bar{M}$ is a compact Kähler orbifold and D is a neat, almost ample and admissible divisor on $\bar{M}$. This follows the analysis of [21]. In [17] he also proved on open manifold M there are complete Kähler–Einstein metrics with negative scalar curvature if the adjoint canonical bundle $K_M + D$ is ample. This too follows the analysis of [21].

Several years ago, the second author [5] proposed the following questions.

Problem 1.1 Let M be a compact Kähler manifold and N be a subvariety with codimension bigger than or equal to 2, how to find a complete canonical metric on the noncompact Kähler manifold $M - N$?

In the Ricci-flat case, this was answered in [22] based on a theorem proved in [19] for volume of complete noncompact Riemannian manifolds. In order to handle this question, the authors (in [4,5]) introduced the concept of complete metrics with Poincaré–Mok–Yau (PMY) asymptotic property, and constructed many constant scalar curvature Kähler metrics with PMY asymptotic property on some special types of noncompact Kähler manifolds. Since these PMY type metrics are not Kähler–Einstein, naturally one can ask the following question.

Problem 1.2 Can $M - N$ be endowed with complete Kähler–Einstein metrics?

In fact, little is known about the obstruction for the existence of complete Kähler–Einstein on noncompact Kähler manifolds. The unique outstanding result is due to Mok and Yau's main theorem in [11] which states that a bounded domain Ω admits a complete Kähler–Einstein if and only if Ω is a domian of holomorphy. If N is a higher codimenion subvariety of a bounded domain Ω, then $\Omega - N$ is not a domain of holomorphy. This implies on $\Omega - N$ there are no complete Kähler metrics with nonpositve Ricci curvature. In our discussions on Problem 1.2, the senior author proposed that the answer should be negative and the original work in [11] will give hints. Here we follow this idea and it turns out that we can obtian the following theorem.

Theorem 1.3 *There are no complete Kähler metrics ω on $M - N$ satisfying $-\lambda \leq Ric_\omega \leq 0$ with nonnegative constant λ.*

Since $M - N$ is noncompact, according to Bonnet–Myer's compactness theorem $M - N$ cannot admit Kähler–Einstein metrics with positive Ricci curvature. Immediately, we find a negative answer for Problem 1.2 that

Corollary 1.4 *There are no complete Kähler–Einstein metrics ω on $M - N$.*

If N was allowed to have singularities, Corollary 1.4 would imply there are no complete Kähler–Einstein metric on $\bar{M} - D$, where $\bar{M}$ is a compact Kähler manifold that desingularizes M and D the exceptional locus of $\bar{M}$. By Hironaka's theorem on resolution of singularities, such desingularizations always exist over a field of characteristic 0, also true for analytical varieties. More precisely, blowing up M along N, one obtains a new compact Kähler manifold $\bar{M} = Bl_N(M)$. Then $M - N$ is biholomorphic to $Bl_N(M) - D$ where D is the exceptional set. And it's clear $\bar{M} - D$ has a complete Kähler–Einstein metric if and only if $M - N$ has one.

Similarly one could ask about the inverse operation, whether there is a complete Kähler–Einstein metric on $M - E$ if E is an exceptional divisor which blows down to a singularity. It is natural to expect that the answer depends on the type of singularities produced for example by a divisorial contraction. In fact, by a local analysis we can show the following.

Theorem 1.5 *Let $f : M \to X$ be two complex normal varieties which are birational, and E the sum of the exceptional divisors. Then if X has only canonical or terminal singularities, and the codimension two singular locus is non-empty in X, there are no complete Kähler–Einstein metrics on $M - E$.*

This result holds in general dimension. But it is difficult to relax the assumption on codimension of the singular locus in X and the non-emptiness of codimension two locus is a necessary condition. We also thank Chenglong Yu for discussion on this point. As a corollary, we obtain in dimension two

Corollary 1.6 *If M is a complex surface and the singularities of X are of type A-D-E, then there are no complete Kähler–Einstein metrics on $M - E$.*

We can generalize Theorem 1.5 to the case of pairs (M, D) with boundary divisor D. As our analysis essentially depends only on the curvature condition, it can be carried out for these open cases as well. For pairs, we have

Theorem 1.7 *If the pair (X, D) is* klt*, then there are no complete Kähler–Einstein metrics on $M - E$, where E is the exceptional divisor of a log resolution $f : M \to X$ where $E \cap f^{-1}(\mathrm{supp}(D))$ is simple normal crossing.*

It is clear any open manifold (non-complete variety) birational to these pairs also cannot admit complete Kähler–Einstein metrics.

In Theorems 1.5 and 1.7 the adjoint canonical bundle $K_M + E$ is generally not ample. It demonstrates that the ampleness condition in [15–18] is necessary. Besides, various generalizations of Kähler–Einstein metrics to the singular setting for pairs have been proposed in the literature, see e.g. [1,2,12] etc. Our theorems assert immediately that these singular metrics for klt pairs are not complete, and it has been shown for klt pairs these metrics have cone singularities [7].

The rest of this note is devoted to the proof of Theorems 1.3, 1.5 and 1.7.

2 The Proof of Theorem 1.3

The Proof of Theorem 1.3 We prove Theorem 1.3 by contradiction and the key point is to show the boundedness of volume element nearby N. Let $\dim_{\mathbb{C}} M = n$ and

$\dim_{\mathbb{C}} N = l$ and $l \leq n-2$. We assume that over $M - N$ there is a complete Kähler metric ω with Ricci curvature

$$-\lambda \leq Ric_\omega \leq 0$$

with a nonnegative constant λ. Locally, let $p \in N$, there is a local open ball $B_\xi = \{(z_1, z_2, z_3, \ldots, z_n), |z_i| < \xi\}$ such that $p = \{0, \ldots, 0\}$ and $N \cap B_\xi$ is

$$z_1 = z_2 = \cdots = z_{n-l} = 0.$$

On this open set $B_\xi - N$, we assume that $\omega = \sqrt{-1} g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. Choose a two dimension polydisc $\Delta_p(\xi)$ to be $\left\{(z_1, z_2, 0, \ldots, 0)\big||z_1| \leq \xi, |z_2| \leq \xi\right\}$, then the logarithm of the volume form of the metric, i.e. $\det g_{i\bar{j}} = e^{\log\det(g_{i\bar{j}})}$ is plurisubharmonic over $\Delta_p(\xi) - p$ according to

$$-Ric = \partial\bar{\partial}(\log\det(g_{i\bar{j}})) \geq 0.$$

Applying the maximum principle,

$$\det(g_{i\bar{j}})\big|_{D_{z_2}} \leq \sup_{|z_2|=\xi} \det(g_{i\bar{j}}).$$

where D_{z_2} is the disc defined by

$$D_{z_2} = \{(z_1, z_2, 0, \ldots, 0) \text{ with } |z_1| \text{ is fixed}\} \subset \Delta_p(\xi) - p.$$

Since the set $\{(z_1, z_2, 0, \ldots, 0)$ with $|z_2| = \xi\}$ in $\overline{\Delta_p(\xi)}$ is a close subset of $M - N$, we have

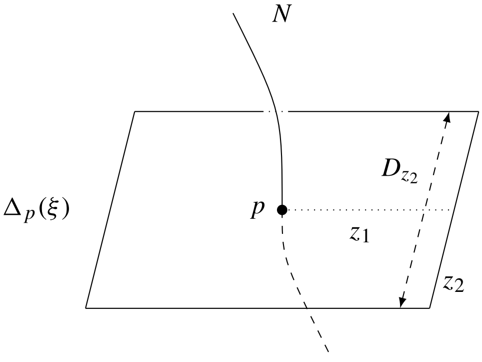
$$\det(g_{i\bar{j}}(z_1, z_2)) \leq \sup_{|z_2|=\xi} \det(g_{i\bar{j}}) \leq C_1 \quad \text{for } 0 < |z_1| \leq \xi, \quad |z_2| \leq \xi, \tag{2.1}$$

with constant C_1. Similarly, the following inequality is satisfied for some constant C_2

$$\det(g_{i\bar{j}}(z_1, z_2)) \leq \sup_{|z_1|=\xi} \det(g_{i\bar{j}}) \leq C_2 \quad \text{for } |z_1| \leq \xi, \quad 0 < |z_2| < \xi. \tag{2.2}$$

Combining (2.1) and (2.2), we have

$$det(g_{i\bar{j}})\big|_{\Delta_p(\xi)-p} \leq \max C_1, C_2. \tag{2.3}$$



the diagram of $\Delta_p(\xi)$

On the other hand, we *claim*

$$\omega > C_3 dz_i \wedge d\bar{z}_i \tag{2.4}$$

on the ball $B_{\frac{\xi}{2}} - N$ with some constant C_3 and $B_{\frac{\xi}{2}} = \{(z_1, \dots, z_n), |z_i| < \frac{\xi}{2}\}$.

If the claim (2.4) is true, then $\det(g_{i\bar{j}})\big|_{B_{\frac{\xi}{2}} - N} > C$. In particular, (2.3) and (2.4) yield

$$g_{1\bar{1}}\big|_{\Delta_p(\frac{\xi}{2}) - p} \le C$$

for some constant C. Let $z_1 = x_1 + \sqrt{-1} y_1$, the length of the real line segment in space $\{(z_1, 0, \dots, 0)\}$ from $(\frac{\xi}{2}, 0, \dots, 0)$ to p is given now by

$$\int_0^{\xi} \sqrt{g_{1\bar{1}}} dx_1 \le \sqrt{C}\xi \tag{2.5}$$

which contradicts the completeness of the metric ω nearby N, and the proof is complete.

It remains to show the claim (2.4). In fact, put Poincaré metric ω_P on B_ξ, and let $u = \text{trace}_\omega \omega_P$. It is clear that

$$\omega_P \le u\,\omega. \tag{2.6}$$

Now we just need to show the boundedness of supremum $\sup_{B_{\frac{\xi}{2}} - N} u$. There are two cases that u tends to its supremum $\sup_{B_{\frac{\xi}{2}} - N} u$.

Case 1 Denote $\bar{B}_{\frac{\xi}{2}} = \{(z_1, \dots, z_n), |z_i| \le \frac{\xi}{2}\}$, if there is a point $p_0 \in \bar{B}_{\frac{\xi}{2}} - N$ such that there is a sequence $p_k \in B_{\frac{\xi}{2}}$ satisfying

$$\sup_{B_{\frac{\xi}{2}} - N} u = \lim_{p_k \to p_0} u,$$

then one has

$$\sup_{B_{\frac{\xi}{2}}-N} u = \lim_{p_k \to p_0} u < \infty$$

since $p_0 \notin N$, and ω_P and ω are both finite at p_0.
Case 2 Otherwise, u achieves its supremum nearby $B_{\frac{\xi}{2}} \cap N$. Here one cannot apply Yau's Schwartz lemma to get the boundness of u directly since ω is not complete over the entire $B_{\frac{\xi}{2}} - N$. However, we can follow the method in [20] to verfy it. In fact, applying Chern–Lu inequality [20], we have

$$\Delta u \geq -\lambda u + cu^2 \tag{2.7}$$

where Δ is the Laplace operator under the metric ω and $-c$ is the upper bound of the bisectional curvature of ω_P. Now, we need the following lemma

Lemma 2.1 *Let f be a smooth function which is bounded on $B_{\frac{\xi}{2}} - N$ and f achieves its supremum nearby N, then for all $\varepsilon > 0$, there exists a point $p \in B_{\frac{\xi}{2}} - N$ such that at p,*

$$|\nabla f| < \varepsilon, \quad \Delta f < \varepsilon \quad \textit{and} \quad f(p) > \sup_{B_{\frac{\xi}{2}}-N} f - \varepsilon. \tag{2.8}$$

The proof of this lemma follows Theorem 1 in [19]. We give a concise description. Let $\mathbb{R}$ be the real line, consider the graph $\Gamma = (f(z), z)$ as a submanifold of $\mathbb{R} \times (B_{\frac{\xi}{2}} - N)$, where $\mathbb{R} \times (B_{\frac{\xi}{2}} - N)$ is the product manifold with the product metric. Fixing a point $p_0 \in B_{\frac{\xi}{2}} - N$, consider the point $p_k = (k, p_0)$ where k is a positive integer. Let g_k be a geodesic segment from p_k to the graph Γ such that the length of g_k is the distance between p_k and Γ. Let $(f(q_k), q_k)$ be another end point of g_k, and one can check $(f(q_k), q_k)$ is not a conjugate point of p_k. Let l_k be the arclength of g_k, and let $r(z)$ be the geodesic distance between p_0 and z. By the geodesic equation of g_k, for each k and q_k, one has

$$l_k^2 = (k - f(q_k))^2 + r(q_k)^2.$$

Then we can show

$$\lim_{k\to\infty} \sup f(q_k) = \sup_{B_{\frac{\xi}{2}}-N} f(z).$$

In fact, if one can find some $\delta > 0$ and z_0 such that

$$f(z_0) > \lim_{k\to\infty} \sup f(q_k) + \delta,$$

then for k large enough

$$(k - f(z_0))^2 + r(z_0)^2 < (k - f(q_k))^2 + r(q_k)^2 = l_k^2$$

which contradicts the minimality of the geodesic distance between p_k and Γ. In this sense, we can choose a subsequence still denoted by q_k such that

$$\sup_{k\to\infty} f(q_k) = \sup_{B_{\frac{\xi}{2}}-N} f(z). \tag{2.9}$$

For simple, we assume $f(p_0)=0$. By the boundenss of f and the assumption $r(q_k)\geq 1$, then there is no difficulty to verify the followings (or see page 205 in [19]):

(i) $\Delta f(q_k)\leq (Q+\frac{2}{r(q_k)})\frac{f(q_k)}{r(q_k)}$ for some constant Q (not dependent on k),
(ii) $|\nabla f(q_k)|\leq \frac{2f(q_k)}{r(q_k)}$.

Since f attains its supremum nearby N and ω is complete nearby N, thus $r(p_k)$ goes to infinity as k tends to infinity. Consequently, together with (i), (ii) and (2.9), we can obtain Lemma 2.1.

Then let K be any positive number, a direct computation shows

$$\Delta\left(-\frac{1}{\sqrt{u+K}}\right)=\frac{\Delta u}{2(u+K)^{3/2}}-\frac{3}{(u+K)^{5/2}}|\nabla u|^2 \tag{2.10}$$

From (2.7),

$$\frac{1}{\sqrt{u+K}}\Delta\left(-\frac{1}{\sqrt{u+K}}\right)+\frac{3}{(u+K)^3}|\nabla u|^2\geq\frac{-\lambda u+cu^2}{2(u+K)^2} \tag{2.11}$$

Then by Lemma 2.1 and let $f=-\frac{1}{\sqrt{u+K}}$, there are sequences p_k such that

$$\lim_{k\to\infty}\Delta\left(-\frac{1}{\sqrt{u(q_k)+K}}\right)=0,\quad \lim_{k\to\infty}\frac{|\nabla u|^2}{(u(q_k)+K)^3}=0,\quad \lim_{k\to\infty}f(q_k)=\sup_{B_{\frac{\xi}{2}}-N}f(z).$$

When $f(q_k)$ goes to its supremum, u goes to its supremum $\sup_{B_{\frac{\xi}{2}}-N}u$. Obviously, $\lim\sup_{B_{\frac{\xi}{2}}-N}u\neq\infty$, otherwise the term on the right hand in (2.11) tends to $\frac{c}{2}$ which leads to a contradiction. Therefore

$$\sup_{B_{\frac{\xi}{2}}-N}u\leq\frac{\lambda}{c}.$$

In short, from Case (1) and Case (2) u is bounded and the claim (2.4) is obtained. □

Remark 2.2 From the proof above, we see that all analysis is local (that is, there are no Kähler–Einstein metrics ω on $B_\xi - N$ such that ω is complete nearby N). It means Theorem 1.3 is still true when M is not compact which therefore allows us to prove similar statements in cases when more than one smooth subvariety of codimension 2 or higher are removed.

3 The Proof of Theorem 1.5

Although our proof shall be local and mainly differential geometric in nature, it is useful to phrase the theorem in the right algebraic setting. The notations are standard in the literature, the readers can consult the standard [9] for further detail.

3.1 Canonical singularities

First we recall some properties of the class of singularities known as *canonical singularity*. We do not restrict to any specific number of dimension of the variety.

Unless otherwise stated, we will be considering normal varieties. Then by Zariski's main theorem which applies to normal varieties, we know that the fundamental locus of all birational morphisms are closed subvarieties of codimension at least 2.

Denote by $M_{\text{reg}} = M - M_{\text{sing}}$ the complement of the singular set M_{sing}, we then have the open immersion $M_{\text{reg}} \overset{i}{\hookrightarrow} M$. The canonical sheaf $\omega_M = i_* \omega_{M_{\text{reg}}}$ exists by Hartog's extension theorem and M being normal. We note that the fundamental locus is generally not contained in M_{sing}, and vice versa. The former clearly depends on the choice of a birational morphism.

Definition 3.1 (*Canonical singularity*) A n-dimensional normal variety M with $\mathbb{Q}$-Cartier canonical divisor K_M is said to have only canonical singularities, if there exists a birational morphism $f : Y \to M$ from a smooth variety Y such that in the ramification formula

$$K_Y = f^* K_M + \sum a_i E_i$$

$a_i \geq 0$ for all divisors E_i which are exceptional.

We shall need later the following result (3.4)(A) from [13].

Theorem 3.2 (Canonical $\Rightarrow$ Du Val in codim 2) *Let M be a n-fold with canonical singularities, not necessarily isolated, then M is isomorphic to the following form analytically*

$$M \cong (\textit{Du Val sing.}) \times \mathbb{A}^{n-2}$$

in the neighborhood of a general point of any codimension 2 stratum.

Where surface canonical singularities are Du Val singularities, also called A-D-E singularities. This can be easily proved using the 'general section theorem' for canonical singularities, see e.g. theorem (1.13) in [14].

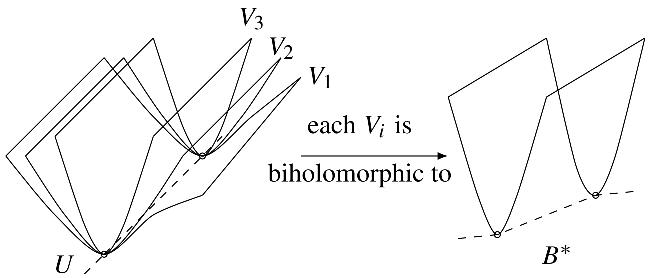
3.2 The Proof of Theorem 1.5

To proceed we need also the following lemma which proves a version of Theorem 1.5 for a canonical (Du Val) singularity in two dimensions.

Lemma 3.3 (Du Val surface singularity) *If M is a normal complex surface with only canonical singularities, then there does not exist a complete Kähler–Einstein metric on the complement of a finite number of points on M.*

The proof of the lemma Here we will show the case that there is only one exceptional divisor E and $\bar{M}$ is smooth. If $f(E)$ consists purely of smooth points of M, we get form Theorem 1.3 that there cannot exist complete Kähler metrics on $M - f(E)$ with Ricci curvature satisfying $-\lambda \leq Ric \leq 0$. So we assume $f(E) \subset M_{\text{sing}}$.

Recall a Du Val singularity is given locally by a polynomial in $\mathbb{A}^3$, i.e. a hypersurface. By Weierstrass preparation theorem, near p, $M - p$ can be covered by finitely many disjoint open sets B_i^* such that each B_i^* is biholomorphic to standard punctured disc $B^* = \{(z_1, \ldots, z_n), 0 < \sum_{i=1}^n |z|^2 < 1\}$, $n = 2$. Then if there is a complete Kähler–Einstein metric ω on $M - p$, each B_i^* admits a Kähler–Einstein metric $\omega\big|_{B_i^*}$. It implies B^* can be endowed with a Kähler–Einstein metric that is complete at the punctured point. But this is impossible from the Proof of Theorem 1.3. This finishes the proof. □



Du Val singularities are also quotients of $\mathbb{C}^2$ by finite subgroups of $SL(2, \mathbb{C})$. By Cartan's lemma any quotient singularity is isomorphic to $\mathbb{C}^n/G$ with $G \subset GL(n, \mathbb{C})$ a finite group. The following theorem applies to quotient singularities in all dimensions $n \geq 2$ by small subgroups G of $GL(n, \mathbb{C})$. They are fixed point free in codimension 1.

Proposition 3.4 (Quotient singularities) *Let $G \subset GL(n, \mathbb{C})$ be a small finite group acting on $\mathbb{C}^n$, then there does not exist a complete Kähler–Einstein metric on the complement of the singular locus in $\mathbb{C}^n/G$.*

Proof Let $S = \{x \in \mathbb{C}^n | x = g(x) \text{ for some } g \neq 1\}$. It is standard that the singular locus is $\widetilde{S} = S/G$. Let $V^* = B - \widetilde{S}$ with $B = \{(z_1, \ldots, z_n), |z_i|^2 < 1\}$ be an open neighborhood of the singular locus. For small enough V^*, it is covered by a disjoint union of open sets V_i $(i = 1, \ldots, l)$ for some finite integer $l \leq |G|$. If on $\mathbb{C}^n/G - \widetilde{S}$ there is a complete Kähler–Einstein metric ω, then there is a G-invariant Kähler–Einstein metric ω^G, whose restriction $\omega^G|_{V_i}$ to each V_i is a Kähler–Einstein metric which is complete at $S \cap V_i$. From the proof of Theorem 1.3 this is impossible. □

The Proof of Theorem 1.5 Here is the main's proof. We make use of Theorem 3.2, by which the claim reduces to showing non-existence of Kähler–Einstein metrics for the complement of the singular points which are Du Val on the codimension two strata. This stratum is non-empty by assumption and the statement then follows from Lemma 3.3. □

4 The Proof of Theorem 1.7

4.1 Log terminal singularities

For simplicity, we will only consider normal varieties. Then by Zariski's main theorem, we know that the fundamental locus of all birational morphisms involved in our setup are closed subvarieties of codimension at least 2. Denote by $M_{\text{reg}} = M - M_{\text{sing}}$ the complement of the singular set M_{sing}, we then have the open immersion $M_{\text{reg}} \overset{i}{\hookrightarrow} M$. The canonical sheaf $\omega_M = i_*\omega_{M_{\text{reg}}}$ exists by Hartog's extension theorem and M being normal. We note that the fundamental locus is generally not contained within M_{sing}, and vice versa.

To measure the local valuative property of a singularity, the notion of discrepancy was introduced (see e.g. [10]). This notion can be defined for $\mathbb{R}$-linear combinations of Weil divisors $D = \sum a_i D_i$ but often $\mathbb{Q}$-linear is sufficient. Since Weil divisors are not always pulled back, we assume $K_M + D$ is $\mathbb{Q}$-Cartier. Let $f : Y \to M$ be a birational (bimeromorphic) morphism, and Y normal, we let

$$K_Y + {f^{-1}}_*(D) = f^*(K_M + D) + \sum a_E(M, D)E$$

where E are distinct prime divisors of Y and $a_E(M, D) \in \mathbb{Q}$. This is now a statement regarding linear equivalence of divisors and not local. As non-exceptional divisors appear in this formal sum, the right hand side needs some explanation.

We demand as in [10] that for a non-exceptional divisor E, the coefficient $a_E(M, D) \neq 0$ iff $E = (f^{-1})_*(D_i)$ for some i, in which case we set $a_{D_i}(M, D) = -a_i$. This defines $a_E(M, D) \in \mathbb{R}$, called the *discrepancy* of E with respect to the pair (M, D).

A more global measure of singularity of the pair is defined by taking the inf of $a_E(M, D)$ over distinct primes $E \subset Y$.

Definition 4.1

$$\text{discrp}(M, D) := \inf_E \{a_E(M, D) | \text{ E} \subset \text{ Y exceptional with center } \neq \varnothing \text{ on M}\}$$
$$\text{discrp}_{\text{total}}(M, D) := \inf_E \{a_E(M, D) | \text{ E} \subset \text{ Y has non-empty center on M}\}$$

We are ready to recall the definition of a log terminal 'singularity'. The following is def. (2.34) in [9].

Definition 4.2 Let (M, D) be as above, then we say (M, D) or $K_M + D$ is

$$\left.\begin{array}{c} terminal \\ canonical \\ klt \\ lc \end{array}\right\} \text{ if discrp(M,D)} \left\{\begin{array}{c} >0 \\ \geq 0 \\ > -1 \text{ and } \lfloor D \rfloor \leq 0 \\ \geq -1 \end{array}\right. \tag{4.1}$$

The running of MMP preserves both the Kawamata log terminal (klt) and log canonical (lc) property of pairs. This means the statements we make may easily be extended to the context of birational geometry.

4.2 Local property of klt pairs

We will need the following lemma characterizing klt singularity structures in codimension 2. This comes from a cutting by hypersurface technique similar to the case of complete varieties, and for details we refer to [6].

Lemma 4.3 (Property in codimension 2) *Let (X, D) be a klt pair. Then there exists a closed subset $N \subset X$ with* $\text{codim}_X N \geq 3$ *such that $X \backslash N$ has quotient singularities.*

This is proposition (9.3) in [6]. However, using [9], Corollary (2.39), this will follow as a corollary of Theorem 3.2. In that case, we can give a proof of Theorem 1.7 based on Theorem 1.5, since we may consider the pair $(X, \varnothing)$. However, we will end up with a redundant restriction on the codimension of the singular locus. The proof of Theorem 1.7 now parallels that of Theorem 1.5 in the previous section.

The Proof of Theorem 1.7 By resolution of singularities the log resolution claimed in the theorem exists. Let $S = f(E) \cap X_{\text{sing}}$. The existence of a complete Kähler–Einstein metric on the complement of the exceptional divisor $M \backslash E$ is equivalent to the existence of a complete Kähler–Einstein metric on $X \backslash S$. By Lemma 4.3, we only have to show the case when $\text{codim}_X S = 2$, in which case each irreducible component of S is a finite quotient. Clearly S has only finitely many irreducible components, and the non-existence of a complete Kähler–Einstein metric on $X \backslash S$ follows from Proposition 3.4. □

A simple application of our results is in the context of desingularization of the Satake compactifications [8]. Without using any detail on modular forms, we can conclude that the compactification of locally symmetric spaces must proceed by adding boundary components which contain singular points, i.e. cusps. This follows from the simple fact that arithmetic quotients admit complete Kähler–Einstein metrics.

Acknowledgements We thank Chenglong Yu for discussions on material in Sect. 2, where he made a critical observation for Lemma 2.1. P. Gao and S.-T. Yau are supported by NSF Grants DMS-1308244 and PHY-0937443, and W. Zhou is partially supported by China Postdoctoral Science Foundation Grant No. 2015M571479.

References

1. Berman, R.J., Guenancia, H.: Kähler–Einstein metrics on stable varieties and log canonical pairs. Geom. Funct. Anal. **24**(6), 1683–1730 (2014)
2. Boucksom, S., Eyssidieux, P., Guedj, V., Zeriahi, A.: Monge–Ampére equations in big cohomology classes. Acta Math. **205**(2), 199–262 (2010)
3. Cheng, S.-Y., Yau, S.T.: On the existence of a complete Kähler metric in non-compact complex manifolds and the regularity of Fefferman'ss equation. Commun. Pure Appl. Math. **33**, 507–544 (1980)
4. Fu, J., Yau, S.-T., Zhou, W.: Complete cscK metrics on the local models of the conifold transition. Commun. Math. Phys. **335**, 1215–1233 (2015)
5. Fu, J., Yau, S.-T., Zhou, W.: On complete cscK metrics with Poincaré-Mok-Yau asymptotic property. Commum. Anal. Geom. **24**(3), 521–557 (2016)
6. Greb, D., Kebekus, S., Kovacs, S., Peternell, T.: Differential Forms on Log Canonical Spaces. Publ. Math. Inst. Hautes Études Sci. No. **114**, 87–169 (2011)
7. Guenancia, H.: Kähler–Einstein metrics with cone singularities on klt pairs. Int. J. Math. **24**(05), 1350035 (2013)
8. Igusa, J.: On the desingularization of Satake compactifications. In: Algebraic Groups and Discontinuous Subgroups, Proceedings of Symposia in Pure Mathematics. American Mathematical Society, pp. 301–305 (1966)
9. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties, Cambridge Tracts in Mathematics, vol. 134. Cambridge University Press, Cambridge (1998)
10. Kollár, J.: Singularities of pairs. arXiv:alg-geom/9601026 (preprint)
11. Mok, N., Yau, S.-T.: Completeness of the Kähler–Einstein metric on bound domains and the Characterization of domain of holomorphy by curvature condition. In: Proceedings of Symposia in Pure mathematics, vol. 39, Part 1, pp. 41–59 (1983)
12. Philippe, E., Guedj, V., Zeriahi, A.: Singular Käler–Einstein metrics. J. Am. Math. Soc. **22**(3), 607–639 (2009)
13. Reid, M.: Young person's guide to canonical singularities. Proc. Symp. Pure Math. **46**, 343–416 (1987)
14. Reid, M.: Canonical 3-folds, Journées de Géométrie Algébrique d'Angers. In: Beauville, A. (ed.) Sijthoff and Noordhoff, Alphen aan den Rijn, pp. 273–310 (1980)
15. Tian, G., Yau, S.T.: Complete Kähler manifolds with zero Ricci curvature. I. J. Am. Math. Soc. **3**, 579–609 (1990)
16. Tian, G., Yau, S.T.: Complete Kähler manifolds with zero Ricci curvature. II. Invent. Math. **106**, 27–60 (1991)
17. Tian, G., Yau, S.T.: Existence of Kähler–Einstein metrics on complete Kähler manifolds and their applications to algebraic geometry. In: Advances in Theoretical and Mathematical Physics. Mathematical aspects of string theory (San Diego, Calif, 1986), pp. 574–628 (1987)
18. Wu, D.: Kähler–Einstein metrics of negative Ricci curvature on general quasi-projective manifolds. Commun. Anal. Geom. **16**(2), 395–435 (2008)
19. Yau, S.-T.: Some function theoretic properties of complete Riemannian manifolds and their application to geometry. Indiana Univ. Math. J. **25**, 659–670 (1976)
20. Yau, S.-T.: A general Schwarz lemma for Kähler manifolds. Am. J. Math. **100**(1), 197–203 (1978)
21. Yau, S.-T.: On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation. I. Commun. Pure. Appl. Math. **31**, 339–411 (1978)
22. Yau, S.-T.: The role of partial differential equations in differential geometry. In: Proceedings of the International Congress of Mathematicians, Helsinki (1978)