Topology and its Applications 267 (2019) 106877

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Homology of path complexes and hypergraphs

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ARTICLE INFO

Article history:
Received 22 May 2019
Received in revised form 30 August 2019
Accepted 31 August 2019
Available online 5 September 2019

MSC:
05C65
05C25
55U05
18G60
55N35
55U10

Keywords:
Path complex
Path homology of hypergraph
Graph homology
Homotopy of path complexes
Simplicial complex

ABSTRACT

The path complex and its homology were defined in previous papers of the authors. The notion of a path complex is a natural discrete generalization of the notion of a simplicial complex. The theory of path complexes contains homotopy invariant homology theory of digraphs and (nondirected) graphs.

In the paper we study the homology theory of path complexes. In particular, we describe functorial properties of paths complexes, introduce the notion of homotopy for path complexes and prove the homotopy invariance of path homology groups. We prove also several theorems that are similar to the results of classical homology theory of simplicial complexes. Then we apply this approach to construct homology theories on various categories of hypergraphs. We describe basic properties of these homology theories and relations between them. As a particular case, these results give new homology theories on the category of simplicial complexes.

1. Introduction

In this paper we study functorial and homotopy properties of *path complexes* that were introduced in previous papers of the authors as a natural discrete generalization of the notion of a simplicial complex. Now we systematically describe properties of path complexes and provide new definitions, including notion of homotopy, and prove theorems that are similar to the results of simplicial homology theory. As an application we construct homology theories of various categories of hypergraphs (see [6]).

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https://doi.org/10.1016/j.topol.2019.106877

Note that the particular case of the theory, the *path homology theory* of digraphs and (nondirected) graphs was investigated in [8], [9], [10], and [11]. For the case of (nondirected) graphs, the path homology coincides with the graph homology defined in [3] and [4] which is closely connected with the A-homotopy theory. See [1], [2], and [3]. The Künneth formulae for the Cartesian product and for the join of path complexes were proved in [7] and [12].

In Section 2, we recall the notion of the path complex on a finite set V and the definition of path homology. We describe also functorial properties of path homology.

In Section 3, we introduce the notion of homotopy for path complexes and prove the homotopy invariance of path homology groups.

In Section 4, we introduce the notion of a sub-complex of a path complex and corresponding relative homology groups. We construct also several natural homology exact sequences.

In Section 5, we apply obtained results to construct homology theories on various categories of hypergraphs and describe theirs properties. We provide also several examples of explicit computations of path homology groups of hypergraphs.

Acknowledgments

The first and the third authors were partially supported by SFB 1283 of German Research Council. The second author was partially supported by the CONACyT Grant 284621. The fourth author was partially supported by the grant “Geometry and Topology of Complex Networks”, no. FA-9550-13-1-0097.

2. Path complexes on finite sets

In this section we recall the notion of the path complex on a finite set V and define the path homology theory of such complexes. Then we introduce a notion of morphism for path complexes and describe functorial properties of path homology groups.

Let V be an arbitrary non-empty finite set. We call a *vertex* any element $v \in V$. An *elementary n-path* on a set V is a sequence $i_0 \dots i_n$ of $n \geq 0$ vertices from V.

For a unitary commutative ring K, consider a free K-module $\Lambda_n = \Lambda_n(V)$ generated over K by elements $e_{i_0 \dots i_n}$ where $i_0 \dots i_n$ is an elementary n-path. The elements of Λ_n for $n \geq 0$ are called *n-paths* on V. Set also $\Lambda_{-1} = K$ and $\Lambda_{-2} = 0$. Each n-path $v \in \Lambda_n$ for $n \geq 0$ has a unique representation of the form

$$v = \sum_{i_0, \dots, i_n \in V} v^{i_0 \dots i_n} e_{i_0 \dots i_n},$$

where $v^{i_0 \dots i_n} \in K$.

For $n \geq 1$, define the *boundary operator*

$$\partial : \Lambda_n \to \Lambda_{n-1}$$

as a linear operator that acts on elementary paths by

$$\partial e_{i_0 \dots i_n} = \sum_{s=0}^{n} (-1)^s e_{i_0 \dots \widehat{i_s} \dots i_n}, \tag{2.1}$$

where the hat $\widehat{i_s}$ means omission of the index i_s. For $n = 0, -1$ we define $\partial : \Lambda_0 \to \Lambda_{-1} = K$ as the augmentation homomorphism ε given by

$$\varepsilon \left(\sum k_p i_p \right) = \sum k_p, \ k_p \in K, \ i_p \in V,$$

and define $\partial : \Lambda_{-1} \to \Lambda_{-2} = 0$ to be zero.

It is an easy exercise to check that $\partial^2 = 0$ [7] and, hence, $\Lambda_* = \{\Lambda_n\}$ is a chain complex.

An elementary path $i_0...i_n$ is called *non-regular* if $i_{k-1} = i_k$ for some $k = 1, \dots, n$, and *regular* otherwise.

For $n \geq 2$, let $I_n = I_n(V) \subset \Lambda_n(V)$ be a submodule generated by non-regular paths. For $n = -2, -1, 0, 1$ we put $I_n = 0$. Then for $n \geq -1$, the restriction of ∂ to I_n satisfies the condition $\partial^2 = 0\colon I_n \to I_{n-2}$ and hence a chain complex I_* is defined. Thus we obtain a quotient chain complex

$$\mathcal{R}_* = \mathcal{R}_*(V) := \Lambda_*(V)/I_*(V),$$

and we continue to denote by ∂ the induced differential. It is clear, that for $n \geq 0$ we have isomorphisms

$$\mathcal{R}_n \cong \operatorname{span}\left\{e_{i_0...i_p} : i_0...i_p \text{ is regular}\right\}.$$

The elements of $\mathcal{R}_p$ are called *regular* p-paths.

Let V, V' be two finite set. Any map $f : V \to V'$ induces a morphism of chain complexes

$$f_* : \Lambda_*(V) \to \Lambda_*(V')$$

given on the basis of $\Lambda_n(V)$ for $n \geq 0$ by the rule

$$f_*\left(e_{i_0...i_n}\right) = e_{f(i_0)...f(i_n)} \text{ and } f_* \text{ is an isomorphism for } n = -1, -2. \tag{2.2}$$

Since $f_*(I_n(V)) \subset I_n(V')$, the morphism f_* induces a morphism of chain complexes of regular paths

$$\mathcal{R}_*(V) \to \mathcal{R}_*(V') \tag{2.3}$$

which we continue to denote f_*.

Definition 2.1. [7] A *path complex* over a set V is a collection $P = P(V)$ of elementary paths on V such that:

- $i \in P$ for any $i \in V$,
- if $i_0...i_n \in P$ then $i_0...i_{n-1} \in P$ and $i_1...i_n \in P$.

For $n \geq 0$, the set of all n-paths from P is denoted by P_n. The $(n-1)$-paths $i_0...i_{n-1}$ and $i_1...i_n$ are called the *truncated paths* of the n-path $i_0...i_n$. The elements of P are called *allowed elementary paths*, and the elementary paths that are not in P are called *non-allowed*. Note that $P_0 = V$. The elements of P_1 are called *edges* of P. It follows immediately from Definition 2.1 that for n-path $i_0...i_n \in P (n \geq 1)$, all 1-paths $i_{k-1}i_k$ are edges.

Definition 2.2. We say, that a map $f : V \to V'$ induces a morphism of path complexes P and P' if, for any path $v \in P$, the path $f_*(v)$ defined in (2.2) lies in P'. We denote this morphism as

$$f_\bullet = (f, f_*)\colon (V, P) \to (V', P').$$

Let $(V, P), (V', P'), (W, S)$ be path complexes and $f\colon V \to V'$, $g\colon V' \to W$ be maps of sets that define the morphisms

$$f_\bullet = (f, f_*)\colon (V, P) \to (V', P') \text{ and } g_\bullet = (g, g_*)\colon (V', P') \to (W, S)$$

of path complexes. Then we define the composition $g_\bullet f_\bullet$ as

$$g_\bullet f_\bullet \colon = (gf, g_* f_*).$$

The map $gf\colon V \to W$ defines a morphism of path complexes

$$(gf)_\bullet = (gf, (gf)_*)\colon (V, P) \to (W, S)$$

that evidently coincides with the morphism $g_\bullet f_\bullet$.

In this way we obtain a category $\mathcal{P}$ whose objects are path complexes and whose morphisms are morphisms of path complexes.

For any integer $n \geq 0$, define the K-module $\mathcal{A}_n(P)$ that is spanned by all the elementary n-paths from P:

$$\mathcal{A}_n = \mathcal{A}_n(P) = \langle e_{i_0 \dots i_n} | i_0 \dots i_n \in P_n \rangle,$$

and we put $\mathcal{A}_{-1} = K, \mathcal{A}_{-2} = 0$. The elements of $\mathcal{A}_n$ are called *allowed* n-paths of the path complex P. Thus we have a natural inclusion of modules $\mathcal{A}_n(P) \subset \Lambda_n(V)$ for $n \geq 1$ and $\mathcal{A}_n(P) = \Lambda_n(V)$ for $n = -2, -1, 0$.

Note that the set of all paths on the set V gives a path complex which we denote by P_V. We shall call P_V *a full path complex* on the set V. For this path complex we have $\mathcal{A}_n(P_V) = \Lambda_n(V)$.

For some path complexes it can happen that $\partial \mathcal{A}_n \subset \mathcal{A}_{n-1}$ (see [9], [7], [11]), but in the general case this is not true.

Define a submodule $\Omega_n(P) \subset \mathcal{A}_n(P)$ as follows. For $n = -2, -1, 0, 1$ we put $\Omega_n(P) = \mathcal{A}_n(P)$, and for $n \geq 2$ we put

$$\Omega_n = \Omega_n(P) = \{v \in \mathcal{A}_n | \partial v \in \mathcal{A}_{n-1}\}.$$

It is clear that $\partial(\Omega_n) \subset \Omega_{n-1}$, since $\partial^2 = 0$. The elements of Ω_n are called ∂-*invariant* n-paths, and we obtain a *reduced* chain complex:

$$0 \leftarrow K \leftarrow \Omega_0 \leftarrow \dots \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow \dots \tag{2.4}$$

where the boundary maps are induced by ∂. The corresponding *non-reduced* chain complex has the form

$$0 \leftarrow \Omega_0 \leftarrow \dots \leftarrow \Omega_{n-1} \leftarrow \Omega_n \leftarrow \Omega_{n+1} \leftarrow \dots \tag{2.5}$$

Homology groups of (2.5) are referred to as the *path homology groups* of the path complex P and are denoted by $H_n(P), n \geq 0$. The homology groups of (2.4) are called the *reduced path homology groups* of P and are denoted by $\widetilde{H}_n(P), n \geq -1$.

Now we introduce *regular path homology groups* of a path complex P. We have a commutative diagram of inclusions of K-modules

$$\begin{array}{ccc} \mathcal{A}_n(P) & \longrightarrow & \Lambda_n(V) \\ \uparrow & & \uparrow \\ \mathcal{A}_n(P) \cap I_n & \longrightarrow & I_n \end{array}$$

which induces homomorphisms of K-modules

$$\mathcal{R}_n(P) = \mathcal{A}_n(P)/\{\mathcal{A}_n(P) \cap I_n\} \to \Lambda_n(V)/I_n = \mathcal{R}_n(V)$$

for $n \geq -2$. Denote the image of this homomorphism as

$$\mathcal{R}_n^{reg}(P) \subset \mathcal{R}_n(V).$$

Note, that $\mathcal{R}_n(P) \cong \mathcal{R}_n^{reg}(P)$.

Define a submodule $\Omega_n^{reg}(P) \subset \mathcal{R}_n^{reg}(P)$ as follows. For $n = -2, -1, 0$ we put $\Omega_n^{reg}(P) = \mathcal{R}_n^{reg}(P)$, and for $n \geq 1$ we put

$$\Omega_n^{reg} = \Omega_n^{reg}(P) = \left\{ v \in \mathcal{R}_n^{reg}(P) | \partial v \in \mathcal{R}_{n-1}^{reg}(P) \text{ where } \partial \colon \mathcal{R}_n(V) \to \mathcal{R}_{n-1}(V) \right\}.$$

The elements of Ω_n^{reg} are called *∂-invariant regular n-paths*, and we obtain a *reduced regular* chain complex:

$$0 \leftarrow K \leftarrow \Omega_0^{reg} \leftarrow \ldots \leftarrow \Omega_{n-1}^{reg} \leftarrow \Omega_n^{reg} \leftarrow \Omega_{n+1}^{reg} \leftarrow \ldots \tag{2.6}$$

where the boundary maps are induced by ∂. The corresponding *non-reduced* complex has the form

$$0 \leftarrow \Omega_0^{reg} \leftarrow \ldots \leftarrow \Omega_{n-1}^{reg} \leftarrow \Omega_n^{reg} \leftarrow \Omega_{n+1}^{reg} \leftarrow \ldots \tag{2.7}$$

Homology groups of (2.7) are referred to as the *regular path homology groups* of the path complex P and are denoted by $H_n^{reg}(P), n \geq 0$. The homology groups of (2.6) are called the *reduced regular path homology groups* of P and are denoted by $\widetilde{H}_n^{reg}(P), n \geq -1$.

Some properties of the introduced chain complexes, various special types of path complexes, and examples of computing homology groups are given in [12] and [7]. From now on we shall consider only non-reduced chain complexes and non-reduced homology groups if otherwise is not state.

Let $\mathcal{C}_*$ denote the category whose objects are chain complexes and whose morphisms are chain maps.

Now we discuss morphisms of chain complexes which are induced by morphisms of path complexes. The proofs of following statements follow immediately from definitions.

Lemma 2.3. (i) *Let Λ_* and Λ'_* be chain complexes with the differentials ∂ and ∂', respectively, and $f_*\colon \Lambda_* \to \Lambda'_*$ be a morphism. Let the submodules $\mathcal{A}_n \subset \Lambda_n$ and $\mathcal{A}'_n \subset \Lambda'_n$ be given in a such way that $f_*(\mathcal{A}_n) \subset \mathcal{A}'_n$.*

Define the modules Ω_n and Ω'_n in such a way

$$\Omega_n = \{v \in \mathcal{A}_n | \partial v \in \mathcal{A}_{n-1}\} \quad \text{and} \quad \Omega'_n = \left\{v \in \mathcal{A}'_n | \partial v \in \mathcal{A}'_{n-1}\right\}. \tag{2.8}$$

(i) *Then*

$$\partial(\Omega_n) \subset \Omega_{n-1}, \quad \partial(\Omega'_n) \subset \Omega'_{n-1} \tag{2.9}$$

and f_* induces a morphism of chain complexes $\Omega_* \to \Omega'_*$ which we continue to denote by f_*.

(ii) *If $f_*\colon \Lambda_* \to \Lambda'_*$ is a monomorphism, then $f_*\colon \Omega_* \to \Omega'_*$ is a monomorphism, too.*

Proposition 2.4. *Any morphism $f_\bullet\colon (V,P) \to (V',P')$ of path complexes induces a morphism f_* of chain complexes*

$$\Omega_*(f_\bullet) = f_* : \Omega_*(P) \to \Omega_*(P') \tag{2.10}$$

and a morphism of regular chain complexes

$$\Omega_*^{reg}(f_\bullet) = f_* : \Omega_*^{reg}(P) \to \Omega_*^{reg}(P') \tag{2.11}$$

and, consequently, homomorphisms

$$f_*\colon H_*(P) \to H_*(P'),\ f_*\colon H_*^{reg}(P) \to H_*^{reg}(P'), \tag{2.12}$$

of homology and regular homology groups.

Corollary 2.5. *We have functors Ω_* and Ω_*^{reg} from the category $\mathcal{P}$ of path complexes to the category $\mathcal{C}_*$ of chain complexes.*

Proposition 2.6. *For any path complex P we have a morphism of chain complexes*

$$\Omega_*(P) \to \Omega_*^{reg}(P)$$

and hence the homomorphisms of homology groups

$$f_*\colon H_*(P) \to H_*^{reg}(P).$$

Now we give yet one definition we shall need in the next section (see [12] and [7]).

Definition 2.7. A path complex P is called *regular* if all the paths $i_0...i_n \in P$ are regular.

It is possible to give a weaker definition of a morphism of path complexes that induces a morphism of regular chain complexes [7].

Definition 2.8. [7] We say, that a map $f : V \to V'$ of sets provides a *weak morphism* of path complexes P and P' if, for any path $v \in P$, the path $f_*(v)$ defined in (2.2) lies in P' or it is an irregular path in the full path complex $P_{V'}$. In this case we shall write

$$f_\circ\colon (V, P) \to (V', P').$$

Let

$$f_\circ\colon (V, P) \to (V', P') \text{ and } g_\circ\colon (V', P') \to (W, S)$$

be a weak morphism of path complexes. Similarly to the case of morphism of path complexes, the composition $g_\circ f_\circ$ is defined. Thus we obtain a category $\mathcal{PW}$ whose objects are path complexes and whose morphisms are weak morphisms of path complexes.

Proposition 2.9. *Any weak morphism $f_\circ\colon (V, P) \to (V', P')$ of path complexes induces a morphism of chain complexes*

$$\Omega_*^{reg}(f_\circ)\colon \Omega_*^{reg}(P) \to \Omega_*^{reg}(P') \tag{2.13}$$

and, consequently, a homomorphism

$$H_*^{reg}(P) \to H_*^{reg}(P') \tag{2.14}$$

of regular homology groups.

Proof. Let $I_n' \subset \Lambda_n(V')$ be a submodule generated by non-regular paths. The result follows from the natural isomorphism

$$\mathcal{A}_n(P')/\{\mathcal{A}_n(P')\cap I'_n\} \xrightarrow{\cong} \mathcal{A}_n(P'\cup I'_n)/\{\mathcal{A}_n(P'\cup I'_n)\cap I'_n\}$$

and from Lemma 2.3 as above.

Consider a commutative diagram

$$\begin{array}{ccccc}
\mathcal{A}_n(P) & \overset{f_*}{\to} & \mathcal{A}_n(P'\cup I'_n) & \leftarrow & \mathcal{A}_n(P') \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{A}_n(P)/\{\mathcal{A}_n(P)\cap I_n\} & \overset{f_*}{\to} & \mathcal{A}_n(P'\cup I'_n)/\{\mathcal{A}_n(P'\cup I'_n)\cap I'_n\} & \overset{\cong}{\leftarrow} & \mathcal{A}_n(P')/\{\mathcal{A}_n(P')\cap I'_n\} \\
|| & & \downarrow\cong & & || \\
\mathcal{R}_n(P) & \overset{f_*}{\to} & \mathcal{R}_n(P') & = & \mathcal{R}_n(P') \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{R}_n^{reg}(P) & \overset{f_*}{\to} & \mathcal{R}_n^{reg}(P') & = & \mathcal{R}_n^{reg}(P') \\
\downarrow & & \downarrow & & \\
\mathcal{R}_*(V) & \to & \mathcal{R}_*(V') & &
\end{array}$$

where the bottom vertical arrows are natural inclusions, and the morphism f induces the morphism of quotients

$$f_*\colon \mathcal{R}_*(V) \to \mathcal{R}_*(V').$$

Now the morphism (2.13) follows from Lemma 2.3 and the rest of the claim is standard. □

Corollary 2.10. *We have that Ω_*^{reg} is a functor from the category $\mathcal{PW}$ of path complexes to the category $\mathcal{C}_*$ of chain complexes.*

3. Homotopy theory for path complexes

In this section we construct a homotopy theory for path complexes and prove the homotopy invariance of homology groups introduced above.

Let $I = \{0,1\}$ be a set with two elements. For any set $V = \{0,\dots,n\}$, let $V\times I$ be the Cartesian product. Let V' be a copy of the set V and we denote such a set by $V' = \{0',\dots,n'\}$ where $i'\in V'$ corresponds to $i\in V$. Then we can identify $V\times I$ with $V\cup V'$ in such a way that $(i,0)$ corresponds to i and $(i,1)$ corresponds to i' for any $i\in V$. Thus V is identified with $V\times\{0\}\subset V\times I$ and V' is identified with $V\times\{1\}\subset V\times I$.

The natural isomorphism $V\cong V'$ defines a path complex P' on the set V' by the following condition: $i'_0\dots i'_n\in P'$ iff $i_0\dots i_n\in P$.

Define a path complex $P\times I$ as a path complex on $V\times I = V\cup V'$ by

$$P\times I = \{w|w\in P\}\cup\{w'|w'\in P'\}\cup\{\widehat{w} = i_0\dots i_k i'_k\dots i'_n | i_0\dots i_k i_{k+1}\dots i_n\in P\} \tag{3.1}$$

where $0\le k\le n$. It follows from (3.1) that we have natural morphisms

$$i_\bullet\colon (V,P)\to(V\times I, P\times I)$$

and

$$j_\bullet\colon (V',P')\to(V\times I, P\times I)$$

which are induced by natural inclusions $i\colon V\to V\times I = V\cup V'$ and $j\colon V'\to V\times I = V\cup V'$.

Now, we define the notion of homotopy in the category of path complexes. Let P be a path complex on the set V and S be a path complex on the set W. Note that any map $f\colon V\to W$ defines naturally a unique map $f'\colon V' \to W$, and similarly, any morphism $f_\bullet\colon (V,P) \to (W,S)$ defines naturally a unique morphism $f'_\bullet\colon (V',P') \to (W,S)$.

Definition 3.1. i) We call two morphisms $f_\bullet, g_\bullet\colon (V,P) \to (W,S)$ of path complexes *one step homotopic* (and write $f_\bullet \simeq_1 g_\bullet$) if there exists a morphism $F_\bullet\colon (V\times I, P\times I) \to (W,S)$ of path complexes such that at least one of the two following conditions are satisfied.

1. $F_\bullet|_{(V,P)} = f_\bullet, \quad F_\bullet|_{(V',P')} = g'_\bullet.$
2. $F_\bullet|_{(V,P)} = g_\bullet, \quad F_\bullet|_{(V',P')} = f'_\bullet.$

ii) We call two morphisms $f_\bullet, g_\bullet\colon (V,P) \to (W,S)$ of path complexes *homotopic* and write $f_\bullet \simeq g_\bullet$ if there exists a sequence of morphisms

$$f_{i\bullet}\colon (V,P) \to (W,S)$$

such that $f_\bullet = f_{0\bullet} \simeq_1 f_{1\bullet} \simeq_1 \cdots \simeq_1 f_{n\bullet} = g_\bullet$.

iii) Two path complexes (V,P) and (W,S) are *homotopy equivalent* if there exist morphisms

$$f_\bullet\colon (V,P) \to (W,S), \quad g_\bullet\colon (W,S) \to (V,P)$$

such that

$$f_\bullet g_\bullet \simeq \mathrm{Id}_{W\bullet}, \quad g_\bullet f_\bullet \simeq \mathrm{Id}_{V\bullet}$$

where $\mathrm{Id}_V\colon V \to V$ and $\mathrm{Id}_W\colon W \to W$ are the identity morphisms. In this case, we shall write $(V,P) \simeq (W,S)$ and shall call the morphisms $f_\bullet$, $g_\bullet$ *homotopy inverses* to each other.

It follows directly from Definition 3.1, that the relation "to be homotopic" is an equivalence relation on the set of morphisms between two path complexes, and homotopy equivalence is an equivalence relation on the set of path complexes. Moreover, we will denote by $\mathcal{P}'$ the category whose objects are path complexes and morphisms are the classes of homotopic morphisms of path complexes.

Let V be a set. For $n \geq 0$, define a homomorphism

$$\tau\colon \Lambda_n(V) \to \Lambda_{n+1}(V\times I)$$

on elementary n-paths $v = e_{i_0\ldots i_n} \in \Lambda_n(V)$ by

$$\tau(v) = \sum_{k=0}^{n} (-1)^k \, e_{i_0\ldots i_k i'_k \ldots i'_n}, \tag{3.2}$$

an extending to $\Lambda_n(V)$ by K-linearity. Recall that we consider non-reduced complexes, hence $\Lambda_i = 0$ for $i \leq -1$ and we define $\tau = 0\colon \Lambda_{-1}(V) \to \Lambda_0(V\times I)$. Recall that we have a natural isomorphism $\Lambda_n(V) \overset{\cong}{\to} \Lambda_n(V')$ of submodules of $\Lambda_n(V\times I)$. It is given on the basis elements by $e_{i_0\ldots i_n} \to e_{i'_0\ldots i'_n}$ and extending by linearity to $\Lambda_n(V)$. We shall denote by $v' \in \Lambda_n(V') \subset \Lambda_n(V\times I)$ the image of the element $v \in \Lambda_n(V) \subset \Lambda_n(V\times I)$

Lemma 3.2. *For $n \geq 0$ and any path $v \in \Lambda_n(V)$ we have*

$$\partial\tau(v) + \tau(\partial v) = v' - v. \tag{3.3}$$

Proof. It is sufficient to prove the statement for basis elements $v = e_{i_0\dots i_n}$. For $n = 0$ and $v = e_{i_0} \in \Lambda_0(V)$, we have

$$\partial\tau(e_{i_0}) = \partial(e_{i_0 i'_0}) = e_{i'_0} - e_{i_0}, \quad \tau(\partial v) = \tau(0) = 0,$$

and the condition (3.3) is satisfied. Consider a path $v = e_{i_0\dots i_n} \in \Lambda_n(V)$ with $n \geq 1$. We have

$$\begin{aligned}\partial(\tau(v)) &= \partial\left(\sum_{k=0}^{n}(-1)^k e_{i_0\dots i_k i'_k\dots i'_{n-1} i'_n}\right)\\ &= \sum_{k=0}^{n}(-1)^k\left[\left(\sum_{m=0}^{k}(-1)^m e_{i_0\dots \widehat{i}_m\dots i_k i'_k\dots i'_n}\right) + \left(\sum_{m=k}^{n}(-1)^{m+1} e_{i_0\dots\dots i_k i'_k\dots \widehat{i}'_m\dots i'_n}\right)\right]\\ &= \sum_{0\leq m\leq k\leq n}(-1)^{k+m} e_{i_0\dots \widehat{i}_m\dots i_k i'_k\dots i'_n} + \sum_{0\leq k\leq m\leq n}(-1)^{k+m+1} e_{i_0\dots\dots i_k i'_k\dots \widehat{i}'_m\dots i'_n}\end{aligned}$$

and

$$\begin{aligned}\tau(\partial v) &= \tau\left(\sum_{m=0}^{n}(-1)^m e_{i_0\dots \widehat{i}_m\dots i_n}\right)\\ &= \sum_{m=0}^{n}(-1)^m\left[\left(\sum_{k=0}^{m-1}(-1)^k e_{i_0\dots i_k i'_k\dots \widehat{i}'_m\dots i'_n}\right) + \left(\sum_{k=m+1}^{n}(-1)^{k-1} e_{i_0\dots\dots \widehat{i}_m\dots i_k i'_k\dots i'_n}\right)\right]\\ &= \sum_{0\leq k< m\leq n}(-1)^{k+m} e_{i_0\dots i_k i'_k\dots \widehat{i}'_m\dots i'_n} + \sum_{0\leq m< k\leq n}(-1)^{k+m-1} e_{i_0\dots\dots \widehat{i}_m\dots i_k i'_k\dots i'_n}\end{aligned}$$

Hence

$$\begin{aligned}\partial\tau(v) + \tau(\partial v) &= \sum_{0\leq k\leq n}(-1)^{k+k} e_{i_0\dots \widehat{i}_k i'_k\dots i'_n} + \sum_{0\leq k\leq n}(-1)^{k+k+1} e_{i_0\dots\dots i_k \widehat{i}'_k\dots i'_n}\\ &= \sum_{0\leq k\leq n} e_{i_0\dots i_{k-1} i'_k\dots i'_n} - \sum_{0\leq k\leq n} e_{i_0\dots\dots i_k i'_{k+1}\dots i'_n}\\ &= e_{i'_0\dots i'_n} + \sum_{1\leq k\leq n} e_{i_0\dots i_{k-1} i'_k\dots i'_n} - \left(\sum_{0\leq k\leq n-1} e_{i_0\dots\dots i_k i'_{k+1}\dots i'_n} + e_{i_0\dots i_n}\right)\\ &= e_{i'_0\dots i'_n} - e_{i_0\dots i_n} + \left(\sum_{0\leq k-1\leq n-1} e_{i_0\dots i_{k-1} i'_k\dots i'_n} - \sum_{0\leq k\leq n-1} e_{i_0\dots\dots i_k i'_{k+1}\dots i'_n}\right)\\ &= e_{i'_0\dots i'_n} - e_{i_0\dots i_n} = v' - v. \quad \square\end{aligned}$$

Remark 3.3. It is easy to transfer results of Lemma 3.2 to the regular paths. The module $\mathcal{R}_n(V)$ has the basis $\{e_{i_0\dots i_n} | i_0 \dots i_n$ is regular path on $V\}$. Then we can define

$$\tau \colon \mathcal{R}_n(V) \to \mathcal{R}_{n+1}(V \times I) \tag{3.4}$$

by the same formulae as for $\tau \colon \Lambda_n(V) \to \Lambda_{n+1}(V \times I)$.

Theorem 3.4. (i) *Let*

$$f_\bullet \simeq g_\bullet \colon (V, P) \to (W, S)$$

be homotopic morphisms of path complexes. Then these morphisms induce chain homotopic morphisms of reduced and non-reduced chain complexes

$$f_* \simeq g_* \colon \Omega_*(P) \to \Omega_*(S) \quad \textit{and} \quad f_* \simeq g_* \colon \Omega_*^{reg}(P) \to \Omega_*^{reg}(S)$$

and hence the same homomorphism of corresponding homology groups, respectively.

(ii) *If the path complexes* (V, P) *and* (W, S) *are homotopy equivalent, then they have isomorphic homology groups. Furthermore, if the homotopy equivalence is provided by homotopy inverse morphisms* $f_\bullet$ *and* $g_\bullet$ *(as in (iii) of Definition 3.1) then the induced homomorphisms* f_* *and* g_* *provide mutually inverse isomorphisms of reduced and non-reduced homology groups of* (V, P) *and* (W, S)*.*

Proof. At first we prove the statement for non-regular and non-reduced chain complexes. It is sufficiently to consider only the first case of one-step homotopy $F_\bullet$ between $f_\bullet$ and $g_\bullet$ as in (i) case 1 of Definition 3.1. By Definition 3.1 we have

$$F_\bullet i_\bullet = f_\bullet \colon \Omega_*(P) \to \Omega_*(S)$$

and

$$F_\bullet j_\bullet = g'_\bullet \colon \Omega_*(P') \to \Omega_*(S)$$

which we identify naturally with the morphism

$$g_\bullet \colon \Omega_*(P) \to \Omega_*(S).$$

By Proposition 2.4, morphisms $f_\bullet$ and $g'_\bullet$ induce morphisms of chain complexes

$$f_*, g_* \colon \Omega_*(P) \to \Omega_*(S),$$

and $F_\bullet$ induces a morphism

$$F_* \colon \Omega_*(P \times I) \to \Omega_*(S)$$

such that

$$F_*|_{\Omega_*(P)} = f_*, \quad F_*|_{\Omega_*(P')} = g'_*.$$

In order to prove that f_* and g_* induce the same homomorphism $H_*(P) \to H_*(S)$, it suffices by [14, Theorem 2.1, p. 40] to construct a chain homotopy

$$L_n \colon \Omega_n(P) \to \Omega_{n+1}(S)$$

such that

$$\partial L_n + L_{n-1}\partial = g_* - f_*$$

For $n \geq 0$, define a homomorphism

$$\tau \colon \mathcal{A}_n(P) \to \mathcal{A}_{n+1}(P \times I)$$

on elementary n-paths $v = e_{i_0 \dots i_n} \in \mathcal{A}_n(P)$ by formulae (3.2) and extend it to $\mathcal{A}_n(P)$ by K-linearity. We put also $\mathcal{A}_i = 0$ for $i = -1$ and define $\tau = 0\colon \mathcal{A}_{-1}(P) \to \mathcal{A}_0(P \times I)$. We prove now that if $v \in \Omega_n(P)$ then $\tau(v) \in \Omega_{n+1}(P \times I)$. Let $v \in \Omega_n(P)$ that is $v \in \mathcal{A}_n(P) \subset \Omega_n(V)$ and $\partial v \in \mathcal{A}_{n-1}(P) \subset \Omega_{n-1}(V)$. Hence, by (3.2) and definition of $\mathcal{A}_*(P \times I)$, we have $\tau(v) \in \mathcal{A}_{n+1}(P \times I)$ and $\tau(\partial v) \in \mathcal{A}_n(P \times I)$. By Lemma 3.2, we have

$$\partial \tau(v) = -\tau(\partial v) + v' - v$$

where the right summands lie in $\mathcal{A}_n(P \times I)$. It follows that if $v \in \Omega_n(P)$ then $\tau(v) \in \Omega_{n+1}(P \times I)$.

For $n \geq 0$, define the homomorphism

$$L_n(v)\colon = F_*(\tau(v)) : \Omega_n(P) \to \Omega_{n+1}(S) \ \text{ for all } \ v \in \Omega_n(P).$$

We obtain

$$\begin{aligned}
(\partial L_n + L_{n-1}\partial)(v) &= \partial(F_*(\tau(v))) + F_*(\tau(\partial v)) && \text{(by definition } L_*) \\
&= F_*(\partial \tau(v)) + F_*(\tau(\partial v)) && \text{(since } F_* \text{ is a chain map) 3.1)} \\
&= F_*(\partial \tau(v) + \tau(\partial v)) && \text{(since } F_* \text{ is a homomorphism)} \\
&= F_*(v' - v) && \text{(by Lemma 3.2)} \\
&= g_*(v) - f_*(v) && \text{(by definition of } F).
\end{aligned}$$

Thus the case (i) for reduced non-regular chain complexes is proved. The proof of (ii) in this case is standard.

For the case of non-reduced non-regular chain complexes the proof is similar. Recall that K is a ring of coefficients. In this case $f_*|_K = g_*|_K = F_*|_K$ is the identity map, and define $\tau\colon K \to \Omega_0(P \times I)$ as the trivial homomorphism.

In the case of reduced regular chain complexes the proof is similar. It is necessary to use the Remark 3.3 instead of Lemma 3.2. □

Now consider two full path complexes P_V and P_W. In this case any map of sets $f\colon V \to W$ defines the morphism $f_\bullet\colon (V, P_V) \to (W, P_W)$ of path complexes.

Proposition 3.5. *Any two morphisms*

$$f_\bullet, g_\bullet\colon (V, P_V) \to (W, P_W)$$

of full path complexes are one step homotopic and, hence, any full path complex is homotopy equivalent to the full path complex $(*, P_*)$ on the one point set $*$. The similar statement is true for any regular full path complex.

Proof. Define the map $F\colon V \times I = V \cup V' \to W$ by $F|_V = f,\ F|_{V'} = g'$. This map defines a morphism of path complexes

$$F_\bullet\colon (V \times I, P_V \times I) \to (W, P_W)$$

which satisfies evidently the conditions on one-step homotopy from Definition 3.1. The Proposition is proved. □

Corollary 3.6. *For any full path complex* (V, P_V) *we have*

$$H_n(P_V) = H_n^{reg}(P_V) = \begin{cases} K, & \text{for } n = 0, \\ 0, & \text{for } n \geq 1. \end{cases}$$

The proof is trivial.

Now we return to arbitrary path complexes.

Definition 3.7. i) We call two weak morphisms $f_\circ, g_\circ\colon (V, P) \to (W, S)$ of path complexes *weak one step homotopic* (and write $f_\circ \sim_1 g_\circ$) if there exists a weak morphism $F_\circ\colon (V \times I, P \times I) \to (W, S)$ of path complexes such that at least one of the two following conditions are satisfied.

1. $F_\circ|_{(V,P)} = f_\circ, \quad F_\circ|_{(V',P')} = g'_\circ.$
2. $F_\circ|_{(V,P)} = g_\circ, \quad F_\circ|_{(V',P')} = f'_\circ.$

ii) We call two weak morphisms $f_\circ, g_\circ\colon (V, P) \to (W, S)$ of path complexes *weak homotopic* and write $f_\circ \sim g_\circ$ if there exists a sequence of weak morphisms

$$f_{i_\circ}\colon (V, P) \to (W, S)$$

such that $f_\circ = f_{0_\circ} \sim_1 f_{1_\circ} \sim_1 \cdots \sim_1 f_{n_\circ} = g_\circ$.

iii) Two path complexes (V, P) and (W, S) are *weak homotopy equivalent* if there exist weak morphisms

$$f_\circ\colon (V, P) \to (W, S), \quad g_\circ\colon (W, S) \to (V, P)$$

such that

$$f_\circ g_\circ \sim \mathrm{Id}_{W\circ}, \quad g_\circ f_\circ \sim \mathrm{Id}_{V\circ}$$

where $\mathrm{Id}_V\colon V \to V$ and $\mathrm{Id}_W\colon W \to W$ are the identity morphisms. In this case, we shall write $(V, P) \sim (W, S)$ and shall call the weak morphisms $f_\circ$, $g_\circ$ *weak homotopy inverses* to each other.

It follows directly from Definition 3.7, that the relation "to be weak homotopic" is an equivalence relation on the set of weak morphisms between two path complexes, and weak homotopy equivalence is an equivalence relation on the set of path complexes. Moreover, we will denote by $\mathcal{PW}'$ the category whose objects are path complexes and morphisms are the classes of weak homotopic weak morphisms of path complexes.

Theorem 3.8. (i) *Let*

$$f_\circ \sim g_\circ\colon (V, P) \to (W, S)$$

be weak homotopic morphisms of path complexes. Then these morphisms induce the chain homotopic morphisms of regular chain complexes

$$f_* \simeq g_*\colon \Omega_*^{reg}(P) \to \Omega_*^{reg}(S)$$

and hence the same homomorphism of corresponding homology groups.

(ii) *If the path complexes* (V, P) *and* (W, S) *are weak homotopy equivalent, then they have isomorphic regular homology groups. Furthermore, if the weak homotopy equivalence is provided by homotopy inverse morphisms* $f_\circ$ *and* $g_\circ$ *then the induced homomorphisms* f_* *and* g_* *provide mutually inverse isomorphisms of reduced and non-reduced homology groups of* (V, P) *and* (W, S).

Proof. The proof is similar to the proof of Theorem 3.4. □

4. Relative path homology groups

In this section we introduce *relative path homology groups* and construct several exact sequences with these groups.

Let (W, S) be a path complex and $V \subset W$.

Definition 4.1. A path complex (V, P) over a set V is a *path subcomplex* of the path complex (W, S) if any elementary path $p \in P$ lies in S. We write in this case $(V, P) \subset (W, S)$ or, to simplify notations, $P \subset S$.

For a subcomplex $P \subset S$, the inclusion morphism $i_\bullet\colon (V, P) \to (W, S)$ induces a monomorphism $i_*\colon \Lambda_*(V) \to \Lambda_*(W)$ of chain complexes such that $i_*(\mathcal{A}_n(P)) \subset \mathcal{A}_n(S)$. By Lemma 3.2, this implies that $i_\bullet$ induces a monomorphism of chain complexes $i_*\colon \Omega_*(P) \to \Omega_*(S)$. Thus we obtain a short exact sequence of (reduced and non-reduced) chain complexes

$$0 \longrightarrow \Omega_*(P) \longrightarrow \Omega_*(S) \longrightarrow \Omega_*(S)/\Omega_*(P) \longrightarrow 0. \tag{4.1}$$

For the reduced chain complexes, the homomorphism i_* in dimension -1 is the identity homomorphism $K \to K$. Hence the factor-complex $\Omega_*(S)/\Omega_*(P)$ will be the same for the reduced and non-reduced chain complexes. We define homology groups $H_*(S, P) = H_*(\Omega_*(S)/\Omega_*(P))$, that are called the *relative path homology groups*.

The same line of arguments show that we have also a short exact sequence of (reduced and non-reduced) regular chain complexes

$$0 \longrightarrow \Omega_*^{reg}(P) \longrightarrow \Omega_*^{reg}(S) \longrightarrow \Omega_*^{reg}(S)/\Omega_*^{reg}(P) \longrightarrow 0. \tag{4.2}$$

Proposition 4.2. *Let (V, P) be a path subcomplex of (W, S). There are long exact sequences of homology groups*

$$0 \leftarrow H_0(S, P) \leftarrow H_0(S) \leftarrow H_0(P) \leftarrow H_1(S, P) \leftarrow H_1(P) \leftarrow \dots$$

and

$$0 \leftarrow H_0^{reg}(S, P) \leftarrow H_0^{reg}(S) \leftarrow H_0^{reg}(P) \leftarrow H_1^{reg}(S, P) \leftarrow H_1^{reg}(P) \leftarrow \dots$$

and similarly for reduced homology groups.

Proof. Follows from (4.1) and (4.2). □

5. The path homology of hypergraphs

In this section we apply the above results to construct a homology theory on the category of hypergraphs. The homology theory based on the theory of path complexes for the particular case of digraphs and (nondirected) graphs was constructed in [9], [10], [11]. As before, we fix a commutative ring K with a unity as a ring of the coefficients.

Definition 5.1. [5] (i) A finite *hypergraph* is a pair $G = (V, E)$ where V is a non-empty set of *vertices* and E is a family $\{e_1, \dots, e_k\}$ of non-empty and non-ordered subsets of V such that

$$\bigcup_{i=1}^{k} e_i = V. \quad (5.1)$$

The elements of E are called *edges*. A *loop* is an edge e_i that has exactly one vertex v that is $e_i = \{v\}$. *Degree* $|e_i|$ of an edge e_i is defined as the number of containing in e_i vertices.

The hypergraph is called *simple* if all edges are distinct. A hypergraph is called *h-homogeneous* if $|e_i| = h \geq 1$ for all $e_i \in E$.

(ii) We say that a hypergraph $G = (V_G, E_G)$ is a *sub-hypergraph* of a hypergraph $H = (V_H, E_H)$ if $V_G \subset V_H$ and $E_G \subset E_H$. If for any $e = \{i_1 \dots i_n\} \in E_H$ and $i_1, \dots, i_n \in V_G$, we have $e \in E_G$, we call the sub-hypergraph G an *induced sub-hypergraph*.

It follows immediately from Definition 5.1 that simple 2-homogeneous hypergraph is a graph without isolated vertices.

Definition 5.2. In a hypergraph $G = (V, E)$, two vertices are said to be *adjacent* if there is an edge e that contains both of these vertices. Two edges are said to be adjacent if their intersection is not empty. Two edges are said to be *h-adjacent* ($h \geq 1$) if their intersection contains at least h vertices.

A *walk* in G is an alternating sequence $v_1, e_1, v_2, e_2, \dots, e_n, v_{n+1}$ of vertices v_i and edges e_j of G such that: $v_i \neq v_{i+1}$ and $v_i, v_{i+1} \in e_i$ for $1 \leq i \leq n$. In this case we say that there exists a walk from v_1 to v_{n+1}.

We call a hypergraph G connected if there is a walk between any two vertices of G.

Now we introduce the notion of a hypergraph morphism and a product of hypergraphs (as in [6]). These morphism and product have good categorical properties and they are effective to construct natural path complexes of hypergraphs.

Let V be a finite set. Denote by $\mathbb{S}(V)$ the set of all non-empty non-ordered subsets of V. By Definition 5.1, for a hypergraph $G = (V, E)$, we have a natural map

$$\varphi_G \colon E \to \mathbb{S}(V) \setminus \emptyset.$$

Note that any map $f \colon V \to W$ induces a map

$$\mathbb{S}_f \colon \mathbb{S}(V) \setminus \emptyset \to \mathbb{S}(W) \setminus \emptyset.$$

For example, $f(0) = 0, f(1) = 0, f(2) = 1$ implies, that $\mathbb{S}_f\{0, 1, 2\} = \{0, 1\}$.

Definition 5.3. A *morphism* $f \colon G \to H$ of a hypergraph $G = (V_G, E_G)$ to a hypergraph $H = (V_H, E_H)$ is given by a pair of maps (f_V, f_E) where $f_V \colon V_G \to V_H$ and $f_E \colon E_G \to E_H$ provided the diagram

$$\begin{array}{ccc} E_G & \xrightarrow{\varphi_G} & \mathbb{S}(V_G) \setminus \emptyset \\ \downarrow f_E & & \downarrow \mathbb{S}_f \\ E_H & \xrightarrow{\varphi_H} & \mathbb{S}(V_H) \setminus \emptyset \end{array}$$

is commutative. The set of morphisms from G to H we shall denote by $\operatorname{Hom}(G, H)$.

Let $\mathcal{H}$ denote the category whose objects are hypergraphs, and whose morphisms are morphisms of hypergraphs defined above.

Definition 5.4. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two hypergraphs. Following [6,13], define the product $G_1 \times G_2 = G = (V, E)$ as follows:

$$V = V_1 \times V_2 \ \text{ is the direct product with the natural projections } \ p_i\colon V \to V_i,$$

and E is a family of triples $\{(A, e_1, e_2)\}$ where $e_1 \in E_1, e_2 \in E_2$ and A is a subset of the direct product

$$\varphi_{G_1}(e_1) \times \varphi_{G_2}(e_2) \subset V_1 \times V_2,$$

for which $p_i(A) = \varphi_{G_i}(e_i)$. Note, that $\varphi_G(A, e_1, e_2) = A$.

Definition 5.5. For a hypergraph $G = (V, E)$ define a *path complex* $(V, P^q(G))$ *of density* $q \geq 1$ on the set V of vertices in the following way. A path p of the length $n \geq 0$ and density q is defined as the sequence p of $n + 1$ vertices $i_0 \dots i_n$ such that any q consecutive vertices of p lie in some edge of G.

For example, condition (5.1) implies that the path complex $P^1(G)$ coincides with the full path complex $\mathcal{P}_V$ defined in Section 2.

In what follows, we shall consider only non-reduced path-complexes and non-reduced homology groups. All results can be immediately transferred to the case of reduced homology. Thus, for a hypergraph G and any $q \geq 1$ we have chain complexes

$$\Omega_*(G, q)\colon = \Omega_*(P^q(G)) \ \text{ and } \ \Omega_*^{reg}(G, q)\colon := \Omega_*^{reg}(P^q(G))$$

which have the following homology groups

- $H_*(G, q)\colon = H_*(\Omega_*(G, q)) = H_*(P^q(G))$,
- $H_*^{reg}(G, q)\colon = H_*(\Omega_*^{reg}(G, q)) = H_*^{reg}(P^q(G))$.

These homologies we call *path q-homology* and *regular path q-homology*, respectively. Recall that this homology groups depend also from the ring of coefficients K.

For example, Corollary 3.6 implies that for any hypergraph G we have

$$H_n(G, 1) = H_n^{reg}(G, 1) = \begin{cases} K, & \text{for } n = 0, \\ 0, & \text{for } n \geq 1. \end{cases}$$

In the case, when G is a simple graph without loops (in this case all edges consist of two vertices) the homology $H_*(G, 2)$ coincide with the graph homology from [3], [4], [9].

We call an edge $e \in E$ of a simple hypergraph $G = (V, E)$ *maximal* if there is no an edge $e_1 \in E(e_1 \neq e)$ such that $\phi_G(e) \subset \phi_G(e_1)$.

Proposition 5.6. *The path homologies of a hypergraph depend only on the set of its maximal edges.*

Proof. For any simple hypergraph $G = (V, E)$ define a sub-hypergraph $G_M = (V_M, E_M)$ as follows. We put $V_M = V$ and E_M will be subset of E consisting only of maximal edges. We have a natural inclusion $m\colon G \to G_M$ that is the identity map on the set of vertices. For $q \geq 1$, the map m induces the identity map of path complexes $m_\bullet\colon P^q(G) \to P^q(G_M)$ and the identity morphism of chain complexes $\Omega_*(G, q) \to \Omega_*(G_M, q)$. Hence $H_*(G, q) = H_*(G_M, q)$. $\square$

Proposition 5.7. *For $q \geq 1$, any morphism of hypergraphs $f\colon G \to H$ induces a morphism*

$$P^q(f)\colon = (f_V, f_{V*})\colon (V_G, P^q(G)) \to (V_H, P^q(H))$$

of path complexes, and thus we have a functor $P^q\colon \mathcal{H} \to \mathcal{P}$.

Proof. The result follows directly from Definition 5.3. □

Theorem 5.8. *Let $q \geq 1$. Any morphism of hypergraphs $f\colon G_1 \to G_2$ induces morphisms*

$$f_* : \Omega_*(G_1, q) \to \Omega_*(G_2, q) \quad \text{and } f_* : \Omega_*^{reg}(G_1, q) \to \Omega_*^{reg}(G_2, q) \tag{5.2}$$

of chain complexes and, consequently, homomorphisms

$$H_*(G_1, q) \to H_*(G_2, q) \quad \text{and} \quad H_*^{reg}(G_1, q) \to H_*^{reg}(G_2, q)$$

of homology and regular homology groups, respectively.

Proof. The result follows from Propositions 2.4 and 5.7. □

Corollary 5.9. *For $q \geq 1$, we have functors $\Omega_*(?, q)$ and $\Omega_*^{reg}(?, q)$ from the category $\mathcal{H}$ to the category $\mathcal{C}_*$.*

Proposition 5.10. (i) *For any hypergraph G and $q \geq 2$, we have a morphism of path complexes $\Delta_G^q\colon P^q(G) \to P^{q-1}(G)$ and, hence, induced morphisms of chain complexes*

$$\Omega_*(G, q) \longrightarrow \Omega_*(G, q-1), \quad \Omega_*^{reg}(G, q) \longrightarrow \Omega_*^{reg}(G, q-1)$$

that define homomorphisms

$$H_*(G, q) \to H_*(G, q-1), \qquad H_*^{reg}(G, q) \to H_*^{reg}(G, q-1)$$

of homology groups.

(ii) *Let $f\colon G \to H$ be a morphism of hypergraphs. Then the morphisms Δ_G^q and Δ_H^q fit into the commutative diagram:*

$$\begin{array}{ccc} \Omega_*(G, q) & \xrightarrow{f_*} & \Omega_*(H, q) \\ \downarrow \Delta_G^q & & \downarrow \Delta_H^q \\ \Omega_*(G, q-1) & \xrightarrow{f_*} & \Omega_*(H, q-1) \end{array}$$

and there is a similar commutative diagram for regular chain complexes.

Proof. The condition $p \in P^q(G)$ implies evidently that $p \in P^{q-1}(G)$ and we obtain a morphism of path complexes. Now the result follows from Proposition 2.4 and Theorem 5.8. □

Definition 5.11. A simple hypergraph $G = (V, E)$ is *simplicial* if the condition

$$e = \{v_0, v_1, \ldots, v_n\} \in E$$

implies that any non-empty subset e_1 of $\{v_0, v_1, \ldots, v_n\}$ is also an edge of G.

It follows immediately from this definition, that any simplicial complex Δ with the set of vertices V defines a simplicial hypergraph $\Gamma(\Delta)$ with the same set of vertices and with the edges that are given by simplexes of Δ (and vice versa, a simplicial hypergraph defines an unique simplicial complex).

Let $\mathcal{S}$ be the category whose objects are finite simplicial complexes and whose morphisms are simplicial maps. Consider a simplicial map $\psi\colon \Delta_1 \to \Delta_2$. The map ψ defines a morphism $\Gamma(\psi)\colon \Gamma(\Delta_1) \to \Gamma(\Delta_2)$ of hypergraphs by a natural way. It is easy to see that we obtained a functor $\Gamma\colon \mathcal{C} \to \mathcal{H}$. In what follows, we shall consider only finite simplicial complexes. For a simplicial complex Δ, we shall use the following notations

$$\Omega_*(\Delta, q)\colon = \Omega_*(\Gamma(\Delta), q) \text{ and } \Omega_*^{reg}(\Delta, q)\colon = \Omega_*^{reg}(\Gamma(\Delta), q).$$

As follows from the consideration above, for any $q \geq 1$ we have a functor $\Omega_*(?, q)$ from the category $\mathcal{S}$ of simplicial complexes to the category $\mathcal{C}$ of chain complexes. Thus (see Definition 5.12 below) we obtain a new collection of *path homology theories on the category* $\mathcal{S}$ *of simplicial complexes.*

Definition 5.12. For any simplicial complex Δ and $q \geq 1$ define q-labelled *path homology groups*

$$H_*(\Delta, q)\colon = H_*(\Omega_*(\Delta, q)) = H_*(P^q(\Gamma(\Delta)))$$

and

$$H_*^{reg}(\Delta, q)\colon = H_*(\Omega_*^{reg}(\Delta, q)) = H_*^{reg}(P^q(\Gamma(\Delta))).$$

It follows from this definition that the path homology groups depend functorially on the simplicial complex. Now we state a proposition which describes the dependence on q and follows immediately from the consideration above. Fix $q \geq 1$.

Proposition 5.13. (i) *For any simplicial complex* Δ *and* $q \geq 2$, *we have a morphism of path complexes* $P^q(\Gamma(\Delta)) \to P^{q-1}(\Gamma(\Delta))$ *which induce morphisms of chain complexes*

$$\Omega_*(\Delta, q) \longrightarrow \Omega_*(\Delta, q-1), \quad \Omega_*^{reg}(\Delta, q) \longrightarrow \Omega_*^{reg}(\Delta, q-1),$$

that defines homomorphisms

$$H_*(\Delta, q) \to H_*(\Delta, q-1), \quad H_*^{reg}(\Delta, q) \to H_*^{reg}(\Delta, q-1)$$

of homology groups.

(ii) *A simplicial map* $\psi\colon \Delta_1 \to \Delta_2$ *induces a commutative diagram of chain complexes*

$$\begin{array}{ccc} \Omega_*(\Delta_1, q) & \xrightarrow{\psi_*} & \Omega_*(\Delta_2, q) \\ \downarrow & & \downarrow \\ \Omega_*(\Delta_1, q-1) & \xrightarrow{\psi_*} & \Omega_*(\Delta_2, q-1) \end{array} \tag{5.3}$$

and a similar diagram for regular chain complexes. These diagrams imply commutative diagrams of homology groups.

Now we apply path homology theory to the construction homology theories on the category of *directed hypergraphs.*

A partially ordered set V is called a *linearly ordered* if for any two distinct elements $a, b \in V$ the one of the conditions $a < b$ or $b < a$ is satisfied.

We shall consider only finite directed hypergraphs without double edges. We shall use the bold fonts for designations directed hypergraphs and their morphisms, vertices, and edges.

Definition 5.14. A *directed hypergraph* is a couple $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ where $\mathbf{V}$ is a non-empty finite set of *vertices*, and $\mathbf{E} = \{\mathbf{e_1}, \dots, \mathbf{e_k}\}$ is a set of *directed edges* consisting of non-empty and distinct subsets of $\mathbf{V}$ such that $\bigcup \mathbf{e_i} = \mathbf{V}$ and the elements of any edge $\mathbf{e} = \{\mathbf{i_0}, \mathbf{i_1}, \dots \mathbf{i_n}\} \in \mathbf{E}$ are distinct and linearly ordered. Without restriction of generality we suppose that $\mathbf{i_0} < \mathbf{i_1} < \cdots < \mathbf{i_n}$. We do not suppose that the order of elements are agree on the intersection of edges. If the order of the elements agree on the intersection of the edges, we call such a hypergraph *strong directed*.

Definition 5.15. A *morphism* $\mathbf{f}: \mathbf{G} \to \mathbf{H}$ of directed hypergraphs is given by a pair of maps $\mathbf{f_V}: \mathbf{V_G} \to \mathbf{V_H}$, $\mathbf{f_E}: \mathbf{E_G} \to \mathbf{E_H}$ that satisfy the following properties:

i) there is a commutative diagram

$$\begin{array}{ccc} \mathbf{E_G} & \xrightarrow{\varphi_{\mathbf{G}}} & \mathbb{S}(\mathbf{V_G}) \setminus \emptyset \\ \downarrow \mathbf{f_E} & & \downarrow \mathbb{S}_{\mathbf{f}} \\ \mathbf{E_H} & \xrightarrow{\varphi_{\mathbf{H}}} & \mathbb{S}(\mathbf{V_H}) \setminus \emptyset, \end{array}$$

ii) for any edge $\mathbf{e} = \{\mathbf{i_0}, \mathbf{i_1}, \dots \mathbf{i_n}\} \in \mathbf{E_G}$ we have $\mathbf{f_V}(\mathbf{i_0}) \le \mathbf{f_V}(\mathbf{i_1}) \le \cdots \le \mathbf{f_V}(\mathbf{i_n})$. The set of morphisms from $\mathbf{G}$ to $\mathbf{H}$ we shall denote by $\mathrm{Hom}(\mathbf{G}, \mathbf{H})$.

It is clear that the set of directed hypergraphs with the defined above morphisms form a category which we denote by $\mathcal{H}^+$.

To any directed hypergraph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ we can assign the hypergraph $\mathcal{F}(\mathbf{G}) = G = (V, E)$ putting $V = \mathbf{V}$ and to any ordered edge $\mathbf{e} \in \mathbf{E}$ we assign the edge e consisting from the same non-ordered elements as $\mathbf{e}$. From Definitions 5.3 and 5.15 it follows that we have a (forgetting the order) functor $\mathcal{F}: \mathcal{H}^+ \to \mathcal{H}$. A directed hypergraph $\mathbf{G}$ is h-homogeneous if the hypergraph G is h-homogeneous (connected, without loops) if the hypergraph $\mathbf{G}$ is homogeneous (connected, without loops).

Definition 5.16. For a directed hypergraph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ define a *path complex* $(\mathbf{V}, P^q(\mathbf{G}))$ *of density* $q \ge 1$ on the set $\mathbf{V}$ of vertices by the following way. The path $\mathbf{p}$ of length $n \ge 0$ is defined as the sequence of $n+1$ vertices $\mathbf{i_0}\mathbf{i_1} \dots \mathbf{i_n} = \mathbf{p}$ such that any q or less consequent vertices from p give an ordered subsequence of an edge $\mathbf{e} \in \mathbf{E}$.

Thus, for a directed hypergraph $\mathbf{G}$ and any $q \ge 1$ we have chain complexes

$$\Omega_*(\mathbf{G}, q) := \Omega_*(P^q(\mathbf{G})), \quad \Omega_*^{reg}(\mathbf{G}, q) := \Omega_*^{reg}(P^q(\mathbf{G}))$$

that define the homology groups of $\mathbf{G}$ as follows:

$$H_*(\mathbf{G}, q) := H_*(\Omega_*(\mathbf{G}, q)), \quad H_*^{reg}(\mathbf{G}, q) := H_*(\Omega_*^{reg}(\mathbf{G}, q)).$$

Let $\mathbf{G}$ be a 2-homogeneous directed hypergraph. Then $\mathbf{G}$ defines a simple digraph (in this case, all edges consist of two ordered vertices). The homology $H_*^{reg}(\mathbf{G}, 2)$ coincides with the path homology of simple digraphs from [9], [11], [8].

For a directed hypergraph $\mathbf{G}$, we have by Corollary 3.6

$$H_n(\mathbf{G}, 1) \cong H_n(G, 1) \cong H_n^{reg}(\mathbf{G}, 1) \cong H_n^{reg}(G, 1) \cong \begin{cases} K, & \text{for } n = 0, \\ 0, & \text{for } n \ge 1. \end{cases} \tag{5.4}$$

Let $\mathbf{G}_i = (\mathbf{V}_i, \mathbf{E}_i), (i = 1, 2)$ be directed hypergraphs. Analogously to Proposition 5.7, for $q \ge 1$, any morphism $\mathbf{f}: \mathbf{G}_1 \to \mathbf{G}_2$ induces a morphism

$$P^q(\mathbf{f}): (\mathbf{V}_1, P^q(\mathbf{G}_1)) \to (\mathbf{V}_1, P^q(\mathbf{G}_2))$$

of path complexes, and thus we have a functor $P^q: \mathcal{H}^+ \to \mathcal{P}$.

Theorem 5.17. *Any morphism of directed hypergraphs* $\mathbf{f}\colon \mathbf{G}_1 \to \mathbf{G}_2$ *induces morphisms of chain complexes*

$$\mathbf{f}_* : \Omega_*(\mathbf{G}_1, q) \to \Omega_*(\mathbf{G}_2, q), \quad \mathbf{f}_* : \Omega_*^{reg}(\mathbf{G}_1, q) \to \Omega_*^{reg}(\mathbf{G}_2, q)$$

and, consequently, homomorphisms

$$\mathbf{f}_*\colon H_*(\mathbf{G}_1, q) \to H_*(\mathbf{G}_2, q), \quad \mathbf{f}_*\colon H_*^{reg}(\mathbf{G}_1, q) \to H_*^{reg}(\mathbf{G}_2, q)$$

of homology groups.

Proof. Similar to the proof of Theorem 5.8. □

The next statement is similar to Proposition 5.10.

Proposition 5.18. *For any directed hypergraph* $\mathbf{G}$ *and* $q \geq 2$, *we have a morphism of path complexes* $P^q(\mathbf{G}) \to P^{q-1}(\mathbf{G})$ *and, hence, morphisms of chain complexes*

$$\Omega_*(\mathbf{G}, q) \longrightarrow \Omega_*(\mathbf{G}, q-1), \quad \Omega_*^{reg}(\mathbf{G}, q) \longrightarrow \Omega_*^{reg}(\mathbf{G}, q-1)$$

that are natural relative to morphism of directed hypergraphs and that define homomorphisms

$$H_*(\mathbf{G}, q) \to H_*(G, q-1), \quad H_*^{reg}(\mathbf{G}, q) \to H_*^{reg}(\mathbf{G}, q-1)$$

of homology groups.

For any directed hypergraph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ define a simple digraph $\mathcal{R}(\mathbf{G}) = \mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows. We put $\mathcal{V} = \mathbf{V}$ and we have an arrow $(\mathbf{v} \to \mathbf{w}) \in \mathcal{E}$ for $\mathbf{v}, \mathbf{w} \in \mathcal{V}$ iff there is at least one edge $\mathbf{e} \in \mathbf{E}$ such that $\mathbf{v}, \mathbf{w} \in \mathbf{e}$ and $\mathbf{v} < \mathbf{w}$. It is easy to see that we have a functor $\mathcal{R}$ from the category $\mathcal{H}^+$ to the category $\mathcal{D}$ of digraphs defined in [9, Section 2].

Consider several examples in which the ring of coefficients is $\mathbb{Z}$.

Example 5.19. i) Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ be a directed connected hypergraph with

$$\mathbf{V} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}, \mathbf{6}\} \text{ and } \mathbf{E} = \{\mathbf{e}_i | 1 \leq i \leq 7\}$$

where the vertices in the edges

$$\begin{gathered} \mathbf{e}_1 = \{\mathbf{1}, \mathbf{2}, \mathbf{5}\},\ \mathbf{e}_2 = \{\mathbf{1}, \mathbf{4}, \mathbf{5}\},\ \mathbf{e}_3 = \{\mathbf{2}, \mathbf{3}, \mathbf{5}\},\ \mathbf{e}_4 = \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}, \\ \mathbf{e}_5 = \{\mathbf{1}, \mathbf{2}, \mathbf{6}\},\ \mathbf{e}_6 = \{\mathbf{1}, \mathbf{4}, \mathbf{6}\},\ \mathbf{e}_7 = \{\mathbf{2}, \mathbf{3}, \mathbf{6}\} \end{gathered} \tag{5.5}$$

have the natural order. Now we compute the homology groups $H_*^{reg}(\mathbf{G}, q)$ for all $q \geq 1$.

Note, that for $q \geq 1$, the module $\Omega_0^{reg}(\mathbf{G}, q)$ is generated by the set of vertices $\mathbf{V}$.

The homology in the case $q = 1$ are given by (5.13) and we have

$$H_n^{reg}(\mathbf{G}, 1) = \begin{cases} \mathbb{Z}, & \text{for } n = 0, \\ 0, & \text{for } n \geq 1. \end{cases}$$

Let $q = 2$, the module $\Omega_n^{reg}(\mathbf{G}, q)$ is generated by the set of paths $e_{\mathbf{i}_0 \ldots \mathbf{i}_n}$ in which any pair of consequent induces $\mathbf{i}_j \mathbf{i}_{j+1}$ is an ordered subset of one from the edges in (5.5). Thus the regular chain complex $\Omega_*^{reg}(\mathbf{G}, 2)$

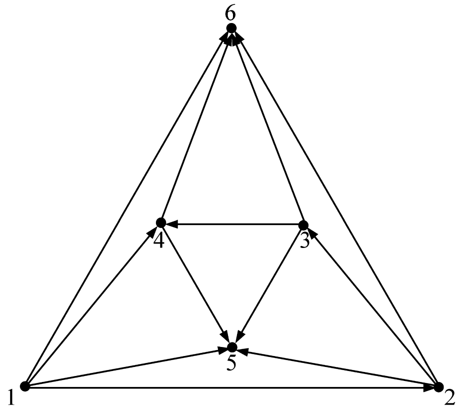

Fig. 1. The planar digraph $\mathcal{G}$ from Example 5.19.

is isomorphic to the regular chain complex (see, [8], [9], [10], and [11]) for path homology groups of the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = \mathbf{V}$ and $\mathcal{E}$ consists of the edges

$$1 \to 2,\ 1 \to 4,\ 1 \to 5,\ 1 \to 6,\ 2 \to 3,\ 2 \to 5,\ 2 \to 6,$$
$$3 \to 4,\ 3 \to 5,\ 3 \to 6,\ 4 \to 5,\ 4 \to 6.$$

The digraph $\mathcal{G}$ is presented on Fig. 1.

Thus we have

$$\Omega_i^{reg}(\mathbf{G}, 2) = \begin{cases} \langle e_{\mathbf{i}} | \mathbf{i} \in \mathcal{V} \rangle, & \text{for } i = 0 \\ \langle e_{\mathbf{ij}} | \mathbf{i} \to \mathbf{j} \in \mathcal{E} \rangle, & \text{for } i = 1 \\ \langle e_{\mathbf{346}}, e_{\mathbf{ijk}} | \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} = \mathbf{e}_m, 1 \leq m \leq 7 \rangle, & \text{for } i = 2 \\ 0, & \text{for } i \geq 3, \end{cases} \tag{5.6}$$

where $\langle \dots \rangle$ means the free abelian group generated by the elements in the angle brackets. Now it is easy to compute the homology group directly, using definition (2.1) of the of differentials. The that the planar digraph of Fig. 1 is the suspension SC over the following digraph C:

$$\begin{array}{ccc} 4 & \longleftarrow & 3 \\ \uparrow & & \uparrow \\ 1 & \longrightarrow & 2 \end{array}$$

and, hence, its regular homology groups are the following (see [8, Theorem 4.13])

$$H_n^{reg}(\mathbf{G}, 2) = H_i(SC, \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{for } n = 0, 2 \\ 0, & \text{for others cases.} \end{cases}$$

For $q \geq 3$, we have

$$\Omega_i^{reg}(\mathbf{G}, q) \cong \begin{cases} \Omega_i^{reg}(\mathbf{G}, 2), & \text{for } i = 0, 1 \\ \langle e_{\mathbf{ijk}} | \{\mathbf{i}, \mathbf{j}, \mathbf{k}\} = \mathbf{e}_m, 1 \leq m \leq 7 \rangle, & \text{for } i = 2 \\ 0, & \text{for } i \geq 3. \end{cases} \tag{5.7}$$

We note the groups $\Omega_2^{reg}(\mathbf{G}, 2)$ and $\Omega_2^{reg}(\mathbf{G}, q)$ differs only by one generator $e_{\mathbf{346}}$. We have

$$\partial e_{\mathbf{346}} = e_{\mathbf{46}} - e_{\mathbf{36}} + e_{\mathbf{34}} = \\ = \partial[e_{\mathbf{125}} - e_{\mathbf{145}} + e_{\mathbf{235}} + e_{\mathbf{345}} - e_{\mathbf{126}} + e_{\mathbf{146}} - e_{\mathbf{236}}]. \tag{5.8}$$

Comparing (5.15) and (5.7) we see that the images of

$$\partial\colon \Omega_3^{reg}(\mathbf{G}, q) \to \Omega_2^{reg}(\mathbf{G}, q) \tag{5.9}$$

are the same for $q = 2$ and for $q \geq 3$. Hence $H_1^{reg}(\mathbf{G}, q) = 0$ for $q \geq 3$. Now we check directly (or using Euler characteristic of chain complex), that the differential in (5.9) is a monomorphism. Hence $H_2^{reg}(\mathbf{G}, q) = 0$ for $q \geq 3$.

i) Let $\mathbf{G_1} = (\mathbf{V}, \mathbf{E})$ be directed connected hypergraph with the same set of vertices as $\mathbf{G}$ in the previous example and the set of edges

$$\mathbf{e}_2 = \{\mathbf{1}, \mathbf{4}, \mathbf{5}\},\ \mathbf{e}_3 = \{\mathbf{2}, \mathbf{3}, \mathbf{5}\},\ \mathbf{e}_4 = \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}, \\ \mathbf{e}_5 = \{\mathbf{1}, \mathbf{2}, \mathbf{6}\},\ \mathbf{e}_6 = \{\mathbf{1}, \mathbf{4}, \mathbf{6}\},\ \mathbf{e}_7 = \{\mathbf{2}, \mathbf{3}, \mathbf{6}\} \tag{5.10}$$

that obtained from the set of edges $\mathbf{G}$ by deleting the edge $\mathbf{e}_1$.

Using the same line of arguments as in the previous example we obtain that

$$H_n^{reg}(\mathbf{G_1}, q) = H_n^{reg}(\mathbf{G}, q) \quad \text{for} \quad q = 0, 1, 2$$

and, for any $q \geq 3$,

$$H_n^{reg}(\mathbf{G_1}, q) = \begin{cases} \mathbb{Z}, & \text{for } n = 0 \\ \mathbb{Z}, & \text{for } n = 1 \\ 0, & \text{for others cases.} \end{cases}$$

Now we discuss the application of homotopy theory to the categories $\mathcal{H}^+$ and $\mathcal{H}$ of hypergraphs.

Let $I = (V_I, E_I)$ be the hypergraph with two vertices $V = \{0, 1\}$ and the set of edges $E_I = \{f_0 = \{0\}, f_1 = \{1\}, f_2 = \{0, 1\}\}$. For any hypergraph $G = (V, E)$ with the set of edges $E = \{e_1, \dots, e_k\}$, let $H = G \times I$ be the product of hypergraphs as in Definition 5.4. The hypergraph H has, in particular, the edges

$$(A_s, e_s, f_0) \text{ with } p_1(A_s) = \phi_G(e_s),\ p_2(A_s) = f_0$$

and

$$(A_s, e_s, f_1) \text{ with } p_1(A_s) = \phi_G(e_s),\ p_2(A_s) = f_1.$$

It is clear that $p_1|_{A_s}$ is the bijection in both cases.

Consider two natural inclusions of the hypergraphs $i = (i_V, i_E)\colon G \to G \times I$ where

$$i_V(v) = (v, 0) \text{ and } i_E(e_s) = (A_s, e_s, f_0), v \in V, e_s \in E$$

and $j = (j_V, j_E)\colon G \to G \times I$ where

$$j_V(v) = (v, 1) \text{ and } j_E(e_s) = (A_s, e_s, f_1), v \in V, e_s \in E.$$

Thus by means i we shall identify G with the sub-hypergraph of $G \times I$ that we shall call the *bottom boundary* of $G \times I$ and by means j we shall identify G with the sub-hypergraph of $G \times I$ that we shall call the *top boundary* of $G \times I$.

Definition 5.20. (i) We call two morphisms $f, g\colon G \to H$ of hypergraphs *one-step homotopic* and write $f \simeq_1 g$ if there exists a morphism $F\colon G \times I \to H$ of hypergraphs such that the restriction of F to the bottom boundary is f and the restriction of F to the top boundary is g.

ii) We call two morphisms $f, g\colon G \to H$ of hypergraphs homotopic and write $f \simeq g$ if there exists a sequence of morphisms of hypergraphs $f_i\colon G \to H$ such that $f = f_0 \simeq_1 f_1 \simeq_1 \cdots \simeq_1 f_n = g$.

iii) Two hypergraphs G and H are homotopy equivalent if there exist morphisms

$$f\colon G \to H, \quad g\colon H \to G \ \text{ such that } \quad fg \simeq \mathrm{Id}_H, \quad gf \simeq \mathrm{Id}_G$$

where $\mathrm{Id}_G\colon G \to G$ and $\mathrm{Id}_H\colon H \to H$ are the identity morphisms. In this case, we write $G \simeq H$ and call the morphisms f, g homotopy inverses to each other.

Theorem 5.21. i) *Let*

$$f \simeq g\colon G_1 \to G_2$$

be homotopic morphisms of hypergraphs. Then for any $q \geq 1$, these morphisms induce chain homotopic morphisms $f_* \simeq g_*$ of chain complexes

$$\Omega_*(G_1, q) \to \Omega_*(G_2, q) \ \ \textit{and} \ \ \Omega_*^{reg}(G_1, q) \to \Omega_*^{reg}(G_2, q)$$

and, hence, the same homomorphism $f_* = g_*$

$$H_*(G_1, q) \to H_*(G_2, q) \ \ \textit{and} \ \ H_*^{reg}(G_1, q) \to H_*^{reg}(G_2, q)$$

of homology groups.

ii) *If the hypergraphs G_1 and G_2 are homotopy equivalent, then for $q \geq 1$, then theirs homology and regular homology groups are isomorphic:*

$$H_*(G_1, q) \cong H_*(G_2, q) \ \ \textit{and} \ \ H_*^{reg}(G_1, q) \cong H_*^{reg}(G_2, q).$$

Furthermore, if the homotopy equivalence is provided by the homotopy inverse morphisms f and g then theirs induced maps f_* and g_* provide mutually inverse isomorphisms of homology groups.

Proof. It is sufficiently to consider the case of one-step homotopy. For $q \geq 1$, the morphisms i, j, f, g, and F induce morphisms of path complexes

$$f_\bullet, g_\bullet\colon P^q(G_1) \to P^q(G_2),$$
$$i_\bullet, j_\bullet\colon P^q(G_1) \to P^q(G_1 \times I)$$

and

$$F_\bullet\colon P^q(G_1 \times I) \to P^q(G_2)$$

such that

$$F_\bullet \circ i_\bullet = f_\bullet \ \text{ and } \ F_\bullet \circ j_\bullet = g_\bullet.$$

Consider the path complex $P^q(G_1) \times I$ as in (3.1) of Section 3. Any path of density q from $P^q(G_1) \times I$ that is in P or in P' (as in (3.1) has the form

$$w = v_0 \dots v_k \text{ and } w' = v'_0 \dots v'_k,$$

respectively. These paths define evidently the unique paths

$$\tau(w) = i_\bullet(w) = (v_0, 0) \dots (v_k, 0), \quad \tau(w') = j_\bullet(w') = (v_0, 1) \dots (v_k, 1) \tag{5.11}$$

in $P^q(G_1 \times I)$. Consider the path $\widehat{w} = v_0 \dots v_k v'_k \dots v'_n \in P^q \times I$ where $0 \le k \le n$ and $w = v_0 \dots v_k \dots v_n \in P^q(G_1)$ (similarly to (3.1)).

We state that the path

$$\tau(\widehat{w}) = (v_0, 0) \dots (v_k, 0)(v_k, 1) \dots (v_n, 1) \tag{5.12}$$

lies in $P^q(G_1 \times I)$. Consider q or less consequent elements of the path $\tau(\widehat{w})$. If the set of these elements is the subset of $(v_0, 0) \dots (v_k, 0)$ or the set $(v_k, 1) \dots (v_n, 1)$ the statement directly follows from the hypothesis that $w \in P^q(G_1)$. If these consequent elements have the form $(v_m, 0) \dots (v_k, 0)(v_k, 1) \dots (v_l, 1)$ with $0 \le k \le l \le n$ the statement also trivial, since the elements $v_m \dots v_k \dots v_l$ lies in some edge e_i of G and hence the elements $(v_m, 0) \dots (v_k, 0)(v_k, 1) \dots (v_l, 1)$ lies in the edge (A, e_i, f_2) of $G_1 \times I$. Hence (5.11) and (5.12) define natural inclusion

$$\tau \colon P^q(G_1) \times I \to P^q(G_1 \times I).$$

Hence the composition

$$F_\bullet \circ \tau \colon P^q(G_1) \times I \longrightarrow P^q(G_2)$$

gives one-step homotopy $f_\bullet \simeq_1 g_\bullet$. Now the result follows from Theorem 3.4. □

Remark 5.22. For directed hypergraphs Theorem 5.21 remains valid.

Now we present several results about homology groups of hypergraphs.

Define a *hypergraph path* $I_n^q = (V, E)$ *of density* $q \ge 1$ *and of length* $n \ge 0$ as follows

$$V = \{0, 1, \dots, n\}, \ E = \{e_1, e_2, \dots, e_k | k = n - q + 2\}$$

where $e_1 = \{0, 1, \dots, q-1\}, e_2 = \{1, 2, \dots, q\}, \dots, e_k = \{n - q + 1, n - q + 2, \dots, n\}$.

Proposition 5.23. *For any simplicial hypergraph* $G = (V, E)$ *we have an one-to-one correspondence* $P_n^q(G) \longleftrightarrow \mathrm{Hom}(I_n^q, G)$ *between the paths of length* $n \ge 0$ *from* $P^q(G)$ *and the set of hypergraph morphisms* $\mathrm{Hom}(I_n^q, G)$.

Proof. Let $p = i_0 i_1 \dots i_n \in P^q(G)$. We define the morphism $(f_V, f_E) \colon I_n^q \to G$ putting $f_V(j) = i_j$ for $0 \le j \le n$ and for $1 \le j \le n - q + 2$ we define $f_E(e_j)$ as the unique minimal edge in G that consists of the vertices $\{f_V(j-1), \dots, f_V(q + j - 2)\}$. Such edge exists as follows from definition p and since G is a simplicial hypergraph. It follows from Definition 5.3 of hypergraph morphism that any $f \in \mathrm{Hom}(I_n^q, G)$ defines a path $p \in P^q(G)$ by the rule $p = f_V(0) \dots f_V(n)$. □

Now we describe one more relation of our construction to the simplicial theory. For a simple hypergraph $H = (V, E)$, let $\widehat{H} \colon = (\widehat{V}, \widehat{E})$ denote the simplicial hypergraph constructed in the following way. We put $\widehat{V} = V$ and we add a minimal number of edges to E to obtain $\widehat{E}$ of the simplicial hypergraph $\widehat{H}$. We shall say that the simplicial hypergraph $\widehat{H}$ is *associated* with the hypergraph H.

Lemma 5.24. *Any morphism $f\colon G\to H$ of simple hypergraphs induces a morphism $\widehat{f}\colon \widehat{G}\to\widehat{H}$ of associated simplicial hypergraphs in such a way that $f_V=\widehat{f}_V$.*

Proof. Any morphism f of simple hypergraphs is defined by the restriction $f_V\colon V_G\to V_H$ and, hence, by the restriction f_V to the maximal edges of G. Consider the restriction of $f_V|_e$ on a maximal edge $e=\{i_0,\dots,i_n\}$. By Definition 5.3, we have $f_E(e)=e'\in E_H$ where

$$e'=\{j_0,\dots,j_k\} \text{ and } f_V\left(\{i_0,\dots,i_n\}\right)=\{j_0,\dots,j_k\}.$$

The map $f_V|_e$ gives a unique map of all subsets of the set $\{i_0,\dots,i_n\}$ to the all subsets of the set $\{j_0,\dots,j_k\}$ which automatically provide the well defined map f_E on any edge $e_1\subset e$ since f is the hypergraph map by assumption. Now the result follows. □

Proposition 5.25. *Let H be a simple hypergraph and $\widehat{H}$ be an associated simplicial hypergraph. Then the identity map on the set of vertices V for any $q\geq 1$ gives the inclusion $s\colon H\to\widehat{H}$ of hypergraphs that induces identity map of path complexes*

$$s_*\colon P^q(H)\xrightarrow{=} P^q(\widehat{H}),$$

in particular, for any $q\geq 1$ and $n\geq 0$ we have

$$P_n^q(H)\overset{=}{\to} P_n^q(\widehat{H}).$$

Moreover, any morphism $f\colon G\to H$ induces a morphism $\widehat{f}\colon P^q(\widehat{G})\to P^q(\widehat{H})$ such that the diagram

$$\begin{array}{ccc} P^q(H) & \underset{=}{\xrightarrow{s_*}} & P^q(\widehat{H}) \\ \downarrow \widehat{f}_* & & \downarrow f_* \\ P^q(G) & \underset{=}{\xrightarrow{s_*}} & P^q(\widehat{G}) \end{array} \tag{5.13}$$

is commutative.

Proof. The first statement is trivial, since the collection of maximal edges of H and $\widehat{H}$ is the same. The second statement follows from Lemma 5.24. □

Lemma 5.26. *Let $G_1=(V_1,E_1),G_2=(V_2,E_2)$ be two simplicial hypergraphs. Then the product $G_1\times G_2$ is also a simplicial hypergraph.*

Proof. It is enough to prove that for any two edges $e_1\in G_1$, $e_2\in G_2$ with $e_1=\{i_0,\dots,i_n\}$, $e_2=\{j_0,\dots,j_k\}$ any subset A of the direct product

$$\{i_0,\dots,i_n\}\times\{j_0,\dots,j_k\}$$

of the sets defines an edge in the hypergraph $G_1\times G_2$. Let $p_i(A)=a_i\subset e_i\subset V_i (i=1,2)$ where p_i are natural projections. Then a_i is an edge of G_i, since G_i is simplicial. Hence the triple (A,a_1,a_2) is an edge in $G_1\times G_2$ by Definition 5.4 and the result follows. □

Proposition 5.27. *Let G_1,G_2 be two simplicial hypergraphs. Then for any $q\geq 1$ we have an one-to-one correspondence*

$$P^q(G_1\times G_2)\longleftrightarrow P^q(G_1)\times P^q(G_2) \tag{5.14}$$

between the path complexes which induces for $n \geq 0$ an isomorphism of modules

$$\Omega_*(G_1 \times G_2, q) \cong \Omega_n(G_1, q) \otimes \Omega_n(G_2, q). \tag{5.15}$$

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