Energy-momentum surface density and quasilocal mass in general relativity

Mu-Tao Wang and Shing-Tung Yau

April 7, 2008

Abstract

We define new energy momentum surface density and quasilocal mass for the boundary surface of a spacelike region in spacetime. Isometric embeddings of such a surface into the Minkowski space are used as reference surfaces in the definition. When the reference surface lies in a flat three-dimensional space slice of the Minkowski space, our expression recovers the Brown-York and Liu-Yau quasilocal mass. Moreover, we prove the positivity of the new quasilocal mass under the dominant energy condition and show that it is zero for surfaces in the Minkowski space.

1 Introduction

The notion of quasi-local energy for a bounded space-like region is essential to several major unsettled problems in general relativity. In [3], Brown and York proposed a quasi-local energy-momentum based on the Hamiltonian formulation of general relativity. Liu and the second author [8] (see also Kijowski [6]) found a gauge independent definition based on attempts to understanding the second author's work on the existence of black holes due to boundary effect. The Brown York mass is proved to be positive in the time-symmetric case by Shi and Tam [13]; Liu and the second author also proved the positivity of their mass under the more physical and desirable dominant energy condition. However, it was shown that surfaces exist in $\mathbb{R}^{3,1}$ for which the Liu-Yau mass (as well as the Brown-York mass) is strictly positive [9]. In the definitions of the Brown-York mass and the Liu-Yau mass, the boundary surface is assumed to have positive Gauss curvature and the (essentially unique) isometric embedding into $\mathbb{R}^3$ as a flat three-dimensional space slice in $\mathbb{R}^{3,1}$ is used as a reference. In this letter, we define new notions of energy-momentum surface density and quasi-local mass using isometric embeddings of surfaces into $\mathbb{R}^{3,1}$ as references. This involves the precise description of isometric embeddings into $\mathbb{R}^{3,1}$ in terms of the time function on Σ and canonical choice of gauges in $\mathbb{R}^{3,1}$ and the physical space so that the expansions of the surfaces in the corresponding directions are the same. We prove the new quasi-local mass is positive under the dominant energy condition and show that it is zero for surfaces in $\mathbb{R}^{3,1}$.

2 Hamiltonian formulation revisited

We recall (e.g. (E.1.42) [16]) that the action for a spacetime region M with boundary ∂M is given by

$$S = \frac{1}{16\pi}\int_M \mathcal{R} + \frac{1}{8\pi}\oint_{\partial M} K + S_m$$

where $\mathcal{R}$ is the scalar curvature of the spacetime Lorentzian metric, K is the trace of the extrinsic curvature of ∂M, and S_m is the matter action. We consider a spacetime region M that is foliated by a family of spacelike hypersurface Ω_t for t in the time interval $[t', t'']$. The boundary of M consists of $\Omega_{t'}$, $\Omega_{t''}$, and ${}^3B$. Let u^μ denote the future pointing timelike unit normal to Ω_t. We assume u^μ is tangent to ${}^3B$. Denote the boundary of Ω_t by Σ_t which is the intersection of Ω_t and ${}^3B$. Let v^μ denote the outward pointing spacelike unit normal of Σ_t such that $u_\mu v^\mu = 0$. Denote by k the trace of the two-dimensional extrinsic curvature of Σ_t in Ω_t in the direction of v^μ. Our choice makes $k = 2$ for the sphere of radius one in $\mathbb{R}^3$. Denote the Riemannian metric and the extrinsic curvature on Ω_t by $g_{\mu\nu}$ and $K_{\mu\nu} = \nabla_\mu u_\nu$, respectively. Both $g_{\mu\nu}$ and $K_{\mu\nu}$ are purely spatial and can be viewed alternatively as tensors on Ω_t, denoted by g_{ij} and K_{ij}, where the indices i, j refer to coordinates on Ω. $K = g^{ij}K_{ij}$ is the trace of the extrinsic curvature. Let t^μ be a timelike vector field satisfying $t^\mu \nabla_\mu t = 1$. t^μ can be decomposed into the lapse function and shift vector $t^\nu = Nu^\nu + N^\nu$. The Hamiltonian at t is then given by

$$H = -\frac{\partial S}{\partial t}.$$

The calculation in Brown-York [3] (see also Hawking-Horowitz [5]) leads to

$$H = \int_{\Omega_t} (N\mathcal{H} + N^\mu \mathcal{H}_\mu) - \frac{1}{8\pi} \int_{\Sigma_t} [Nk - N^\mu v^\nu (K_{\mu\nu} - K g_{\mu\nu})] \tag{2.1}$$

where $\mathcal{H}$ is the Hamiltonian constraint and $\mathcal{H}_\mu$ the momentum constraint. On a solution M of the Einstein equation both $\mathcal{H}$ and $\mathcal{H}_\mu$ vanish. Equation (2.1) is the same as equation (2.11) in [5] (see also equation (4.13) in [3] in which the reference Hamiltonian is already subtracted). Notice that our k is the same as in [8] and the ${}^2K$ in [5], but is the negative of the k in [3]. To define the quasi-local energy, we need to find a reference action S_0 that corresponds to fixing the metric on ${}^3B$ and compute the corresponding reference Hamiltonian H_0. The energy is then

$$E = \frac{\partial}{\partial t}(S - S_0) = H - H_0.$$

In Brown-York's definition of quasilocal energy, the reference is taken to be an isometric embedding of Σ into $\mathbb{R}^3$, considered as a flat three-dimensional slice with $K_{\mu\nu} = 0$ in a flat spacetime. Choosing $N = 1$ and $N^\mu = 0$, the Brown-York quasilocal energy is

$$\frac{1}{8\pi} \int_{\Sigma_t} (k_0 - k).$$

A similar choice of reference leads to the expression in (2.14) of [5]. References such as surfaces in the light cones ([2] and [7]) and other conditions [6] have been proposed. However, the Brown-York energy for the examples [9] of surfaces in the Minkowski are in general non-zero for all these references.

We shall use a different reference to define our quasilocal energy. We suppress the subindex t and rewrite the Hamiltonian (2.1) on a solution as

$$H = \frac{1}{8\pi} \int_{\Sigma} [k u^\mu + v^\nu (K^\mu_\nu - K \delta^\mu_\nu)] t_\mu. \tag{2.2}$$

This is an integral on Σ that depends on the choices of a future pointing timelike unit normal u^μ (v^μ is then determined by the outward pointing and the orthogonal $u^\mu v_\mu = 0$ conditions) and a time-like vector field t^μ along Σ. Given a space-like surface Σ in a spacetime M, we shall define two types of quasi-local energy using isometric embeddings into $\mathbb{R}^{3,1}$, the first type corresponds to an arbitrary choice of u^μ while the second type corresponds to a canonical choice of u^μ. When the embedding falls in a flat slice $\mathbb{R}^3$, the first type recovers the Brown-York quasi-local energy and the second type recovers the Liu-Yau quasi-local energy. In both cases, the t^μ is obtained by transplanting a Killing vector field in $\mathbb{R}^{3,1}$ back to the physical space through the isometric embedding of Σ.

3 Definition in general gauge

We first look at the case when Σ bounds a space-like hypersurface Ω and u^ν is the future-pointing time-like unit normal of Ω. Consider the four vector field

$$ku^\nu + v^\mu(K_\mu^\nu - K\delta_\mu^\nu) \tag{3.1}$$

on Σ. The definition of this vector field only depends on the two normals u^ν and v^ν along Σ. In particular, only the first order information of Ω along Σ is needed. The normal component (with respect to Σ) of (3.1) is

$$j^\nu = ku^\nu - pv^\nu,$$

where $p = K - K_{\mu\nu}v^\mu v^\nu$. j^ν, as well as the *mean curvature vector field*

$$h^\nu = -kv^\nu + pu^\nu,$$

is defined independent of the choice of gauge u^ν and v^ν. Given an isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ of Σ. We fix a constant timelike unit vector t_0^ν in $\mathbb{R}^{3,1}$ and choose a preferred pair of normals u_0^ν and v_0^ν along $i(\Sigma)$ in the following way. Take a space-like hypersurface Ω_0 with $\partial\Omega_0 = i(\Sigma)$, and such that the outward pointing spacelike unit normal v_0^ν of $\partial\Omega_0$ satisfies $t_\nu v_0^\nu = 0$. Let u_0^ν be the future pointing timelike unit normal of Ω_0 along $i(\Sigma)$. We can similarly form

$$k_0 u_0^\nu + v_0^\mu((K_0)_\mu^\nu - K_0\delta_\mu^\nu)$$

in terms of the corresponding geometric quantities on Ω_0 and $i(\Sigma)$. $({u_0}^\nu, {v_0}^\nu)$ along $i(\Sigma)$ in $\mathbb{R}^{3,1}$ is the reference normal gauge we shall fix and it depends on the choice of the pair (i, t_0^ν). Four-vectors in $\mathbb{R}^{3,1}$ and M, along $i(\Sigma)$ and Σ respectively, can be identified through

$${u_0}^\nu \to u^\nu, {v_0}^\nu \to v^\nu, \tag{3.2}$$

and the identification of tangent vectors on $i(\Sigma)$ and Σ. For example, the four-vector field $t_0^\nu = N_0 u_0^\nu + N_0^\nu$ in $\mathbb{R}^{3,1}$ with lapse function N_0 and shift vector N_0^ν is identified with the four-vector field $t^\nu = N_0 u^\nu + N_0^\nu$ in M with the same lapse function and shift vector.

Definition 3.1 *The energy-momentum surface density vector in the u^ν gauge is defined to be*

$$\frac{1}{8\pi G}[ku^\nu + v^\mu(K_\mu^\nu - K\delta_\mu^\nu) - k_0 u_0^\nu + v_0^\mu((K_0)_\mu^\nu - K_0\delta_\mu^\nu)]$$

as a four vector field of M along Σ through the identification (3.2).

Definition 3.2 *The quasi-local energy-momentum in the u^ν gauge is defined to be*

$$\frac{1}{8\pi G}\int_\Sigma [ku^\nu + v^\mu(K_\mu^\nu - K\delta_\mu^\nu) - k_0 u_0^\nu + v_0^\mu((K_0)_\mu^\nu - K_0\delta_\mu^\nu)](t_0)_\nu \qquad (3.3)$$

where the identification (3.2) is used.

When the reference isometric embedding lies in a flat space slice on which the time-function t is a constant, $t_0^\nu = u_0^\nu$ and (3.3) reduces to the Brown-York quasi-local energy $\frac{1}{8\pi G}\int_\Sigma (k_0 - k)$.

4 Definition in the canonical gauge

When the mean curvature vector h^ν of Σ in M is spacelike, a reference isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ and $t_0^\nu \in \mathbb{R}^{3,1}$ determines a canonical future-directed time-like normal vector field $\bar{u}^\nu$ in M along Σ. Indeed, there is a unique $\bar{u}^\nu$ that satisfies

$$h_\nu \bar{u}^\nu = (h_0)_\nu {u_0}^\nu \qquad (4.1)$$

where h_0^ν is the mean curvature vector of $i(\Sigma)$ and u_0^ν is the same as in the previous section. Physically, (4.1) means the expansions of $\Sigma \subset N$ and $i(\Sigma) \subset \mathbb{R}^{3,1}$ along the respective directions $\bar{u}^\nu$ and ${u_0}^\nu$ are the same. This condition corresponds to fixing the metric on ${}^3B$ up to the first order in choosing the reference Hamiltonian in §2. $\bar{u}^\nu$ shall be called the *canonical gauge* with respect to the pair (i, t_0^ν). For any surface in the Minkowski space, the canonical gauge is the same as the u_0^ν gauge we choose in the previous section. We can take $\bar{v}^\nu$ to be the space-like normal vector that is orthogonal to $\bar{u}^\nu$ and satisfies $\bar{v}^\nu h_\nu < 0$. Shrink Σ in the direction $\bar{v}^\nu$ to get a space-like hypersurface $\bar{\Omega}$ that is locally defined near Σ, we can similarly form

$$\bar{k}\bar{u}^\nu + \bar{v}^\mu(\bar{K}_\mu^\nu - \bar{K}\delta_\mu^\nu)$$

where $\bar{K}_\mu^\nu$, $\bar{K}$, and $\bar{k}$ are the corresponding data on $\bar{\Omega}$. The trace of the two-dimensional extrinsic curvature $\bar{k}$ of Σ with respect to $\bar{v}^\nu$ is then given by

$$\bar{k} = -\bar{v}^\nu h_\nu > 0.$$

Definition 4.1 *The energy-momentum surface density vector in the canonical gauge is defined to be*

$$\frac{1}{8\pi}[\bar{k}\bar{u}^\nu + \bar{v}^\mu(\bar{K}_\mu^\nu - \bar{K}\delta_\mu^\nu) - k_0 u_0^\nu - v_0^\mu((K_0)_\mu^\nu - K_0\delta_\mu^\nu)]$$

where $\bar{u}^\nu$ is given by (4.1) and the identification (3.2) is used.

Definition 4.2 *The quasi-local energy-momentum of Σ in the canonical gauge with respect to (i, t_0^ν) is defined to be*

$$\frac{1}{8\pi}\int_\Sigma [\bar{k}\bar{u}^\nu + \bar{v}^\mu(\bar{K}_\mu^\nu - \bar{K}\delta_\mu^\nu) - k_0 u_0^\nu - v_0^\mu((K_0)_\mu^\nu - K_0\delta_\mu^\nu)](t_0)_\nu, \qquad (4.2)$$

where the identification (3.2) is used.

The mean curvature vector h^ν being spacelike is equivalent to $\rho\mu > 0$ where ρ and μ are the expansion along the future and past outer null-normals of Σ, respectively (so-called the Newman-Penrose spin coefficients, see [11] or [12]). Indeed the Lorentzian norm of the mean curvature vector is $h_\nu h^\nu = 8\rho\mu$. When the reference isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ has its image $i(\Sigma)$ in a flat space slice, we have $t_0^\nu = u_0^\nu$. The canonical gauge $\bar{u}^\nu = \frac{1}{\sqrt{8\rho\mu}} j^\nu$, $\bar{v}^\nu = -\frac{1}{\sqrt{8\rho\mu}} h^\nu$, and $\bar{k} = \sqrt{8\rho\mu}$. In this case, (4.2) recovers the Liu-Yau quasi-local mass $\int_\Sigma (k_0 - \sqrt{8\rho\mu})$.

5 Positivity of quasi-local mass

Unlike Brown-York or Liu-Yau, we do not require the surface Σ to have positive Gauss (intrinsic) curvature and apply the isometric embedding theorem of Weyl. Instead, we prove a uniqueness and existence theorem of isometric embeddings into the Minkowski space under a more general convexity condition.

Definition 5.1 *Let t_0^ν be a constant timelike unit vector in $\mathbb{R}^{3,1}$. An isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ is said to have convex shadow in the direction of t_0^ν if the projection of $i(\Sigma)$ onto the orthogonal complement $\mathbb{R}^3$ of t_0^ν is a convex surface.*

The set of isometric embedding with convex shadows is parametrized by functions satisfying an convexity condition.

Theorem 5.1 *Let σ_{ab} be a Riemannian metric on a two-sphere Σ. Given any function τ with*

$$\kappa + (1 + \sigma^{ab}\nabla'_a\tau\nabla'_b\tau)^{-1}\frac{\det(\nabla'_a\nabla'_b\tau)}{\det\sigma_{ab}} > 0 \tag{5.1}$$

where κ is the Gauss curvature and ∇' is the covariant derivative of the metric σ_{ab}. Then there exists a unique space-like isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ such that the time function restricts to τ on Σ.

Proof. We prove the uniqueness part first. Suppose there are two such isometric embeddings i_1 and i_2 with the same time function. It is not hard to check that the condition (5.1) implies the projections of $i_1(\Sigma)$ and $i_2(\Sigma)$ onto the orthogonal complement of the time direction are isometric as convex surfaces in $\mathbb{R}^3$. By Cohn-Vossen's rigidity theorem, the projections are congruent by a rigid motion of $\mathbb{R}^3$. Since they have the same time functions, $i_1(\Sigma)$ and $i_2(\Sigma)$ are congruent by a Lorentizian rigid motion of $\mathbb{R}^{3,1}$. Now we turn to the existence part. The condition (5.1) implies the metric $\sigma_{ab} + \nabla'_a\tau\nabla'_b\tau$ has positive Gauss curvature, and thus can be isometrically embedded into $\mathbb{R}^3$. We may assume this $\mathbb{R}^3$ is a space-slice in the Minkowski space, so the induced metric on the graph of τ is exactly σ_{ab}. □

In order the recognize a surface in the Minkowski space, we solve the Dirichlet boundary value problem for the Jang'equation. Given a hypersurface (Ω, g_{ij}, K_{ij}) in M, we recall that the Jang's equation asks for a solution

f of

$$\sum_{i,j=1}^{3}(g^{ij} - \frac{f^i f^j}{1 + g^{ij} D_i f D_j f})(\frac{D_i D_j f}{(1 + g^{ij} D_i f D_j f)^{1/2}} - K_{ij}) = 0, \tag{5.2}$$

where D is the covariant derivative of g_{ij}. The graph of f in the space $\Omega \times \mathbb{R}$ is denoted by $\widetilde{\Omega}$. In this article, we are interested in the case when $\partial\Omega = \Sigma$ and the prescribed value of f on the boundary Σ is given. Notice that if Σ is in $\mathbb{R}^{3,1}$, and if we take the time function τ as the boundary value to solve the Jang's equation, then $\widetilde{\Omega}$ will be a flat domain in $\mathbb{R}^3$.

Definition 5.2 *Given a space-like two surface Σ in a spacetime M, an isometric embedding $i : \Sigma \hookrightarrow \mathbb{R}^{3,1}$, and a constant timelike unit vector $t_0^\nu \in \mathbb{R}^{3,1}$. Let τ denote the time function restricted on Σ. (i, t_0^ν) is said to be an admissible pair for Σ if the followings are satisfied*
(1) i has convex shadow in the direction of t_0^ν.
(2) Σ bounds a space-like domain Ω in M such that the Jang's equation (5.2) with the Dirichlet boundary data τ is solvable on Ω (with possible apparent horizons in the interior).
(3) Suppose f is the solution of the Jang's equation in (2) and v^ν is the outward unit normal of Σ that is tangent to Ω, and u^ν is the future-directed time-like normal of Ω in N. Consider the new gauge u'^ν given by

$$u'^\nu = \sinh\phi v^\nu + \cosh\phi u^\nu, \text{ and } v'^\nu = \cosh\phi v^\nu + \sinh\phi u^\nu$$

where

$$\sinh\phi = \frac{f_v}{\sqrt{1 + \sigma^{ab}\nabla'_a\tau\nabla'_b\tau}},$$

and f_v is the normal derivative of f in the direction of v^ν. We require that

$$k'N_0 - N_0^\nu v'^\mu (K'_{\mu\nu} - K' g'_{\mu\nu}) > 0, \tag{5.3}$$

where k', $g'_{\mu\nu}$, $K'_{\mu\nu}$, and K' are the corresponding data on Ω' spanned by v'^ν, and N_0 and N_0^ν are the lapse function and shift vector of $t_0^\nu = N_0 u_0^\nu + N_0^\nu$.

Remark 5.1 *By a barrier argument, we show that Ω satisfies (2) if on Σ, $k > \frac{1}{t^a t_a(1+t^a t_a)}(K_{ab}t^a t^b) + K_{ab}u^a u^b$ where u^a is a two-vector such that $t^a u_a = 0$ and $u^a u_a = 1$, and $t^a = \pi^a_\nu t_0^\nu$ is the projection of t_0^ν onto Σ. Also by elliptic estimates, (3) will be satisfied if (5.3) holds for u^ν and v^ν and $\sigma^{ab}\nabla_a\tau\nabla_b\tau$ is small enough. In particular, if Σ has space-like mean curvature vector in M, then any isometric embedding i whose image lies in an $\mathbb{R}^3$ is admissible.*

We emphasize that although the definition of admissible pairs involves solving the Jang's equation, the results only depend on the solvability but not on the specific solution. The expression of quasi-local energy only depends on the canonical gauge $\bar{u}^\nu$.

Theorem 5.2 *Suppose M is a time-orientable spacetime that satisfies the dominant energy condition. Suppose Σ has spacelike mean curvature vector in M. Then the quasi-local energy-momentum (4.2) with respect to any admissible pair (i, t_0^ν) is positive.*

Proof. We take the time function τ on $i(\Sigma)$ and consider the Dirichlet problem of the Jang's equation (5.2) over (Ω, g_{ij}, K_{ij}) with $f = \tau$ on Σ. Condition (2) guarantees the equation is solvable on Ω. Denote by $\widetilde{\Omega}$ the graph of the solution of the Jang's equation. Schoen and Yau [14] showed that if M satisfies the dominant energy condition, there exists a vector filed X on $\widetilde{\Omega}$ such that

$$R \geq 2|X|^2 - 2divX \tag{5.4}$$

where R is the scalar curvature of $\widetilde{\Omega}$.

Let $\widetilde{\Sigma}$ be the graph of τ over Σ and denote the outward normal of $\widetilde{\Sigma}$ with respect to $\widetilde{\Omega}$ by $\tilde{v}^i$ and the mean curvature of $\widetilde{\Sigma}$ with respect to $\tilde{v}^i$ by $\tilde{k}$. We show that the condition (3) guarantees that $\tilde{k} - \tilde{v}^i X_i > 0$. Then we make use of another important property of the canonical gauge that

$$\int_{\widetilde{\Sigma}} (\tilde{k} - \tilde{v}^i X_i) \geq - \int_\Sigma [\bar{k}\bar{u}^\nu + \bar{v}^\mu(\bar{K}_\mu^\nu - \bar{K}\delta_\mu^\nu)](t_0)_\nu. \tag{5.5}$$

This is why the eventual definition of the quasi-local energy momentum is independent of the solution of the Jang's equation. On the other hand, it is not hard to check that $-\int_\Sigma [k_0 {u_0}^\nu + {v_0}^\mu ({K_0}_\mu^\nu - K_0 \delta_\mu^\nu)](t_0)_\nu = \int_{\widehat{\Sigma}} \widehat{k}$ (we were motivated by Gibbon's paper [4] to study this expression, however, the equality he obtained is different from ours). Here $\widehat{\Sigma}$ is the image of the projection of $i(\Sigma)$ onto the orthogonal complement of t_0^ν and $\widehat{k}$ is the mean curvature of $\widehat{\Sigma}$. Therefore the proof is reduced to the inequality

$$\int_{\widehat{\Sigma}} \widehat{k} \geq \int_{\widetilde{\Sigma}} (\tilde{k} - \tilde{v}^i X_i).$$

We notice that the Riemannian metrics on $\widetilde{\Sigma}$ and $\widehat{\Sigma}$ are the same. The proof will be completed by the following comparison theorem for the solution of

Jang's equation. Suppose $\widetilde{\Omega}$ is a Riemannian three-manifold with boundary $\widetilde{\Sigma}$ and suppose there exists a vector field X on $\widetilde{\Omega}$ such that (5.4) holds on $\widetilde{\Omega}$ and

$$\tilde{k} > \tilde{v}^i X_i$$

on $\widetilde{\Sigma}$. Suppose the Gauss curvature of $\widetilde{\Sigma}$ is positive and k_0 is the mean curvature of the isometric embedding of $\widetilde{\Sigma}$ into $\mathbb{R}^3$. Then

$$\int_{\widetilde{\Sigma}} k_0 \geq \int_{\widetilde{\Sigma}} (\tilde{k} - \tilde{v}^i X_i).$$

When $X = 0$, the theorem was proved by Shi-Tam [13]. In the general case, Liu-Yau [8] essentially proved the theorem by conformal changing the scalar curvature to zero. The proof of Theorem 6.2 in [15] gives a direct proof without conformal change in a slightly different setting. □

Definition 5.3 *The quasi-local mass of* Σ *in* M *is defined to be the infimum of the quasi-local energy-momentum (4.2) among all admissible pairs* (i, t_0^ν).

Under the assumption of Theorem 5.2, we obtain

Corollary 5.1 *If the set of admissible pairs is nonempty, then the quasi-local mass of* Σ *in* M *is positive. In particular, this is the case when* Σ *has positive Gauss curvature.*

Proof. The first part is clear from the definition. When Σ has positive Gauss curvature, we can use Weyl's isometric embedding theorem to embed Σ into a flat space-slice $\mathbb{R}^3$ on which the time function in $\mathbb{R}^{3,1}$ is a constant. Thus the admissible set is non-empty by Remark 5.1. □

6 Properties of the new quasi-local mass

The expression (4.2) contains the desired correction term so the examples of surfaces in $\mathbb{R}^{3,1}$ found in [9] have zero quasi-local mass. In calculating the large or small sphere limits, only the asymptotic expansions of the geometric data on the isometric embedding are needed. Therefore the analysis in [2], [3], and [5] apply to the current situation and the mass has the desired limits. To summarize, the new quasi-local mass given in Definition 5.3 has the following properties:

1. Suppose Σ is a space-like two-surface which bounds a spacelike hypersurface in a spacetime M. The quasi-local mass is defined when the mean curvature vector of Σ in M is spacelike (or $\rho\mu > 0$ in terms of the Newman-Penrose spin coefficients, see [11] or [12]). If M satisfies the dominant energy condition and Σ has positive intrinsic curvature, then the quasi-local mass is positive.

2. The definition of the quasi-local mass is independent of whichever spacelike hypersurface Σ bounds in M.

3. Any space-like two-surface in $\mathbb{R}^{3,1}$ with convex shadow in some time-direction (see Definition 5.1) has zero quasi-local mass.

4. The quasi-local mass has the small sphere limits recovering the matter energy-momentum tensor in the presence of matter and the Bel-Robinson tensor in vacuo, and the large sphere limits approaching the ADM mass in the asymptotically flat case and the Bondi mass in the asymptotically null case.

We remark that the admissible pairs indeed form an open subset of the set of functions τ on Σ that satisfies (5.1). The condition that the admissible set is nonempty in Corollary 5.1 is a very mild assumption and the quasilocal mass should be positive regardless of the sign of the intrinsic curvature of Σ.

The argument that the vanishing of the quasi-local mass implies M is flat along Σ requires some further study of the regularity of the minimizing isometric embedding which will be discussed later. The Euler-Lagrange equation for the minimizing isometric embedding $(x, y, z, \tau) : \Sigma \hookrightarrow \mathbb{R}^{3,1}$ is:

$$\nabla'_a x \nabla'_b x + \nabla'_a y \nabla'_b y + \nabla'_a z \nabla'_b z - \nabla'_a \tau \nabla'_b \tau = \sigma_{ab}, \ a, b = 1, 2$$

$$\begin{aligned} &\sigma^{ab} \nabla'_a \left[\nabla'_b \theta + \frac{\nabla'_b \tau}{\sqrt{1 + \sigma^{cd} \nabla'_c \tau \nabla'_d \tau}} \cosh \theta \sqrt{8\rho\mu} + \alpha_b \right] \\ &= (\widehat{k} \hat{\sigma}^{ab} - \hat{\sigma}^{ae} \hat{\sigma}^{bf} \hat{k}_{ef}) \frac{\nabla'_a \nabla'_b \tau}{\sqrt{1 + \sigma^{cd} \nabla'_c \tau \nabla'_d \tau}} \end{aligned} \tag{6.1}$$

where $\sinh \theta = \frac{\sigma^{ab} \nabla'_a \nabla'_b \tau}{\sqrt{8\rho\mu} \sqrt{1 + \sigma^{cd} \nabla'_c \tau \nabla'_d \tau}}$, $\hat{\sigma}_{ab} = \sigma_{ab} + \nabla'_a \tau \nabla'_b \tau$ is the metric on the projection $\widehat{\Sigma}$, $\hat{k}_{ab}$ is the two-dimensional extrinsic curvature of $\widehat{\Sigma}$, $\alpha_b =$

$\frac{-1}{8\rho\mu}\pi^{\mu}_{b}h_{\nu}\nabla_{\mu}j^{\nu}$, and π^{μ}_{b} is the projection onto Σ. This is a fourth-order elliptic system with four equations and four unknown functions. When a Σ in spacetime is given, we can take the data σ_{ab}, $\rho\mu$, and α_b and solve the elliptic system for the minimizing isometric embedding. Once the minimizing isometric embedding is obtained, a quasi-local energy-momentum four-vector can be defined.

The new quasi-local mass on a concentric round sphere of the Schwarzschild solution is the same as the Brown-York mass and the Liu-Yau mass. Although this does not give the usual Schwarzschild mass and is monotone decreasing as the sphere becomes larger, it was pointed out in [10] the answer coincides with the binding energy of spherical stars and is perhaps the only quasi-local mass that can tell us about the possible interiors enclosed by the surface.

We wish to thank Richard Hamilton for helpful discussions on isometric embeddings.

References

[1] J. D. Brown and J. W. York, in Mathematical aspects of classical field theory (Seattle, WA, 1991), 129–142, Contemp. Math., 132, Amer. Math. Soc., Providence, RI, 1992.

[2] J. D. Brown, S.R. Lau and J.W. York, Phys. Rev. D **59** 064028 (1999).

[3] J. D. Brown and J. W. York, Phys. Rev. D (3) **47** (1993).

[4] G. W. Gibbons, Classical Quantum Gravity **13**, (1997).

[5] S. W. Hawking and G. T. Horowitz, Class. Quantum Grav. **13**, (1996).

[6] J. Kijowski, Gen. Relativity Gravitation **29**, (1997).

[7] S. R. Lau, Phys. Rev. D **60** 104034 (1999).

[8] M. C.-C. Liu and S.-T. Yau, Phys. Rev. Lett. **90**, 231102 (2003); J. Amer. Math. Soc. **19** (2006).

[9] N. Ó Murchadha, L.B. Szabados, and K. P. Tod, Phys. Rev. Lett **92**, 259001 (2004).

[10] N. Ó Murchadha, arXiv:0706.1166.

[11] E. T. Newman and R. Penrose, J. Math. Phys. **3**, 566 (1962).

[12] R. Penrose and W. Rindler, *Spinors and spacetime 2: spinor and twistor methods in spacetime geometry* Cambridge: CUP(1986).

[13] Y. Shi and L.-F. Tam, J. Differential Geom. **62** (2002).

[14] R. Schoen and S.-T. Yau, Comm. Math. Phys. **79** (1981).

[15] M.-T. Wang and S.-T. Yau, Comm. Anal. Geom. **15** (2007).

[16] R. Wald, *General Relativity* Chicago, IL: University of Chicago Press (1984) Appendix E.

[17] S-.T. Yau, Adv. Theor. Math. Phys. **5** (2001).