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Spacetime and the Geometry behind it

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Abstract. This is the *Leonardo da Vinci Lecture* given in Milan in March 2006. It is a survey on the concept of space-time over the last 3000 years: it starts with Euclidean geometry, discusses the contributions of Gauss and Riemannian geometry, presents the dynamic concept of space-time in Einstein's general relativity, describes the importance of symmetries, and ends with Calabi-Yau manifolds and their importance in today's string theories in the attempt for a unified theory of physics.

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This is the *da Vinci Lecture* I gave in Milan in March 2006. I would like to thank J. Fu and L. Tseng for their help in preparing the talk.

Einstein: Subtle is the Lord, but malicious he is not.

Geometric problems that reveal the true nature of space-time are always exciting and rewarding, no matter how difficult they are.

In the past three thousand years, our concept of space-time has evolved according to our understanding of nature. Geometry is the basic tool for such investigations. It is often hard to distinguish geometry from physical nature. Over the years, the development of geometry had a deep influence on our understanding of space-time. On the other hand, new concepts of space-time always give breakthroughs in geometry.

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In ancient days, people believed that space was static and flat. This was partially due to the limitation in our understanding of geometry. In fact, Plato and the Greeks did consider geometry to be part of nature. We shall describe how our concept of spacetime evolves according to our understanding of geometry.

Aristole thought that there are four basic elements that made up the universe while Democritus proposed the theory of atoms. In mathematics, axioms for geometry were formulated. Such a unique development in geometry is perhaps based on the belief of the Greek philosopher that the building blocks of nature should be elegant and simple.

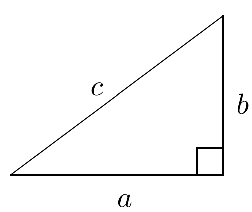
1. Euclidean Geometry

Euclid (325 BC∼265 BC) gave a systematic treatment of geometry which is governed by lines, planes, circles and spheres. The concept of axiom laid the foundation of modern science that complicated phenomena can be understood systematically by simple hypotheses.

There are two most basic theorems:

1. Pythagoras theorem: For the right angle triangle,

$$c^2 = a^2 + b^2$$



2. The sum of inner angles of triangle is π.

The first theorem is the most important statement of geometry. Up to now, modern geometry demands this statement to be true infinitesimally.

The second theorem is a statement that amounts to the fact that the plane is flat and has no curvature. It is equivalent to the following statement which was observed by Legendre.

Euclid's Fifth Postulate: If a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines if produced indefinitely meet on that side on which the angles are less than two right angles.

The following people hoped to prove it from the other axiom of Euclid:

Ptolemy (168), Proclos (410-485), Nasir al din al Tusi (1300), Levi ben Gerson (1288-1344), Cataldi (1548-1626), Giovanni Alfonso Borelli (1608-1679), Giordano Vitale (1633-1711), John Wallis (1616-1703), Geralamo Saccheri(1667-1733), Johann Heinrich Lambert (1728-1777), Adrien Marie Legendre (1752- 1833).

Finally Gauss (1777-1855), Bólyai (1802-1860) and Nikolai Ivanovich Lobachevski (1793-1856) independently discovered hyperbolic geometry: two dimensional surfaces with constant negative curvature. It was a myth that Gauss surveyed a triangle in the Harz Mountains formed by Inselberg, Brocken and Hoher Hagen to see if the sum of its interior angles was 180 degrees. Finally Felix Klein (1849-1925) created an analytic method to describe a model of hyperbolic geometry by giving a formula for distance between any pair of points on the unit disk. (This is called the Klein model.) Euclid's fifth postulate was finally proved to be independent of the other postulates.

In addition to its contribution to the foundation of mathematics, the discovery of hyperbolic geometry shows us a new abstract geometry that is not intuitively clear in day to day life.

In hyperbolic geometry, the sum of inner angles minus π is equal to the curvature of the surface times the area of the triangle. This was generalized by Gauss to an integral formula of curvature. This formula is called Gauss-Bonnet formula and plays a fundamental role in modern geometry and topology as it relates a local quantity (curvature) to a global topological quantity (the Euler number). The formula was eventually generalized to the index formula of Atiyah-Singer, which plays an important role in modern physics.

There is a great deal of limitation for geometric objects built solely by planes and spheres. However, the concept is completely changed once we know how to construct curved geometric objects by infinite process of approximation. This was invented by Archimedes who, for example, computed areas cut by straight lines with a parabola. This infinite process is the seed for the invention of Calculus which was achieved by Newton and Leibniz much later.

2. Analytic geometry

An important breakthrough in geometry is due to Rene Descartes whose invention of analytic geometry allows geometric figures to be described by Cartesian coordinates. Geometry and Algebra are unified through the introduction of coordinate systems. It would be difficult to solve classical problems such as trisecting an angle without using algebra (through Galois theory). Analytic geometry allows us to consider geometric figures without using the classical axioms of Euclid. Only with a coordinate system, we can visualize and calculate higher dimensional geometric figures.

3. Calculus

Combination of analytic geometry and calculus allowed Newton and astronomers to do extensive calculations for the motion of celestial bodies. The description of these bodies is based on a single coordinate system. Newton thought that space is static and time is independent of space.

Newton (*Philosophiae Naturalis Principia Mathematica*): Absolute space, in its own nature and with regard to anything external, always remains similar and unmovable. Relative space is some movable dimension or measure of absolute space, which our senses determine by its position with respect to other bodies, and is commonly taken for absolute space.

Newton's idea was challenged by Leibniz (1646-1716).

The great success of calculus and Newtonian mechanics kept physicists and mathematicians busy until the nineteenth century. Euler was the major contributor to geometry in this period. He was the major founder of the calculus of variations and combinatorial topology.

4. Gauss and Riemannian geometry

When classical geometers (such as Euler) described a surface in three space, they found two directions at each point where curvatures should be measured. For example, on the cylinder, one is the direction along the circle and the other is along the straight line. Gauss (1827) found that the product of these two curvatures has a remarkable property. It is the same even if we deform the surface as long as we do not stretch it. It is called the Gauss curvature.

4.1. Intrinsic geometry of Gauss

Gauss published this theorem in his *Disquisitiones generales circa superficies curvas* where he distinguished the inner properties of a surface, that is the geometry experienced by small flat bugs living in the surface, from its outer properties, the way that it embeds in a higher dimensional space. He said that this inner property is "most worthy of being diligently explored by geometers". The inner geometry is what we call intrinsic geometry.

Gauss noticed that when the Gauss curvature is a positive constant the surface is a piece of the sphere. When the Gauss curvature is minus one, it is a piece of the hyperbolic surface, which was developed by Lobachevsky and himself.

Gauss: I am becoming more and more convinced that the necessity of our geometry cannot be proved, at least not by human reason nor for human reason. Perhaps in another life we will be able to obtain insight into the nature of space which is now unattainable.

Gauss: Until then we must place geometry not in the same class with arithmetic which is purely a priori, but with mechanics.

4.2. Riemannian geometry

While Gauss described his geometry for two dimensional surfaces, it was Georg Friedrich Bernhard Riemann (1826-1866) who presented intrinsic geometry for higher dimensional manifolds in his Göttingen inaugural lecture: *Über die Hypothesen, welche der Geometrie zu Grunde liegen*. Riemann formally introduced the concept of an abstract space which is defined by some infinitesimal form of a metric. The concept of Gauss curvature was then given a proper meaning. This is an important event as we have finally formulated the concept of a space with no reference to flat Euclidean (linear) space.

In his address, Riemann mentioned the influence of Newtonian mechanics and physics on his thinking in the formulation of the abstract space.

Riemann's new discovery did radically change the mathematician's view of geometry. It was followed by Christoffel, Ricci, Levi-Civita, Beltrami. They developed calculus on manifolds: tensor calculus. On the other hand, this was considered to be not interesting by many contemporary mathematicians.

5. Global and local symmetries

When Riemann created his geometry, he had a vague idea of its relation to Newtonian Mechanics and he knew that any meaningful geometric quantity should be independent of the choice of coordinate systems. The concept of the Riemann curvature tensor was introduced for that purpose. Nowadays we say that the group of diffeomorphisms provides a gauge symmetry for geometry.

At around the same time, S. Lie introduced the concept of a continuous group. F. Klein outlined his famous Erlangen program in 1887. They believed that geometry should be dictated by a global group of symmetries. E. Cartan introduced the concept of a connection on fiber bundles. By doing so, he succeeded in merging the concept of global symmetry with Riemannian geometry. Local gauge symmetries started to appear in geometry. Geometers would create quantities invariant with respect to such local symmetries.

In the process, Cartan developed exterior differential calculus based on Grassmann's work on exterior algebra and Frobenius' work on integrability conditions for solving differential equations.

Then Poincaré discovered the Poincaré lemma and de Rham proved the isomorphism of de Rham cohomology with singular cohomology. This enabled Hodge to apply harmonic theory to topology. While Hodge was motivated by equations of two dimensional fluid dynamics and Maxwell's equations, his theory has contributed to an important part of particle physics and string theory.

6. Relativity: Einstein's view of spacetime

The first constructive criticism of Newtonian absolute space was by the Austrian Philosopher Ernst Mach (1836-1916). He put forth the hypothesis that one must take into account the influence of the mass of the earth and the other celestial bodies to determine the inertial frames. This is called Mach's principle.

By 1905, special relativity was found, by Einstein (with contribution by Poincare, Lorentz and Minkowski). A very important fact is that space and time are unified (under Lorentz transformation).

In 1908, Minkowski said that "from now on, we cannot discuss space and time independently."

However, Newton's basic concept of gravity, *action at a distance*, is inconsistent with the basic principle of special relativity that information cannot travel faster than light.

In 1907, Einstein introduced the principle of equivalence of gravitation and inertia. He realized the importance of relative motion to gravity and that there is a similarity of gravity with electricity and magnetism.

Gravity has the effect of making physical bodies accelerate. According to special relativity, when we measure length, it will stretch according to the velocity if the measurement is parallel to the velocity vector. However, the length will not change if the direction is perpendicular to the velocity vector. The changes in the metric according to direction and position gives rise to a metric tensor according to Riemannian geometry.

Einstein tried for about ten years to combine this fundamental concept of general relativity with Newton's theory of gravity. One of the reasons it took some time for him to achieve such a goal was his lack of knowledge of mathematics of abstract space. Until he was informed by his friend Grossmann, he did not realize the powerful meaning of Riemann's curvature tensor, which he later used to describe gravitational force. The equivalence principle of physics requires the laws of gravity to be independent of choice of coordinate systems.

The quantity given by Riemann has exactly this property. The trace of the curvature tensor is called the Ricci tensor which was developed by Ricci. Einstein found that he could construct from the Ricci tensor the kind of tensor he needed to satisfy the classical conservation law of matter. (Bianchi found the identity that Einstein needed.) So the abstract space of Riemann is exactly suitable for description of gravity. (Hilbert found the Hilbert action, which is the integral of scalar curvature, to be the action principle behind the Einstein equation.)

The gravitational field must be identified with the 10 components of the metric tensor of Riemannian space-time geometry. He presented to the Prussian Academy of Sciences a series of papers in which he worked out the metric tensor and calculated the gravitational deflection of light and the precession of the perihelia of Mercury. This was summarized in "the foundation of the general theory of relativity" in Annalen der Phys. 1916.

Hence space-time fits beautifully well into the framework of Riemannian geometry. The effect of gravity can be expressed by curvature. Geometry and gravity cannot be distinguished any more. Since space and time are now merged together, the universe is dynamically driven by gravity constantly and is no more static.

When celestial bodies move, the geometry and topology of space-time move according to the velocity of light. This solves the puzzle of the contradiction which arose in Newtonian mechanics and special relativity.

7. Symmetry dictates dynamics in physics

Besides the inspiration of Mach's principle, Einstein observed the importance of symmetry in formulating dynamical equations in physics. The fact that Maxwell's equations admit the Lorentz group as group of symmetry is a foundation for special relativity. The derivation of Einstein's equation is based on the invariance of coordinate change. Conservation laws in physics are linked to the conserved quantity of continuous groups of symmetries.

8. Quantum Mechanics and a unified theory of Physics

The great discovery of quantum mechanics led to the understanding of particles in high energy physics. It requires spinors and gauge theory to understand the basic building blocks of forces in nature. However, these concepts also appeared in geometry. In fact, E. Cartan studied the theory of fiber bundles (twisted spaces) before physicists. Herman Weyl introduced abelian gauge theory for electromagnetism while Yang-Mills introduced non-abelian gauge theory. Strong and weak interactions are dictated by such gauge theories.

The success of quantum field theory also changed our understanding of the geometry of space-time. However, there is still great a difficulty to understand space-time when the radius is below the Planck scale. Quantum physics contradicts general relativity at small scales. It is clear that space should not be made up merely of points. The very difficult task of quantizing gravity led physicists to build many different models. This is part of the ambitious goal of unifying all forces in nature (a dream that Einstein wanted to accomplish).

8.1. Birth of string theory

The physicist Veneziano [24] found that the Γ functions of Euler can be related to functions appearing in the description of strong force phenomena. Afterwards, Nambu [15], Nielsen [16] and Susskind [22] suggested that if fundamental particles are strings instead of points, the Γ function can indeed be derived in the theory of strong interactions. However, the tremendous success of the standard model show that this interpretation is not needed.

Thus string theory was quiet for a long time although Scherk and Schwarz [18] did propose string theory to include strong forces and gravity in this period. The first breakthrough occured in 1984 when Green and Schwarz [7] found that in a consistent quantized string theory (with supersymmetries), there are only two gauge groups $SO(32)$ and $E_8 \times E_8$ and space-time is ten dimensional. With the anomaly cancellation, the field theory converge in one loop term.

8.2. Kaluza-Klein model

What we observe in the real world is four dimensional, and so we need a mechanism to reduce ten dimensional string theory to an effective four dimensional theory. This was provided by Kaluza-Klein [12, 13] who discovered such mechanism right after the discovery of general relativity. Kaluza and Klein looked at the five dimensional space by thickening standard space-time by a circle.

A good example is a line thickened to be a circular cylinder. If the circle is very tiny, we may think of that cylinder as a straight line.

Kaluza-Klein consider the equation for geometry (with no matter) on this thickened space-time and conclude that when the circle is very small, the vacuum Einstein equation on five-dimensional space-time will give rise to a coupled system of Maxwell's equations with the four-dimensional gravitational equation on the four-dimensional space-time. Hence it gives a unified theory of gravity with electricity and magnetism. Einstein thought highly of this theory. Unfortunately there was an extra scalar particle that did not have a good physical meaning at the time when it was introduced.

9. Geometry of string theory

For a consistent quantized string theory, spacetime is ten-dimensional. We would like to imitate what Kaluza-Klein did by looking at the four-dimensional spacetime thickened with a six-dimensional space. This six-dimensional space would be tiny. String theorists believe that when the energy is very high, there is a symmetry relating bosons and fermions. Such a correspondence is called supersymmetry. It provides parallel spinors on the six-dimensional internal space.

9.1. Calabi-Yau space

The requirement of space-time to be supersymmetric gives a strong constraint on this six dimensional space. It must be a vacuum space with a complex structure. This was discovered by Candelas, Horowitz, Strominger and Witten [3]. It turns out that such a space was constructed by me many years ago. This class of spaces are called Calabi-Yau spaces. In the last twenty years, a great deal of activities have been devoted to the study of such spaces. It has been a major place for interactions for string theory and mathematics.

In this compactification theory of Candelas, Horowitz, Strominger and Witten [3], the space-time is proposed to be

$$R^{3,1} \times M,$$

where the metric is taken to be a product metric. They also took the Yang-Mills gauge bundle to be the tangent bundle of M and the scalar field to be constant.

Supersymmetries of M require M to admit parallel spinors and they observed that M must be Kähler and the holonomy group is $SU(3)$ (Calabi-Yau manifold).

Kählerian means that we have coordinate charts (z_1, z_2, z_3) on M so that the metric tensor can be written as $\sum g_{i\bar{j}} dz_i d\bar{z}_j$ and the $(1,1)$-form $\omega = \sqrt{-1}\sum g_{i\bar{j}} dz_i \wedge d\bar{z}_j$ is closed. The holonomy group being $SU(3)$ means that there is a globally defined holomorphic 3-form

$$\Omega = f dz_1 \wedge dz_2 \wedge dz_3,$$

where f is a non-vanishing holomorphic function. Conversely, given a Kähler manifold with such a holomorphic three form Ω, we want to deform ω to a new Kähler metric $\tilde{g}$ such that

$$\frac{\det(\tilde{g}_{i\bar{j}})}{|\ f\ |^2} = \text{constant}$$

in order for the holonomy group to be $SU(3)$. I was able to prove that $\tilde{g}$ can be deformed uniquely in the same cohomology class of ω. This produces a way to parametrize the space of all $SU(3)$ structures on the complex manifold. They are parametrized by those complex structures which admit holomorphic three forms and also Kähler classes of such manifolds.

10. Calabi-Yau phenomenology

Around 1984, during the first string revolution, string theorists thought that there are only a couple of Calabi-Yau manifolds. And they believed that by the Kaluza-Klein idea, one could compute fundamental constants for particles (Yukawa Coupling) by the topology and complex structure of these manifolds.

In principle, the wave functions $\psi(x, y)$ for Dirac operators on $R^{3,1} \times M$ are written as

$$\sum \overline{\psi}_i(x)\ \overline{\overline{\psi}}_i(y),$$

where $\overline{\psi}_i$ and $\overline{\overline{\psi}}_i$ are eigenfunctions of the Dirac operators on $R^{3,1}$ and M respectively. When M is very small, the non-zero spectrum of the the Dirac operator is very large. They should not be observable and we are only interested in harmonic spinors on M. Since for Calabi-Yau manifolds, the canonical line bundle is trivial, the Dirac spinors can be identified with the Hodge groups of M. The Yukawa Couplings can be computed in terms of geometry of interesections of elements in the Hodge groups [19].

10.1. Many Calabi-Yau's

The theory would be great to explain particle physics if there were only a few models. Then there were some disappointments when I demonstrated that there are large classes of Calabi-Yau manifolds which are complete intersections of products of projection spaces. The program was carried out effectively by Candelas et al [4] by using a computer. I soon realized that toric method can be used to construct more CY manifolds. (This was carried out in my paper with Roan [17] and followed by Batyrev [1] and others.)

It is possible that all CY 3-folds can be constructed from complete intersections of suitable toric varieties. If that is the case, it will point into a good direction to classify all three-dimensional CY manifolds.

10.2. Calabi-Yau standard model

During the Argonne Lab Conference, I constructed a CY manifold in response to a question asked by Strominger and Witten. Find a CY manifold which has the following properties:

1. Euler number $\chi = \pm 6$;
2. Nontrivial fundamental group.

$\frac{1}{2} \mid \chi \mid$ accounts for the number of families of fermions which is known to be three. The fundamental group can be used to construct a flat bundle which can be used to break symmetries. (By embedding the fundamental group into the gauge group $E_8 \times E_8$, the commutant of the discrete group will be a small group.)

During the conference, I constructed such a manifold by taking complete intersections of three hypersurfaces in $\mathbb{C}P^3 \times \mathbb{C}P^3$ given by:

$$\sum_i z_i^3 = 0, \quad \sum_i w_i^3 = 0, \quad \sum_i z_i \, w_i = 0.$$

Then I took a quotient of this manifold by an automorphism of order three. The quotient manifold satisfies the properties (1) and (2).

This manifold was used by B. Greene in his thesis to study string phenomenology. Greene et al [8] demonstrated that such a model is consistent with standard model. Greene also observed that the later models I constructed with Tian can all be deformed to this manifold I constructed. Later there were several constructions related to this manifold. The mirror construction gave another class of manifolds with the opposite Euler number. Tian and I applied flop constructions. However, all of these constructions give rise to the same conformal field theory attached to the manifold. Thus it is possible that the conformal field theory attached to this manifold is rather unique.

10.3. Cosmic strings and black holes

Based on the inputs of string theorists, we now know quite a bit about the mathematics of Calabi-Yau spaces. One hopes that eventually one can compute fundamental constants in particle physics in terms of these spaces. We may construct models of cosmology or black holes based upon continuous evolution among these spaces. A particularly interesting question is related to the construction of a stringy cosmic string due to Greene-Shapere-Vafa-Yau [11]. It is a half $K3$ surface fibered over $\mathbb{C}$ with elliptic curves as fibers. A Ricci flat metric is constructed which is singular along some singular elliptic curves. (the construction of noncompact Calabi-Yau manifolds was outlined by me in my talk in the 1978 Congress of Mathematics.) However, it is still a major question to construct a Calabi-Yau manifold which admits a fibration given by a special Lagrangian torus.

One can construct a ten-dimensional black hole solution by fibering the space over a spherical symmetric spacetime with fibers given by a compact Calabi-Yau metric. This gives rise to a path in the moduli space of Calabi-Yau spaces and hence the attractor mechanism [6]. Many physicists, including Strominger, Vafa, Moore, Horowitz, Ooguri, Donglas, Kalloh and others, contributed to this fascinating subject of stringy black hole physics. A very major work was due to Strominger-Vafa [20] on the verification of the black hole entropy formula of Bekenstein-Hawking.

11. Mirror symmetry

While Calabi-Yau is a building block for the vacuum structure of string theory, it is not thought to be the last microstructure to be found in studying space-time. A very important symmetry for space-time called T duality shows that microstructure of space-time is complicated. The duality says that the quantum field theory based on a circle of radius R is the same as the quantum field theory based on the circle of radius $\frac{1}{R}$. This fact comes from the duality between space and momentum.

On the other hand, T duality gives rise to a concept called Mirror symmetry. It provides a deep mathematical insight into CY manifolds. The concept of conformal field theory defined by CY manifolds led Dixon and Lerche-Vafa-Warner [14] to ask whether one can associate a completely new CY to a given CY manifold so that the Hodge number is interchanged:

$$H^{p,q}(M) \longleftrightarrow H^{n-p,q}(\check{M})$$

and such that the conformal field theory attached to M is isomorphic to the one on $\check{M}$. Such a duality is called mirror symmetry.

11.1. Mirror manifolds

Greene-Plesser [10] and Candelas et al [2] demonstrated a construction of the mirror manifold by the quintic

$$M = \left\{\sum z_i^5 = 0\right\}$$

in $\mathbb{C}P^4$.

They found that the mirror manifold $\check{M}$ is a quotient of M by a group $\mathbb{Z}_5 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

Under the mirror correspondence, the cubic coupling for the deformation of complex structures on M corresponds to the cubic coupling of the symplectic theory on $\check{M}$ with quantum corrections given by pseudoholomorphic curves of various degrees and genus. Since there is a rich classical literature on the computation of the cubic coupling for the deformation of complex structure based on computations of period of integrals, it is possible to compute symplectic invariants by "counting" the number of pseudoholomorphic rational curves.

The spectacular work of Candelas et al was finally established with mathematical rigor by completely different arguments due to the efforts of Kontsevich, Givental and Lian-Liu-Yau. Many more examples of mirror manifolds were later constructed by Batyrev [1]. Large number of calculations were given by various groups led by Candelas, Hosano, Katz, Klemm, Morrison and myself.

11.2. Quantum geometry

Mirror symmetry has produced a major way to calculate strongly coupled quantum field theory over spacetime which allows one to test many interesting possible phenomena. For example, the possibility of topological change for quantum spacetime, due to Greene-Morrison-Strominger [9], is an achievement based on such calculations. Geometric explanation of mirror manifolds due to Strominger-Yau-Zaslow [21] has produced many interesting questions in quantum geometry. Homological mirror symmetry of Kontsevich and the Fukaya category of symplectic geometry is still being explored. One expects many further developments in quantum geometry.

12. Three- and four-manifolds

While quantum geometry is being pursued through the understanding of Black hole physics and the theory of duality, the geometry of space is making big steps forward in the theory of three manifolds led by Hamilton's Ricci flow. The Hamilton-Perelman theory gives a complete structure theorem for three manifolds. This deep work, as was completed by Zhu and his group, should be considered as a crowning achievement of geometric analysis developed in the past thirty years. We expect the geometry of three-dimensional manifold to have deep developments in the near future. Its role in mathematics could be as central as the theory of Riemann surfaces in late nineteenth century and twentieth century.

The geometry of four dimensional manifolds will clearly play an important role. However, it is a subject of much more difficulty as there is no well formulated structure theorem yet. New understanding of space-time will clearly give excitement to this subject. Manifolds with holonomy group G_2 and $Spin(7)$ have arisen from M theory. They should be one of the fundamental building blocks for low dimensional geometry.

Chuang Tzu said: "Heaven and earth and I co-exist; the myriad things and I are one."

References

[1] V. V. Batyrev, *Dual polyhedra and mirror symmetry for Calabi-Yau hypersurfaces in toric varieties.* J. Algebraic Geom. **3**(1994), no. 3, 493–535.

[2] P. Candelas, X. C. De La Ossa, P. S. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal Theory,* Nucl. Phys. B **359**(1991) 21–74.

[3] P. Candelas, G. T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings.* Nucl. Phys. B **258**(1985) 46–74.

[4] P. Candelas, M. Lynker and R. Schimmrigk, *Calabi-Yau Manifolds In Weighted P(4).* Nucl. Phys. B **341** (1990) 383–402.

[5] H. D. Cao, X. P. Zhu, *Ricci flow and its application.*

[6] S. Ferrara, R. Kallosh and A. Strominger, *N=2 extremal black holes.* Phys. Rev. D **52**(1995) 5412–5416.

[7] M. B. Green and J. H. Schwarz, *Anomaly cancellation in supersymmetric D=10 gauge theory and superstring theory* Phys. Lett. B **149**(1984) 117–122.

[8] B. R. Greene, K. H. Kirklin, P. J. Miron and G. G. Ross, *A three generation superstring model. 1. Compactification and discrete symmetries.* Nucl. Phys. B **278**(1986) 667–693.

[9] B. R. Greene, D. R. Morrison and A. Strominger, *Black hole condensation and the unification of string vacua.* Nucl. Phys. B **451**(1995) 109–120.

[10] B. R. Greene and M. R. Plesser, *Duality in Calabi-Yau moduli space.* Nucl. Phys. B **338**(1990) 15–37.

[11] B. R. Greene, A. D. Shapere, C. Vafa and S. T. Yau, *Stringy cosmic strings and noncompact Calabi-Yau manifolds.* Nucl. Phys. B **337**(1990) 1–36.

[12] Th. Kaluza, *On the problem of unity in physics.* Sitz. Preuss. Akad. Wiss. Berlin **K1**(1921) 966–972

[13] O. Klein, *Quantentheorie und fünfdimensionale Relativitätstheorie.* Z. Phys. bf 37(1926) 895–906.

[14] W. Lerche, C. Vafa and N. P. Warner, *Chiral rings in N=2 superconformal theories.* Nucl. Phys. B **324**(1989) 427–474.

[15] Y. Nambu, *Quark model and the factorization of the Veneziano amplitude.* in Symmetries and quark models, ed. R. Chand, Gordon and Breach, (1970) 269.

[16] H. B. Nielsen, *An almost physical interpretation of the integrand of the n-point Veneziano model.* submitted to the 15th International Conference on High Energy Physics, Kiev (1970).

[17] S. S. Roan and S. T. Yau, *On Ricci flat* 3*-fold*, Acta Math. Sinica (N.S.) 3 (1987), 256–288.

[18] J. Scherk and J. H. Schwarz, *Dual models for nonhadrons*. Nucl. Phys. B **81** (1974) 118–144.

[19] A. Strominger, *Yukawa Couplings in Superstring Compactification*. Phys. Rev. Lett. **55**(1985) 2547–2550.

[20] A. Strominger and C. Vafa, *Microscopic origin of the Bekenstein-Hawking entropy*. Phys. Lett. B **379**(1996) 99–104.

[21] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality*. Nucl. Phys. B **479**(1996), no. 1-2, 243–259.

[22] L. Susskind, *Dual symmetric theory of hadrons. 1.* Nuovo Cim. A **69** (1970) 457–496.

[23] G. Tian and S.-T. Yau, *Three-dimensional algebraic manifolds with $C_1 = 0$ and $\chi = -6$*. Mathematical aspects of string theory (San Diego, Calif., 1986), 543–559, Adv. Ser. Math. Phys., **1**, World Sci. Publishing, Singapore, 1987.

[24] G. Veneziano, *Construction of a crossing-symmetric, regge behaved amplitude for linearly rising trajectories*. Nuovo Cim. A **57** (1968) 190.

[25] S.-T. Yau, *Compact three-dimensional Kähler manifolds with zero Ricci curvature*. Symposium on anomalies, geometry, topology (Chicago, Ill., 1985), 395–406, World Sci. Publishing, Singapore, 1985.

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