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Mirror maps, modular relations and hypergeometric series II

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As a continuation of [1], we study modular properties of the periods, the mirror maps and Yukawa couplings for multi-moduli Calabi-Yau varieties. In Part A of this paper, motivated by the recent work of Kachru-Vafa, we degenerate a three-moduli family of Calabi-Yau toric varieties along a codimension one subfamily which can be described by the vanishing of certain Mori coordinate, corresponding to going to the "large volume limit" in a certain direction. Then we see that the deformation space of the subfamily is the same as a certain family of K3 toric surfaces. This family can in turn be studied by further degeneration along a subfamily which in the end is described by a family of elliptic curves. The periods of the K3 family (and hence the original Calabi-Yau family) can be described by the squares of the periods of the elliptic curves. The consequences include: (1) proofs of various conjectural formulas of physicists [2][3] involving mirror maps and modular functions; (2) new identities involving multi-variable hypergeometric series and modular functions – generalizing [1]. In Part B, we study for two-moduli families the perturbation series of the mirror map and the type A Yukawa couplings near certain large volume limits. Our main tool is a new class of polynomial PDEs associated with Fuchsian PDE systems. We derive the first few terms in the perturbation series. For the case of degree 12 hypersurfaces in $\mathbf{P}^4[6,2,2,1,1]$, in one limit the series of the couplings are expressed in terms of the j function. In another limit, they are expressed in terms of rational functions. The latter give explicit formulas for infinite sequences of "instanton numbers" n_d.

This paper is a continuation of [1]. There we study the modular properties of a multi-moduli family of Calabi-Yau varieties degenerated along a dimension one subfamily. In this article, we study properties of the mirror map, periods and the type A Yukawa couplings under a degeneration along a codimension one subfamily, and their perturbations around this subfamily. The problem is clearly motivated by recent developments in the so-called heterotic-type II string duality.

1. PART A: DEGREE 24 HYPERSURFACES IN $\mathbf{P}^4[1,1,2,8,12]$

The mirror symmetry of this family of Calabi-Yau toric varieties X has been studied in detail in [4]. Its Picard-Fuchs system is given by ($\Theta_x = x\frac{d}{dx}$ etc.)

$$\begin{aligned} L_1 &= \Theta_x(\Theta_x - 2\Theta_z) - 12x(6\Theta_x+5)(6\Theta_x+1) \\ L_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_z + 1)(2\Theta_y - \Theta_z) \\ L_3 &= \Theta_z(\Theta_z - 2\Theta_y) - z(\Theta_z - \Theta_x + 1) \\ &\qquad (2\Theta_z - \Theta_x). \qquad (1.1) \end{aligned}$$

The x, y, z are deformation coordinates, which we call the Mori coordinates (see [4] for definition), near the "large volume limit" in the family of Calabi-Yau varieties.

Comparing the type II string compactification along X with a heterotic string theory, Kachru-Vafa suggest that one should study the limit $y \to 0$. When restricted to this subfamily with $y = 0$, a subset of the periods of this subfamily satisfy a new system given by

$$\begin{aligned} L_1 &= \Theta_x(\Theta_x - 2\Theta_z) - 12x(6\Theta_x+5)(6\Theta_x+1) \\ L_3 &= \Theta_z^2 - z(2\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x). \quad (1.2) \end{aligned}$$

This is identical to the Picard-Fuchs system for the family of toric K3 surfaces corresponding to degree 12 hypersurfaces in $\mathbf{P}^3[1,1,4,6]$. (For the relevance of the appearance of K3 surfaces and their moduli spaces in heterotic-type II duality, see the recent papers [3][5].) By further restricting along $z = 0$, the Picard-Fuchs system reduces to a single equation

$$L = \Theta_x^2 - 12x(6\Theta_x+5)(6\Theta_x+1). \qquad (1.3)$$

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This is the Picard-Fuchs operator for a family of elliptic curves in $\mathbf{P}^2[1,2,3]$. This suggests a close relationship between the curves and the above 2-moduli K3 family, and ultimately the 3-moduli family of Calabi-Yau varieties X. Indeed, it is found numerically in [3] that the mirror map defined by the K3 family can be given in terms of the j-function.

In this section, we will prove that the solutions to (1.2) are given by "squares" of solutions to (1.3). This result (1) generalizes a theorem in [1] which we will review briefly; (2) proves some formulas of [3] as a consequence.

Consider the differential operators

$$\begin{aligned}\ell &= \Theta_x^3 - \sum_{i=1}^m \lambda_i x^i(\Theta_x + i/2)(\Theta_x + i/2 + \nu_i) \\ &\qquad (\Theta_x + i/2 - \nu_i) \\ \tilde{\ell} &= \Theta_x^2 - \sum_{i=1}^m \lambda_i x^i(\Theta_x + i/4 + \nu_i/2) \\ &\qquad (\Theta_x + i/4 - \nu_i/2) \qquad (1.4)\end{aligned}$$

where the λ_i, ν_i are arbitrary complex numbers. In [1], we prove that if $\tilde{\ell}f(x) = 0$ then $\ell f(x)^2 = 0$. Since the dimension of solution space of $\tilde{\ell}$ is two, the span of $f(x)^2$ with $\tilde{\ell}f(x) = 0$ has dimension three and so must be the full solution space of ℓ. This result was inspired by the observation in numerous examples (see [1]) that the periods of certain 1-moduli K3 family are nothing but products of periods of some family of elliptic curves. This leads to nontrivial identities involving modular functions and series solutions to an ODE of Fuchsian type. This suggests to us a 2-moduli analogue for the systems (1.2), (1.3): is the solution space to (1.2) given by the span of the $f(x)g(z)$ where $f(x), g(x)$ are solutions to (1.3)? The answer turns out to be no, but almost. Note that the span of the $f(x)g(z)$ is 4 dimensional, which is the same as the dimension of the solution space for (1.2) with at most *Log* singularities.

Theorem 1.1. *Let L_1, L_3, L be as defined in (1.2), (1.3). There exists a rational mapping $\mathbf{C}^2 \to \mathbf{C}^2$, $(R,S) \mapsto (x,z)$, such that if $f(x), g(x)$ are solutions to L, then $f(R(x,z))g(S(x,z))$ is a solution to L_1, L_3, where $(x,z) \mapsto (R(x,z), S(x,z))$ is (any branch of) the inverse mapping.*

Proof: We will construct the mapping using the condition that

$$L_1 f(R(x,z))f(S(x,z)) \equiv 0 \ \ mod\ Lf(R), Lf(S). \qquad (1.5)$$

It will be seen that the mapping we will construct also satisfies the analogous condition for L_2.

Clearly by expanding the expression $L_1 f(R(x,z))f(S(x,z))$ by chain rule, we get a homogeneuos quadratic polynomial of $f^{(i)}(R), f^{(i)}(S)$, whose coefficients are differential polynomials of $R(x,z), S(x,z)$. Upon applying the conditions that $Lf(R) = 0 = Lf(S)$, we can eliminate any appearance of $f''(R), f''(S)$. Thus after the elimination, a sufficient condition for (1.8) to hold is that coefficients of $f(R)f(S), f'(R)f(S), f(R)f'(S), f'(R)f'(S)$, each vanishes identically. Thus we want to solve the conditions of the vanishing of these coefficients, and they are given by:

$$\begin{aligned}
(1)\quad & -RSx + 1728R^2Sx + 1728RS^2x \\
& -2985984R^2S^2x - 2SR_zR_x + 3456S^2R_zR_x \\
& +SR_x^2 - 1728S^2R_x^2 - 432SxR_z^2 \\
& +746496S^2xR_x^2 - 2RS_zS_x \\
& +3456R^2S_zS_x + RS_x^2 \\
& -1728R^2S_x^2 - 432RxS_x^2 + 746496R^2xS_x^2 = 0 \\
(2)\quad & 432RxR_x - 746496R^2xR_x - 2R_zR_x \\
& +5184RR_zR_x + R_x^2 - 2592RR_x^2 \\
& -432xR_x^2 + 1119744RxR_x^2 + 2RR_{xz} \\
& -3456R^2R_{xz} - RR_{xx} + 1728R^2R_{xx} \\
& +432RxR_{xx} - 746496R^2xR_{xx} = 0 \\
(3)\quad & \textit{As in (2) with } S, R \textit{ interchanged.} \\
(4)\quad & -S_zR_x - R_zS_x + R_xS_x - 432xR_xS_x = 0
\end{aligned} \qquad (1.6)$$

where S_x means $\Theta_x S$ etc. This is an overdetermined system of polynomial PDEs. We claim that the following relations define an algebraic solution to (1.6):

$$\begin{aligned} &R + S - 864RS - x = 0 \\ &RS(1-432R)(1-432S) - x^2z = 0. \end{aligned} \qquad (1.7)$$

The proof is by direct computation. (This solution is easy to motivated by the following consideration. Since we propose that the periods of the K3 surfaces in question are symmetric squares of those of the elliptic curves, it is reasoable that the K3 moduli x, z are symmetric functions in the elliptic curve moduli R, S. The above solution makes x, z the simplest kinds of algebraic symmetric functions of R, S.) Note that (1.7) defines a rational mapping $(R, S) \mapsto (x, z)$. It is also easy to check that given this solution, the condition

$$L_2 f(R(x,z))g(S(x,z)) \equiv 0 \;\; mod \; Lf(R), Lg(S) \tag{1.8}$$

holds automatically. This completes our proof. •

As a first consequence, we prove a formula first conjectured to exist in [2] on physical ground, and then found numerically in [3].

Corollary 1.2. *The mirror map for (1.2) is given by*

$$\begin{aligned} x(q_1,q_3) &= 2\frac{1/j(q_1)+1/j(q_1q_3)-1728/(j(q_1)j(q_1q_3))}{1+\sqrt{1-1728/j(q_1)}\sqrt{1-1728/j(q_1q_3)}} \\ z(q_1,q_3) &= \frac{1}{j(q_1)j(q_1q_3)x(q_1,q_3)^2}. \end{aligned} \tag{1.9}$$

Proof: Recall that (1.2) has a *unique* powers series solution w_0 with leading term 1, *unique* solutions w_1, w_3 of the form $w_1 = Log\ x + g_1, w_3 = Log\ z + g_3$ with $g_1, g_3 \to 0$ as $|x|, |z| \to 0$. (This is so because this is a holonomic PDE system with regular singularity at $x = 0, z = 0$. It is also straightforward to check this directly using (1.2) and the resulting recursion relations on the coefficients of the power series.) The mirror map $x(q_1, q_3), z(q_1, q_3)$ for (1.2) is then defined by the inverse of the power series relations:

$$\begin{aligned} q_1 &= e^{w_1/w_0} \\ q_3 &= e^{w_3/w_0}. \end{aligned} \tag{1.10}$$

Similarly, the ODE (1.3) has a power series solution $\tilde{w}_0$ with leading term 1 and a solution $\tilde{w}_1$ of the form $Log\ x + \tilde{g}$ with $g \to 0$ as $|x| \to 0$. The mirror map $r(q)$ for (1.3) is then defined by the inverse of the power series relation $q = e^{\tilde{w}_3(r)/\tilde{w}_0(r)}$. It is also easy to prove that (see [6]) $j(q) = \frac{1}{r(q)(1-432r(q))}$.

By the theorem above, the following are three solutions to (1.2): $\tilde{w}_0(R(x,z))\tilde{w}_0(S(x,z))$, $\tilde{w}_1(R(x,z))\tilde{w}_0(S(x,z))$, $\tilde{w}_0(R(x,z))\tilde{w}_1(S(x,z)) - \tilde{w}_1(R(x,z))\tilde{w}_0(S(x,z))$. It is straightforward to solve (1.7) for R, S as power series in x, z, and we see there are four branches of solutions. One branch has leading terms $R = x + O(h^2), S = xz + O(h^3)$. (Here $O(h^k)$ means terms of total degree k or higher.) The second branch has $R = 1/432 + O(h), S = 1/432 + O(h)$. The third and the fourth branches are obtained by interchanging the roles of R, S in the first two branches. We choose $R(x,z), S(x,z)$ to be given by the first branch. Then it easy to see that the three solutions $\tilde{w}_0(R(x,z))\tilde{w}_0(S(x,z))$, $\tilde{w}_1(R(x,z))\tilde{w}_0(S(x,z))$, $\tilde{w}_0(R(x,z))\tilde{w}_1(S(x,z)) - \tilde{w}_1(R(x,z))\tilde{w}_0(S(x,z))$, have the leading behaviour identical to that of the solutions w_0, w_1, w_3 respectively. By uniqueness, we conclude that

$$\begin{aligned} w_0 &= \tilde{w}_0(R(x,z))\tilde{w}_0(S(x,z)) \\ w_1 &= \tilde{w}_1(R(x,z))\tilde{w}_0(S(x,z)) \\ w_3 &= \tilde{w}_0(R(x,z))\tilde{w}_1(S(x,z)) \\ &\quad -\tilde{w}_1(R(x,z))\tilde{w}_0(S(x,z)). \end{aligned} \tag{1.11}$$

This implies that

$$\begin{aligned} q_1 &= e^{\tilde{w}_1(R(x,z))/\tilde{w}_0(R(x,z))} \\ q_1q_3 &= e^{\tilde{w}_1(S(x,z))/\tilde{w}_0(S(x,z))}. \end{aligned} \tag{1.12}$$

Inverting these, applying (1.7), and using $j(q) = \frac{1}{r(q)(1-432r(q))}$, we see that (1.9) follows. •

Corollary 1.3. *Let E_4 be the normalized Eisenstein series of weight 4. Then*

$$\left(\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\binom{6k+12m}{3k+6m}\binom{3k+6m}{2k+4m}\binom{k+2m}{2m}\binom{2m}{m}\right.$$
$$\frac{1}{j(q_1)^m j(q_2)^m}$$
$$\left.\times\left(2\frac{j(q_1)+j(q_2)-1728}{j(q_1)j(q_2)+\sqrt{j(q_1)(j(q_1)-1728)}\sqrt{j(q_2)(j(q_2)-1728)}}\right)^k\right)^4$$
$$= E_4(q_1)E_4(q_2). \tag{1.13}$$

Proof: Computing the power series solution w_0 to (1.2) with leading term 1, we get

$$w_0(x,z) = \sum_{n\geq 2m\geq 0} \frac{(6n)!}{(3n)!(2n)!(m!)^2(n-2m)!} x^n z^m. \tag{1.14}$$

Now do a change of variable on the summation $n = k+2m$, put $q_2 = q_1 q_3$, and apply the corollary above. We see that the left hand side of (1.13) is $w_0(x(q_1,q_3), z(q_1,q_3))$.

By the first equation in (1.11), it is enough to show that $\tilde{w}_0(r(q))^4 = E_4(q)$ where $\tilde{w}_0(x)$ is the solution to (1.3) regular at $x = 0$. In [1], we have proved that

$$\tilde{w}_0(r(q))^2 = \frac{\Theta_q r(q)}{r(q)(1-432r(q))}. \tag{1.15}$$

But we know that

$$\begin{aligned} j(q) &= \tfrac{1}{r(q)(1-432r(q))} \\ E_4(q) &= \tfrac{(\Theta_q j(q))^2}{j(q)(j(q)-1728)^2}. \end{aligned} \tag{1.16}$$

Combining the three equations, we get $E_4(q) = \tilde{w}_0(r(q))^4$. •

2. GENERALIZATIONS

The technique we have used to study the example above is clearly applicable to a more general class of PDEs. The only step which involves details of the example is the system (1.6). It turns out that even the form of our solution (1.7) to (1.6) has more general applicability as we now show.

Consider the PDE system

$$\begin{aligned} L_1 &= \Theta_x(\Theta_x - 2\Theta_z) - \lambda x(\Theta_x + 1/2 + \nu) \\ &\qquad (\Theta_x + 1/2 - \nu) \\ L_3 &= \Theta_z^2 - z(2\Theta_z - \Theta_x + 1)(2\Theta_z - \Theta_x) \end{aligned} \tag{2.1}$$

and the ODE

$$L = \Theta_x^2 - \lambda x(\Theta_x + 1/2 + \nu)(\Theta_x + 1/2 - \nu) \tag{2.2}$$

where λ, ν are complex numbers. For $(\lambda, \nu) = (432, 1/3)$ we recover the case (1.2) above. We now have the following generalization.

Theorem 2.1. *The rational mapping $\mathbf{C}^2 \to \mathbf{C}^2$, $(R,S) \mapsto (x,z)$, defined by the relations*

$$\begin{aligned} R + S - 2\lambda RS - x = 0 \\ RS(1-\lambda R)(1-\lambda S) - x^2 z = 0 \end{aligned} \tag{2.3}$$

has the following property: if $f(x), g(x)$ are solutions to L, then $f(R(x,z))g(S(x,z))$ is a solution to L_1, L_3, where $(x,z) \mapsto (R(x,z), S(x,z))$ is (any branch of) the inverse mapping.

The proof is vitually word for word similar to the proof of the special case above.

Consider the following two families of toric K3 surfaces corresponding to degree 6 hypersurfaces in $\mathbf{P}^3[1,1,2,2]$ and degree 8 hypersurfaces in $\mathbf{P}^3[1,1,2,4]$ respectively. Their Picard-Fuchs systems are exactly (2.1) with $(\lambda, \nu) = (27, 1/6), (64, 1/4)$ respectively. The corresponding ODEs are (2.2) with those parameter values. It turns out that they are exactly the Picard-Fuchs equations for two families of elliptic curves: degree 3 curves in $\mathbf{P}^2[1,1,1]$ and degree 4 curves in $\mathbf{P}^2[1,1,2]$ respectively. As shown in [6], The mirror maps for these two families of curves are hauptmoduls for the genus zero groups $\Gamma_0(2), \Gamma_0(3)$ respectively.

Finally it is amusing to note that the three examples above with parameter values $(\lambda, \nu) = (27, 1/6), (64, 1/4), (432, 1/3)$ correspond to the so-called simple elliptic singularities of types E_6, E_7, E_8 respectively. (See introduction of [7].) That is, the three families of elliptic curves mentioned above – degrees 3, 4, 6 hypersurfaces in $\mathbf{P}^2[1,1,1], \mathbf{P}^2[1,1,2], \mathbf{P}^2[1,2,3]$ respectively – correspond to singularities of these types. Note that their two dimensional counterparts are the three families of K3 surfaces above – degrees 6, 8, 12 hypersurfaces in $\mathbf{P}^3[1,1,2,2], \mathbf{P}^3[1,1,2,4], \mathbf{P}^3[1,1,4,6]$ respectively.It turns out that there is an explicit relation between a generic member of the K3 family in $\mathbf{P}^3[1,1,2,2]$, and a cubic family in $\mathbf{P}^2[1,1,1]$. That is, if we intersect the hypersurface

$$y_1^6 + y_2^6 + y_3^3 + y_4^3 + a y_1 y_2 y_3 y_4 + b y_1^3 y_2^3 = 0 \tag{2.4}$$

with the hyperplane $y_2 - \lambda y_1 = 0$ in $\mathbf{P}^3[1,1,2,2]$, the locus is the curve $(1+\lambda^6+b\lambda^3)(y_1^2)^3 + y_3^3 + y_4^3 + a\lambda y_1^2 y_3 y_4 = 0$. It is isomorphic to the following cubic curve in $\mathbf{P}[1,1,1]$:

$$(1+\lambda^6+b\lambda^3)x_1^3+x_2^3+x_3^3+a\lambda x_1x_2x_3=0. \quad (2.5)$$

The map is induced from natural isomorphism $\mathbf{P}^2[1,2,2]\to\mathbf{P}^2[1,1,1]$. There is an analogous relation between the K3 family in $\mathbf{P}^3[1,1,2,4]$ and a quartic family in $\mathbf{P}^2[1,1,2]$, and similarly for $\mathbf{P}^3[1,1,4,6]$ and $\mathbf{P}^2[1,2,3]$.

There are also three dimensional Calabi-Yau varieties which bear the same relation to the above K3 families as these K3 surfaces bear with their elliptic curve counterparts. Namely the three K3 families above correspond respectively to the degrees 12, 16, 24 hypersurfaces in $\mathbf{P}^4[1,1,2,4,4], \mathbf{P}^4[1,1,2,4,8], \mathbf{P}^4[1,1,2,8,12]$ respectively. The intersection of the Calabi-Yau hypersurface

$$z_1^{12}+z_2^{12}+z_3^6+z_4^3+z_5^3+az_1z_2z_3z_4z_5+bz_1^6z_2^6=0 \quad (2.6)$$

with the hyperplane $z_2-\lambda z_1=0$ in $\mathbf{P}^4[1,1,2,4,4]$ is isomorphic to a sextic surface in $\mathbf{P}^3[1,1,2,2]$, and similarly for the other two cases.

3. PART B: SCHWARZIAN EQUATIONS FOR LINEAR PDEs

Since the periods, the mirror map and the Yukawa couplings are of fundamental importance for the prediction of the numbers of rational curves via mirror symmetry, one must understand these objects from as many points of views as one can. For example, can one give an analytic characterization for the mirror map? It has been shown in 1-modulus cases that the answer is yes: the mirror map is characterized by some polynomial ODE near the "large volume limit" [6]. Motivated by this problem, we study in this section the analogues in 2-moduli cases. We will construct polynomial PDE systems naturally associated with the Picard-Fuchs systems for the periods of Calabi-Yau varieties. Later we will see that these PDEs give us powerful tool for doing perturbation theory on the periods, the mirror map and the type A Yukawa couplings.

Consider for fixed $m\geq 2$ the following pair of linear partial differential operators:

$$\begin{aligned} L_1 &= \textstyle\sum_{0\leq i+j\leq m} a_{ij}\partial_x^i\partial_y^j \\ L_2 &= \textstyle\sum_{0\leq i+j\leq 2} b_{ij}\partial_x^i\partial_y^j \end{aligned} \quad (3.1)$$

where the a_{ij}, b_{ij} rational function of x,y. We assume that $b_{11}^2-4b_{02}b_{20}$ is not identically zero, and that near $x=0=y$ the system admits unique series solutions w_0,w_1,w_2 with the leading behaviour

$$\begin{aligned} w_0(x,y) &= 1+O(h) \\ w_1(x,y) &= w_0 Log\ x+O(h) \\ w_2(x,y) &= w_0 Log\ y+O(h) \end{aligned} \quad (3.2)$$

where $O(h^k)$ here means terms in powers of x,y which are of total degree at least k. We remark that all known examples arising in mirror symmetry result in Picard- Fuchs systems of this kind, where x,y denotes the Mori coordinates for the complex structure deformation space near the large volume limit. Let $s=w_1/w_0$, $t=w_2/w_0$ as (locally defined) $\mathbf{C}^2$-valued maps of x,y. It is clear that the Jacobian of this map is nonzero. Then inverting this map, we can regard $x(s,t), y(s,t)$ as functions of s,t (or as power series in $q_1:=e^s, q_2:=e^t$). We wish to derive a system of polynomial PDEs for these functions. Recall the transformation laws under a change of variables

$$\begin{aligned} \partial_x f &= \tfrac{[y,f]}{[y,x]} \\ \partial_y f &= \tfrac{[x,f]}{[x,y]} \end{aligned} \quad (3.3)$$

where $[f,g]$ is the "Poisson bracket":

$$[f,g]:=\partial_s f\partial_t g-\partial_s g\partial_t f. \quad (3.4)$$

Under this change of variables, L_1, L_2 becomes

$$\begin{aligned} \mathcal{L}_1 &= \textstyle\sum_{0\leq i+j\leq m} c_{ij}\partial_s^i\partial_t^j \\ \mathcal{L}_2 &= \textstyle\sum_{0\leq i+j\leq 2} d_{ij}\partial_s^i\partial_t^j. \end{aligned} \quad (3.5)$$

From the transformation laws, it is easy to see that up to an overall factor the new coefficients c_{ij}, d_{ij} are differential *polynomials* of $x(s,t), y(s,t)$.

Now observe that

$$\begin{aligned} \mathcal{L}_1(sw_0)-s\mathcal{L}_1w_0 &= \sum ic_{ij}\partial_s^{i-1}\partial_t^j w_0=0 \\ \mathcal{L}_1(tw_0)-t\mathcal{L}_1w_0 &= \sum jc_{ij}\partial_s^i\partial_t^{j-1} w_0=0 \\ \mathcal{L}_2(sw_0)-s\mathcal{L}_2w_0 &= d_{10}w_0+d_{11}\partial_t w_0 \\ &\quad +2d_{20}\partial_s w_0=0 \end{aligned}$$

$$\begin{aligned}\mathcal{L}_2(tw_0) - t\mathcal{L}_2 w_0 &= d_{01}w_0 + d_{11}\partial_s w_0 \\ &\quad +2d_{02}\partial_t w_0 = 0 \\ \mathcal{L}_1 w_0 &= 0 \\ \mathcal{L}_2 w_0 &= 0.\end{aligned} \tag{3.6}$$

It is easy to check that $(d_{11}^2 - 4d_{02}d_{20})[x,y]^2 = b_{11}^2 - 4b_{02}b_{20}$ which is nonzero. Thus we can solve for $\partial_s w_0, \partial_t w_0$ in terms of w_0 and differential expressions in x, y in the third and fourth equations in (3.6). We can hence use this to eliminate all higher derivatives $\partial_s^i \partial_t^j w_0$ in the first and second equations in (3.6). But since w_0 appears linearly everywhere, we can factor it out and obtain a pair of coupled polynomial PDEs in x, y. Their order is at most m. Thus we have

Proposition 3.1. *Given the linear PDE system L_1, L_2 above, there exist a pair of polynomial PDEs for x, y.*

Note that the system (3.6) can be regarded as a (overdetermined) system of polynomial PDEs in x, y, w_0. We will use this system to study the mirror map and the Yukawa couplings by means of perturbation theory in the next section.

As an example, consider the system (2.1) which we can write in the form (3.1), where $m = 2$, hence (3.5) under a change of variables. As we have seen, such a system arises as the Picard-Fuchs system for certain 2-moduli families of toric K3 surfaces. The hypotheses on the uniqueness of series solutions, and that $b_{11}^2 - 4b_{02}b_{20}$ is nonzero can be easily checked. In this case our polynomial PDEs for $x(s,t), y(s,t)$ becomes

$$\begin{aligned}&(b_{11}^2 - 4b_{02}b_{20})(2c_{02}c_{10} - c_{01}c_{11}) = \\ &\quad (a_{11}^2 - 4a_{02}a_{20})(2d_{02}d_{10} - d_{01}d_{11}) \\ &(b_{11}^2 - 4b_{02}b_{20})(2c_{20}c_{01} - c_{10}c_{11}) = \\ &\quad (a_{11}^2 - 4a_{02}a_{20})(2d_{20}d_{01} - d_{10}d_{11}).\end{aligned} \tag{3.7}$$

4. PERTURBATIONS

We have seen a nice description of the periods and the mirror map when one degenerates a certain family of Calabi-Yau threefolds along a codimension one subfamily. We would like to use perturbation theory to study the Calabi-Yau threefolds in a neighborhood of the codimension one subfamily. Consider for example the mirror threefold X of the degree 12 hypersurfaces in $\mathbf{P}^4[6,2,2,1,1]$ which has $h^{1,1} = 2$. Let x, y be the Mori coordinates near the large volume limit of X. It is now known that the periods and the mirror map in this case admit a description in terms of the j function in the limit $y \to 0$. Can one give a similar description near $y = 0$? How about near $x = 0$?

We will give two descriptions in the case of the degree 12 hypersurface above. We show that mirror map and the Yukawa couplings, order by order in q_2, can be described by quadrature in terms of the j function. In fact, we will compute the first few terms. The second description is by means of perturbation theory in the q_1 direction. Here remarkably, the first few terms are purely algebraic, rather than transcendental.

We briefly review what is known for the degree 12 hypersurfaces in $\mathbf{P}^4[1,1,2,2,6]$. The Picard-Fuchs system in the Mori coordinates x, y is given by

$$\begin{aligned}L_1 &= \Theta_x^2(\Theta_x - 2\Theta_y) - 8x(6\Theta_x + 5)(6\Theta_x + 3) \\ &\qquad (6\Theta_x + 1) \\ L_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_x + 1)(2\Theta_y - \Theta_x).\end{aligned} \tag{4.1}$$

This motivates the study of the following family of PDEs, where λ, ν are constants, (cf. [1] and see also Appendix B):

$$\begin{aligned}L_1 &= \Theta_x^2(\Theta_x - 2\Theta_y) \\ &\quad -\lambda x(\Theta_x + 1/2)(\Theta_x + 1/2 - \nu)(\Theta_x + 1/2 + \nu) \\ L_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_x + 1)(2\Theta_y - \Theta_x).\end{aligned} \tag{4.2}$$

As before there are unique solutions near $x = y = 0$ of the form $w_0 = 1 + O(h), w_1 = w_0 Log\ x + O(h), w_2 = w_0 Log\ y + O(h)$, and the coefficients of L_2 satisfies $b_{11}^2 - 4b_{02}b_{20} = -4y^2 \neq 0$. Thus the system (4.1) is of the type (3.1) with $m = 3$. Associated to it is the nonlinear system (3.6). The "mirror map" $(t_1, t_2) \mapsto (x(q_1,q_2), y(q_1,q_2))$ is defined locally by the inverse of the power series relations

$$\begin{aligned}q_1 &= e^{w_1(x,y)/w_0(x,y)} = x(1 + O(h)) \\ q_2 &= e^{w_2(x,y)/w_0(x,y)} = y(1 + O(h)).\end{aligned} \tag{4.3}$$

Thus we can write

$$\begin{aligned} x(q_1,q_2) &= \textstyle\sum_{i=0}^{\infty} x_i(q_1)q_2^i \\ y(q_1,q_2) &= \textstyle\sum_{i=1}^{\infty} y_i(q_1)q_2^i \\ w_0(x(q_1,q_2),y(q_1,q_2)) &= \textstyle\sum_{i=0}^{\infty} g_i(q_1)q_2^i \end{aligned} \quad (4.4)$$

where the $x_i(q_1), y_i(q_1), g_i(q_1)$ are power series. We will use the PDEs (3.6) derived in the last section to compute the x_i, y_i, g_i, $i \leq 2$. The results turn out to have a universal form.

Theorem 4.1. *For x_i, y_i, g_i as defined in (4.4), x_0 is the unique power series solution with $x_0 = q_1 + O(q_1^2)$ to the Schwarzian equation $2Q(x_0)x_0'^2 + \{x_0, s\} = 0$ with $Q(x) = \frac{1+(-\frac{5}{4}+\nu^2)\lambda x+(1-\nu)(1+\nu)\lambda^2x^2}{4x^2(1-\lambda x)^2}$. We also have*

$$\begin{aligned} g_0 &= \tfrac{x_0'}{x_0(1-\lambda x_0)^{1/2}} \\ x_1 &= \tfrac{-x_0^2 y_1}{g_0 x_0'^2}(-2g_0'x_0' + g_0x_0'') \\ y_1 &= \exp\Big(2\int\int\int \tfrac{dsdsds}{1-\lambda x_0}((Log\ g_0)'(Log\ x_0)''- \\ &\quad (Log\ g_0)''(Log\ x_0)' - (Log\ g_0)'^2(Log\ x_0)')\Big) \\ g_1 &= \tfrac{1}{x_0'^3}(x_1 g_0' x_0'^2 + x_0^2 y_1 x_0' g_0'' - x_0^2 y_1 g_0' x_0'') \end{aligned} \quad (4.5)$$

where prime here means Θ_{q_1}.

Proof: For the proof of the statements concerning x_0, g_0, see [1]. We will study the PDE system (3.6) associated to (4.1) up to first order in powers of $q_2 = e^t$. We substitute $x = x_0 + q_2x_1 + O(q_2^2)$, $y = q_2y_1 + O(q_2^2)$, $w_0 = g_0 + q_2g_1 + O(q_2^2)$ into (3.6) where the x_i, y_i, g_i are all power series in $q_1 = e^s$. To lowest order, the first equation in (3.6) is a polynomial ODE in g_0, x_0. Using the result [1] that $w_0(x(q_1), 0)^2 = g_0^2 = \frac{x_0'^2}{x_0^2(1-1728x_0)}$, it is easy to show that this ODE holds identically. Now consider the second equation in (3.6). To leading order it is a complicated polynomial ODE in y_1, g_0, x_0. But after we apply repeatedly the fact that x_0 satisfies the Schwarzian equation to eliminate $x_0''', x_0'''', \ldots$, we see that the equation is solvable. The general solution is exactly the third equation in (4.5). There is a unique particular solution which is a power series in q_1 with leading coefficient 1. Similarly the third and sixth equations in (3.6), to lowest order, gives respectively the second and fourth equations in (4.5). •

We can also use the same system (3.6) to compute the higher order terms. At each order $i \geq 2$ the triple (x_i, y_i, g_i), can now be solved successively in terms of the lower terms by applying the third, fourth and sixth equations in (3.6). It appears that at each order $i \geq 2$, x_i, y_i, g_i are given by some differential *rational* functions of the lower order terms without solving a differential equation. For example, x_2, y_2, g_2 in fact occurs linearly (with no derivative thereof) in the following equations. They are in fact the order $O(q_2^2)$ terms of the third, fourth and sixth equations respectively in (3.6):

$$\begin{aligned}
&-2x_1y_1g_0'x_0'^2 + 4x_0y_1^2g_0'x_0'^2 + 2g_1y_1x_0'^3 \\
&-2g_0y_1^2x_0'^3 - g_0y_2x_0'^3 + 2x_0^2y_1g_0'x_0'y_1' \\
&+g_0x_1x_0'^2y_1' - g_0x_0^2x_0'y_1'^2- \\
&-g_0x_0^2y_1y_1'x_0'' + g_0x_0^2y_1x_0'y_1'' = 0 \\[1em]
&2x_1^2y_1g_0'x_0'^2 - 12x_0x_1y_1^2g_0'x_0'^2 - 2x_0^2y_1y_2g_0'x_0'^2 \\
&-2x_0^2y_1^2g_1'x_0'^2 - 3g_1x_1y_1x_0'^3 - 4g_0x_2y_1x_0'^3 \\
&+4g_1x_0y_1^2x_0'^3 + 6g_0x_1y_1^2x_0'^3 + 3g_0x_1y_2x_0'^3 \\
&+4x_0^2y_1^2g_0'x_0'x_1' + 3g_0x_1y_1x_0'^2x_1' - 4g_0x_0y_1^2x_0'^2x_1' \\
&-4x_0^2x_1y_1g_0'x_0'y_1' - 3g_0x_1^2x_0'^2y_1' + 2g_1x_0^2y_1x_0'^2y_1' \\
&+4g_0x_0x_1y_1x_0'^2y_1' - 2g_0x_0^2y_1x_0'x_1'y_1' + 2g_0x_0^2x_1x_0'y_1'^2 \\
&-g_0x_1^2y_1x_0'x_0'' + g_1x_0^2y_1^2x_0'x_0'' + 6g_0x_0x_1y_1^2x_0'x_0'' \\
&+g_0x_0^2y_1y_2x_0'x_0'' - 3g_0x_0^2y_1^2x_1'x_0'' + 3g_0x_0^2x_1y_1y_1'x_0'' \\
&+g_0x_0^2y_1^2x_0'x_1'' - g_0x_0^2x_1y_1x_0'y_1'' = 0 \\[1em]
&-4x_2y_1g_0'x_0'^3 + 6x_1y_1^2g_0'x_0'^3 + 3x_1y_2g_0'x_0'^3 \\
&-3x_1y_1g_1'x_0'^3 + 4x_0y_1^2g_1'x_0'^3 + 4g_2y_1x_0'^4 - 6g_1y_1^2x_0'^4 \\
&-3g_1y_2x_0'^4 + 3x_1y_1g_0'x_0'^2x_1' - 4x_0y_1^2g_0'x_0'^2x_1' \\
&-3x_1^2g_0'x_0'^2y_1' + 4x_0x_1y_1g_0'x_0'^2y_1' + 2x_0^2y_1g_1'x_0'^2y_1' \\
&+3g_1x_1x_0'^3y_1' - 4g_1x_0y_1x_0'^3y_1' - 2x_0^2y_1g_0'x_0'x_1'y_1' \\
&+2x_0^2x_1g_0'x_0'y_1'^2 - 2g_1x_0^2x_0'^2y_1'^2 + x_1^2y_1x_0'^2g_0'' \\
&-6x_0x_1y_1^2x_0'^2g_0'' - x_0^2y_1y_2x_0'^2g_0'' + 2x_0^2y_1^2x_0'x_1'g_0'' \\
&-2x_0^2x_1y_1x_0'y_1'g_0'' - x_0^2y_1^2x_0'^2g_1'' - x_1^2y_1g_0'x_0'x_0'' \\
&+6x_0x_1y_1^2g_0'x_0'x_0'' + x_0^2y_1y_2g_0'x_0'x_0'' + x_0^2y_1^2g_1'x_0'x_0'' \\
&-3x_0^2y_1^2g_0'x_1'x_0'' + 3x_0^2x_1y_1g_0'y_1'x_0'' - g_1x_0^2y_1x_0'y_1'x_0'' \\
&+x_0^2y_1^2g_0'x_0'x_1'' - x_0^2x_1y_1g_0'x_0'y_1'' + g_1x_0^2y_1x_0'^2y_1'' = 0
\end{aligned} \quad (4.6)$$

We have also checked that the order $O(q_2^3)$ term of the third, fourth and sixth equations in (3.6) are linear in x_3, y_3, g_3 (with no derivative thereof), which determines this triple in terms of the lower order terms. We emphasize that the ODEs above are universal in the sense that they are independent of the parameters λ, ν of our linear PDEs.

Corollary 4.2. *For the case of the degree 12 hypersurface in* $\mathbf{P}^4[6,2,2,1,1]$, *the* x_i, y_i, g_i, $i \le 3$, *are given in terms of the* j *function by quadrature.*

Proof: In this case, we have $(\lambda, \nu) = (1728, 1/3)$ and we check that the unique solution to the Schwarzian equation $2Qx_0'^2 + \{x_0, s\} = 0$ is given by $x_0 = 1/j$. By theorem above and the remarks following it, we see that the x_i, y_i, g_i, $i \le 3$ can be expressed (explicitly!) in terms of j by quadrature. •

4.1 Remarks

1. When $(\lambda, \nu) = (1728, 1/3)$, if we use the Schwarzian equation for j to further simplifies things, we get

$$\begin{aligned} x_0 =& \; 1/j \\ g_0 =& \; E_4^{1/2} \\ x_1 =& \; y_1 \tfrac{x_0(2x_0'^2 - 5184 x_0 x_0' - x_0 x_0'' + 1728 x_0^2 x_0'')}{(1-1728x_0)x_0'^2} \\ y_1 =& \; \exp\Big(\int\int\int -((1-1728x_0)^2 x_0^2 x_0''^2 \\ & -2(1-3456x_0)(1-1728x_0)x_0 x_0'^2 x_0'' + \\ & (1-6672x_0+10782720x_0^2)x_0'^4)/ \\ & (x_0^3(1-1728x_0)^3 x_0') ds ds ds\Big) \end{aligned} \quad (4.7)$$

2. It turns out that in the example above, the $O(q_2^k)$ coefficient g_k of $w_0(x(q_1,q_2), y(q_1,q_2))$ is always a rational function of the lower order coefficients for x, y. More precisely, we have:

Lemma 4.3. *For each* k, g_k *as a power series is a rational function of* g_0, x_0, x_0', x_0'', $x_1, .., x_k, y_1, .., y_k$, *which is polynomial in* $x_1, .., x_k, y_1, .., y_k$.

Proof: Since $g_0^2 = \frac{x_0'^2}{x_0^2(1-1728x_0)}$ and since $x_0 = 1/j$ satisfies a third order Schwarzian equation, a rational function in $g_0, g_0', ..., x_0, x_0', ...$ can be reduced to one in g_0, x_0, x_0', x_0''. It is enough then to show that $\frac{\partial^k w_0}{\partial q_2^k}|_{q_2=0} = k! g_k$ lives in the ring:

$$\mathcal{R} = \mathbf{C}(g_0, g_0', ..., x_0, x_0', ...)[x_1, .., x_k, y_1, .., y_k]. \quad (4.8)$$

Observe that

$$\begin{aligned} w_0(x,y) &= \sum_{n \ge 2m \ge 0} c(n,m) x^n y^m \\ &= \sum_{m \ge 0} \tfrac{1}{m!^2} y^m x^{2m} f^{(2m)}(x) \text{ where} \\ c(n,m) &:= \tfrac{(6n)!}{(3n)! n!^2 m!^2 (n-2m)!} \\ f(x) &:= \sum_{n \ge 0} c(n,0) x^n \\ g_0 &= f(x_0(q_1)). \end{aligned} \quad (4.9)$$

It follows that $\frac{\partial^k w_0}{\partial q_2^k}|_{q_2=0}$ is a sum of terms of the form $\frac{\partial^a y^m}{\partial q_2^a} \frac{\partial^b x^{2m}}{\partial q_2^b} \frac{\partial^c f^{(2m)}(x)}{\partial q_2^c}|_{q_2=0}$, with $0 \le a, b, c \le k$. But $\frac{\partial^a y^m}{\partial q_2^a}|_{q_2=0}$ is zero for all $m > a$, because $y^m = q_2^m(1 + O(h))$, and is a polynomial of $y_1, .., y_k$ for $m \le a \le k$, hence is in $\mathcal{R}$. Similarly, the $\frac{\partial^b x^{2m}}{\partial q_2^b}|_{q_2=0}$ are polynomials of $x_0, x_1, .., x_k$, hence are in $\mathcal{R}$. Finally (by $\frac{\partial}{\partial q_2} = \frac{\partial x}{\partial q_2}\frac{d}{dx}$, $x|_{q_2=0} = x_0$) the $\frac{\partial^c f^{(2m)}(x)}{\partial q_2^c}|_{q_2=0}$ are clearly polynomials in $x_0, x_1, .., x_k$ and $f(x_0), f'(x_0), ...$ But since $g_0 = f(x_0(q_1))$ and $f'(x_0) = g_0'/x_0' \in \mathbf{C}(g_0, x_0, x_0')$, it follows that these polynomialsare also in $\mathcal{R}$. •

3. One of the consequences of the fact that the restriction of the mirror map is given by the j function is that the mirror map cannot be algebraic. More precisely, there is no nontrivial polynomial relations

$$\begin{aligned} P(x,y,q_1,q_2) =& \; 0 \\ Q(x,y,q_1,q_2) =& \; 0 \end{aligned} \quad (4.10)$$

along the graph of the mirror map $(q_1, q_2) \mapsto (x, y)$. To see this, suppose both P, Q are irreducible. Then from the resultants of the two polynomials we obtain two irreducible polynomial relations, along the graph:

$$\begin{aligned} \tilde{P}(x,q_1,q_2) =& \; 0 \\ \tilde{Q}(y,q_1,q_2) =& \; 0. \end{aligned} \quad (4.11)$$

By irreducibility, the polynomial in two variables $\tilde{P}(a,b,0)$ is not *identically* zero. But under the mirror map we have $(q_1, 0) \mapsto (1/j(q_1), 0)$, implying that $\tilde{P}(1/j(q_1), q_1, 0) = 0$ identically, which is absurd.

However it turns out that the mirror map in this case is very close to being algebraic in the sense we shall explain in the next section.

4. What we have effectively described above is a perturbation method for computing, order by order in one of the Kahler coordinates q_2, the mirror map given by $x(q_1,q_2), y(q_1,q_2)$, and the special period given by w_0. The perturbation series (4.4) turn out to be useful also for computing the Yukawa couplings via mirror symmetry. Recall that the type A couplings of a Calabi-Yau variety X is given in terms of the type B coupling of the mirror variety Y via the formulas [8][4][9]:

$$K^A_{ijk} = \frac{1}{w_0^2} \sum_{l,m,n} \frac{\partial x_l}{\partial t_i} \frac{\partial x_m}{\partial t_j} \frac{\partial x_n}{\partial t_k} K^B_{lmn}(x) \qquad (4.12)$$

where the K^B are rational functions. In the 2-moduli example above, these rational functions have been computed in [4] (up to multiplicative constants):

$$\begin{aligned} K^B_{111} &= \tfrac{4}{\bar{x}^3 D} \\ K^B_{112} &= \tfrac{2(1-\bar{x})}{\bar{x}^2\bar{y}D} \\ K^B_{122} &= \tfrac{2\bar{x}-1}{\bar{x}\bar{y}(1-\bar{y})D} \\ K^B_{222} &= \tfrac{1-\bar{x}+\bar{y}-3\bar{x}\bar{y}}{2\bar{y}^2(1-\bar{y})^2 D} \end{aligned} \qquad (4.13)$$

where $D = (1-\bar{x})^2 - \bar{x}^2\bar{y}$, $\bar{x} = 1728x, \bar{y} = 4y$. It follows that in this case the K^A are computable order by order in terms of modular functions simply by computing the x, y, w_0 using the perturbation method above. Up to order $O(q_2^2)$, the answers are given in Appendix A.

Now on the other hand the type A coupling takes the form

$$K^A_{ijk} = K^0_{ijk} + \sum_{d_1+d_2>0} \frac{n_{d_1,d_2} d_i d_j d_k q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}} \qquad (4.14)$$

where the K^0_{ijk} are the classical cubic intersection numbers of X, and the n_{d_1,d_2} is the mirror symmetry prediction for the number of rational curves of degrees (d_1, d_2). If we write

$$K^A_{ijk} = \sum_{m=0}^{\infty} K_{ijk}[m] q_2^m \qquad (4.15)$$

where the $K_{ijk}[m]$ are power series in q_1, then using $(m > 0)$

$$\frac{\partial^m}{\partial q_2^m}\Big|_{q_2=0} \frac{q_1^{d_1} q^{d_2}}{1 - q_1^{d_1} q_2^{d_2}} = \begin{cases} m! q_1^{\frac{d_1 m}{d_2}} & \textit{if } d_2 | m \\ 0 & \textit{otherwise} \end{cases} \qquad (4.16)$$

it is easy to show that

$$\begin{aligned} K_{ijk}[0] &= K^0_{ijk} + \delta_{i,1}\delta_{j,1}\delta_{k,1} \sum_{d_1>0} \tfrac{n_{d_1,0} d_1^3 q_1^{d_1}}{1-q_1^{d_1}} \\ K_{ijk}[m] &= \sum_{d_1 \geq 0, d_2 | m} n_{d_1,d_2} d_i d_j d_k q_1^{\frac{d_1 m}{d_2}} \\ & \qquad \textit{for } m > 0. \end{aligned} \qquad (4.17)$$

Thus using perturbation theory each of the $K_{ijk}[m]$ can now be expressed in terms of modular functions.

5. PERTURBATIONS AROUND $x = 0$

We now interchange the roles of (x, q_1) and (y, q_2). One might expect that the discussion above would carry over with few changes. It turns out that while all the techniques carry over, the results have vast simplifications in this case. This consideration is motivated by a few observations.

First note that the Picard-Fuchs system (4.1) is highly asymmetric in x, y. Thus it is reasonable that the two limits along $y = 0$ and $x = 0$ are qualitatively different. Second note that along $x = 0$, the solutions w_0, w_2 degenerate to elementary functions

$$\begin{aligned} w_0(0,y) &= 1 \\ w_2(0,y) &= Log(1 - \sqrt{1-4y} - 2y) - Log(2y), \end{aligned} \qquad (5.1)$$

and they are solutions to $\Theta_y^2 - 2y(2\Theta_y + 1)\Theta_y$. It is then easy to compute the mirror map restricted along $x = 0$: $y(0, q_2) = \frac{q_2}{(1+q_2)^2}$ which is rational rather than transcendental! Third from the definition of the series x, y, w_0, we can write

$$x(q_1,q_2) = \sum_{i=1}^{\infty} X_i(q_2) q_1^i$$
$$y(q_1,q_2) = \sum_{i=0}^{\infty} Y_i(q_2) q_1^i$$
$$w_0(x(q_1,q_2), y(q_1,q_2)) =$$
$$\sum c(n,m) x(q_1,q_2)^n y(q_1,q_2)^m = \sum_{i=0}^{\infty} G_i(q_2) q_1^i \tag{5.2}$$

where the X_i, Y_i, G_i are power series. Let's describe the G_i in terms of the X_i, Y_i. The $c(n,m)$ are such that $c(n,m) = 0$ for all $2m > n$. (The same argument below applies to any other 2-moduli family of Calabi-Yau toric varieties with fundamental period having the form $w_0(x,y) = \sum c(n,m) x^n y^m$ such that for each n, $c(n,m) = 0$ for $m >> n$; see Appendix B for more examples.) Since $x(q_1,q_2)^n = q_1^n(1 + O(h))$ and since $c(n,m) = 0$ for $m >> n$, at most finitely many terms in the sum $\sum c(n,m) x(q_1,q_2)^n y(q_1,q_2)^m$ contribute to a given G_k. Thus it is a *finite* linear sum of $X_i Y_j$. In particular, if the X_i, Y_j are algebraic then so are the G_k. As seen above,

$$\begin{aligned} G_0 &= 1 \\ Y_1 &= \frac{q_2}{(1+q_2)^2}. \end{aligned} \tag{5.3}$$

In fact applying perturbation theory along the q_1 direction on the PDE system (3.6), we compute the first few terms. The emerging pattern is clear evidence that the X_i, Y_i, G_i are in fact rational.

Proposition 5.1. *Denote $q := q_2$. Then*

$$\begin{aligned}
G_0 &= 1 \\
G_1 &= 120(1+q) \\
G_2 &= 360(-17 + 268q - 17q^2) \\
G_3 &= 480(1+q)(1537 + 135866q + 1537q^2) \\
G_4 &= 120(-893747 + 362384432q \\
&\quad +1610384580q^2 + 362384432q^3 - 893747q^4) \\
G_5 &= 720(1+q)(24145921 + 38170176314q \\
&\quad +411770251626q^2 + 38170176314q^3 \\
&\quad +24145921q^4) \\
\\
X_1 &= 1+q \\
X_2 &= -24(31 + 82q + 31q^2) \\
X_3 &= 36(1+q)(9907 + 6130q + 9907q^2) \\
X_4 &= -64(2193143 + 8342176q + 9151506q^2 \\
&\quad +8342176q^3 + 2193143q^4) \\
X_5 &= 30(1+q)(1644556073 - 1014171566q \\
&\quad -26082465678q^2 - 1014171566q^3 \\
&\quad +1644556073q^4) \\
\\
Y_0 &= q/(1+q)^2 \\
Y_1 &= -240q(1-q)^2/(1+q)^3 \\
Y_2 &= -360q(1-q)^2(37 + 554q + 37q^2)/(1+q)^4 \\
Y_3 &= -320q(1-q)^2(7747 + 393600q \\
&\quad +1117306q^2+ \\
&\quad +393600q^3 + 7747q^4)/(1+q)^5 \\
Y_4 &= -60q(1-q)^2(14352887 + 1931431324q \\
&\quad +10227963073q^2 + 17727689272q^3 \\
&\quad +10227963073q^4 + 1931431324q^5 \\
&\quad +14352887q^6)/(1+q)^6.
\end{aligned} \tag{5.4}$$

Proof: X_i, Y_i, G_i, $i \le 3$, can be computed by solving the differential equations (3.6) order by order as we have done before. But this will be hard without first knowing the answers. So we use the following slightly different approach. Numerically it is easy to compute x, y, w_0 as a power series in q_1, q_2 up total order say $O(h^{15})$. We first guess an ansatz (the list above) for the X_i, Y_i, G_i based on the numerical results. Then we check that our ansatz satisfies our differential equations derived from (3.6) (up to $O(q_2^6)$) governing the X_i, Y_i, G_i. Observe also that the ODEs for the X_i, Y_i, G_i derived from (3.6) can have three as the highest order in derivatives. This means that the differential equations together with the first three coefficients of each of the X_i, Y_i, G_i determines the whole series X_i, Y_i, G_i uniquely. The first three coefficients of our ansatz are easily check to be correct. •

Clearly we can apply our perturbation argument to the Yukawa couplings K^A_{ijk} as in the previous section, with the roles of q_1, q_2 interchanged. Thus we can write

$$K^A_{ijk} = \sum_{m=0}^{\infty} K_{ijk}[m] q_1^m \tag{5.5}$$

where now the $K_{ijk}[m]$ are power series in q_2. Then

$$\begin{aligned} K_{ijk}[0] &= K^0_{ijk} + \delta_{i,2}\delta_{j,2}\delta_{k,2} \sum_{d_2>0} \frac{n_{0,d_2} d_2^3 q_2^{d_2}}{1-q_2^{d_2}} \\ K_{ijk}[m] &= \sum_{d_2 \geq 0, d_1|m} n_{d_1,d_2} d_i d_j d_k q_2^{\frac{d_2 m}{d_1}} \quad \text{for } m > 0. \end{aligned} \tag{5.6}$$

If the X_i, Y_i, G_i, for $i \leq l$, are rational, then so are the couplings $K_{ijk}[m]$, for small m. In fact using the proposition above, we have

$$\begin{aligned} K^A_{111} = {} & 4 + 2496(1+q_2)q_1 + 1152(1556 + 13481 q_2 \\ & +1556 q_2^2) q_1^2 + 4768(1358353 + 46666143 q_2 \\ & +46666143 q_2^2 + 1358353 q_2^3) q_1^3 + O(q_1^4) \\ K^A_{112} = {} & 2 + 2496 q_2 q_1 + 576 q_2 (13481 + 3112 q_2) q_1^2 \\ & +768 q_2 (15555381 + 31110762 q_2 \\ & +1358353 q_2^2) q_1^3 + O(q_1^4) \\ K^A_{122} = {} & 2496 q_2 q_1 + 288 q_2 (13481 + 6224 q_2) q_1^2 \\ & +768 q_2 (5185127 + 20740508 q_2 \\ & +1358353 q_2^2) q_1^3 + O(q_1^4) \\ K^A_{222} = {} & \tfrac{2q_2}{1-q_2} + 2496 q_2 q_1 + 144 q_2 (13481 \\ & +12448 q_2) q_1^2 + 256 q_2 (5185127 \\ & +41481016 q_2 + 4075059 q_2^2) q_1^3 + O(q_1^4). \end{aligned} \tag{5.7}$$

These formulas give infinitely many n_{d_1,d_2} simultaneously! For example, we have

$$n_{0,d_2} = 2\delta_{d_2,1}. \tag{5.8}$$

For another example, for at least $d_1 = 0, 1, .., 4$, we have $n_{d_1,d_2} = 0$ for all but finitely many d_2. The nonzero ones can be computed immediately from the formulas above.

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6. Appendix A

In this appendix, we compute perturbatively the type A Yukawa couplings K^A_{ijk} (see discussions in section 4) for the family of Calabi-Yau toric varieties corresponding to the degree 12 hypersurface in $\mathbf{P}^4[6,2,2,1,1]$. We perturb in the neighborhood of the codimension one subfamily with $y = 0$ up to $O(y^2)$ or equivalently $O(q_2^2)$. We give $K^A_{ijk}[0], K^A_{ijk}[1]$, as differential rational functions in x_i, y_i, g_i, $i \leq 2$, which in turn have been given explicitly in terms of the j function by quadrature in section 4. The computation for higher order terms is straightforward but tedious. The $K^A_{ijk}[0]$ have already been considered in [2][10].

$$\begin{aligned} K^A_{111}[0] &= \frac{4x_0'}{x_0(1-1728x_0)} + 6\frac{y_1'}{y_1} \\ K^A_{111}[1] &= \Big(\frac{-2g_1}{g_0^3(1-1728x_0)^2} \\ &\quad + \frac{3456(x_1 - 1728x_0x_1 + 3456x_0^2y_1)}{g_0^2(-1+1728x_0)^4}\Big) \\ &\quad \Big(\frac{4x_0'^3}{x_0^3} + \frac{6(1-1728x_0)x_0'^2y_1'}{x_0^2y_1}\Big) \\ &\quad +\Big(\frac{12x_0'^2(-(x_1x_0') + x_0x_1')}{x_0^4} \\ &\quad + \frac{12(-1+3456x_0)x_0'y_1'^2}{x_0y_1} \\ &\quad + \frac{2(1-1728x_0)y_1'^3}{y_1^2} \\ &\quad + \frac{12(-1+864x_0)x_0'^2x_1y_1'}{x_0^3y_1} \\ &\quad + \frac{6(-1+1728x_0)x_0'^2y_2y_1'}{x_0^2y_1^2} \\ &\quad + \frac{12(1-1728x_0)x_0'x_1'y_1'}{x_0^2y_1} \\ &\quad + \frac{6(1-1728x_0)x_0'y_2'}{x_0^2y_1}\Big)/(g_0^2(1-1728x_0)^2) \\ K^A_{112}[0] &= 2 \\ K^A_{112}[1] &= 2(2g_1y_1x_0'^2 - 6912g_1x_0y_1x_0'^2 \\ &\quad +5971968g_1x_0^2y_1x_0'^2 - 5184g_0x_1y_1x_0'^2 \\ &\quad +8957952g_0x_0x_1y_1x_0'^2 \\ &\quad -11943936g_0x_0^2y_1^2x_0'^2 \end{aligned}$$

$$
\begin{aligned}
&-g_0 y_2 x_0'^2 + 3456 g_0 x_0 y_2 x_0'^2 \\
&-2985984 g_0 x_0^2 y_2 x_0'^2 - 2 g_0 y_1 x_0' x_1' \\
&+5971968 g_0 x_0^2 y_1 x_0' x_1' - 2 g_0 x_1 x_0' y_1' \\
&+6912 g_0 x_0 x_1 x_0' y_1' - 5971968 g_0 x_0^2 x_1 x_0' y_1' \\
&+4 g_0 x_0 y_1 x_0' y_1' - 20736 g_0 x_0^2 y_1 x_0' y_1' \\
&+23887872 g_0 x_0^3 y_1 x_0' y_1' - g_0 x_0^2 y_1'^2 \\
&+3456 g_0 x_0^3 y_1'^2 - 2985984 g_0 x_0^4 y_1'^2)/ \\
&(g_0^3 x_0^2 (-1 + 1728 x_0)^3 y_1)
\end{aligned}
$$

$$
\begin{aligned}
K_{122}^A[0] &= 0 \\
K_{122}^A[1] &= 2(-2x_1 x_0' + 3456 x_0 x_1 x_0' + 2 x_0 y_1 x_0' \\
&\quad -6912 x_0^2 y_1 x_0' - x_0^2 y_1' + 1728 x_0^3 y_1'))/ \\
&\quad ((-1 + 1728 x_0) x_0'^2) \\
K_{222}^A[0] &= 0 \\
K_{222}^A[1] &= \frac{2 y_1}{g_0^2 (1 - 1728 x_0)}
\end{aligned}
\tag{6.1}
$$

7. Appendix B

In this appendix, we will study using perturbation technique introduced above, the perturbations around $y = 0$ ($q_2 = 0$) and $x = 0$ ($q_1 = 0$) for the following five families Calabi-Yau toric varieties, all of which have $h^{1,1} = 2$:

$$
\begin{array}{ll}
a. & \textit{degree 8 in } \mathbf{P}^4[2,2,2,1,1] \\
b. & \textit{degrees } (6,4) \textit{ in } \mathbf{P}^5[2,2,2,2,1,1] \\
c. & \textit{degrees } (4,4,4) \textit{ in } \mathbf{P}^6[2,2,2,2,2,1,1] \\
d. & \textit{degrees 12 in } \mathbf{P}^4[4,3,2,2,1] \\
e. & \textit{degrees 14 in } \mathbf{P}^4[7,2,2,2,1].
\end{array}
\tag{7.1}
$$

Throughout we will use the same notations as in our previous discussion above. We can use the Picard-Fuchs systems and their associated polynomial PDEs to compute the first few coefficients x_i, y_i, g_i of the perturbation series of the mirror map $(q_1, q_2) \mapsto (x, y)$ and the fundamental period w_0 around $q_2 = 0$ (cf. Theorem 4.1). It turns out that theorem Theorem 4.1 also covers the cases a, b, c. Hence all the formulas proved in that section apply here. We won't go into the details of cases d, e, which are a bit more tedious. We will give the coefficients X_i, Y_i, G_i of the perturbation series around $q_1 = 0$ (cf. Proposition 5.1). Note that one can now also compute the type A Yukawa coupling K_{ijk}^A order by order near either $q_2 = 0$ or $q_1 = 0$, by substituting the x_i, y_i, g_i (or X_i, Y_i, G_i) into (4.12) (see section 4).

The respective Picard-Fuchs systems for the five families above are given by [4]:

$$
\begin{aligned}
a.\ L_1 &= \Theta_x^2(\Theta_x - 2\Theta_y) - 4x(4\Theta_x + 3) \\
&\quad (4\Theta_x + 2)(4\Theta_x + 1) \\
L_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_x + 1)(\Theta_y - \Theta_x) \\
b.\ L_1 &= \Theta_x^2(\Theta_x - 2\Theta_y) - 6x(2\Theta_x + 1) \\
&\quad (3\Theta_x + 2)(3\Theta_x + 1) \\
L_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_x + 1)(\Theta_y - \Theta_x) \\
c.\ L_1 &= \Theta_x^2(\Theta_x - 2\Theta_y) - 8x(2\Theta_x + 1)^3 \\
L_2 &= \Theta_y^2 - y(2\Theta_y - \Theta_x + 1)(\Theta_y - \Theta_x) \\
d.\ L_1 &= \Theta_x^2(3\Theta_x - 2\Theta_y) - 36x(6\Theta_x + 5) \\
&\quad (6\Theta_x + 1)(\Theta_y - \Theta_x + 2y(1 \\
&\quad +6\Theta_x - 2\Theta_y)) \\
L_2 &= \Theta_y(\Theta_y - \Theta_x) - y(3\Theta_x - 2\Theta_y - 1) \\
&\quad (3\Theta_x - 2\Theta_y) \\
e.\ L_1 &= \Theta_x^2(7\Theta_x - 2\Theta_y) - 7x(y(28\Theta_x - 4\Theta_y + 18) \\
&\quad +\Theta_y - 3\Theta_x - 2) \times (y(28\Theta_x - 4\Theta_y + 10) \\
&\quad +\Theta_y - 3\Theta_x - 1)(y(28\Theta_x - 4\Theta_y + 2) \\
&\quad +\Theta_y - 3\Theta_x) \\
L_2 &= \Theta_y(\Theta_y - 3\Theta_x) - y(7\Theta_x - 2\Theta_y - 1) \\
&\quad (7\Theta_x - 2\Theta_y)
\end{aligned}
\tag{7.2}
$$

As shown in [3] using the results of [6], in cases a, b, c, the $x_0(q_1)$ are hauptmoduls for the following genus zero groups: $\Gamma_0(2)+, \Gamma_0(3)+, \Gamma_0(4)+$. Using a very similar argument as for Lemma 4.3, it is easy to show that the lemma holds for these three cases as well. The analogue in cases d, e are even easier because the Fourier coefficients $c(n, m)$ for the fundamental period $w_0(x, y)$ here have the properties that for fixed n (or fixed m) all but finitely many $c(n, m)$ vanish. It follows that the g_k are finite sums of $x_i y_j$ in cases d, e (cf. argument in section 5). Similarly in all cases, the G_i are finite sums of $X_i Y_j$.

case a.

$$
\begin{aligned}
G_0 &= 1\\
G_1 &= 24(1+q)\\
G_2 &= 24(1+116q+q^2)\\
G_3 &= 96(1+q)(1+3002q+q^2)\\
G_4 &= 24(1+1226480q+4864468q^2\\
&\quad +1226480q^3+q^4)\\
G_5 &= 48(1+q)(3\\
&\quad +60632494q+547690558q^2\\
&\quad +60632494q^3+3q^4)\\
X_1 &= 1+q\\
X_2 &= -8(13+38q+13q^2)\\
X_3 &= 36(1+q)(179+178q+179q^2)\\
X_4 &= -64(4871+25120q+28658q^2+25120q^3\\
&\quad +4871q^4)\\
X_5 &= 2(1+q)(6509415-12918578q\\
&\quad -176313170q^2-12918578q^3\\
&\quad +6509415q^4)\\
Y_0 &= q/(1+q)^2\\
Y_1 &= -48(-1+q)^2q/(1+q)^3\\
Y_2 &= -24(-1+q)^2q(11+310q+11q^2)/(1+q)^4\\
Y_3 &= -64(-1+q)^2q(115+7680q+28954q^2\\
&\quad +7680q^3+115q^4)/(1+q)^5\\
Y_4 &= -12(-1+q)^2q(37587+5539852q\\
&\quad +32302789q^2+62448408q^3\\
&\quad +32302789q^4+5539852q^5\\
&\quad +37587q^6)/(1+q)^6
\end{aligned}
\tag{7.3}
$$

case b.

$$
\begin{aligned}
G_0 &= 1\\
G_1 &= 12(1+q)\\
G_2 &= 36(1+16q+q^2)\\
G_3 &= 12(1+q)(1+2132q+q^2)\\
G_4 &= 12(7+94760q+347256q^2\\
&\quad +94760q^3+7q^4)\\
G_5 &= 36(1+q)(2+1368046q+10903287q^2\\
&\quad +1368046q^3+2q^4)\\
X_1 &= 1+q\\
X_2 &= -6(7+22q+7q^2)\\
X_3 &= 9(1+q)(109+148q+109q^2)\\
X_4 &= -4(4247+28450q+33606q^2\\
&\quad +28450q^3+4247q^4)\\
X_5 &= 3(1+q)(81410-367682q\\
&\quad -3523185q^2-367682q^3+81410q^4)\\
Y_0 &= q/(1+q)^2\\
Y_1 &= -24(-1+q)^2q/(1+q)^3\\
Y_2 &= -36(-1+q)^2q(1+50q+q^2)/(1+q)^4\\
Y_3 &= -8(-1+q)^2q(55+4890q+23494q^2\\
&\quad +4890q^3+55q^4)/(1+q)^5\\
Y_4 &= -6(-1+q)^2q(2279+378364q\\
&\quad +2348113q^2+5049976q^3+\\
&\quad 2348113q^4+378364q^5+2279q^6)/(1+q)^6
\end{aligned}
\tag{7.4}
$$

case c.

$$
\begin{aligned}
G_0 &= 1\\
G_1 &= 8(1+q)\\
G_2 &= 8(3+28q+3q^2)\\
G_3 &= 32(1+q)(1+186q+q^2)\\
G_4 &= 8(3+20176q+69500q^2+20176q^3+3q^4)\\
G_5 &= 48(1+q)(1+88890q+641386q^2\\
&\quad +88890q^3+q^4)\\
X_1 &= 1+q\\
X_2 &= -8(3+q)(1+3q)\\
X_3 &= 20(1+q)(15+26q+15q^2)\\
X_4 &= -64(41+352q+430q^2+352q^3+41q^4)\\
X_5 &= 2(1+q)(9063\\
&\quad -82738q-620882q^2--82738q^3+9063q^4)\\
Y_0 &= q/(1+q)^2\\
Y_1 &= -16(-1+q)^2q/(1+q)^3\\
Y_2 &= -8(-1+q)^2q(1+98q+q^2)/(1+q)^4\\
Y_3 &= -64(-1+q)^2q(1+128q+766q^2\\
&\quad +128q^3+q^4)/(1+q)^5\\
Y_4 &= -4(-1+q)^2q(377+72740q\\
&\quad +471887q^2+1126728q^3+\\
&\quad 471887q^4+72740q^5+377q^6)/(1+q)^6
\end{aligned}
\tag{7.5}
$$

case d.

$$
\begin{aligned}
G_0 &= 1\\
G_1 &= 360q(1+q)\\
G_2 &= 1080q^2(211+872q+211q^2)\\
G_3 &= 720q^2(1+q)(2565+828158q\\
&\quad +4510066q^2+828158q^3+2565q^4)\\
X_1 &= (1+q)^3\\
X_2 &= 12(1+q)^2(5-196q-642q^2\\
&\quad -196q^3+5q^4)\\
X_3 &= 18(1+q)(-85-6755q+78932q^2\\
&\quad +349843q^3+682082q^4+\\
&\quad 349843q^5+78932q^6-6755q^7-85q^8)\\
Y_0 &= q/(1+q)^2\\
Y_1 &= -60(-1+q)^2q(1+10q+q^2)/(1+q)^3\\
Y_2 &= 90(-1+q)^2q(57+274q-7341q^2\\
&\quad -22796q^3-7341q^4+\\
&\quad 274q^5+57q^6)/(1+q)^4\\
Y_3 &= 40(-1+q)^2q(-16844-22047q\\
&\quad +1066354q^2-38340920q^3\\
&\quad -242849702q^4-428992850q^5\\
&\quad -242849702q^6-38340920q^7\\
&\quad +1066354q^8-22047q^9\\
&\quad -16844q^{10})/(1+q)^5
\end{aligned}
$$

case e.

$$
\begin{aligned}
G_0 &= 1\\
G_1 &= 840q^3(1+q)\\
G_2 &= 840q^4(-2+28q\\
&\quad +1491q^2+3960q^3+1491q^4+28q^5-2q^6)\\
X_1 &= (1+q)^7\\
X_2 &= 2(1+q)^6(3-46q+434q^2\\
&\quad -2562q^3-11466q^4-2562q^5\\
&\quad +434q^6-46q^7+3q^8)\\
Y_0 &= q/(1+q)^2\\
Y_1 &= 2(-1+q)^2q(-1+13q-113q^2\\
&\quad -638q^3-113q^4+13q^5-q^6)/(1+q)^3
\end{aligned}
\tag{7.6}
$$

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