Hermitian connections on holomorphic bundles and its brief history

Shing-Tung Yau

Harvard University

Dedicated to Fedor Bogomolov on his 70th Birthday

Courant Institute, NYU
2016

I have known Fedor for forty years. Actually I have heard about many important works of his before I met him. Around 1976, I was visiting UCLA. I got married there and I proved the Calabi conjecture right after the marriage. I started to derive corollaries of the theorem. Three years before that, I was trying to give counterexamples to the Calabi conjecture. The approach was to derive consequences of the conjecture and gave examples that the consequences were wrong. Well, once we know the conjecture is in fact right, all the consequences are in fact theorems. I was pleased that it worked out that way.

Just at that time, my friend David Gieseker in UCLA told me that Mumford is visiting him and would give two seminars: one in UCLA and one in UC Irvine. The topic in UCLA was on integrable system and Abelian varities. The one in Irvine was on Bogomolov's work. I found the title attractive and decided to drive two and a half hours to listen to Mumford in Irvine. I did not understand the talk completely. But Mumford declared that the work of Bogomolov is great and challenged the audience to prove the best Chern number inequality for algebraic surfaces of general type : $3C_2 \geq C_1^2$.

I found that attractive as I knew already that this follows from the Calabi conjecture. But I was not sure about the constant. I told Mumford about it. It was clear that he did not believe I can do it: as I come from nowhere as far as the subject of algebraic geometry is concerned. But I went home and worked out the constant precisely. I sent a letter to Mumford telling him that not only the constant can be achieved but also that it can only be achieved when the algebraic surface is covered by the ball. Mumford was very pleased and I think that was the reason that Harvard offered me a job two years later.

My friend Gieseker then showed me a manuscript circulated by Reid on the inequality of Bogomolov. He told me that there is idea of concept of stability which needs to be studied further. He wrote a paper about reproving the inequality of Bogomolov for stable bundles which I found fascinating and in fact I was very much puzzled by it. The Chern number inequality is very natural in term of curvature representation of Chern classes. Why would algebraic method be useful to derive it? Clearly stability of bundles plays an important role. Fedor's originality in pioneering this direction excited me.

Since I have already used Kähler-Einstein metric to improve his Chern number inequality for manifolds, I was sure that there is a version of Kähler-Einstein metric for bundles I learnt from Singer about the self-dual connection on bundles over four manifolds and this seems to be a natural replacement of Kähler-Einstein metric. At the same time, I learnt from C.N. Yang on how to rewrite the anti-self dual equations in terms of Hermitian connections for holomophic bundles. The formulation of the equation is readily generalized to holomorphic bundles over Kähler manifolds. There was no question in my mind that these equations and the concept of stability are linked.

The proof of the existence for the equation turned out to be highly nontrivial. I spent a lot of time with Karen Uhlenbeck to solve the problem. The global idea and some reasonable detail was obtained by 1980. However the complete proof was achieved only a couple of years later. In the meanwhile, Simon Donaldson used the Bott-Chern form to solve the problem for stable bundles over Kähler surfaces. But his argument cannot be generalized to higher dimension. Upon seeing my paper with Uhlenbeck in higher dimension, he used hypeplane section theorem to handle bundles over algebraic manifolds. But up to now, the approach of Uhlenbeck-Yau is still the only one that can handle Higgs bundle and general complex manifolds.

When the manifold is quasi-projective with a complete Kähler metric with finite volume and behaves like Poincaré metric at the divisor at infinity, Simpson solved the existence for the curve case and O. Biquard proved it when the divisor is nonsingular while T. Mochizuki settled the case when the divisor is normal crossing.

A very important consequence of analytic approach to the algebro-geometric problem is that some rather transcendental part of the manifold or bundle can be detected. In the case of Kähler manifold, I was able to derive that the equality of the Chern number inequality for manifolds with ample canonical line bundle implies that the manifold must be covered by the ball. Up to present day, analytic method is still the only way to prove such a statement.

In the case of the bundle, the corresponding statement says that polystable bundles are projective flat if the Chern number inequality becomes equality. When the first Chern class is zero, the bundle is unitary flat. The special case of curve was proved, using algebraic method, by Narasimhan-Sesadri in the 1960s.

An important consequence of Uhlenbeck-Yau theorem is that for a complex manifold, its fundamental group has nontrivial projective representation whenever we can find nontrivial polystable holomorphic bundles over the manifold. Since representation theory concerns unitary representation in an infinite dimensional Hilbert space, it is natural to consider holomorphic bundle with infinite dimensional fiber equipped with a unitary connection. The original argument of Uhlenbeck-Yau can be carried out without much difficulty. The interesting question is how to construct such bundles over a Kähler manifold.

A natural question is the following: Suppose M is a holomorphic fiber space with base N and a generic fiber F such that for some subvariety B of N, M is a topological fiber bundle over $N - B$. For a unitary representation R of the fundament group of F, we look at a topological map from $N - B$ to the moduli space of unitary representation of the fundamental group of F that is deformation of R.

We need to deform the map to be holomorphic map such that it can be extended to a holomorphic map from N to the compactification of the moduli space. From this map, we form a holomorphic bundle over M whose restriction to each fiber is an unitary flat bundle. On the other hand, we can also form a unitary flat bundle over B which can be pulled back to a unitary flat bundle over M. It is interesting to see how we use methods of algebraic geometry to form an extension of the first bundle by this last one. Hopefully the resulting holomorphic bundle can be made to be stable with trivial second Chern class. In that case, we have a unitary flat representation of the fundament group of M.

The point here is to use methods of algebraic geometry to build unitary representation of fundamental groups of Kähler manifolds. One can extend unitary representation to representation of more general group. The concept of Higgs field can be used. This was used by Carlos Simpson in his thesis for handling local Hermitian symmetric space. Simpson used the theory of Higgs bundle to study deformation of Hodge structures. About 14 years ago, I suggested Kang Zuo and Viehweg to continue this study in relationship to my theorem on characterization of local Hermitian symmetric space where symmetric power of bundle were used. This idea went back to Bogomolov's study of Chern number inequalities.

In any case, moduli space of curves has many similar property as local Hermitian symmetric space. In particular, it has Deligne-Mumford compactification and we can study log version of the Chern number inequality for holomorphic bundles over moduli space which can be extended as coherent sheaf to the compactification. If it is stable and if the relative first and second Chern classes vanish, we can construct a unitary flat representation of the modular group. The relationship between such representations when genus of the curves changes would be an interesting topic to study.

When the detail of the proof of the Uhlenbeck-Yau theorem was finished in 1984, I proposed to Edward witten to use it to be part of heterotic string theory, and this was carried out by him in a very important paper in 1986. Since then, a great deal of physics literature was devoted to explore heterotic string theory based on the Uhlenbeck-Yau theorem. Because of the supersymmetric nature of the Hermitian Yang-Mills connection, it has become a central tool in string theory. The discovery of mirror symmetry in string theory has enriched the understanding of Hermitian Yang-Mills connection.

In fact, in 1996, Strominger-Yau-Zaslow proposed a geometric interpretation of the mysterious mirror symmetry that appears among 3-dimensional Calabi-Yau manifolds. The interpretation is based on duality of a special Lagrangian torus fibration of the Calabi-Yau manifolds. The SYZ picture shows that, under mirror transformation, stable holomorphic bundles will map to stable Lagrangian submanifolds while Hermitian Yang-Mills connection becomes special Lagrangian condition on the submanifold.

This is a totally unexpected picture as classical geometry does not show any such connections. But the connection is very fruitful because it gives more interesting structures for either holomorphic bundles or Lagrangian cycles . An important reason is that the duality between stable holomorphic bundle and special Lagrangian cycle is only exact at “large radius limit” of the Calabi-Yau manifolds.

In 1986, Jun Li and I generalized the work of Uhlenbeck-Yau to non-Kähler manifolds which are Gauduchon. For holomorphic bundles, one can define concept of stability. Stability is not easy to be checked. But in some extreme case when there is no curves in a compact complex surface, one can check that the tangent bundle is always stable. If equality of the Chern number inequality holds, tangent bundle admits projective flat connection. Hence for complex surfaces of class VII_0 which admits no curves, Li-Yau-Zheng were able to give a proof of the classification of Fedor on such surfaces.

The problem of classification of complex surfaces of class VII_0 would be complete if we can classify those class VII_0 surfaces with (finite number of) curves. Many years ago, I proposed that the argument of Li-Yau-Zheng should be generalized to connections with poles on the curves. So far, the approach has not been carried out to finish the classification of class VII_0 surfaces. But I still hope this program can be furnished.

The theory of Hermitian Yang-Mills connections has been greatly enriched after the idea of Mirror symmetry appeared in string theory. In 1996, Strominger-Yau-Zaslow gave the first geometric interpretation of the concept of mirror symmetry between Calabi-Yau manifolds. We proposed that each Calabi-Yau manifold that has another Mirror partner must have a fibration whose fibers are special Lagrangian tori. (These are middle dimension subtori of the CY manifold, each of which are Lagrangian and the holomorphic 3-form restricted to it is constant multiple of the volume form. The constant is a complex number with norm one.)

SYZ proposed that the mirror of the CY manifold is obtained by taking the dual torus of each of the fiber of the fibration. This geometric picture is very attractive as it helps to explain many mysterious questions appeared in the theory of mirror symmetry. This is especially true when the CY manifold is at the "large radius limit". In that case, the torus is supposed to be linear and mirror symmetry for CY manifold is simply the T-duality along the torus.

In fact, in 1999, Leung-Yau-Zaslow studied this picture in more detail. When there is a special Lagragian submanifold intersecting the special lagrangian torus at one point generically, we know that this submanifold is mirror to a holomorphic line bundle in the mirror CY manifold. This map is rather explicit. And we were able to find a equation for an Hermitian connection defined on this holomorphic line bundle which came from the equation of special Lagrangian submanifold. Using the Fourier-Mukai transform, the equation can be written as:

$$\mathrm{Im}(\omega - F)^n = \tan\hat{\theta}\,\mathrm{Re}(\omega - F)^n,$$

where ω is the Kähler form of the manifold, F is the curvature of the connection A and θ is the phase of the special Lagrangian. This equation can be interpreted as nonlinear instantons which are supersymmetric, according to Marino, Minasian, Moore and Strominger.

We can also rewrite this equation in the following way:

$$\Theta_\alpha(\omega) = \sum_{i=1}^{n} \arctan(\lambda_i) = \hat{\Theta}.$$

This equation in this form can be generalized to any compact Kähler manifold without assuming it is CY. Jacob and I used a parabolic flow to study the existence of solutions to this equation when the Kähler manifold has positive bisectional curvature, and the initial data is sufficiently positive.

Later, Collins-Jacob-Yau provided more in-depth understanding of the above equation. First of all, we observe that in the above equation, we can take high power of the line bundle L. Letting the power going to infinity, we obtain, after normalization, a limiting equation which can be written as

$$c\omega^n = n\omega^{n-1} \wedge \alpha,$$

for $\omega \in c_1(L)$ with c a topological constant.

This turns out to be an equation discovered by Donaldson in 1999. He called it J-equation.

Building on previous works of Weinkove, Song-Weinkove in 2008 showed that the existence of a solution to the J-equation is equivalent to the existence of a Kähler metric $\chi \in [\omega]$ with

$$c\chi^{n-1} - (n-1)\chi^{n-2} \wedge \alpha$$

in the sense of $(n-1, n-1)$ forms.

The J-equation was studied by Weinkove and Song-Weinkove. Recently, Lejmi-Székelyhidi introduced a notion of K-stability and made the following conjecture:

> *If V is a p-dimensional irreducible subvariety of X, define*
>
> $$c_V = \frac{p \int_V \omega^{p-1} \wedge \alpha}{\int_V \omega^p}.$$
>
> *Then there exists a solution to the J-equation if and only if $c_V > c_X$ for all p-dimensional proper irreducible subvariety V of X with $p > 0$.*

Collins-Székelyhidi proved this conjecture in the case of toric Kähler manifolds.

Hence we made the following conjecture:

For every irreducible subvariety V of X, define

$$\Theta_V := \operatorname{Arg} \int_V (\alpha + i\omega)^{\dim V}$$

where we defined $\operatorname{Arg} \int_V \alpha^{\dim V} = 0$. We conjecture that if $\Theta_X > (n-2)\frac{\pi}{2}$, then there exists a solution to the deformed Hermitian- Yang-Mills equation

$$\operatorname{Im}(e^{-i\Theta}(\alpha + i\omega)^n) = 0$$

if and only if, for all irreducible analytic subvariety V of X we have

$$\Theta_V > \Theta_X - \operatorname{codim}(V)\frac{\pi}{2}.$$

We proved that stability is necessary for the existence of a solution. We verified the conjecture for Kähler surfaces. There is preliminary evidence that this stability condition can be used to define a Bridgeland stability condition on the derived category of coherent sheaves and hence fit the general picture proposed in Kontsevich-Soibelman. We hope to extend these ideas to higher rank bundles and to formulate precisely the role of Bridgeland stability.

Suppose now that $\dim_{\mathbb{C}} X = 3$. One important part of a Bridgeland stability condition is the assignment to (semi)-stable objects of a phase $\theta(L) \in \mathbb{R}$, which should be determined by numerical data. We expect that, in the above setting, we should have

$$\mathrm{Im}(\alpha + i\omega)^3 = \tan(\theta(L))\mathrm{Re}(\alpha + i\omega)^3$$

but this only determines $\theta(L)$ mod 2π. By simple algebra, we can locally write

$$\theta(L) := \theta(h) = \sum_{i=1}^{3} \arctan(\lambda_i)$$

where λ_i are the eigenvalues of the relative endomorphism $\alpha^{-1}\omega$. But this is not obviously numerical.

Proposition. If L has a solution of dHYM, and pointwise we have $\theta(L) \in 2[\pi/2; 3\pi/2)$, then $\theta(L)$ is determined numerically by the path

$$Z(t) = -\int_X e^{-t\sqrt{-1}\alpha} ch(L)$$

where $t \in [1, \infty)$, and $\mathrm{Arg}_{\mathbb{R}} Z(\infty) = \mathrm{Arg}_{p.v.} Z(\infty) = -\pi/2$. Specifically, we have $\theta(L) = \mathrm{Arg}_{\mathbb{R}} Z(1) + \pi/2$, and $Z(1)$ lies in the upper half-plane.

The statement of this proposition contains a hidden Chern number inequality. Namely, $Z(t)$ must never pass through the origin in order for the angle to be well-defined.

We have

$$Z(t) = -\int_X e^{-t\sqrt{-1}\alpha} ch(L)$$
$$= \left(\frac{t^2}{2}\alpha^2 . ch_1(L) - ch_3(L)\right) + \sqrt{-1}\left(t\alpha . ch_2(L) - \frac{t^3}{6}\alpha^3\right).$$

If this path passes through the origin at some time $T \geq 1$, then since the real part is zero we have

$$T^2 = \frac{2ch_3(L)}{ch_1(L).\alpha^2}$$

and plugging this into the equation for the imaginary part being zero we get

$$ch_3(L)\alpha^3 - 3(ch_2(L).\alpha)(ch_1(L).\alpha^2) = 0.$$

Proposition. If L has a solution of dHYM, and pointwise we have $\theta(L) \in [\pi/2, 3\pi/2)$, then we have the Chern number inequality

$$(\alpha^3)(ch_3(L)) < 3(ch_2(L).\alpha)(ch_1(L).\alpha^2)$$

What about stability conditions? If $C \subset X$ is a surface, then some simple linear algebra shows that the metric $h|_C$ on the bundle $L|_C$ has

$$\theta(h_C) \geq 0.$$

From the above discussion, and the Hodge index theorem we get

If $C \subset X$ is a curve or a surface, and L is as above, the path

$$Z(t) = \int_X e^{-t\sqrt{-1}\alpha} ch(L \otimes \mathcal{O}_C)$$

never passes through the origin, and hence the angle $\mathrm{Arg}_{\mathbb{R}} Z(1) \in \mathbb{R}$ is well-defined, and we have

$$\mathrm{Arg}_{\mathbb{R}} Z(L \otimes \mathcal{O}_C) > \mathrm{Arg}_{\mathbb{R}} Z(L).$$

This statement corresponds to a Bridgeland-type stability inequality coming from the exact sequence

$$0 \to L \otimes \mathcal{I}_C \to L \to L \otimes \mathcal{O}_C \to 0.$$

Thank you!