OPEN GROMOV-WITTEN INVARIANTS AND SUPERPOTENTIALS FOR SEMI-FANO TORIC SURFACES

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Abstract. In this paper, we compute the open Gromov-Witten invariants for every compact toric surface X which is semi-Fano (i.e. the anticanonical line bundle K_X is nef). Unlike the Fano case, this involves non-trivial obstructions in the corresponding moduli problem. As a consequence, an explicit formula for the Lagrangian Floer superpotential W is obtained, which in turn gives an explicit presentation of the small quantum cohomology ring of X. We also provide a computational verification of the conjectural ring isomorphism between the small quantum cohomology of X and the Jacobian ring of W.

Contents

1. Introduction

Let X be a compact toric manifold of complex dimension n. The mirror for X is given by a so-called *Landau-Ginzburg model* which consists of a noncompact complex n-dimensional manifold $\check{X}$ together with a holomorphic function $W : \check{X} \to \mathbb{C}$ called the *superpotential*. From the perspective of the Strominger-Yau-Zaslow conjecture [21], the manifold $\check{X}$, which is a bounded domain in $(\mathbb{C}^*)^n$, is given by

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taking the fiberwise torus dual of the moment map restricted to the complement of toric divisors in X [1, 5]. Furthermore, as shown by the work of Cho-Oh [6] and Fukaya-Oh-Ohta-Ono [9, 10], the superpotential W comes from Lagrangian Floer theory for the moment map fibers. More precisely, the coefficients of W are generating functions of genus zero *open Gromov-Witten invariants* which are virtual counting of Maslov index two holomorphic stable disks bounded by the Lagrangian torus fibers of the moment map.

In this paper, we investigate the computation of open Gromov-Witten invariants and superpotentials for a class of toric surfaces. Similar problems have been studied by various authors. In [6], Cho and Oh classified all embedded holomorphic disks in a compact toric manifold X with boundary lying in Lagrangian torus fibers. In case X is Fano, since the moduli spaces of holomorphic stable disks do not contain any bubbling configurations, Cho-Oh's results imply that all open Gromov-Witten invariants are equal to one, and hence they obtained an explicit formula for the superpotential W, which agrees with the one predicted by Hori and Vafa [14]. For non-Fano toric manifolds, however, bubbling configurations *do* contribute to open Gromov-Witten invariants (Fukaya-Oh-Ohta-Ono [9, 10]), so there are "quantum corrections" to Hori-Vafa's formula for the superpotential. In this situation, the obstruction theory is non-trivial and this makes explicit computations of open Gromov-Witten invariants much more difficult than the Fano case.

There are very few known computations for non-Fano toric manifolds: In [2], by using toric degenerations and studying the wall-crossing phenomenon for disk counting, Auroux was able to compute open Gromov-Witten invariants and write down explicitly the superpotentials for the Hirzebruch surfaces $\mathbb{F}_2$ and $\mathbb{F}_3$. Later, Fukaya, Oh, Ohta and Ono [11], again making use of toric degenerations, gave another proof for the example $\mathbb{F}_2$. More recently, the first author [4] established a formula relating open and closed Gromov-Witten invariants. Applying this formula, the open Gromov-Witten invariants for all toric Calabi-Yau surfaces and certain toric Calabi-Yau threefolds (including the total space of the canonical line bundles of any toric Del Pezzo surface) were computed in the joint works [17, 18] of the second author with Leung and Wu.

The purpose of this paper is to compute all genus zero open Gromov-Witten invariants and hence obtain an explicit formula for the superpotential for any semi-Fano toric surface X. We call a compact toric surface *semi-Fano* if its anti-canonical bundle is nef (or equivalently, if every toric prime divisor has self-intersection at least -2). To state our main result, let $\mathbf{T}$ be a Lagrangian torus fiber of the moment map for a semi-Fano toric surface X. Let $b \in \pi_2(X, \mathbf{T})$ be a relative homotopy class of Maslov index two.

Definition 1.1. *We call a Maslov index two class $b \in \pi_2(X, \mathbf{T})$* admissible *if and only if b is of the form*

$$b = \beta + \sum_{k=-m}^{n} s_k D_k,$$

where

(1) $\beta \in \pi_2(X, \mathbf{T})$ is a class represented by an embedded holomorphic disk $D^2 \subset X$ with boundary $\partial D^2 \subset \mathbf{T}$ which intersects a unique irreducible toric divisor D_0 with multiplicity one; such a class is called a basic disk class*;*

(2) D_k's are toric prime divisors which form a chain of (-2)-curves in X;

(3) m, n are non-negative integers, and both $s_0 \geq s_1 \geq s_2 \geq \cdots \geq s_n$ and $s_0 \geq s_{-1} \geq s_{-2} \geq \cdots \geq s_{-m}$ are nondecreasing integer sequences with $|s_k - s_{k+1}| = 0$ or 1 for each k, and the last term of each sequence is not greater than one.

We can now state our main result:

Theorem 1.2. *Let X be a compact semi-Fano toric surface. Let $b \in \pi_2(X, \mathbf{T})$ be a relative homotopy class of Maslov index two. Then the genus zero one-pointed open Gromov-Witten invariant n_b is either one or zero according to whether b is admissible or not. As a consequence, the superpotential for the mirror of X is given explicitly by*

$$W = \sum_{\substack{b \text{ admissible} \\ b \in \pi_2(X, \mathbf{T})}} Z_b,$$

where Z_b is an explicit holomorphic function (in fact a monomial) on $\check{X}$ defined by Equation (2).

The proof of Theorem 1.2 is based on the comparison between open and closed Gromov-Witten invariants in [4] (and its generalization in [17]) and the computation on local Gromov-Witten invariants obtained by Bryan and Leung [3]. The idea is similar to the proof of Theorem 4.2 in [18].

As a consequence of Theorem 1.2, we can write down an explicit formula for the superpotential for a semi-Fano toric surface (see the tables in Appendix A). We apply this to verify the conjectural ring isomorphism between the small quantum cohomology $QH^*(X)$ of a semi-Fano toric surface X and the Jacobian ring $Jac(W)$ of its superpotential W via direct computations.

Corollary 1.3. *Let X be a compact semi-Fano toric surface, and W its superpotential. Then there is a natural ring isomorphism*

$$QH^*(X) \cong \mathrm{Jac}(W). \tag{1}$$

In a recent work [7], Fukaya, Oh, Ohta and Ono proved a much stronger result than Corollary 1.3: For every compact toric manifold X and $\mathbf{b} \in H_*(X)$, there is a ring isomorphism

$$QH^*_{\mathbf{b}}(X) \cong \mathrm{Jac}(W_{\mathbf{b}})$$

where $QH^*_{\mathbf{b}}(X)$ is the *big* quantum cohomology ring and $W_{\mathbf{b}}$ is the superpotential bulk-deformed by $\mathbf{b}$. We remark that their proof uses their big machinery of Lagrangian Floer theory and does not involve explicit computations of open Gromov-Witten invariants.

On the other hand, via the isomorphism (1), our explicit formula for the supepotential W leads to an explicit presentation of the small quantum cohomology ring $QH^*(X)$ for a semi-Fano toric surface X. Indeed we can achieve more:

Corollary 1.4. *Let X be a compact semi-Fano toric surface and $\mathbf{b} = D + aX$ be a linear combination of toric cycles, where D is a toric divisor and $a \in \mathbb{C}$. Then the bulk-deformed superpotential is*

$$W_{\mathbf{b}} = a + \sum_{\beta \text{ admissible}} \exp(\langle \beta, D \rangle) Z_\beta.$$

Then by using the results of Fukaya, Oh, Ohta and Ono mentioned above, an explicit ring presentation of $QH^*_{\mathbf{b}}(X)$ can be obtained for $\mathbf{b} \in H_2(X) \oplus H_4(X)$.

Remark 1.5. *Fukaya-Oh-Ohta-Ono* [9, 10, 7] *used a Novikov ring instead of* $\mathbb{C}$ *as the coefficient ring, which is more appropriate in general. Throughout this paper we stick to the tradition of using* $\mathbb{C}$ *as the coefficient ring because it turns out that the superpotential* W*, which is a priori a formal power series, is a finite sum for any toric semi-Fano surface* X*. All the statements in this paper remains unchanged if* $\mathbb{C}$ *is replaced by a Novikov ring.*

The rest of this paper is arranged as follows. Section 2 is a brief review on toric manifolds and their Landau-Ginzburg mirrors. In Section 3 we compute the open Gromov-Witten invariants for compact semi-Fano toric surfaces and prove Theorem 1.2. In Section 4, we outline our computational proof of the isomorphism $QH^*(X) \cong Jac(W)$ and demonstrate the explicit calculations by several examples. Corollary 1.4 is proved in Section 5. We end by some further discussions on bulk-deformation by points.

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2. Landau-Ginzburg models as mirrors for toric manifolds

The purpose of this section is to set up some notations and give a review on certain basic facts in toric geometry and mirror symmetry for toric manifolds that we will need in this paper.

2.1. **A quick review on toric manifolds.** Let $N \cong \mathbb{Z}^n$ be a lattice of rank n. For simplicity we will always use the notation $N_R := N \otimes R$ for a $\mathbb{Z}$-module R. Let X_Σ be a compact complex toric n-fold X_Σ defined by a fan Σ supported in $N_{\mathbb{R}}$. X_Σ admits an action by the complex torus $N_{\mathbb{C}}/N \cong (\mathbb{C}^\times)^n$, whence its name "toric manifold". There is an open orbit in X_Σ on which $N_{\mathbb{C}}/N$ acts freely, and by abuse of notation we shall also denote this orbit by $N_{\mathbb{C}}/N \subset X_\Sigma$.

We denote by M the dual lattice of N. Every lattice point $\nu \in M$ gives a nowhere-zero holomorphic function $\exp(\nu, \cdot) : N_{\mathbb{C}}/N \to \mathbb{C}$ which extends to a meromorphic function on X_Σ. Its zero and pole sets define a toric divisor which is linearly equivalent to the zero divisor. (By a toric divisor X_Σ we mean a divisor $D \subset X$ which is invariant under the action of $N_{\mathbb{C}}/N$.)

If we further equip X_Σ with a toric Kähler form ω, then the action of $N_{\mathbb{R}}/N$ on X_Σ induces a moment map

$$\mu_0 : X_\Sigma \to M_{\mathbb{R}},$$

whose image is a polytope $P \subset M_{\mathbb{R}}$ defined by a system of inequalities

$$(v_i, \cdot) \geq c_i, \ i = 1, \ldots, d,$$

where v_i are all primitive generators of rays of Σ, and $c_i \in \mathbb{R}$ are some suitable constants.

P admits a natural stratification by its faces. Each codimension-one face $T_i \subset P$ which is normal to $v_i \in N$ gives an irreducible toric divisor $D_i = \mu_0^{-1}(T_i) \subset X_\Sigma$ for $i = 1, \ldots, d$, and all other toric divisors are generated by $\{D_i\}_{i=1}^d$. For example, the anti-canonical divisor of X_Σ is given by $\sum_{i=1}^d D_i$.

2.2. **Gromov-Witten invariants.** First we recall the definition of closed Gromov-Witten invariants for a projective manifold. Let $\beta \in H_2(X, \mathbb{Z})$ be a 2-cycle in a smooth projective variety X. Let $\overline{M}_{g,k}(X, \beta)$ be the moduli space of stable maps

$$f : (C; x_1, \cdots x_k) \longrightarrow X,$$

where C is a genus g nodal curve with k marked points and $f_*[C] = \beta$. Let $\mathrm{ev}_i : \overline{M}_{g,k}(X, \beta) \to X$ $(i = 1, \ldots, k)$ be the evaluation maps $f \mapsto f(x_i)$.

Definition 2.1. *Given cohomology classes $\gamma_i \in H^*(X)$, $1 \leq i \leq k$, the closed Gromov-Witten invariant $GW_{g,k}^{X,\beta}(\gamma_1, \cdots, \gamma_k)$ is defined by*

$$GW_{g,k}^{X,\beta}(\gamma_1, \cdots, \gamma_k) := \int_{[\overline{M}_{g,k}(X,\beta)]^{\mathrm{vir}}} \prod_{i=1}^{k} \mathrm{ev}_i^*(\gamma_i),$$

where $[\overline{M}_{g,k}(X, \beta)]^{\mathrm{vir}}$ denotes the virtual fundamental class of the moduli space $\overline{M}_{g,k}(X, \beta)$.

For toric manifolds, Fukaya-Oh-Ohta-Ono [9] defined open Gromov-Witten invariants as follows. Let $X = X_\Sigma$ be a toric manifold defined by a fan Σ. For a moment map Lagrangian torus fiber $\mathbf{T} \subset X$, let $\pi_2(X, \mathbf{T})$ be the group of homotopy classes of maps

$$u : (\Delta, \partial\Delta) \longrightarrow (X, \mathbf{T})$$

where $\Delta := \{z \in \mathbb{C} : |z| \leq 1\}$ denotes the standard closed unit disk in $\mathbb{C}$. Then $\pi_2(X, \mathbf{T})$ is generated by the *basic disk classes* $\beta_i \in \pi_2(X, \mathbf{T})$ which correspond to the primitive generators $v_i \in N$ of rays in Σ for $i = 1, \ldots, d$. The two most important classical symplectic invariants associated to $\beta \in \pi_2(X, \mathbf{T})$ are its symplectic area $\int_\beta \omega$ and its Maslov index $\mu(\beta)$.

Now for $\beta \in \pi_2(X, \mathbf{T})$, let $\overline{M}_k(\mathbf{T}, \beta)$ be the moduli space of stable maps from a bordered Riemann surface of genus zero with k boundary marked points respecting the cyclic order of the boundary in the class β. Notice that the bordered Riemann surface could have disk or sphere bubbles. It is known that $\overline{M}_k(\mathbf{T}, \beta)$ has expected dimension $n + \mu(\beta) + k - 3$. Let $[\overline{M}_k(\mathbf{T}, \beta)]^{\mathrm{vir}}$ be its virtual fundamental chain constructed in [9]. We let

$$\mathrm{ev}_i : \overline{M}_k(\mathbf{T}, \beta) \longrightarrow \mathbf{T}$$

be the evaluation maps defined by $\mathrm{ev}_i([u; p_0, \ldots, p_{k-1}]) = u(p_i)$ for $0 \leq i \leq k-1$.

Consider the case $k = 1$ and $\mu(\beta) = 2$. Note that the virtual dimension of $\overline{M}_1(\mathbf{T}, \beta)$ is equal to $\dim \mathbf{T} = n$ if and only if $\mu(\beta) = 2$. Since the minimal Maslov index is two, the virtual fundamental chain $[\overline{M}_k(\mathbf{T}, \beta)]^{\mathrm{vir}}$ becomes a cycle when $\mu(\beta) = 2$. Hence we can define:

Definition 2.2 ([9])**.** *Given a Lagrangian torus fiber $\mathbf{T} \subset X$ and $\beta \in \pi_2(X, \mathbf{T})$, the genus zero one-pointed open Gromov-Witten invariant n_β is defined as*

$$n_\beta := \mathrm{ev}_{0*}([\overline{M}_1(\mathbf{T}, \beta)]^{\mathrm{vir}}) \in H_n(\mathbf{T}; \mathbb{Q}) \cong \mathbb{Q}.$$

It was shown in [9] that the number n_β is independent of the perturbations used to define the virtual fundamental cycle and hence the above indeed defines an invariant. One should view the invariant $n_\beta \in \mathbb{Q}$ as the virtual number of holomorphic stable disks representing the class β such that their boundaries pass through a fixed generic point in $\mathbf{T}$.

Let us consider the situation where $X = X_\Sigma$ is semi-Fano, i.e. with nef anti-canonical line bundle. By the classification result of Cho-Oh [6], a class $\beta \in \pi_2(X, \mathbf{T})$ represented by a holomorphic stable disk must be of the form $\beta = \beta' + \alpha$, where β' is disk class represented by several embedded holomorphic disks and $\alpha \in H_2(X)$ is represented by a rational curve. The Maslov index of β' is $2k$ where k is the number of disk components, and the first Chern number $c_1(\alpha) := \int_\alpha c_1(X)$ of α must be non-negative since X is semi-Fano. This shows that any holomorphic stable disk with Maslov index two must be of the form $\beta_i + \alpha$ where β_i is a basic disk class and $\alpha \in H_2(X)$ is an effective curve class with first Chern number $c_1(\alpha) = 0$.

2.3. **The LG mirror of toric manifolds.** The mirror of a toric manifold $X = X_\Sigma$ is a Landau-Ginzburg model $(\check{X}, W)$, which consists of a noncompact complex manifold $\check{X}$ together with a holomorphic function $W : \check{X} \to \mathbb{C}$ called the superpotential. From the perspective of Lagrangian Floer theory, the superpotential W comes from the obstruction classes for Lagrangian torus fibers, and can be written down in terms of Kähler parameters and open Gromov-Witten invariants of X [6, 9, 10, 1]. The following is a brief review of this procedure from the SYZ viewpoint. See [5] for more details.

First of all, we recall that the *semi-flat mirror* of X is

$$\check{X}_0 := \left\{(\mathbf{T}_r, \nabla) : r \in P^{\text{int}}, \nabla \text{ is a flat } U(1)\text{-connection on } \mathbf{T}_r\right\},$$

where $\mathbf{T}_r \subset X$ denotes the moment-map fiber over r and P^{int} denotes the interior of P. It is well known that $\check{X}_0$ can be equipped with the so-called semi-flat complex structure, making it into a complex manifold [19]. In this toric case, $\check{X}_0$ is simply $P^{\text{int}} \times M_{\mathbb{R}}/M$ equipped with the standard complex structure.

Let Λ^* be the lattice bundle over B_0 whose fiber at $r \in P^{\text{int}}$ is $\Lambda^*_r = \pi_1(\mathbf{T}_r)$. For each $\lambda \in \Lambda^*$, we may consider the following weighted count of stable holomorphic disks:

$$\mathcal{F}(\lambda) := \sum_{\partial\beta = \lambda} n_\beta \exp\left(-\int_\beta \omega\right).$$

This defines a function $\mathcal{F} : \Lambda^* \to \mathbb{R}$. Applying fiberwise Fourier transform on $\mathcal{F}$, we obtain the superpotential

$$\begin{aligned} W &: \check{X}_0 \to \mathbb{C}, \\ W(\mathbf{T}_r, \nabla) &= \sum_{\beta \in \pi_2(X, \mathbf{T}_r)} n_\beta \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta). \end{aligned}$$

Notice that the above expression can be an infinite series. Nevertheless we will see that for semi-Fano toric surfaces, this is just a finite sum and hence there are no convergence issues for all the examples we consider in this paper. In general, assuming convergence, W is a holomorphic function on $\check{X}_0$.

For $\beta \in \pi_2(X, \mathbf{T}_r)$, we define a function $Z_\beta : \check{X}_0 \to \mathbb{C}$ by

$$Z_\beta(\mathbf{T}_r, \nabla) := \exp\left(-\int_\beta \omega\right) \mathrm{Hol}_\nabla(\partial\beta), \tag{2}$$

so that the superpotential can be written in the form $W = \sum_{\beta \in \pi_2(X,T_r)} n_\beta Z_\beta$. Note that Z_β is holomorphic and in fact it is a monomial in terms of the standard coordinates on $M_{\mathbb{C}^*}$.

It is already known by Cho-Oh [6] that $n_{\beta_i} = 1$ for all the basic disk classes β_i. When X is semi-Fano, as we have seen above, the moduli space $\overline{M}_1(\mathbf{T}, \beta)$ is non-empty only when $\beta = \beta_i + \alpha$ for some $i = 1, \ldots, d$ and $\alpha \in H_2(X)$ represented by a rational curve of Chern number zero. Thus we may write

$$W = W_0 + \sum_{i=1}^{d} \sum_{\alpha \neq 0, c_1(\alpha)=0} n_{\beta_i+\alpha} Z_{\beta_i+\alpha},$$

where $W_0 = \sum_{i=1}^d Z_{\beta_i}$. In general it is very hard to compute $n_{\beta_i+\alpha}$ starting from the definition. In the following section, we will give a method to compute these invariants when X is a semi-Fano toric surface.

3. Disk counting and GW invariants

3.1. **A fact on toric surfaces.** In this subsection we discuss some elementary results on toric surfaces, which will be needed in the proof of Theorem 1.2. These are probably well-known facts among experts; but for convenience of the reader, we include their proofs here.

We start with the well-known formula for the self-intersection number of a toric prime divisor in a compact toric surface. Let $X = X_\Sigma$ be a smooth toric surface defined by a fan Σ in $\mathbb{Z}^2$. Suppose $D \subset X$ is a compact toric prime divisor. Then D corresponds to a ray $\tau \in \Sigma$, so that $\tau = \sigma^- \cap \sigma^+$ for two 2-dimensional cones $\sigma^-, \sigma^+ \in \Sigma$. (See Figure 1).

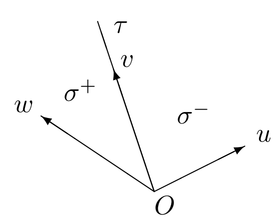

Figure 1. Cones corresponding to a compact divisor.

Let τ be generated by $v \in \mathbb{Z}^2$, σ^- be generated by u, v and σ^+ be generated by v, w such that u, v, w are placed in a counterclockwise fashion. Then the self-intersection of D is given by

$$D^2 = -\begin{vmatrix} u_1 & w_1 \\ u_2 & w_2 \end{vmatrix}$$

where

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

Proposition 3.1. *Let $D = \cup_{i=1}^l D_i$ be a connected union of compact toric prime divisors with $D_i^2 = -2$, and τ_i be the ray corresponding to D_i. Suppose $\sigma_i \in \Sigma$ are 2-dimensional cones so that $\tau_i = \sigma_{i-1} \cap \sigma_i$. Then the cone $\cup_{i=0}^n \sigma_i$ is strictly convex.*

Proof. Suppose τ_i is generated by $v_i \in \mathbb{Z}^2$. Without loss of generality, we can assume v_i are labeled in a counterclockwise order as vectors in $\mathbb{R}^2$. We further let σ_0 be generated by v_0, v_1; and σ_n be generated by v_n, v_{n+1}.

Let
$$v_i = \begin{pmatrix} a_i \\ b_i \end{pmatrix}.$$
Since D_i is a (-2)-curve, we have
$$\begin{vmatrix} a_{i-1} & a_{i+1} \\ b_{i-1} & b_{i+1} \end{vmatrix} = 2.$$
In other words, the area of the triangle spanned by v_{i-1} and v_{i+1} is 1.

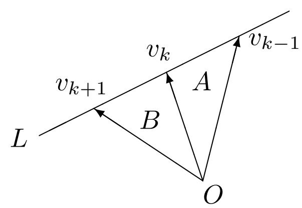

FIGURE 2. -2 toric divisors.

On the other hand, let A be the triangle spanned by vectors v_{i-1} and v_i; and let B be the triangle spanned by v_i and v_{i+1}. Since X is smooth, the areas of A and B are $\frac{1}{2}$. Now because the sum of areas of A and B is 1, which is equal to the area of the triangle spanned by v_{i-1} and v_{i+1}, we know the heads of the vectors v_{i-1}, v_i and v_{i+1} are on the same line L. Moreover,
$$v_i = \frac{1}{2}(v_{i-1} + v_{i+1}).$$
Now since the heads of all vectors v_i are on the same line, the cone $\cup_{i=0}^n \sigma_i$ must be strictly convex. $\square$

3.2. **Proof of Theorem 1.2.** We are now in a position to give a proof of the main result (Theorem 1.2) of this paper.

Let X be a compact semi-Fano toric surface. Let $D_1, \cdots, D_d$ denote the toric prime divisors of X. Let $\mathbf{T}$ be a Lagrangian torus fiber and let $\beta_i \in \pi_2(X, \mathbf{T})$ be the basic disk class such that $\beta_i \cdot D_j = \delta_{ij}$. Recall that, given any $b \in \pi_2(X, \mathbf{T})$ of Maslov index two, the moduli space $\overline{M}_1(\mathbf{T}, b)$ of stable maps from bordered Riemann surfaces of genus zero with one boundary marked point to X in the class b is empty unless $b = \beta_i$, or $b = \beta_i + \alpha$ for some $i \in \{1, \ldots, d\}$ and $\alpha \in H_2(X, \mathbb{Z})$ with $c_1(\alpha) = 0$. Moreover, such an α must be of the form $\alpha = \sum s_k D_k$ where all D_k have self-intersection -2.

Our goal is to compute the open Gromov-Witten invariant n_b for all Maslov index two classes $b \in \pi_2(X, \mathbf{T})$. To state the result, we need the following definitions.

Definition 3.2. *Let $m_1, m_2 \in \mathbb{Z}$. We call a sequence $\{s_k : m_1 \leq k \leq m_2\}$ admissible with center l if each s_k is a positive integer, and*

(1) $s_i \leq s_{i+1} \leq s_i + 1$ when $i < l$;
(2) $s_i \geq s_{i+1} \geq s_i - 1$ when $i \geq l$;
(3) $s_{m_1}, s_{m_2} \leq 1$.

For any toric prime divisor D_i with self-intersection -2, we have a maximal chain $D_i^{\max}$ of compact toric (-2)-divisors containing D_i. Given a sequence $\{s_k\}$, we have an induced sequence $\{\tilde{s}_k\}$ with respect to D_i, defined as $\tilde{s}_j = s_j$ if $D_j \subset D_i^{\max}$ and $s_j = 0$ otherwise.

Definition 3.3. *Let $b = \beta_i + \alpha$ with $\alpha = \sum s_k D_k$. We say b is admissible if $D_i^2 = -2$ and the sequence $\{s_k\}$ is identical to its induced sequence with respect to D_i, and $\{s_k\}$ is admissible with center i.*

To prove Theorem 1.2, we recall the computations of local Gromov-Witten invariants for a configuration of $\mathbf{P}^1$'s in a complex surface which was obtained by Bryan and Leung in [3] as follows. Let $L(n)$ be a genus 0 nodal curve consisting of a linear chain of $2n+1$ smooth components $L_{-n}, \cdots, L_n$ with an additional smooth component L_* meeting L_0. So we have $L_n \cap L_m = \emptyset$ unless $|n-m| = 1$ and $L_* \cap L_n = \emptyset$ unless $n = 0$. It was shown in [3] that $L(n)$ can be embedded into a smooth surface S so that all L_i are (-2)-curves and L_* is a (-1)-curve, where S can be taken as a certain blowup of $\mathbf{P}^2$ along points.

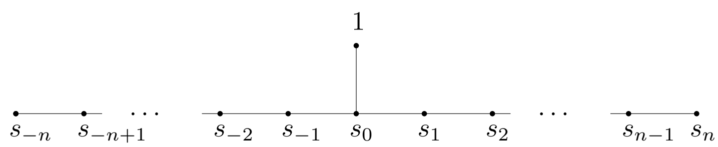

FIGURE 3. The graph of $L(n)$.

The local Gromov-Witten invariants of $L(n)$ is well-defined, at least for curve classes

$$L_* + \sum_{k=-n}^{n} s_k L_k, \quad s_k \geq 0.$$

Theorem 3.4. [3] *The genus zero local Gromov-Witten invariants $N(s_k)$ of $L(n)$ for classes $L_* + \sum_{k=-n}^{n} s_k L_k$ is given by*

$$N(s_k) = \begin{cases} 1 & \text{if } \{s_k\} \text{ is admissible with center } 0. \\ 0 & \text{otherwise.} \end{cases}$$

We remark that here admissible with center 0 is an equivalent term for 1-admissible used in [3].

Proof of Theorem 1.2. Given a semi-Fano toric surface X defined by a fan Σ, we would like to compute the open Gromov-Witten invariant n_b for $b \in \pi_2(X, \mathbf{T})$. First of all, by [6, 9], n_b is non-zero only when $b = \beta_i + \alpha$ for some i and $\alpha \in H_2(X, \mathbb{Z})$ represented by rational curves with $c_1(\alpha) = 0$. It is already known that $n_b = 1$ when $\alpha = 0$, so it suffices to consider $\alpha \neq 0$.

Suppose $n_{\beta_i+\alpha} \neq 0$ and $\alpha \neq 0$. Then D_i must have self-intersection -2, and α must be of the form $\alpha = \sum_{k \in I} s_k D_k$, where I is the index set containing all the natural numbers k such that $D_k \subset D_i^{\max}$, and $s_i \neq 0$. We want to show that the sequence $\{s_k\}$ is admissible, and in such cases $n_b = 1$.

This is done by equating the open Gromov-Witten invariant n_b to a closed Gromov-Witten invariant of yet another toric manifold Y, which is a toric modification of X. The modification is constructed as follows. Let v_i be the primitive generator of the ray of Σ corresponding to D_i, and let Σ_1 be the refinement of Σ by adding the ray generated by $v_\infty := -v_i$ (and then completing it into a convex fan). In general the corresponding toric variety X_{Σ_1} may not be smooth. If this is the case, then we take a toric desingularization Y of X_{Σ_1} by adding rays which are adjacent to v_∞. By abuse of notations we still denote the divisors in Y corresponding to v_l's by D_l, and $\alpha = \sum_{k\in I} s_k D_k$ is regarded as a homology class in Y. We remark that the above procedure does nothing if the ray generated by v_∞ is already in Σ.

Notice that in Σ, the ray generated by v_∞ cannot be adjacent to those generated by v_k for $k \in I$ (I is the index set introduced above) by using the fact that D_k's have self-intersection (-2). Then the newly added rays are not adjacent to any v_k for $k \in I$, and thus each $D_k \subset Y$ for $k \in I$ still has self-intersection number (-2). Let $f \in H_2(Y)$ be the fiber class, that is, $f = \beta_i + \beta_\infty$, where β_∞ is the disk class corresponding to v_∞.

By Theorem 1.1 in [4] and its generalization in [17], we have the following equality between open and closed Gromov-Witten invariants:

$$n_b = GW_{0,1}^{Y,\alpha+f}([\mathrm{pt}]).$$

The proof was by showing that the open moduli space $\overline{M}_1^{\mathrm{ev}=p}(X,b)$ and the closed moduli space $\overline{M}_1^{\mathrm{ev}=p}(Y,f+\alpha)$ are isomorphic as Kuranishi spaces. We refer the reader to [4, 17] for details.

Next we identify $GW_{0,1}^{Y,\alpha+f}([\mathrm{pt}])$ with the local Gromov-Witten invariant of a configuration of $\mathbf{P}^1$'s. Let $\tilde{Y}$ be the blowup of Y at a generic point p. Then, by the result of Hu [15] and Gathmann [12], which relates Gromov-Witten invariants of blowups along points, we know that the Gromov-Witten invariant of Y for a class γ with one point constraint is equal to that of $\tilde{Y}$ for the strict transform of γ without this point constraint. More precisely, we have

$$GW_{0,1}^{Y,\alpha+f}([\mathrm{pt}]) = GW_{0,0}^{\tilde{Y},\alpha+f'},$$

where f' is the strict transform f, which is the class of a (-1)-curve.

Because $\alpha = \sum s_k D_k$, with all D_k have self-intersection -2, it is easy to see that every curve in $\alpha + f'$ is a tree of $\mathbf{P}^1$'s, with the same configuration as $L(n)$, up to a relabeling of its index. Therefore, $GW_{0,0}^{\tilde{Y},\alpha+f'}$ is precisely the local Gromov-Witten invariant of $L(n)$. Theorem 1.2 now follows from Theorem 3.4. □

Theorem 1.2 allows us to explicitly compute the superpotential for any compact semi-Fano toric surface. Since these surfaces can be completely classified (there are totally 16 such surfaces, 5 of which are Fano), we can give explicit formulas for all their superpotentials; a list of which is given in the appendix.[1]

4. Small quantum cohomology and Jacobian ring

For a toric Fano manifold X, the map

$$\psi : QH^*(X) \to Jac(W),\ D_i \mapsto Z_{\beta_i},$$

[1] In a very recent work Fukaya-Oh-Ohta-Ono [8], our explicit formula for the superpotential W of the semi-Fano toric surface X_{11} in the table was used in their proof of the existence of a continuum of mutually disjoint non-displaceable Lagrangian tori in a cubic surface.

gives a canonical ring isomorphism between the small quantum cohomology $QH^*(X)$ of X and the Jacobian ring $Jac(W)$ of the superpotential W [5, 9]. Recall that the Jacobian ring of W is defined as

$$Jac(W) = \mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]/\langle \partial_1 W, \ldots, \partial_n W \rangle,$$

where ∂_j denotes $z_j \frac{\partial}{\partial z_j}$ and $n = \dim X$. In the non-Fano case, it is expected that we still have an isomorphism $QH^*(X) \cong Jac(W)$,[2] but the map $\psi : QH^*(X) \to Jac(W)$ needs to be modified by quantum corrections.

In the following, we briefly recall the definition of the corrected map following Fukaya, Oh, Ohta and Ono [9, 10]. As before, X is a compact toric manifold and $\mathbf{T}$ is a Lagrangian torus fiber. Consider the moduli space $\overline{M}_{k,l}(\mathbf{T}, \beta)$ of stable maps from genus 0 bordered Riemann surfaces to (X, L) with k boundary marked points and l interior marked point in the class β. We have evaluation maps

$$\mathrm{ev}^{\mathrm{int}} : \overline{M}_{k,l}(\mathbf{T}, \beta) \to X^l, \ [u; p_0, p_1, \ldots, p_{k-1}; z_1, \ldots, z_l] \mapsto (u(z_1), \ldots, u(z_l)),$$

and

$$\mathrm{ev}_i : \overline{M}_{k,1}(\mathbf{T}, \beta) \to \mathbf{T}, \ [u; p_0, p_1, \ldots, p_{k-1}; z] \mapsto u(p_i),$$

$i = 0, 1, \ldots, k-1$, at the interior and boundary marked points respectively.

Let $V_1, \ldots, V_l \subset X$ be toric subvarieties. Consider the fiber product

$$\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l) := \overline{M}_{1,l}(\mathbf{T}, \beta)_{\mathrm{ev}^{\mathrm{int}}} \times_{X^l} \left(\prod_{j=1}^{l} V_j \right).$$

More precisely, $\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l)$ is the set of all elements

$$([u; p_0; z_1, \ldots, z_l], x_1, \ldots, x_l) \in \overline{M}_{1,l}(\mathbf{T}, \beta) \times \prod_{j=1}^{l} V_j$$

such that $u(z_1, \ldots, z_l) = (x_1, \ldots, x_l)$. The expected dimension of $\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l)$ is given by $n + \mu(\beta) + 2l - 2 - \sum_{j=1}^{l} \mathrm{codim}_{\mathbb{R}}(V_j)$.

Definition 4.1 ([10, 11])**.** *The genus zero open Gromov-Witten invariant* $n(\beta; V_1, \ldots, V_l)$ *is defined as*

$$n(\beta; V_1, \ldots, V_l) = \mathrm{ev}_{0*}([\overline{M}_{1,l}(\mathbf{T}, \beta; V_1, \ldots, V_l)]^{\mathrm{vir}}) \in \mathbb{Q}.$$

It is non-zero only when

$$\mu(\beta) = 2 - 2l + \sum_{j=1}^{l} \mathit{codim}_{\mathbb{R}}(V_j).$$

By Lemma 6.8 in [10], the number $n(\beta; V_1, \ldots, V_l) \in \mathbb{Q}$ is independent of the auxiliary perturbation data used to define $[\overline{M}_{1,1}(\mathbf{T}, \beta; V)]^{\mathrm{vir}}$ and hence gives an invariant. Definition 2.2 is the special case when $l = 0$.

Choose an additive basis $\{T_i = \mathrm{PD}[V_i]\}$ of $H^*(X, \mathbb{C})$ represented by the Poincaré duals of fundamental classes of toric subvarieties $V_i \subset X$.

---

[2]This is now proved in the recent work [7] of Fukaya, Oh, Ohta and Ono (as a special case of their main result).

Definition 4.2 ([10, 11])**.** *Define an additive map $\psi : QH^*(X) \to Jac(W)$ by setting*

$$\psi(T_i) = \sum_{\beta:\mu(\beta)=codim_{\mathbb{R}}(V_i)} n(\beta; V_i) Z_\beta,$$

and extending linearly.

Remark 4.3. *Fukaya, Oh, Ohta and Ono* [10] *also study the so-called potential function with bulk of a toric manifold X, by incorporating deformations of Floer cohomology by cycles on the ambient space X. (In contrast, the superpotential, or what Fukaya, Oh, Ohta and Ono called the potential function, W just encodes deformations of Floer cohomology by the cycles on L.) In the recent work* [7]*, they proved that the Jacobian ring of the potential function with bulk is canonically isomorphic to the big quantum cohomology ring of X. The map $\psi : QH^*(X) \to Jac(W)$ we discuss here is a special case of this isomorphism, when the bulk deformation is set to zero. We will also discuss the potential function with bulk in Section 5.*

Now, for the toric prime divisors $D_1, \ldots, D_d$, the map ψ is given by

$$D_i \mapsto \sum_{\beta:\mu(\beta)=2} n(\beta; D_i) Z_\beta.$$

A special case of Lemma 9.2 in [10] gives the following analogue of the divisor equation for open Gromov-Witten invariants.

Proposition 4.4 ([10])**.** *If D is a toric divisor, then we have the following equality*

$$n(\beta; D) = (D \cdot \beta) n_\beta.$$

Combining with our Theorem 1.2, we can compute the map $\psi : QH^*(X) \to Jac(W)$ on toric divisors for any compact semi-Fano toric surface. As an application, we outline a proof of Corollary 1.3 in the following.

To begin with, recall that the cohomology ring $H^*(X, \mathbb{C})$ of a compact toric manifold X is generated by the divisor classes $D_1, \ldots, D_d \in H^2(X, \mathbb{C})$. Moreover, a presentation of $H^*(X, \mathbb{C})$ is given by

$$H^*(X, \mathbb{C}) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}),$$

where $\mathcal{L}$ is the ideal generated by linear equivalences among divisors and $\mathcal{SR}$ is the Stanley-Reisner ideal generated by primitive relations.

By a result of Siebert and Tian [20], when X is semi-Fano, the small quantum cohomology $QH^*(X)$ is also generated by the divisor classes $D_1, \ldots, D_d$ and a presentation of $QH^*(X)$ is given by replacing each relation in $\mathcal{SR}$ by its quantum counterpart, i.e. denoting the quantum Stanley-Reisner ideal by $\mathcal{SR}_Q$, then we have

$$QH^*(X) = \mathbb{C}[D_1, \ldots, D_d]/(\mathcal{L} + \mathcal{SR}_Q).$$

Consider the case when $X = X_\Sigma$ is a semi-Fani toric surface. We also assume that X is not $\mathbb{P}^2$. Then any primitive collection is of the form $\mathfrak{P} = \{v_i, v_j\}$ so that v_i, v_j do not generate a cone in Σ. To compute $\mathcal{SR}_Q$, we need to calculate $D_i * D_j$, where $*$ denotes the small quantum product. Choose dual bases $\{D_m\}$, $\{D^m\}$ of $H^2(X)$, both represented by toric divisors. Then, by the divisor equation and a straightforward manipulation, we have

$$\begin{aligned} D_i * D_j &= \sum_{\alpha: c_1(\alpha)=2} (D_i \cdot \alpha)(D_j \cdot \alpha) GW_{0,1}^{X,\alpha}([\mathrm{pt}]) q^\alpha \\ &+ \sum_m \left(\sum_{\alpha: c_1(\alpha)=1} (D_i \cdot \alpha)(D_j \cdot \alpha)(D^m \cdot \alpha) GW_{0,0}^{X,\alpha} q^\alpha \right) D_m. \end{aligned}$$

The Gromov-Witten invariants $GW_{0,1}^{X,\alpha}([\mathrm{pt}]), GW_{0,0}^{X,\alpha}$ can be computed using the results of Bryan-Leung [3] as follows. To compute $GW_{0,1}^{X,\alpha}([\mathrm{pt}])$, note that we have $c_1(\alpha) = 2$ so that $\alpha^2 = 0$. Such an α must be of the form $\alpha' + f$ where α' is represented by a chain of (-2)-toric prime divisors and f is a fiber class. We are therefore in exactly the same situation as in the proof of Theorem 1.2. Hence, $GW_{0,1}^{X,\alpha}([\mathrm{pt}])$ can be computed as before.

As for $GW_{0,0}^{X,\alpha}$, we have $c_1(\alpha) = 1$, so that α is represented by a chain $\sum_{k=-p}^{q} s_k D_{i_k}$ of toric prime divisors such that $D_{i_k}^2 = -2$ for all $k \neq 0$, $D_{i_0}^2 = -1$ and $s_0 = 1$. The results of Bryan and Leung also apply in this situation: namely, the Gromov-Witten invariant $GW_{0,0}^{X,\alpha} = 1$ if both the chains $\sum_{k=-p}^{0} s_k D_{i_k}$ and $\sum_{k=0}^{q} s_k D_{i_k}$ are admissible with center 0 and $GW_{0,0}^{X,\alpha} = 0$ otherwise.

Let us give an example to illustrate the explicit computations.

Example. Let Σ be the fan whose rays are generated by

$$v_1 = (1,0), v_2 = (0,1), v_3 = (-1,-1), v_4 = (0,-1), v_5 = (1,-1), v_6 = (2,-1).$$

This determines a toric surface X. We equip X with a toric Kähler form such that the polytope P is given by

$$\begin{aligned} P &= \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, 0 \leq x_2 \leq t_1 + t_3 + 2t_4, x_1 + x_2 \leq t_1 + t_2 + 2t_3 + 3t_4, \\ &\qquad t_1 + t_4 + x_1 - x_2 \geq 0, t_1 + 2x_1 - x_2 \geq 0\}, \end{aligned}$$

where $t_i > 0$ are the Kähler parameters.

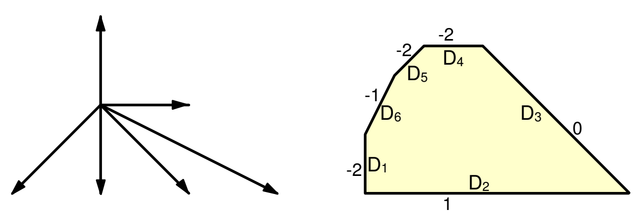

FIGURE 4. The fan Σ and the polytope P defining X. The numbers beside the divisors indicate their self-intersection numbers.

The linear equivalences among divisors are generated by the following two relations

$$D_1 - D_3 + D_5 + 2D_6 = 0,$$
$$D_2 - D_3 - D_4 - D_5 - D_6 = 0.$$

Hence, $H^2(X)$ is of rank 4. We choose the dual bases $\{D^m\}$ and $\{D_m\}$ to be $\{D_1, D_4, D_5, D_6\}$ and $\{D_2, D_3, D_4 + 2D_3, D_1 + 2D_2\}$ respectively.

We can now start to compute the primitive relations. For example, we want to compute $D_2 * D_4$. We need to look for all curve classes with $c_1 = 1, 2$ which intersect both D_2 and D_4 non-trivially. There are two such classes with $c_1 = 2$: the classes represented by D_3 and $D_3 + D_4$, and also two with $c_1 = 1$: the classes represented by $D_1 + D_5 + D_6$ and $D_1 + D_4 + D_5 + D_6$. Since all these configurations are admissible, the corresponding Gromov-Witten invariants are all equal to one, by the above discussion. Hence, we get

$$\begin{aligned} D_2 * D_4 &= q_1q_3q_4^2 - q_1q_2q_3q_4^2 + q_1q_3q_4(-D_2 + D_3 - (D_4 + 2D_3) + (D_1 + 2D_2)) \\ &\quad - q_1q_2q_3q_4(-D_2 - D_3 + (D_1 + 2D_2)) \\ &= q_1q_3q_4^2 - q_1q_2q_3q_4^2 + q_1q_3q_4(D_1 + D_5 + D_6) \\ &\quad - q_1q_2q_3q_4(D_1 + D_4 + D_5 + D_6), \end{aligned}$$

where we have used linear equivalences to get the second equality. Similarly, we can compute all other primitive relations.

Having computed all the primitive relations, we can go on to show the following

Lemma 4.5. *The map*

$$\psi : \mathbb{C}[D_1, \ldots, D_d] \to \mathbb{C}[z_1^{\pm 1}, z_2^{\pm 1}],\ D_i \mapsto \sum_{\beta:\mu(\beta)=2} n(\beta; D_i) Z_\beta$$

defines a ring homomorphism $\psi : QH^*(X) \to Jac(W)$.

Sketch of proof. First of all, we show that the ideal $\mathcal{L}$ of linear equivalences is mapped to the ideal $\langle \partial_1 W, \ldots, \partial_n W\rangle$ by ψ. Linear equivalences are generated by the relations $\sum_{i=1}^d v_i^j D_i = 0$, $j = 1, 2$, where we write $v_i = (v_i^1, v_i^2)$ in coordinates. By Proposition 4.4, we have

$$\begin{aligned} \psi(D_i) &= \sum_{k=1}^{d} \sum_{\alpha:c_1(\alpha)=0} n(\beta_k + \alpha; D_i) Z_{\beta_k+\alpha} \\ &= \sum_{k=1}^{d} \sum_{\alpha:c_1(\alpha)=0} (D_i \cdot (\beta_k + \alpha)) n(\beta_k + \alpha) Z_{\beta_k+\alpha}. \end{aligned}$$

Hence, we have

$$\begin{aligned} \psi\left(\sum_{i=1}^d v_i^j D_i\right) &= \sum_{i=1}^d v_i^j \left(\sum_{k=1}^{d} \sum_{\alpha:c_1(\alpha)=0} (D_i \cdot (\beta_k + \alpha)) n(\beta_k + \alpha) Z_{\beta_k+\alpha}\right) \\ &= \sum_{k=1}^{d} \sum_{\alpha:c_1(\alpha)=0} \left(\sum_{i=1}^d v_i^j (\delta_{ik} + D_i \cdot \alpha)\right) n(\beta_k + \alpha; D_i) Z_{\beta_k+\alpha} \\ &= \sum_{k=1}^{d} \sum_{\alpha:c_1(\alpha)=0} v_k^j n(\beta_k + \alpha; D_i) Z_{\beta_k+\alpha} \\ &= \partial_j W. \end{aligned}$$

Next, we need to show that each primitive relation is mapped by ψ to a relation in the ideal $\langle \partial_1 W, \ldots, \partial_n W\rangle$. This can be done by explicit computations. Again, we illustrate this by an example.

Consider X in the previous example. By Theorem 1.2, we can compute the superpotential explicitly. The result is given by

$$\begin{aligned} W &= (1+q_1)z_1 + z_2 + \frac{q_1 q_2 q_3^2 q_4^3}{z_1 z_2} + (1+q_2+q_2q_3)\frac{q_1 q_3 q_4^2}{z_2} \\ & \quad +(1+q_3+q_2q_3)\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}, \end{aligned}$$

where $q_l = \exp(-t_l)$, $l = 1, \ldots, 4$. We can also compute the images of the divisors D_i under ψ:

$$\begin{aligned} \psi(D_1) &= (1-q_1)z_1, \\ \psi(D_2) &= z_2 + q_1 z_1, \\ \psi(D_3) &= \frac{q_1 q_2 q_3^2 q_4^3}{z_1 z_2} + (q_2+q_2q_3)\frac{q_1 q_3 q_4^2}{z_2} + \frac{q_1 q_2 q_3 q_4 z_1}{z_2}, \\ \psi(D_4) &= (1-q_2)(\frac{q_1 q_3 q_4^2}{z_2} + \frac{q_1 q_3 q_4 z_1}{z_2}), \\ \psi(D_5) &= (1-q_3)(\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 q_2 q_3 q_4^2}{z_2}), \\ \psi(D_6) &= \frac{q_1 z_1^2}{z_2} + q_1 z_1 + (q_3+q_2q_3)\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 q_2 q_3^2 q_4^2}{z_2}. \end{aligned}$$

Using what we have computed before,

$$\begin{aligned} D_2 * D_4 &= q_1 q_3 q_4^2 - q_1 q_2 q_3 q_4^2 + q_1 q_3 q_4 (D_1 + D_5 + D_6) \\ & \quad - q_1 q_2 q_3 q_4 (D_1 + D_4 + D_5 + D_6) \\ &= q_1 q_3 q_4 [(1-q_2)(q_4 + D_1 + D_5 + D_6) - q_2 D_4]. \end{aligned}$$

This is mapped by ψ to

$$\begin{aligned} & q_1 q_3 q_4 [(1-q_2)(q_4 + z_1 + \frac{q_1 z_1^2}{z_2} + (1+q_2q_3)\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 q_2 q_3 q_4^2}{z_2}) \\ & \quad - q_2(1-q_2)(\frac{q_1 q_3 q_4^2}{z_2} + q_3 \frac{q_1 q_4 z_1}{z_2})] \\ = \; & q_1 q_3 q_4 (1-q_2)(q_4 + z_1 + \frac{q_1 z_1^2}{z_2} + \frac{q_1 q_4 z_1}{z_2}), \end{aligned}$$

which is exactly $\psi(D_2) \cdot \psi(D_4)$.

Similarly, we can show that $\psi(\mathcal{SR}_Q) = \{0\} \subset Jac(W)$. Hence, ψ defines a ring homomorphism $\psi : QH^*(X) \to Jac(W)$. $\square$

Corollary 1.3 now follows from the following lemma.

Lemma 4.6. *For generic choices of the Kähler parameters q_l, $\psi : QH^*(X) \to Jac(W)$ is a bijective map.*

Sketch of proof. Having computed the superpotential W and the images of the divisors D_i under ψ, we can check surjectivity of ψ in a straightforward way. For instance, for the surface X in the previous example, we have

$$z_1 = \psi((1-q_1)^{-1} D_1), z_2 = \psi(D_2 - q_1(1-q_1)^{-1} D_1),$$
$$z_2^{-1} = \psi([q_1 q_3 q_4^2 (1-q_2)(1-q_2q_3)]^{-1} D_4 - [q_1 q_4^2 (1-q_3)(1-q_2q_3)]^{-1} D_5).$$

Also, since we have the relation $\partial_1 W = 0$ which gives

$$z_1^{-1} = (q_1 q_2 q_3^2 q_4^3)^{-1}[(1+q_1)z_1 z_2 + (1+q_3+q_2 q_3)q_1 q_4 z_1 + 2q_1 z_1^2],$$

and ψ is a homomorphism, z_1^{-1} also lies in the image of ψ. The surjectivity of ψ for all other examples can be checked in this way.

On the other hand, by Proposition 3.7 and Lemma 3.9 in Iritani [16] (which were proved by using Kouchnirenko's results), we have $\dim H^*(X) = \dim Jac(W)$ for generic choices of the Kähler parameters q_l. Hence, $\psi : QH^*(X) \to Jac(W)$ is bijective. □

5. The big quantum cohomology

5.1. **The potential with bulk.** For a Lagrangian torus fiber $\mathbf{T}$ in a compact toric manifold X and $\mathbf{b} \in \mathcal{A}$, where $\mathcal{A} := \mathbb{C}\langle \text{toricinvariantcycles} \rangle$, Fukaya, Oh, Ohta and Ono [10] defined the *potential with bulk* $W_{\mathbf{b}}$ as

$$W_{\mathbf{b}} := \sum_{\substack{\beta \in \pi_2(X,\mathbf{T}) \\ l \geq 0}} \frac{1}{l!} n_l(\beta; \underbrace{\mathbf{b}, \ldots, \mathbf{b}}_{l}) Z_\beta$$

where the open Gromov-Witten invariants $n(\beta; V_1, \ldots, V_l)$ (see Definition 4.1) extend multilinearly to give a function $n_l : \pi_2(X,\mathbf{T}) \times \mathcal{A}^{\otimes l} \to \mathbb{C}$. In a recent preprint [7] they proved that

$$QH^*_{\mathbf{b}}(X) \cong \mathrm{Jac}(W_{\mathbf{b}}).$$

Thus an explicit expression of $W_{\mathbf{b}}$ would give an explicit presentation of the big quantum cohomology ring $QH^*_{\mathbf{b}}(X)$.

In the previous section, we have given an explicit expression of $W_{\mathbf{b}}$ when $\mathbf{b} = 0$ for a semi-Fano toric surface X. We consider its potential with bulk in this section. For the purpose of computing $QH^*_{\mathbf{b}}(X)$, it is enough to consider $\mathbf{b} = aX + D + cp$, where D is a toric divisor, p is the intersection point of two toric prime divisors (say D_1 and D_2), and $a, c \in \mathbb{C}$.

Proposition 5.1 (Restatement of Corollary 1.4). *Let X be a semi-Fano toric surface, and $\mathbf{b} = aX + D + cp$ as described above. Then*

$$W_{\mathbf{b}} = a + \sum_{\beta \neq 0} \exp(\langle \beta, D \rangle) \left(\sum_{k=0}^{\infty} \frac{c^k}{k!} n_k(\beta; p, \ldots, p) \right) Z_\beta.$$

In particular, when $c = 0$,

$$W_{\mathbf{b}} = a + \sum_{\beta \text{ admissible}} \exp(\langle \beta, D \rangle) Z_\beta.$$

Proof. When $\beta \neq 0$,

$$n_k(\beta; [X], \gamma_1, \ldots, \gamma_{k-1}) = 0$$

for all $k \geq 1$ and $\gamma_1, \ldots, \gamma_{k-1} \in H_*(X)$ due to dimension reason. Thus

$$\begin{aligned} W_{\mathbf{b}} &:= \sum_{\substack{\beta \in \pi_2(X,\mathbf{T}) \\ l \geq 0}} \frac{1}{l!} n_l(\beta; \mathbf{b}, \ldots, \mathbf{b}) Z_\beta \\ &= \sum_{l \geq 0} \frac{1}{l!} n_l(0; \mathbf{b}, \ldots, \mathbf{b}) + \sum_{\substack{\beta \neq 0 \\ l \geq 0}} \frac{1}{l!} n_l(\beta; D + cp, \ldots, D + cp) Z_\beta. \end{aligned}$$

Moreover, $n_1(0;X) = 1$ ($\overline{M}_{1,1}(\mathbf{T},0;X)$ contains the constant map only) and $n_1(0;p) = n_1(0;D) = 0$ (the corresponding moduli spaces are empty). Also by dimension counting, $n_l(0;\gamma_1,\ldots,\gamma_l) = 0$ for all $l \neq 1$. Thus the first term is

$$\sum_{l\geq 0} \frac{1}{l!} n_l(0;\mathbf{b},\ldots,\mathbf{b}) = a.$$

Using the divisor equation for open Gromov-Witten invariants ([10]; see Proposition 4.4), the second term is

$$\begin{aligned}
\sum_{\substack{\beta\neq 0\\ l\geq 0}} \frac{1}{l!} n_l(\beta;D+cp,\ldots,D+cp)Z_\beta &= \sum_{\substack{\beta\neq 0\\ l\geq 0}} \frac{1}{l!}\sum_{k=0}^{l} \mathrm{C}^l_k c^k n_l(\beta;\underbrace{D,\ldots,D}_{l-k},\underbrace{p,\ldots,p}_{k})Z_\beta \\
&= \sum_{\substack{\beta\neq 0\\ l\geq 0}} \frac{1}{l!}\sum_{k=0}^{l} \mathrm{C}^l_k c^k (\langle\beta,D\rangle)^{l-k} n_k(\beta;p,\ldots,p)Z_\beta \\
&= \sum_{\substack{\beta\neq 0\\ j,k\geq 0}} \frac{c^k}{j!k!}(\langle\beta,D\rangle)^j n_k(\beta;p,\ldots,p)Z_\beta \\
&= \sum_{\beta\neq 0} \exp(\langle\beta,D\rangle)\left(\sum_{k=0}^{\infty}\frac{c^k}{k!} n_k(\beta;p,\ldots,p)\right)Z_\beta.
\end{aligned}$$

When $c = 0$,

$$W_{\mathbf{b}} = a + \sum_{\beta\neq 0}\exp(\langle\beta,D\rangle) n_\beta Z_\beta.$$

By Theorem 1.2, $n_\beta = 1$ when β is admissible, and 0 otherwise. Thus

$$W_{\mathbf{b}} = a + \sum_{\beta \text{ admissible}} \exp(\langle\beta,D\rangle)Z_\beta.$$

□

5.2. **Speculations and discussions.** In Proposition 5.1, $n_l(\beta;p,\ldots,p)$ ($l \geq 1$) has not been computed. In the following we give an informal discussion concerning these invariants.

One of the issues involved in computing these invariants is the presence of "ghost bubbles" in the moduli space $\overline{M}_{1,l}(\mathbf{T},\beta;p,\ldots,p)$ (see Figure 5) when p is chosen to be a toric fixed point. On the other hand, if we consider $p_1,\ldots,p_l \in X$ in generic position, which is the approach taken by Gross [13] where he used tropical geometry to define the superpotential with bulk, the moduli space $\overline{M}_{1,l}(\mathbf{T},\beta;p_1,\ldots,p_l)$ does not involve disk bubbling (when β has the suitable Maslov index $\mu(\beta) = 2 - 2l + \sum_{j=1}^{l}\mathrm{codim}_{\mathbb{R}}(V_j)$ so that the moduli has expected dimension $n = \dim \mathbf{T}$), and also ghost bubbles are not present. The invariant $n_l(\beta;p_1,\ldots,p_l)$ can still be defined, and it is easier to compute.

This motivates us to consider $p' \in D_1$ which is *not* fixed by the torus action, and define the invariant $n_l(\beta;p',\ldots,p')$ by taking a generic perturbation of the l points around p'.

Example 5.2 (The Hirzebruch surface $\mathbf{F}_2$). *Consider The Hirzebruch surface* $\mathbf{F}_2$ *whose polytope picture is shown in Figure 6. If we take the above approach, then*

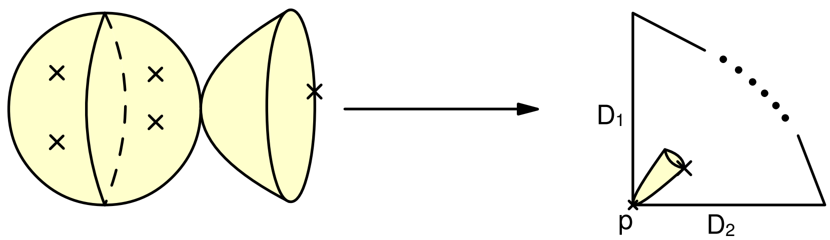

FIGURE 5. Ghost bubbles in $\overline{M}_{1,4}(\mathbf{T},\beta;p,p,p,p)$. The whole sphere bubble is contracted to the toric fixed point p. The disk class is taken such that $\overline{M}_{1,4}(\mathbf{T},\beta;p,p,p,p)$ has expected dimension 2. However the actual dimension is bigger than 2 since the interior marked points are free to move in the bubble.

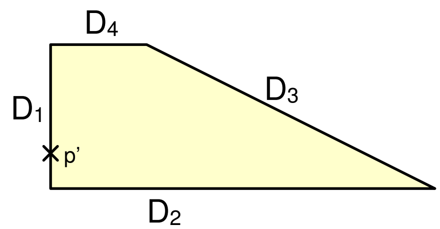

FIGURE 6. The polytope of the Hirzebruch surface $\mathbf{F}_2$.

$n_l(\beta;p',\ldots,p')$ *equals to* 1 *when* $\beta = l\beta_1 + \beta_i$ *for* $i = 2,3,4$ *or* $\beta = l\beta_1 + \beta_4 + D_4$, *and* 0 *otherwise. Then for* $\mathbf{b} = a[X] + D + cp$,

$$\begin{aligned}
W_{\mathbf{b}} &= a + \sum_{\beta\neq 0} \exp(\langle\beta, D\rangle)\left(\sum_{k=0}^{\infty}\frac{c^k}{k!}n_k(\beta;p,\ldots,p)\right)Z_\beta\\
&= a + \exp(\langle\beta_1, D\rangle)Z_{\beta_1} + \sum_{i=2}^{4}\exp(c\,\mathbf{e}^{\langle\beta_1,D\rangle}Z_{\beta_1})\exp(\langle\beta_i, D\rangle)Z_{\beta_i}\\
&\quad + \exp(c\,\mathbf{e}^{\langle\beta_1,D\rangle}Z_{\beta_1})\exp(\langle\beta_4 + D_4, D\rangle)q_4 Z_{\beta_4}.
\end{aligned}$$

The above consideration is tentative, and we are still investigating whether this idea is in the right direction.

APPENDIX A. LIST OF SUPERPOTENTIALS FOR SEMI-FANO TORIC SURFACES

Using the fact that any smooth compact toric surface is a blowup of either $\mathbb{P}^2$ or a Hirzebruch surface $\mathbb{F}_m$ ($m \geq 0$) at torus fixed points, it is easy to see that there are finitely many isomorphism classes of semi-Fano toric surfaces. In fact, all except $\mathbb{F}_2$ and $\mathbb{P}^1\times\mathbb{P}^1$ are blowups of $\mathbb{P}^2$; there are 16 of such surfaces, five of which are Fano (namely, $\mathbb{P}^2$, $\mathbb{P}^1\times\mathbb{P}^1$ and the blowup of $\mathbb{P}^2$ at 1, 2 or 3 points).

By using Theorem 1.2, we can compute the superpotentials for all these surfaces explicitly. In this appendix, we provide a list of the superpotentials for the 11 semi-Fano but non-Fano toric surfaces. We enumerate them as $X_1,\ldots,X_{11}$, and each surface is specified by the primitive generators $\rho(\Sigma)$ of rays of its fan and the defining inequalities of its polytope. Also, in the following tables, the t_l's are positive numbers and $q_l = \exp(-t_l)$ are the Kähler parameters.

	$\rho(\Sigma)$	polytope P	superpotential W
X_1	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,-2)$ $v_4 = (0,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $2t_1 + t_2 - x_1 - 2x_2 \geq 0$ $t_1 - x_2 \geq 0$	$z_1 + z_2 + \frac{q_1^2 q_2}{z_1 z_2^2} + (1+q_2)\frac{q_1}{z_2}$
X_2	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,-1)$ $v_4 = (0,-1)$ $v_5 = (1,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_1+t_2+2t_3-x_1-x_2 \geq 0$ $t_1 + t_3 - x_2 \geq 0$ $t_1 + x_1 - x_2 \geq 0$	$z_1 + z_2 + \frac{q_1 q_2 q_3^2}{z_1 z_2} + (1+q_2)\frac{q_1 q_3}{z_2} + \frac{q_1 z_1}{z_2}$
X_3	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,-1)$ $v_4 = (0,-1)$ $v_5 = (1,-1)$ $v_6 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_1+t_2+2t_3+3t_4-x_1-x_2 \geq 0$ $t_1 + t_3 + 2t_4 - x_2 \geq 0$ $t_1 + t_4 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1+q_1)z_1 + z_2 + \frac{q_1 q_2 q_3^2 q_4^3}{z_1 z_2} + (1 + q_2 + q_2 q_3)\frac{q_1 q_3 q_4^2}{z_2} + (1 + q_3 + q_2 q_3)\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$
X_4	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,0)$ $v_4 = (0,-1)$ $v_5 = (1,-1)$ $v_6 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2 + t_3 + t_4 - x_1 \geq 0$ $t_1 + t_3 + 2t_4 - x_2 \geq 0$ $t_1 + t_4 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1+q_1)z_1 + z_2 + \frac{q_2 q_3 q_4}{z_1} + \frac{q_1 q_3 q_4^2}{z_2} + (1+q_3)\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$
X_5	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,0)$ $v_4 = (-1,-1)$ $v_5 = (0,-1)$ $v_6 = (1,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2 + t_3 + t_4 - x_1 \geq 0$ $t_1+t_3+2t_4-x_1-x_2 \geq 0$ $t_1 + t_4 - x_2 \geq 0$ $t_1 + x_1 - x_2 \geq 0$	$z_1 + z_2 + \frac{q_2 q_3 q_4}{z_1} + \frac{q_1 q_3 q_4^2}{z_1 z_2} + (1 + q_3)\frac{q_1 q_4}{z_2} + \frac{q_1 z_1}{z_2}$
X_6	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,0)$ $v_4 = (-1,-1)$ $v_5 = (0,-1)$ $v_6 = (1,-1)$ $v_7 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2+t_3+t_4+t_5-x_1 \geq 0$ $t_1+t_3+2t_4+3t_5-x_1-x_2 \geq 0$ $t_1 + t_4 + 2t_5 - x_2 \geq 0$ $t_1 + t_5 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1+q_1)z_1 + z_2 + \frac{q_2 q_3 q_4 q_5}{z_1} + \frac{q_1 q_3 q_4^2 q_5^3}{z_1 z_2} + (1 + q_3 + q_3 q_4)\frac{q_1 q_4 q_5^2}{z_2} + (1 + q_4 + q_3 q_4)\frac{q_1 q_5 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$

	$\rho(\Sigma)$	polytope P	superpotential W
X_7	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,1)$ $v_4 = (-1,0)$ $v_5 = (0,-1)$ $v_6 = (1,-1)$ $v_7 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2 + t_3 - t_1 - t_5 - x_1 + x_2 \geq 0$ $t_3 + t_4 + t_5 - x_1 \geq 0$ $t_1 + t_4 + 2t_5 - x_2 \geq 0$ $t_1 + t_5 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1 + q_1)z_1 + z_2 + \frac{q_2 q_3 z_2}{q_1 q_5 z_1} + \frac{q_3 q_4 q_5}{z_1} + \frac{q_1 q_4 q_5^2}{z_2} + (1 + q_4)\frac{q_1 q_5 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$
X_8	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,0)$ $v_4 = (-2,-1)$ $v_5 = (-1,-1)$ $v_6 = (0,-1)$ $v_7 = (1,-1)$ $v_8 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2 + t_3 + t_4 + t_5 + t_6 - x_1 \geq 0$ $t_1 + t_3 + 2t_4 + 3t_5 + 4t_6 - 2x_1 - x_2 \geq 0$ $t_1 + t_4 + 2t_5 + 3t_6 - x_1 - x_2 \geq 0$ $t_1 + t_5 + 2t_6 - x_2 \geq 0$ $t_1 + t_6 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1 + q_1)z_1 + z_2 + (1 + \frac{q_1 q_5 q_6^2}{q_2^2 q_3})\frac{q_2 q_3 q_4 q_5 q_6}{z_1} + \frac{q_1 q_3 q_4^2 q_5^3 q_6^4}{z_1^2 z_2} + (1 + q_3 + q_3 q_4 + q_3 q_4 q_5)\frac{q_1 q_4 q_5^2 q_6^3}{z_1 z_2} + (1 + q_4 + q_3 q_4 + q_4 q_5 + q_3 q_4 q_5 + q_3 q_4^2 q_5)\frac{q_1 q_5 q_6^2}{z_2} + (1 + q_5 + q_4 q_5 + q_3 q_4 q_5)\frac{q_1 q_6 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$
X_9	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,1)$ $v_4 = (-1,0)$ $v_5 = (-1,-1)$ $v_6 = (0,-1)$ $v_7 = (1,-1)$ $v_8 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2 + 2t_3 + t_4 - t_1 - t_6 - x_1 + x_2 \geq 0$ $t_3 + t_4 + t_5 + t_6 - x_1 \geq 0$ $t_1 + t_4 + 2t_5 + 3t_6 - x_1 - x_2 \geq 0$ $t_1 + t_5 + 2t_6 - x_2 \geq 0$ $t_1 + t_6 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1 + q_1)z_1 + z_2 + \frac{q_2 q_3^2 q_4 z_2}{q_1 q_6 z_1} + (1 + q_2)\frac{q_3 q_4 q_5 q_6}{z_1} + \frac{q_1 q_4 q_5^2 q_6^3}{z_1 z_2} + (1 + q_4 + q_4 q_5)\frac{q_1 q_5 q_6^2}{z_2} + (1 + q_5 + q_4 q_5)\frac{q_1 q_6 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$
X_{10}	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,1)$ $v_4 = (-2,1)$ $v_5 = (-1,0)$ $v_6 = (0,-1)$ $v_7 = (1,-1)$ $v_8 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2 + t_3 + t_4 - t_1 - t_6 - x_1 + x_2 \geq 0$ $2t_4 + t_5 - t_1 - t_3 - 2x_1 + x_2 \geq 0$ $t_4 + t_5 + t_6 - x_1 \geq 0$ $t_1 + t_5 + 2t_6 - x_2 \geq 0$ $t_1 + t_6 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1 + q_1)z_1 + z_2 + (1 + \frac{q_1 q_5 q_6^2}{q_2^2 q_3})\frac{q_2 q_3 q_4 z_2}{q_1 q_6 z_1} + \frac{q_4^2 q_5 z_2}{q_1 q_3 z_1^2} + (1 + q_3)\frac{q_4 q_5 q_6}{z_1} + \frac{q_1 q_5 q_6^2}{z_2} + (1 + q_5)\frac{q_1 q_6 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$
X_{11}	$v_1 = (1,0)$ $v_2 = (0,1)$ $v_3 = (-1,2)$ $v_4 = (-1,1)$ $v_5 = (-1,0)$ $v_6 = (-1,-1)$ $v_7 = (0,-1)$ $v_8 = (1,-1)$ $v_9 = (2,-1)$	$x_1 \geq 0$ $x_2 \geq 0$ $t_2 + 2t_3 + 3t_4 + t_5 - 2t_1 - t_6 - 3t_7 - x_1 + 2x_2 \geq 0$ $t_3 + 2t_4 + t_5 - t_1 - t_7 - x_1 + x_2 \geq 0$ $t_4 + t_5 + t_6 + t_7 - x_1 \geq 0$ $t_1 + t_5 + 2t_6 + 3t_7 - x_1 - x_2 \geq 0$ $t_1 + t_6 + 2t_7 - x_2 \geq 0$ $t_1 + t_7 + x_1 - x_2 \geq 0$ $t_1 + 2x_1 - x_2 \geq 0$	$(1 + q_1 + \frac{q_2 q_3^2 q_4^3 q_5}{q_1 q_6 q_7^3})z_1 + (1 + \frac{q_2 q_3^2 q_4^3 q_5}{q_1^2 q_6 q_7^3} + \frac{q_2 q_3^2 q_4^3 q_5}{q_1 q_6 q_7^3})z_2 + \frac{q_2 q_3^2 q_4^3 q_5 z_2^2}{q_1^2 q_6 q_7 z_1} + (1 + q_2 + q_2 q_3)\frac{q_3 q_4^2 q_5 z_2}{q_1 q_7 z_1} + (1 + q_3 + q_2 q_3)\frac{q_4 q_5 q_6 q_7}{z_1} + \frac{q_1 q_5 q_6^2 q_7^3}{z_1 z_2} + (1 + q_5 + q_5 q_6)\frac{q_1 q_6 q_7^2}{z_2} + (1 + q_6 + q_5 q_6)\frac{q_1 q_7 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}$

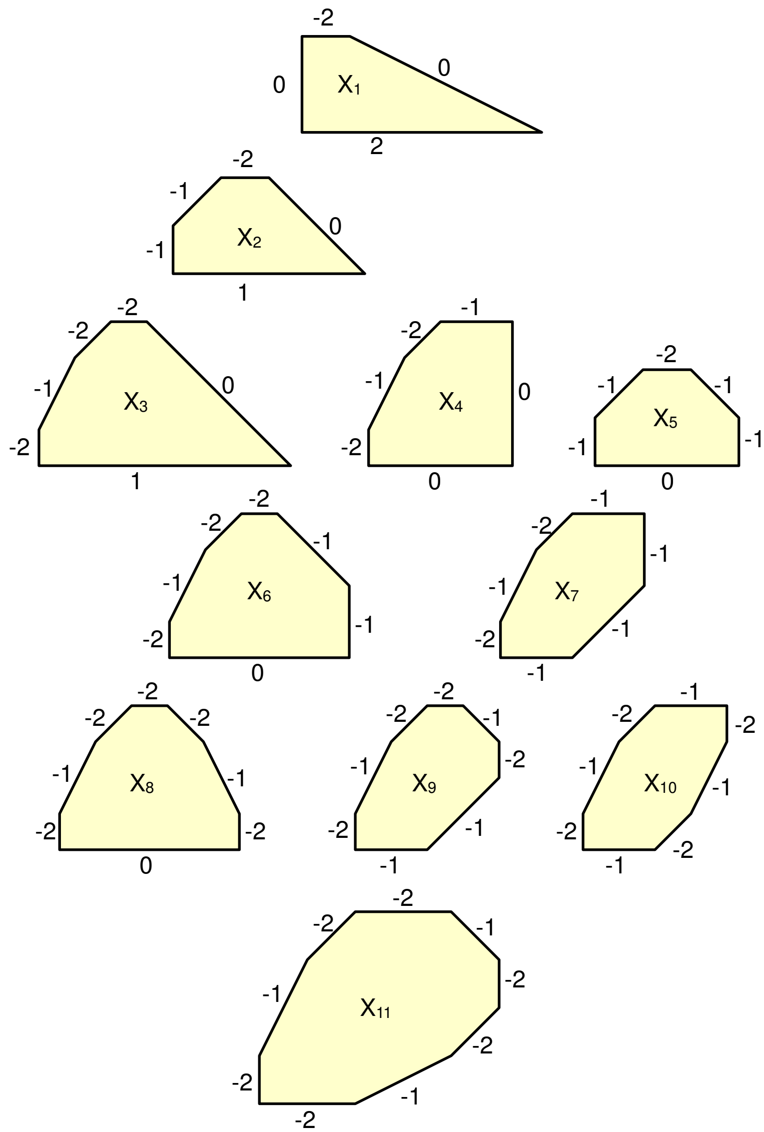

FIGURE 7. Polytopes defining the semi-Fano but non-Fano toric surfaces. The numbers indicate the self-intersection numbers of the toric divisors.

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