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Deformability of Lie Algebra Bundles and Geometry of Rational Surfaces

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When $X = X_n$ is a blowup of $\mathbb{P}^2$ at n points $x_1, \ldots, x_n$ with $n \le 9$, there is a canonical (affine) Lie algebra bundle $\mathcal{E}_0^{E_n}$ over it, where E_9 is the affine E_8. In this paper, we will give a detail study of the relationship between the geometry of X_n and the deformability of $\mathcal{E}_0^{E_n}$.

1 Introduction

A rational surface is a surface birationally equivalent to the projective plane, its minimal model is the projective plane $\mathbb{P}^2$ or the Hirzebruch surface $\mathbb{F}_m$ for $m = 0$ or $m \ge 2$. In this paper, we consider $X = X_n$, a blowup of $\mathbb{P}^2$ at n points $x_1, \ldots, x_n$ with $n \le 9$.

When $n \le 8$, $\langle K_{X_n} \rangle^\perp \subset \mathrm{Pic}(X_n)$ is isomorphic to Λ_{E_n}, the root lattice of the simple Lie algebra E_n, then we have a root system Φ_n of E_n and we can associate a Lie algebra bundle $\mathcal{E}_0^{E_n}$ over X_n [4, 13, 14],

$$\mathcal{E}_0^{E_n} := O_X^{\oplus n} \oplus \bigoplus_{\alpha \in \Phi_n} O_X(\alpha).$$

By restriction, we have an E_n-bundle over any anti-canonical curve Σ in X_n. Note that Σ is always an elliptic curve. For a fixed elliptic curve Σ, the above construction gives a

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bijection between del Pezzo surfaces containing Σ and E_n-bundles over Σ [5, 6, 8, 13, 18]. Such an identification was predicted by the F-theory/string duality in physics [8]. This was generalized to all simple Lie algebras in [13, 14].

When $n=9$, X_9 is not Fano and $E_9=\hat{E}_8$ is an affine Lie algebra. Corresponding results for the $\hat{E}_8$-bundle over X_9 are obtained in [12].

In this paper, we explain how the geometry of X_9 can be reflected by the deformability of the $\hat{E}_8$-bundle $\mathcal{E}_0^{\hat{E}_8}$ over it. Similar results for X_n and $\mathcal{E}_0^{E_n}$ with $n\leq 8$ can be easily deduced from this case. Among other things, we obtained the following results.

Theorem 1 (Theorem 8). $\mathcal{E}_0^{\hat{E}_8}$ is totally nondeformable if and only if the nine blowup points in $\mathbb{P}^2$ are in general position. □

Theorem 2 (Theorem 9). Suppose $-K_{X_9}$ is nef, then

(i) X_9 admits an elliptic fibration with a multiple fiber of multiplicity m ($m\geq 1$) if and only if $\mathcal{E}_0^{\hat{E}_8}$ is deformable in $(-mK)$-direction but not in $(-m+1)K$-direction.

(ii) X_9 has a (maximal) ADE curve C of type $\mathfrak{g}$ if and only if $\mathcal{E}_0^{\hat{E}_8}$ is (maximal) $\mathfrak{g}$-deformable. Q1

(iii) X_9 has a (maximal) affine ADE curve C of type $\hat{\mathfrak{g}}$ if and only if $\mathcal{E}_0^{\hat{E}_8}$ is (maximal) $\hat{\mathfrak{g}}$-deformable. □

Here ADE means Lie algebras of types A_n, D_n, E_6, E_7 and E_8, where n can be any nature numbers.

The organization of this paper is as follows. Section 2 gives the construction of the (affine) ADE Lie algebra bundles over X_n. In Section 3, we state the definition of deformability and the main theorems. Section 4 studies the negative curves in X_9. The proofs of the main theorems are in Section 5.

2 E_n-Bundle Over X_n with $n\leq 9$

The Picard group $\mathrm{Pic}(X_n)\cong H^2(X_n,\mathbb{Z})$ is a rank $n+1$ lattice with generators $h, l_1,\ldots,l_n$, where h is the class of lines in $\mathbb{P}^2$ and l_i is the exceptional class of the blow-up at x_i. So $h^2=1=-l_i^2$ and $h\cdot l_i=0=l_i\cdot l_j$, $i\neq j$. Thus, $H^2(X_n,\mathbb{Z})\cong\mathbb{Z}^{1,n}$. The canonical class is $K_{X_n}=-3h+l_1+\cdots+l_n$. Denote

$$\Phi_n:=\{\alpha\in H^2(X_n,\mathbb{Z})|\alpha^2=-2,\alpha\cdot K=0\}.$$

Then Φ_n is a root system of type E_n when $n \leq 8$ and Φ_9 is an affine real root system of $\hat{E}_8$ (also denoted as E_9). More explicitly, $\Phi_{\hat{E}_8} := \Phi_9 \cup \{mK_{X_9} | m \neq 0, m \in \mathbb{Z}\}$ forms a root system of (untwisted) affine E_8-type (i.e., $\hat{E}_8$-type) with $\Phi^{re}_{\hat{E}_8} := \Phi_9$ the set of real roots and $\Phi^{im}_{\hat{E}_8} := \{mK_{X_9} | m \neq 0, m \in \mathbb{Z}\}$ the set of imaginary roots (see [10] or [12]). We have an $\hat{E}_8$-bundle $\mathcal{E}_0^{\hat{E}_8}$ over X_9:

$$\mathcal{E}_0^{\hat{E}_8} = O^{\oplus 9} \oplus \bigoplus_{\alpha \in \Phi^{re}_{\hat{E}_8}} O(\alpha) \bigoplus_{\beta \in \Phi^{im}_{\hat{E}_8}} O(\beta).$$

The Lie algebra structure on $\mathcal{E}_0^{\hat{E}_8}$ is explained in [12]. When $n \leq 8$, $\mathcal{E}_0^{E_n} = O^{\oplus n} \oplus \bigoplus_{\alpha \in \Phi_n} O(\alpha)$ is an E_n-bundle over X_n.

Remark 3. We can construct an $\hat{E}_8$-bundle over a blowup of $\mathbb{F}_m$ (Hirzebruch surface) at eight points similarly [3]. □

Definition 4. A curve $C = \cup C_i$ in a surface X is called an ADE (respectively, affine ADE) curve of type $\mathfrak{g}$ (respectively, $\hat{\mathfrak{g}}$) if each C_i is a smooth (-2)-curve in X and the dual graph of C is a Dynkin diagram of the corresponding type. □

Suppose $C = \cup C_i$ is an (affine) ADE curve of type $\mathfrak{g}$ in X_n, then C_i's generates a subroot system Φ inside Φ_n since $C_i \cdot K = 0$ for every i. Therefore, the corresponding bundle $\mathcal{E}_0^{\mathfrak{g}}$ is a Lie algebra subbundle of $\mathcal{E}_0^{E_n}$.

Suppose $\mathcal{E}_0^{\mathfrak{g}}$ is a $\mathfrak{g}$-bundle over a surface X corresponding to a root system $\Lambda_{\mathfrak{g}} \subset \mathrm{Pic}(X)$ of type $\mathfrak{g}$.

Definition 5. A Lie algebra subbundle $\mathcal{F}$ of $\mathcal{E}_0^{\mathfrak{g}}$ is called strict, if there exists a subroot lattice Λ of $\Lambda_{\mathfrak{g}}$ such that $\mathcal{F}$ is a direct sum of line bundles corresponding to the roots in Λ. □

In order to describe $\mathcal{E}_0^{\hat{E}_8}$ as a central extension of a loop Lie algebra bundle over X_9, we pick any smooth (-1)-curve l in X_9, then we have

$$\mathcal{E}_0^{\hat{E}_8} \cong \mathcal{E}_0^{E_8} \otimes \left(\bigoplus_{n \in \mathbb{Z}} O(nK_{X_9}) \right) \oplus O,$$

where $\mathcal{E}_0^{E_8}$ is the pull-back of the E_8-bundle over X_8 via $\pi : X_9 \to X_8$, the blow down map of l. The next proposition describes the converse.

Proposition 6. When $\mathcal{E}_0^{\hat{E}_8}$ is a central extension of a loop E_8-subbundle over X for some strict E_8-bundle $\mathcal{F}_0^{E_8}$ over X_9, that is,

$$\mathcal{E}_0^{\hat{E}_8} \cong \mathcal{F}_0^{E_8} \otimes \left(\bigoplus_{n\in\mathbb{Z}} O(nK_{X_9}) \right) \oplus O,$$

as a Lie algebra bundle isomorphism, then there is a unique (possibly reducible) (-1)-curve l in X such that $\mathcal{F}_0^{E_8}$ is constructed from those $\alpha \in \Lambda^{re}$ satisfying $\alpha \cdot l = 0$.

Proof. Denote $\Delta_{E_8} = \{\alpha_1, \ldots, \alpha_8\}$ as a root base of the corresponding E_8 Lie algebra from $\mathcal{F}_0^{E_8}$, we need to find a unique (-1)-curve l in X such that $l \cdot \alpha_i = 0$ for any α_i in Δ_{E_8}. Since $\{\pm 1\} \times W(\hat{E}_8)$ acts on the set of all root bases of $\hat{E}_8$ simply transitively [11] and $W(\hat{E}_8)$ acts on the set of (-1)-curves [12], we only need to find l for one particular root base of any E_8 in $\hat{E}_8$ and show that such a l is unique. For example, if we take $\alpha_1 = h - l_1 - l_2 - l_3, \alpha_k = l_{k-1} - l_k$ for $k = 2, \ldots 8$, then we can take $l = l_9$ and by the condition that $l \cdot \alpha_i = 0, l^2 = -1 = l \cdot K$, we know such a l is unique. ■

3 Deformability of such $\mathcal{E}_0^{\hat{E}_8}$

In this section, we will describe relationships between the geometry of X_9 and the deformability of $\mathcal{E}_0^{\hat{E}_8}$.

Recall when Pic(X) contains a lattice Λ isomorphic to a root lattice $\Lambda_{\mathfrak{g}}$, then we have a $\mathfrak{g}$-bundle $\mathcal{E}$ over X [5, 8, 12–14].

$$\mathcal{E} := O^{\oplus r} \oplus \bigoplus_{\alpha\in\Phi} O(\alpha).$$

Infinitesimal deformations of holomorphic structures on $\mathcal{E}$ are parameterized by $H^1(X, End(\mathcal{E}))$, and those which also preserve the Lie algebra structure are parameterized by $H^1(X, ad(\mathcal{E})) = H^1(X, \mathcal{E})$ since $\mathfrak{g}$ is simple. Hence we introduce the following definitions.

Definition 7.

(i) $\mathcal{E}$ is called fully deformable if there exists a base $\Delta \subset \Phi$ such that $H^1(X, O(\alpha)) \neq 0$ for any $\alpha \in \Delta$.

(ii) $\mathcal{E}$ is called $\mathfrak{h}$-deformable if there exists a strict $\mathfrak{h}$ Lie algebra subbundle $\mathcal{E}^{\mathfrak{h}} \subseteq \mathcal{E}$ which is fully deformable.

(iii) $\mathcal{E}$ is called deformable in α-direction for $\alpha \in \Phi$ if $H^1(X, O(\alpha)) \neq 0$.

(iv) $\mathcal{E}$ is called totally nondeformable if $H^1(X, O(\alpha)) = 0$ for any $\alpha \in \Phi$. □

After the definition of deformability, we state the main results of this paper in the following two theorems.

Theorem 8. $\mathcal{E}_0^{\hat{E}_8}$ over X_9 is totally nondeformable if and only if the nine blowup points in $\mathbb{P}^2$ are in general position. □

Let us recall some facts about elliptic fibrations on X_9 [15, 17]. Any elliptic fibration on X_9 must be relatively minimal, that is, there is no (-1)-curves in any of its fibrations, as there is no elliptic fibration on X_8, this is because the Euler characteristic of any elliptic surface is a multiple of 12 [7] and also $\chi(X_9) = 12$. There is at most one multiple fiber [9], say of multiplicity m. This happens precisely when there exists an irreducible pencil of degree $3m$ in $\mathbb{P}^2$ with nine base points, each of multiplicity m and X_9 is the blow up of $\mathbb{P}^2$ at these nine points. We can characterize the existence of such an elliptic fibration on X_9 in terms of deformability of $\mathcal{E}_0^{\hat{E}_8}$ along imaginary root directions. For instance, X_9 with $-K_{X_9}$ nef admits an elliptic fibration (without multiple fiber) if and only if $\mathcal{E}_0^{\hat{E}_8}$ is deformable in $(-mK)$-direction for some $m \in \mathbb{N}$ (with $m = 1$). Deformability of $\mathcal{E}_0^{\hat{E}_8}$ can also detect the existence of ADE or affine ADE curves in X.

Theorem 9. Suppose $-K_{X_9}$ is nef, then

(i) X_9 admits an elliptic fibration with a multiple fiber of multiplicity m $(m \geq 1)$ if and only if $\mathcal{E}_0^{\hat{E}_8}$ is deformable in $(-mK)$-direction but not in $(-m+1)K$-direction.

(ii) X_9 has an (maximal) ADE curve C of type $\mathfrak{g}$ if and only if $\mathcal{E}_0^{\hat{E}_8}$ is (maximal) $\mathfrak{g}$-deformable.

(iii) X_9 has a (maximal) affine ADE curve C of type $\hat{\mathfrak{g}}$ if and only if $\mathcal{E}_0^{\hat{E}_8}$ is (maximal) $\hat{\mathfrak{g}}$-deformable. □

Here, we say an ADE or affine ADE curve C is maximal if it is not proper contained in another ADE or affine ADE curve. We say $\mathcal{E}_0^{\hat{E}_8}$ is maximal $\mathfrak{g}$ (or $\hat{\mathfrak{g}}$) deformable if there does not exist another fully deformable (affine) Lie algebra subbundle of $\mathcal{E}_0^{\hat{E}_8}$ containing this $\mathfrak{g}$ (or $\hat{\mathfrak{g}}$) bundle.

4 Negative Curves in X_9

In this section, we study negative rational curves in X_9. We can get corresponding results for X_n with $n \leq 8$ from this $n = 9$ case.

A divisor D in X is called a $(-m)$-class if $D \cdot D = -m$ and $D \cdot K = m-2$. An effective $(-m)$-class is called a $(-m)$-curve. Note when $D = \sum n_i C_i$ is a $(-m)$-curve, we will also denote the corresponding curve $\cup C_i$ as D.

Use the notations in the above section, every effective divisor $D = ah - \sum_{i=1}^{9} a_i l_i \in \mathrm{Pic}(X_9)$ must have $a = D \cdot h \geq 0$. It is well known that all (-1)-classes are effective, and there are infinite number of them in X_9. There are also infinite number of (-2)-classes, but whether they are effective or not depends on the positions of the nine blow-up points.

Definition 10. Let $x_1, \ldots, x_n$ be n distinct points in $\mathbb{P}^2$. These n points are said to be nonspecial with respect to Cremona transformations if for any Cremona transformation T with centers within x_i's, the points $y_1, \ldots, y_n$ corresponding to x_i's under T are distinct points such that no three points among $y_1, \ldots, y_n$ are collinear. □

Definition 11 ([12])**.** Let $x_1, \ldots, x_9$ be nine points in $\mathbb{P}^2$, we say they are in general position if they satisfy the following three conditions:

(i) they are distinct points in $\mathbb{P}^2$;
(ii) they are nonspecial with respect to Cremona transformations;
(iii) there is a unique cubic curve passing through all of them. □

The conditions (i) and (ii) mean that any eight of these nine points are in general position. That is, no lines pass through three of them, no conics pass through six of them, and no cubic curves pass through eight of them with one of the eight points being a double point.

If the 9 blowing up points are in general position, then there is no effective (-2)-class in X_9 [12]. In general, there are at most finite number of $(-m)$-curves with $m \geq 3$.

Lemma 12. Let $D = ah - \sum_{i=1}^{9} a_i l_i$ be a $(-m)$-curve in X_9 with $m \geq 3$, then

(i) $m \leq 9$;
(ii) $0 \leq a \leq 3$;
(iii) $-1 \leq a_i \leq 2$ for all i, and there exists some j with $a_j = 1$;
(iv) there are finite number of such curves. □

Proof. (i) Since D is a $(-m)$-curve, $D \cdot D = -m$ and $D \cdot K = m-2$, that is,

$$\sum a_i^2 = a^2 + m \quad \text{and} \quad \sum a_i = 3a + m - 2.$$

From the above two equations, we have

$$(3a+m-2)^2 = \left(\sum a_i\right)^2 \leq 9\left(\sum a_i^2\right) = 9(a^2+m).$$

Thus, $a \leq \frac{-m^2+13m-4}{6(m-2)}$, also $a \geq 0$ since D is effective, hence $m \leq 12$.

When $m \geq 10$, we must have $a = 0$, that means $\sum a_i^2 = m$ and $\sum a_i = m-2$, hence $\sum a_i^2 - \sum a_i = 2$, which implies every a_i satisfies $|a_i| \leq 1$ and there exists exactly one a_i with $a_i = -1$. But we also have $\sum a_i = m-2 \geq 8$, which is impossible since we only have nine a_i's.

(ii) When $m \geq 4$, $a \leq \frac{-m^2+13m-4}{6(m-2)} \leq \frac{8}{3} < 3$. When $m = 3$, $a \leq \frac{-m^2+13m-4}{6(m-2)} = \frac{13}{3} < 5$. Hence we only need to prove there is no (-3)-curve with $a = 4$.

Suppose not, then there exists a_i's such that $\sum a_i^2 = 19$ and $\sum a_i = 13$. From $\sum a_i^2 - \sum a_i = 6$, we know $-2 \leq a_i \leq 3$. If there is any a_i with $a_i = 3$, then the other a_i's can only be 0 or 1, but we have $\sum a_i = 13$ and there is only nine a_i's, which is impossible. Hence $-2 \leq a_i \leq 2$, from $\sum a_i^2 - \sum a_i = 6$, we can have at most three a_i's equal to 2, which is also impossible since $\sum a_i = 13$.

(iii) From $\sum a_i^2 = a^2+m$, $\sum a_i = 3a+m-2$ and $0 \leq a \leq 3$, we have

$$\sum a_i = 3a+m-2 \geq a^2+m-2 = \sum a_i^2 - 2.$$

Hence $-1 \leq a_i \leq 2$. And there are three cases:

Case 1, one a_i equal to 2, the others equal to 0 or 1;

Case 2, one a_i equal to -1, the others equal to 0 or 1;

Case 3, all a_i's are equal to 0 or 1.

By $\sum a_i = 3a+m-2 \geq 1$, we know in case 2 and case 3, there must exist some a_i with $a_i = 1$. In case 1, if there is no a_i with $a_i = 1$, then $D = ah - 2l_j$. From $\sum a_i^2 = a^2+m$, $\sum a_i = 3a+m-2$, we have $a = 0$, $m = 4$, hence $D = -2l_j$, which is not an effective divisor.

(iv) It is obvious from the above results. ■

From this lemma, we can easily obtain the following as a corollary.

Corollary 13. If there exists a $(-m)$-curve in X_9 with $m \geq 3$, then there also exists a $(-m+1)$-curve in X_9.

Proof. If $D \in |ah - \sum a_i l_i|$ is a $(-m)$-curve in X_9 with $m \geq 3$, then there exists j with $a_j = 1$ by (iii) of Lemma 12. It is easy to check that $D + l_j$ is a $(-m+1)$-curve in X_9. ■

If the nine blowing up points are in general position, then there is no (-2)-curve in X_9, as a consequence, there is also no $(-m)$-curve in X_9 with $m \geq 3$. The following result shows that this happens exactly when X_9 is almost Fano. We include a proof here as we could not find it in the literatures.

Lemma 14. X_9 has no $(-m)$-curve with $m \geq 3$ if and only if $-K_{X_9}$ is nef. □

Proof. If $-K$ is nef, then from $C \cdot K^{-1} = 2 - m \geq 0$ for any $(-m)$-curve C, we know $m \leq 2$.

Conversely, assume X_9 has no $(-m)$-curve with $m \geq 3$. Since X_9 is a blowup of $\mathbb{P}^2$ at nine points $\{x_i\}_{i=1}^9$, we have an effective anti-canonical divisor D. Recall when $D \cdot \Sigma < 0$ for any irreducible curve Σ in X, Σ must be a component of D. So if D is an irreducible curve or a affine ADE curve, then D is nef. We denote the image of D in $\mathbb{P}^2$ as C, which is a cubic curve passing through these 9 blowing up points.

(i) If C is smooth, then we are done as $D \cong C$ and therefore irreducible.

(ii) If C is reduced and irreducible, then it must be a nodal or cuspidal cubic. If $\{x_i\}_{i=1}^9 \cap \operatorname{sing}(C) = \varnothing$ ($\operatorname{sing}(C)$ means the set of singular points on C), then $D \cong C$ and we are done. Otherwise, say $x_1 \in \operatorname{sing}(C)$ and we write the strict and proper transformations of C in $Bl_{x_1}(\mathbb{P}^2)$ as C_1 and $C_1 + E$, respectively. Then the remaining x_i's must have exactly 1 point (respectively, 7 points) lying on E (respectively, C_1) in order to avoid having $(-m)$-curve with $m \geq 3$. Thus, D is a affine ADE curve of type $\hat{A}_1$ or $III(\hat{A}_1)$ for C being a nodal or cuspidal, respectively.

(iii) If C is reduced and reducible, then $C = B \cup H_0$ or $H_1 \cup H_2 \cup H_3$ with B and H_j's are conic and distinct lines in $\mathbb{P}^2$. As before, we must have exactly 6 x_i's on B and 3 x_i's on each H_j and none on $\operatorname{sing}(C)$. Thus, $D \cong C$ is a affine ADE curve of type $\hat{A}_1$, $\hat{A}_2$, $III(\hat{A}_1)$, or $VI(\hat{A}_2)$.

(iv) If C is nonreduced, $C = 3H$, D must have a $(-m)$-curve with $m \geq 3$.

Hence D is an irreducible curve or a affine ADE curve, we are done. ■

In the following two lemmas, we will use [1, Lemma 2.21] to give a criteria of a curve in X_n being an ADE or affine ADE curve. Lemma 2.21 can be reformulated as follows: if $C = \bigcup_{i=1}^r C_i$ is a connected curve in a surface X satisfying: (i) $C_i^2 = -2$ and $C_i \cdot K_X = 0$ for any i; (ii) $C_i \cdot C_j \leq 1$ for any $i \neq j$; (iii) $(C_i \cdot C_j)_{r\times r} \leq 0$. Then when $(C_i \cdot C_j)_{r\times r} < 0$, C is an ADE curve, otherwise, it is an affine ADE curve.

Lemma 15. Suppose $-K_{X_n}$ $(n \leq 8)$ is nef. Let $C = \cup C_i$ be a connected curve in X_n. If $C \cdot K_{X_n} = 0$, then C is an ADE curve. □

Proof. Since $-K_{X_n}$ is nef, $C \cdot K_{X_n} = 0$ implies $C_i \cdot K_{X_n} = 0$ for each i, that is, $[C_i] \in \langle K \rangle^{\perp} \cong \Lambda_{E_n}$. We have $C_i^2 < 0$ and $(C_i + C_j)^2 < 0$ for any i and j. Together with the genus formula, we have $C_i^2 = -2$ and $C_i \cdot C_j \leq 1$ for $i \neq j$. By [1, Lemma 2.21], we know C is an ADE curve. ■

For $n = 9$ case, we have the following lemma.

Lemma 16. Suppose $-K_{X_9}$ is nef. Let $C = \cup C_i$ be a connected curve in X_9. If $C \cdot K_{X_9} = 0$ and $C_i + K_{X_9}$ is not effective for each i, then C is a smooth elliptic curve, an ADE curve or an affine ADE curve. □

Proof. Since $-K_{X_9}$ is nef, $C \cdot K_{X_9} = 0$ implies $C_i \cdot K_{X_9} = 0$ for each i, that is, $[C_i] \in \langle K_{X_9} \rangle^{\perp} \cong \Lambda_{E_9}$. We have $C_i^2 \leq 0$ and $(C_i + C_j)^2 \leq 0$ for any i and j. Moreover, for any effective divisor $D \in \langle K_{X_9} \rangle^{\perp}$, if $D^2 = 0$, then $D \in |mK_{X_9}|$ for some nonzero integer m. From $C_i^2 \leq 0$ and genus formula, we have $C_i^2 = -2$ or 0.

If there exists C_i such that $C_i^2 = 0$, then $C_i \in |mK|$ for some nonzero integer m. Since $C_i + K_{X_9}$ is not effective, we know $m = -1$, that is, $C_i \in |-K|$. If C is not irreducible, then there exists C_j which intersects C_i, which is impossible. So $C = C_i \in |-K|$ is an elliptic curve or an affine A_0 curve by Lemma 14.

If $C_i^2 = -2$ for any i, then $C_i \cdot C_j \leq 2$ for any $i \neq j$. If there exist C_i and C_j such that $C_i \cdot C_j = 2$, then $(C_i + C_j)^2 = 0$, $C_i + C_j \in |mK|$ for some integer m. Hence $C = C_i \cup C_j$ is an affine A_1 curve, this is because if C_k is another irreducible component of C and assume it intersects with C_i, then it must be an irreducible component of C_j, which contradicts to C_j being irreducible. Otherwise, we will have $C_i^2 = -2$ for each i and $C_i \cdot C_j \leq 1$ for $i \neq j$. By [1, Lemma 2.21], we know C is an ADE or affine ADE curve. ■

5 Proof of Theorems 8 and 9

Proof of Theorem 8. If the nine blowup points in $\mathbb{P}^2$ are in general position, then for any $\alpha \in \Phi_9$, we have $h^0(X, O(\alpha)) = 0$ [12]. Since $K \cdot K = 0$, we also have $K - \alpha \in \Phi_9$ and therefore $h^2(X, O(\alpha)) = 0$ by Serre duality. However, the Riemann–Roch formula gives $\chi(X, O(\alpha)) = 1 + \frac{\alpha^2 - \alpha K}{2} = 0$ and therefore $h^1(X, O(\alpha)) = 0$. For the imaginary roots mK's, from [12, Lemma 4 and Proposition 11], we have $h^0(X, O(mK)) = 0$ and $h^0(X, O(-mK)) = 1$ for $m \geq 1$. By Serre duality and Riemann–Roch formula, we have $h^1(X, O(mK)) = 0$ for any imaginary root mK. Hence $\mathcal{E}_0^{\hat{E}_8}$ is totally nondeformable.

Conversely, if $\mathcal{E}_0^{\hat{E}_8}$ is totally nondeformable, then X has no (possibly reducible) (-2)-curve, hence no $(-n)$-curve with $n \geq 2$. By [16, Proposition 10], this implies the nine blowup points are nonspecial with respect to Cremona transformations. Also from $h^1(X, O(mK)) = 0$ for any imaginary root mK, we obtain $h^0(X, O(-K)) = 1$, we have a unique cubic curve in $\mathbb{P}^2$ passing through all of the blow-up points. Hence, the nine blow-up points in $\mathbb{P}^2$ are in general position. ■

Proof of Theorem 9. (i) We have $h^1(X, O(-mK)) = h^0(X, O(-mK)) - 1$ for any m by Riemann–Roch formula. So $\mathcal{E}_0^{\hat{E}_8}$ is deformable in $(-mK)$-direction if and only if $h^0(X, O(-mK)) = 2$.

Let $F_0 \in |-K|$, then by [2, Proposition 2.2], X admits an elliptic fibration with a multiple fiber of multiplicity m if and only if $O_{F_0}(F_0)$ is of order m in $\mathrm{Pic}(F_0)$. But $O_{F_0}(mF_0) \cong O_{F_0}$ if and only if $h^0(O_{F_0}(mF_0)) = 1$ as $O_{F_0}(mF_0)$ is topologically trivial. By the exact sequence

$$0 \longrightarrow O_X \longrightarrow O_X(mF_0) \longrightarrow O_{F_0}(mF_0) \longrightarrow 0$$

together with $h^1(X, O_X) = 0$, we know $h^0(O_X(mF_0)) = 1 + h^0(O_{F_0}(mF_0))$. So $m = \min\{n : h^0(O_{F_0}(nF_0)) = 1\} = \min\{n : h^0(X, O(-nK)) = 2\}$.

(ii) If X has an ADE curve C of type $\mathfrak{g}$, we can use it to construct a fully deformable $\mathfrak{g}$-subbundle of $\mathcal{E}_0^{\hat{E}_8}$ as in Section 3.2. When C is maximal, then this $\mathfrak{g}$-subbundle is not contained in any other fully deformable Lie algebra subbundle of $\mathcal{E}_0^{\hat{E}_8}$.

Conversely, if $\mathcal{E}_0^{\hat{E}_8}$ is maximal $\mathfrak{g}$-deformable, then we can find a base $\Delta \subset \Phi_{\hat{E}_8}$ of $\mathfrak{g}$ such that $h^1(X, O(\alpha)) \neq 0$ for every $\alpha \in \Delta$. Since $\chi(O(\alpha)) = 1 + \frac{\alpha^2 - \alpha \cdot K}{2} = 0$, we must have $h^0(O(\alpha)) \neq 0$ or $h^2(O(\alpha)) = h^0(O(K - \alpha)) \neq 0$, that is either α or $K - \alpha$ is effective. Hence, there must exist some integers m's such that $\alpha + mK$ is effective because $-K$ is effective, we denote the largest such m as m_α.

We claim that for every $\alpha \in \Delta$, $C_\alpha \in |\alpha + m_\alpha K|$ is an irreducible (-2)-curve. If so, then $C = \bigcup_{\alpha \in \Delta} C_\alpha$ is a maximal ADE curve of type $\mathfrak{g}$. If there exists reducible C_α, we write $C_\alpha = \cup D_i$. Then each D_i is perpendicular to K as $-K$ is nef and $C_\alpha \cdot K = 0$. Since $C_\alpha + K$ is not effective, every $D_i + K$ is also not effective and $D_i \notin |-K|$. Hence $D_i^2 = -2$ for any i as $D_i^2 = 0$ will imply $D_i \in |-K|$. We know C_α is connected, this is because if C_α is not connected, then one of its connected component must have self-intersection zero from $C_\alpha^2 = -2$, which contradicts to $C_\alpha + K$ is not effective. Hence $C = \bigcup_{\alpha \in \Delta} C_\alpha$ is an (affine) ADE curve by Lemma 16. It is obvious that this curve strictly contains a $\mathfrak{g}$-curve, which contradicts to $\mathcal{E}_0^{\hat{E}_8}$ being maximal $\mathfrak{g}$-deformable.

(iii) The proof is similar to (ii). ■

Remark 17. If X_9 admits an elliptic fibration, then we can find m such that $h^1(X_9, O(-mK)) \neq 0$. Conversely, if $h^1(X_9, O(-mK)) \neq 0$, we need to add the condition of

$-K$ being nef to show that X admits an elliptic fibration. To see this, we take $x_1, \ldots, x_5$ to be five points on a line $l \subset \mathbb{P}^2$, and another four generic points (not on l) $x_6, \ldots, x_9$ in $\mathbb{P}^2$. Then we have an one parameter family of conics C_t's passing through these four points. If we blow up $\mathbb{P}^2$ at these nine points and denote the strict transforms of l and C_t with same notations, then $l^2 = -4$, $C_t^2 = 0$. Moreover, $C_t + l \in |-K|$ and $h^0(X_9, O(-K)) = 2$. But $-K$ is not nef as $(-K) \cdot l = -2$, which implies that X_9 is not elliptic. □

From the above, we can easily deduce similar results for the E_n-bundle $\mathcal{E}_0^{E_n}$ over X_n when $n \leq 8$, namely

(i) $\mathcal{E}_0^{E_n}$ is totally nondeformable if and only if the n blowup points in $\mathbb{P}^2$ are in general position.

(ii) When $-K_{X_n}$ nef, $\mathcal{E}_0^{E_n}$ is maximal $\mathfrak{g}$-deformable if and only if X_n has a maximal $\mathfrak{g}$ curve.

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