Math. Z.
DOI 10.1007/s00209-016-1645-6

Mathematische Zeitschrift

Affine ADE bundles over surfaces with $p_g = 0$

Yunxia Chen[1] · **Naichung Conan Leung**[2]

Received: 5 June 2014 / Accepted: 7 February 2016

Abstract Given any Kodaira curve C in a complex surface X, we construct a simply-laced affine Lie algebra bundle $\mathcal{E}$ over X. When $p_g(X) = 0$, we construct deformations of holomorphic structures on $\mathcal{E}$ such that the new bundle is trivial over any ADE curve C' inside C and therefore descends to the singular surface obtained by contracting C'.

1 Introduction

Let X be a complex surface and $\Lambda \subset \text{Pic}\,(X)$ be a sublattice. If Λ is isomorphic to the root lattice $\Lambda_{\mathfrak{g}}$ of a simple Lie algebra $\mathfrak{g}$, then we have a root system Φ of $\mathfrak{g}$ and we can associate a Lie algebra bundle $\mathcal{E}_0^{\mathfrak{g}}$ over X [6,10,11]:

$$\mathcal{E}_0^{\mathfrak{g}} := \mathcal{O}_X^{\oplus r} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{O}_X(\alpha).$$

This can be generalized to the affine Lie algebra $\widehat{\mathfrak{g}}$ [9].

There are many instances when this happens. Here we list the following three cases as examples:

(1) When X_n is a del Pezzo surface, namely a blowup of $\mathbb{P}^2$ at $n \leq 8$ points in general position (or $\mathbb{P}^1 \times \mathbb{P}^1$), $\langle K_{X_n} \rangle^{\perp} \subset \text{Pic}\,(X_n)$ is isomorphic to Λ_{E_n}. Thus we have an E_n-bundle over X_n. By restriction, we have an E_n-bundle over any anti-canonical curve Σ in X_n. Notice that Σ is always a genus one curve. For a fixed elliptic curve Σ, the above

✉ Yunxia Chen
yxchen76@ecust.edu.cn

Naichung Conan Leung
leung@math.cuhk.edu.hk

1 School of Science, East China University of Science and Technology, Meilong Road 130, Shanghai, China

2 Department of Mathematics, The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

Published online: 05 April 2016

construction gives a bijection between del Pezzo surfaces containing Σ and E_n-bundles over Σ [4,5,7,10,12,14]. Such an identification was predicted by the F-theory/string duality in physics [7]. This was generalized to all simple Lie algebras in [10,11]. When $n = 9$, X_9 is not Fano and $E_9 = \hat{E}_8$ is an affine Lie algebra. Corresponding results for the $\hat{E}_8$-bundle over X_9 are obtained in [9].

(2) When $\widetilde{X}$ is the canonical resolution of a surface X with a rational double point of type $\mathfrak{g}$, then the corresponding exceptional curve $C = \bigcup C_i$ is an *ADE* curve of type $\mathfrak{g}$. Therefore all these C_i span a sublattice of $\mathrm{Pic}(\widetilde{X})$ which is isomorphic to $\Lambda_{\mathfrak{g}}$, thus giving a $\mathfrak{g}$-bundle $\mathcal{E}_0^{\mathfrak{g}}$ over $\widetilde{X}$. When $p_g(X) = 0$, there exists a deformation $\mathcal{E}_\varphi^{\mathfrak{g}}$ of $\mathcal{E}_0^{\mathfrak{g}}$ such that $\mathcal{E}_\varphi^{\mathfrak{g}}$ is trivial over each C_i, thus it can descend to the singular surface X [2].

(3) When X is a relatively minimal elliptic surface, Kodaira classified all possible singular fibers (see e.g., [1]) and we call such a curve $C = \bigcup C_i$ a Kodaira curve. Its irreducible components C_i span a sublattice of $\mathrm{Pic}\,(X)$ which is isomorphic to the root lattice of an affine root system $\Phi_{\widehat{\mathfrak{g}}}$ and therefore we can construct an affine Lie algebra bundle $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ over X.

The motivations of this paper have three aspects. One is to generalize the results for *ADE* bundles in [2] to affine *ADE* bundles (see Theorem 1 below). The second is the natural question: if the del Pezzo surface X_n (resp. X_9) has a rational double point, does the E_n-bundle (resp. $\hat{E}_8$-bundle) still exist? For this question, Friedman and Morgan gave a positive answer for del Pezzo surfaces [6]. In this paper, the authors will give a positive answer for both cases using a very different method (see Remark 22). The third is the following question: for a complex surface X with $p_g(X) = 0$ and containing a Kodaira curve C, there is a natural affine *ADE* bundle of the corresponding type over it, can we deform this bundle such that it can descend to the singular surface obtained by contracting any *ADE* curve C' inside C (Remark 23)?

Theorem 1 (Lemma 13, Proposition 17 and Theorem 21) *Let X be a complex surface with $p_g = 0$. If X has a Kodaira curve $C = \bigcup_{i=0}^r C_i$ of type $\widehat{\mathfrak{g}}$, then*

(i) *given any*

$$(\varphi_{C_i})_{i=0}^r \in \Omega^{0,1}\left(X, \bigoplus_{i=0}^r \mathcal{O}(C_i)\right)$$

with $\overline{\partial}\varphi_{C_i} = 0$ for every i, it can be extended to

$$\varphi = (\varphi_\alpha)_{\alpha\in\Phi_{\widehat{\mathfrak{g}}}^+} \in \Omega^{0,1}\Big(X, \bigoplus_{\alpha\in\Phi_{\widehat{\mathfrak{g}}}^+} \mathcal{O}(\alpha)\Big)$$

such that $\overline{\partial}_\varphi := \overline{\partial} + \mathrm{ad}(\varphi)$ is a holomorphic structure on $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$. We denote the new bundle as $\mathcal{E}_\varphi^{\widehat{\mathfrak{g}}}$;

(ii) *the new holomorphic structure $\overline{\partial}_\varphi$ is compatible with the Lie algebra structure on $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$;*

(iii) *the new bundle $\mathcal{E}_\varphi^{\widehat{\mathfrak{g}}}$ is trivial on C_i if and only if*

$$[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(C_i, \mathcal{O}_{C_i}(C_i)) \cong \mathbb{C};$$

(iv) *there exists $[\varphi_{C_i}] \in H^1(X, \mathcal{O}(C_i))$ such that $[\varphi_{C_i}|_{C_i}] \neq 0$.*

The organization of this paper is as follows. Section 2 gives the construction of the (affine) *ADE* Lie algebra bundles directly from (affine) *ADE* curves. In Sect. 3, we assume $p_g(X) = 0$.

We construct deformations of the holomorphic structures on these bundles such that the new bundles are trivial over irreducible components of the curve.

Notation for a holomorphic bundle $(\mathcal{E}_0, \overline{\partial}_0)$ with $\mathcal{E}_0 = \bigoplus_i \mathcal{O}(D_i)$, $\overline{\partial}_0$ means the $\overline{\partial}$-operator for the direct sum holomorphic structure. If we construct a new holomorphic structure $\overline{\partial}_\varphi$ on $\mathcal{E}_0$, we denote the resulting bundle as $\mathcal{E}_\varphi$.

2 Affine *ADE* bundles from affine *ADE* curves

2.1 *ADE* and affine *ADE* curves

Definition 2 A curve $C = \bigcup C_i$ in a surface X is called an *ADE* (resp. affine *ADE*) curve of type $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) if each C_i is a smooth (-2)-curve in X and the dual graph of C is a Dynkin diagram of the corresponding type.

It is known that C is an *ADE* curve if and only if C can be contracted to a rational double point. In this case, the intersection matrix $(C_i \cdot C_j)$ is negative definite [1].

If C is an affine *ADE* curve, then the intersection matrix $(C_i \cdot C_j)$ is non positive definite and there exists n_i (these are unique if we ask n_i to be positive integers without common integers) such that $F := \sum n_i C_i$ satisfies $F \cdot F = 0$. Dynkin diagrams of affine *ADE* types are drawn as follows and the corresponding $n_i C_i$ are labelled in the pictures. *ADE* Dynkin diagrams can be obtained by removing the node corresponding to C_0 (Fig. 1).

Remark 3 We will also call a nodal or cuspidal rational curve with trivial normal bundle an $\widehat{A}_0$ curve.

Remark 4 By Kodaira's classification of singular fibers of relative minimal elliptic surfaces, every singular fiber is an affine *ADE* curve unless it is rational with a cusp, tacnode or triplepoint (corresponding to type *II* or $III(\widehat{A}_1)$ or $VI(\widehat{A}_2)$ in Kodaira's notation), which can also be regarded as a degenerated affine *ADE* curve of type $\widehat{A}_0$, $\widehat{A}_1$ or $\widehat{A}_2$ respectively. In this paper, we will not distinguish affine *ADE* curves from their degenerated forms since they have the same intersection matrices. We also call the affine *ADE* curves as Kodaira curves.

Definition 5 A bundle E is called an *ADE* (resp. affine *ADE*) bundle of type $\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) if E has a fiberwise Lie algebra structure of the corresponding type.

In the following two subsections, we will recall an explicit construction of the Lie algebra $\mathfrak{g}$-bundle, loop Lie algebra $L\mathfrak{g}$-bundle and the affine Lie algebra $\widehat{\mathfrak{g}}$-bundle from (affine) *ADE* curves in X.

2.2 *ADE* bundles

Suppose $C = \bigcup_{i=1}^r C_i$ is an *ADE* curve of type $\mathfrak{g}$ in X. We will construct the corresponding *ADE* bundle $\mathcal{E}_0^{\mathfrak{g}}$ over X as follows [2].

Note the rank r of $\mathfrak{g}$ equals the number of C_i. We set $\Phi := \{\alpha = \left[\sum_{i=1}^r a_i C_i\right] \in H^2(X, \mathbb{Z}) | \alpha^2 = -2\}$. Then Φ is a simply-laced root system of $\mathfrak{g}$ with a base $\Delta := \{[C_i] | i = 1, 2, \ldots, r\}$. We have a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots. We define a bundle $\mathcal{E}_0^{(\mathfrak{g},\Phi)}$ over X as follows:

$$\mathcal{E}_0^{(\mathfrak{g},\Phi)} := \mathcal{O}^{\oplus r} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{O}(\alpha).$$

Here $\mathcal{O}(\alpha) = \mathcal{O}\left(\sum_{i=1}^{r} a_i C_i\right)$ where $\alpha = \left[\sum_{i=1}^{r} a_i C_i\right]$. There is an inner product $\langle , \rangle$ on Φ defined by $\langle \alpha, \beta \rangle := -\alpha \cdot \beta$, negative of the intersection form.

For every open chart U of X, we take x_α^U to be a nonvanishing section of $\mathcal{O}_U(\alpha)$ and h_i^U $(1 \leq i \leq r)$ nonvanishing sections of $\mathcal{O}_U^{\oplus r}$. Define a Lie algebra structure $[,]_\Phi$ on $\mathcal{E}_0^{(\mathfrak{g},\Phi)}$ such that $\{x_\alpha, \alpha \in \Phi; h_i, 1 \leq i \leq r\}$ is the Chevalley basis [8], i.e.,

(a) $[h_i^U, h_j^U]_\Phi = 0,\ 1 \leq i,\ j \leq r$.
(b) $[h_i^U, x_\alpha^U]_\Phi = \langle \alpha,\ C_i \rangle x_\alpha^U,\ 1 \leq i \leq r,\ \alpha \in \Phi$.
(c) $[x_\alpha^U, x_{-\alpha}^U]_\Phi = h_\alpha^U$ is a $\mathbb{Z}$-linear combination of h_i^U.
(d) If $\alpha,\ \beta$ are independent roots, and $\beta - p\alpha, \ldots, \beta + q\alpha$ is the α-string through β, then $[x_\alpha^U, x_\beta^U]_\Phi = 0$ if $q = 0$, otherwise $[x_\alpha^U, x_\beta^U]_\Phi = \pm(p+1)x_{\alpha+\beta}^U$.

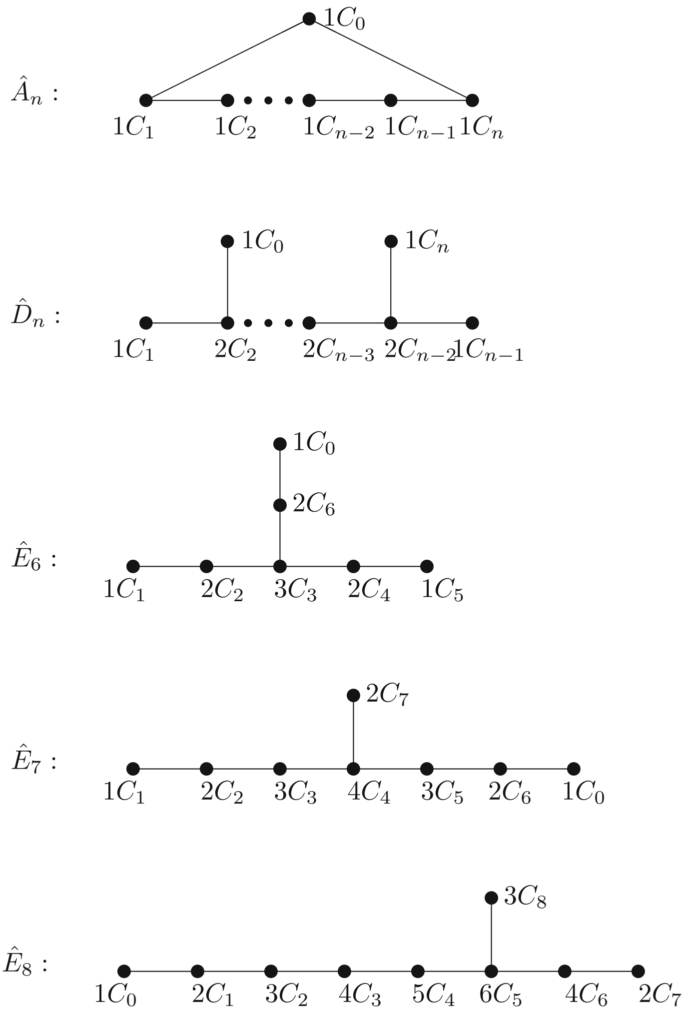

Fig. 1 Dynkin diagrams of affine *ADE* types

Since $\mathfrak{g}$ is a simply-laced Lie algebra, all the roots for $\mathfrak{g}$ have the same length, we have any α-string through β is of length at most 2. So (d) can be written as $[x_\alpha^U, x_\beta^U]_\Phi = n_{\alpha,\beta} x_{\alpha+\beta}^U$, where $n_{\alpha,\beta} = \pm 1$ if $\alpha+\beta \in \Phi$, otherwise $n_{\alpha,\beta} = 0$. It is easy to check that these Lie algebra structures are compatible with different trivializations of $\mathcal{E}_0^{(\mathfrak{g},\Phi)}$ (see page 10 of [10] for more details). Hence $\mathcal{E}_0^{(\mathfrak{g},\Phi)}$ is a Lie algebra bundle of type $\mathfrak{g}$ over X.

2.3 Affine *ADE* bundles

Suppose $C = \bigcup_{i=0}^r C_i$ is an affine ADE curve of type $\widehat{\mathfrak{g}}$ in X. We will construct the corresponding affine ADE bundle $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ of type $\widehat{\mathfrak{g}}$ over X as follows.

First, we choose an extended root of $\widehat{\mathfrak{g}}$, say C_0, then $\mathfrak{g}$ is corresponding to the Dynkin diagram consists of those C_i with $i \neq 0$, i.e.,

$$\Phi := \left\{ \alpha = \left[\sum_{i\neq 0} a_i C_i \right] \in H^2(X,\mathbb{Z}) | \alpha^2 = -2 \right\}$$

is the root system of $\mathfrak{g}$. As above, we have a $\mathfrak{g}$-bundle

$$\mathcal{E}_0^{(\mathfrak{g},\Phi)} = \mathcal{O}^{\oplus r} \oplus \bigoplus_{\alpha\in\Phi} \mathcal{O}(\alpha).$$

We define

$$\mathcal{E}_0^{(L\mathfrak{g},\Phi)} := \bigoplus_{n\in\mathbb{Z}} \left(\mathcal{E}_0^{(\mathfrak{g},\Phi)} \otimes \mathcal{O}(nF) \right)$$

and

$$\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)} := \bigoplus_{n\in\mathbb{Z}} (\mathcal{E}_0^{(\mathfrak{g},\Phi)} \otimes \mathcal{O}(nF)) \oplus \mathcal{O}.$$

We know

$$\Phi_{\widehat{\mathfrak{g}}} := \{\alpha + nF | \alpha \in \Phi, n \in \mathbb{Z}\} \bigcup \{nF | n \in \mathbb{Z}, n \neq 0\}$$

is an affine root system and it decomposes into the union of positive and negative roots, i.e., $\Phi_{\widehat{\mathfrak{g}}} = \Phi_{\widehat{\mathfrak{g}}}^+ \cup \Phi_{\widehat{\mathfrak{g}}}^-$, where

$$\begin{aligned}\Phi_{\widehat{\mathfrak{g}}}^+ &= \left\{ \sum a_i C_i \in \Phi_{\widehat{\mathfrak{g}}} | a_i \geq 0 \ for\ all\ i \right\} \\ &= \left\{ \alpha + nF | \alpha \in \Phi^+, n \in \mathbb{Z}_{\geq 0} \right\} \cup \left\{ \alpha + nF | \alpha \in \Phi^-, n \in \mathbb{Z}_{\geq 1} \right\} \cup \left\{ nF | n \in \mathbb{Z}_{\geq 1} \right\}\end{aligned}$$

and $\Phi_{\widehat{\mathfrak{g}}}^- = -\Phi_{\widehat{\mathfrak{g}}}^+$.

To describe the Lie algebra structures, we proceed as before, for every open chart U of X, we take a local basis e_i^U of $\mathcal{E}_0^{(\mathfrak{g},\Phi)}|_U$ (e_i^U is just h_j^U or x_α^U as above), e_{nF}^U of $\mathcal{O}(nF)|_U$, e_c^U of $\mathcal{O}|_U$, compatible with the tensor product, for example, $e_{nF}^U \otimes e_{mF}^U = e_{(n+m)F}^U$. Then define

$$\left[e_i^U e_{nF}^U, e_j^U e_{mF}^U \right]_{L\mathfrak{g},\Phi} := [e_i^U, e_j^U]_\Phi e_{(n+m)F}^U, \tag{1}$$

$$\left[e_i^U e_{nF}^U + \lambda e_c^U, e_j^U e_{mF}^U + \mu e_c^U \right]_{\widehat{\mathfrak{g}},\Phi} := \left[e_i^U, e_j^U \right]_\Phi e_{(n+m)F}^U + n\delta_{n+m,0} k \left(e_i^U, e_j^U \right) e_c^U. \tag{2}$$

Here $[,]_\Phi$ is the Lie bracket on $\mathcal{E}_0^{(\mathfrak{g},\Phi)}$ and $k(x,y)=\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$ is the Killing form on $\mathfrak{g}$.

Lemma 6 *The above* (1) [*resp.* (2)] *defines a fiberwise loop* (*resp. affine*) *Lie algebra structure which is compatible with any trivialization of* $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ (*resp.* $\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}$).

Proof See Proposition 23 of [9]. □

From the above lemma, we have the following result.

Proposition 7 *If C is an affine ADE curve of type $\widehat{\mathfrak{g}}$ in X, then $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ (resp. $\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}$) is a loop (resp. affine) Lie algebra bundle of type $L\mathfrak{g}$ (resp. $\widehat{\mathfrak{g}}$) over X.*

Note any C_i with $n_i=1$ can be chosen as the extended root.

Proposition 8 *The loop Lie algebra bundle $(\mathcal{E}_0^{(L\mathfrak{g},\Phi)},[,]_{L\mathfrak{g},\Phi})$ does not depend on the choice of the extended root.*

Proof Suppose C_k $(k\neq 0)$ is another root with $n_k=1$. We set

$$\Psi=\left\{\beta=\left[\sum_{i\neq k}b_iC_i\right]\in H^2(X,\mathbb{Z})|\beta^2=-2\right\}.$$

Then Ψ is a root system of $\mathfrak{g}$. As before, we construct the Lie algebra bundle $\mathcal{E}_0^{(\mathfrak{g},\Psi)}$ and $\mathcal{E}_0^{(L\mathfrak{g},\Psi)}$ from Ψ.

We denote $\alpha_0:=\sum_{i\neq 0}n_iC_i=F-C_0$, the longest root in Φ. For any $\alpha=\sum_{i\neq 0}a_i(\alpha)C_i\in\Phi$, $a_k(\alpha)$ can only be 0, ± 1. Hence there is a bijection between Φ and Ψ given by $\alpha\mapsto\beta=\alpha-a_k(\alpha)F$. Then from the definition of $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ and $\mathcal{E}_0^{(L\mathfrak{g},\Psi)}$, we know they are the same as holomorphic vector bundles.

We compare the Lie brackets on them. We choose a local basis of $\mathcal{E}_0^{(L\mathfrak{g},\Psi)}$ compatible with those of $\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$ and define $[,]_{L\mathfrak{g},\Psi}$ similarly as $[,]_{L\mathfrak{g},\Phi}$, i.e.,

(i) when $\beta=\alpha\in\Phi\cap\Psi$, we take $x_\beta=x_\alpha$;
(ii) when $\beta=\alpha+F\in\Psi^+\backslash\Phi$, we take $x_\beta=x_\alpha e_F$;
(iii) when $\beta=\alpha-F\in\Psi^-\backslash\Phi$, we take $x_\beta=x_\alpha e_{-F}$;
(iv) take h_i $(i\neq 0,k)$ as before, take $h_0=-h_{\alpha_0}$ as we want $[x_{C_0},x_{-C_0}]_{L\mathfrak{g},\Psi}=[x_{-\alpha_0+F},x_{\alpha_0-F}]_{L\mathfrak{g},\Phi}$.

It is obvious $[,]_{L\mathfrak{g},\Psi}=[,]_{L\mathfrak{g},\Phi}$ on $\mathcal{E}_0^{(L\mathfrak{g},\Psi)}\cong\mathcal{E}_0^{(L\mathfrak{g},\Phi)}$. □

For the affine case, we recall that the Killing form of $\mathfrak{g}$ is the symmetric bilinear map $k:\mathfrak{g}\times\mathfrak{g}\to\mathbb{C}$ defined by $k(x,y)=\mathrm{Tr}(\mathrm{ad}(x)\,\mathrm{ad}(y))$. It is ad-invariant, that is for $x,y,z\in\mathfrak{g}$, $k([x,y],z)=k(x,[y,z])$.

Lemma 9 *For any simple simply-laced Lie algebra $\mathfrak{g}$ with a Chevalley basis $\{x_\alpha,\alpha\in\Phi;h_i,1\leq i\leq r\}$ and $m^*(\mathfrak{g})$ the dual Coxeter number of $\mathfrak{g}$, we have*

(i) $k(h_i,x_\alpha)=0$ *for any i and α;*
(ii) $k(x_\alpha,x_\beta)=0$ *for any $\alpha+\beta\neq 0$;*
(iii) $k(h_i,h_j)=2m^*(\mathfrak{g})\langle C_i,C_j\rangle$;
(iv) $k(x_\alpha,x_{-\alpha})=2m^*(\mathfrak{g})$ *for any α.*

Proof Directly from the Killing form k being ad-invariant or see [13]. □

Proposition 10 *The affine Lie algebra bundle* $(\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}, [,]_{\widehat{\mathfrak{g}},\Phi})$ *does not depend on the choice of the extended root.*

Proof Follow the notation in Proposition 8, but we will take

$$h_0 = -h_{\alpha_0} + 2m^*(\mathfrak{g})e_c.$$

We will check that $[,]_{\widehat{\mathfrak{g}},\Psi} = [,]_{\widehat{\mathfrak{g}},\Phi}$ on $\mathcal{E}_0^{(\widehat{\mathfrak{g}},\Psi)} = \mathcal{E}_0^{(\widehat{\mathfrak{g}},\Phi)}$:

(a) when $\beta_1 = \alpha_1 + F, \beta_2 = \alpha_2 + F \in \Psi^+\backslash\Phi,\ \alpha_1, \alpha_2 \in \Phi^-\backslash\Psi$ we have

$$[h_{\beta_1} e_{nF}, h_{\beta_2} e_{mF}]_{\widehat{\mathfrak{g}},\Psi} = n\delta_{n+m,0} k(h_{\beta_1}, h_{\beta_2}) e_c,$$

which is the same with

$$[h_{-\alpha_1} e_{nF}, h_{-\alpha_2} e_{mF}]_{\widehat{\mathfrak{g}},\Phi} = n\delta_{n+m,0} k(h_{\alpha_1}, h_{\alpha_2}) e_c,$$

since $k(h_{\beta_1}, h_{\beta_2}) = 2m^*(\mathfrak{g})\langle\beta_1, \beta_2\rangle = 2m^*(\mathfrak{g})\langle F - \alpha_1, F - \alpha_2\rangle = k(h_{\alpha_1}, h_{\alpha_2})$.

(b) For $[h_i e_{nF}, x_\alpha e_{mF}]_{\widehat{\mathfrak{g}},\Phi}$, automatically from $k(h_i, x_\alpha) = 0$ and loop case.

(c) When $\beta = \alpha + F \in \Psi^+\backslash\Phi,\ \alpha \in \Phi^-\backslash\Psi$,

$$[x_\beta e_{nF}, x_{-\beta} e_{mF}]_{\widehat{\mathfrak{g}},\Psi} = h_\beta e_{(n+m)F} + n\delta_{n+m,0} k(x_\beta, x_{-\beta}) e_c,$$

which is the same with

$$[x_{-\alpha} e_{(n+1)F}, x_\alpha e_{(m-1)F}]_{\widehat{\mathfrak{g}},\Phi} = -h_\alpha e_{(n+m)F} + (n+1)\delta_{n+m,0} k(x_\alpha, x_{-\alpha}) e_c,$$

by considering $m + n = 0$ and $m + n \neq 0$ separately.

(d) For $[x_{\alpha_1} e_{nF}, x_{\alpha_2} e_{mF}]_{\widehat{\mathfrak{g}},\Phi}$ with $\alpha_1 + \alpha_2 \neq 0$, automatically from $k(x_{\alpha_1}, x_{\alpha_2}) = 0$ and loop case. □

For simplicity, we will omit Φ in $(\mathfrak{g}, \Phi)$, $(L\mathfrak{g}, \Phi)$ and $(\widehat{\mathfrak{g}}, \Phi)$ when there is no confusion.

3 Trivialization of $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ over C_i after deformations

If $C = \bigcup C_i$ is an affine ADE curve in X, then the corresponding $F = \sum n_i C_i$ satisfies $F \cdot F = 0$, i.e., $\mathcal{O}_F(F)$ is a topologically trivial bundle. If $\mathcal{O}_F(F)$ is trivial holomorphically and $q(X) = 0$, then from the long exact sequence of cohomologies induced by $0 \to \mathcal{O}_X \to \mathcal{O}_X(F) \to \mathcal{O}_F(F) \to 0$, we know $H^0(X, \mathcal{O}_X(F)) \cong \mathbb{C}^2$. Hence F is a fiber of an elliptic fibration on X.

Suppose X is an elliptic surface, i.e., there is a smooth curve B and a surjective morphism $\pi: X \to B$ whose generic fiber F_b ($b \in B$) is an elliptic curve. Assume π is singular at $b_0 \in B$ and $F_{b_0} = \sum n_i C_i$ is a singular fiber of type $\widehat{\mathfrak{g}}$. Hence, we have a $\widehat{\mathfrak{g}}$-bundle $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ over X. The restriction of $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ to any fiber F_b, other than F_{b_0}, is trivial because $F_b \cap C_i = \varnothing$ for any i. However, $\mathcal{E}_0^{\widehat{\mathfrak{g}}}|_{F_{b_0}}$ is not trivial, for instance $\mathcal{O}(-C_i)|_{C_i} \cong \mathcal{O}_{\mathbb{P}^1}(2)$. Nevertheless, we will show that after deformations of holomorphic structures, $\mathcal{E}_0^{\widehat{\mathfrak{g}}}$ will become trivial on every irreducible component of F_{b_0}.

3.1 Review of *ADE* cases

In our earlier paper [2], we showed how to take successive extensions to make the $\mathfrak{g}$-bundle $\mathcal{E}_0^{\mathfrak{g}}$ trivial on every component C_i of the *ADE* curve $C = \bigcup_{i=1}^r C_i$.

Definition 11 Given any $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha\in\Phi^+} \mathcal{O}(\alpha))$, we define $\overline{\partial}_\varphi \colon \Omega^{0,0}(X, \mathcal{E}_0^{\mathfrak{g}}) \longrightarrow \Omega^{0,1}(X, \mathcal{E}_0^{\mathfrak{g}})$ by

$$\overline{\partial}_\varphi := \overline{\partial}_0 + \mathrm{ad}(\varphi) := \overline{\partial}_0 + \sum_{\alpha\in\Phi^+} \mathrm{ad}\,(\varphi_\alpha)\,,$$

More explicitly, if we write $\varphi_\alpha = c_\alpha^U x_\alpha^U$ locally for some one form c_α^U, then $\mathrm{ad}(\varphi_\alpha) = c_\alpha^U\,\mathrm{ad}(x_\alpha^U)$. It is easy to check that $\overline{\partial}_\varphi$ is well-defined and compatible with the Lie algebra structure, i.e., $\overline{\partial}_\varphi[,]_\Phi = 0$. For $\overline{\partial}_\varphi$ to define a holomorphic structure, we need

$$0 = \overline{\partial}_\varphi^2 = \sum_{\alpha\in\Phi^+}\left(\overline{\partial}_0 c_\alpha^U + \sum_{\beta+\gamma=\alpha}\left(n_{\beta,\gamma} c_\beta^U \wedge c_\gamma^U\right)\right)\mathrm{ad}(x_\alpha^U).$$

That is $\overline{\partial}_0\varphi_\alpha + \sum_{\beta+\gamma=\alpha}(n_{\beta,\gamma}\varphi_\beta \wedge \varphi_\gamma) = 0$ for any $\alpha \in \Phi^+$. Explicitly:

$$\begin{cases} \overline{\partial}_0\varphi_{C_i} = 0 & i \in \{1, 2, \ldots, r\} \\ \overline{\partial}_0\varphi_{C_i+C_j} = n_{C_i,C_j}\varphi_{C_i} \wedge \varphi_{C_j} & \text{if } C_i + C_j \in \Phi^+ \\ \vdots \end{cases}$$

Recall $\{C_i\}_{i=1}^r \subset \Phi^+$ is a base.

Proposition 12 *Given any $(\varphi_{C_i})_{i=1}^r \in \Omega^{0,1}(X, \bigoplus_{i=1}^r \mathcal{O}(C_i))$ with $\overline{\partial}\varphi_{C_i} = 0$ for any i, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha\in\Phi^+} \mathcal{O}(\alpha))$ satisfying $\overline{\partial}_\varphi^2 = 0$, so that we have a holomorphic $\mathfrak{g}$-bundle $\mathcal{E}_\varphi^{\mathfrak{g}}$ over X.*

The proof of this proposition uses the following lemma.

Lemma 13 *If $p_g(X) = 0$, then*

(i) *for any $\alpha \in \Phi^+$, $H^2(X, \mathcal{O}(\alpha)) = 0$.*
(ii) *the restriction homomorphism $H^1(X, \mathcal{O}_X(C_i)) \to H^1(X, \mathcal{O}_{C_i}(C_i))$ is surjective.*

Theorem 14 *For any given i, the holomorphic $\mathfrak{g}$-bundle $\mathcal{E}_\varphi^{\mathfrak{g}}$ over X is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$.*

The proof of this theorem can be found in Theorem 9 of [2]. Note that part (ii) of Lemma 13 says that such φ_{C_i} can always be found.

3.2 Trivializations in loop *ADE* cases

Definition 15 Given any $\varphi = (\varphi_\alpha)_{\alpha\in\Phi_{\hat{\mathfrak{g}}}^+} \in \Omega^{0,1}(X, \bigoplus_{\alpha\in\Phi_{\hat{\mathfrak{g}}}^+} \mathcal{O}(\alpha))$, we define $\overline{\partial}_{(\varphi,\Phi)} \colon \Omega^{0,0}(X, \mathcal{E}_0^{L\mathfrak{g}}) \longrightarrow \Omega^{0,1}(X, \mathcal{E}_0^{L\mathfrak{g}})$ by $\overline{\partial}_{(\varphi,\Phi)} := \overline{\partial}_0 + \mathrm{ad}(\varphi)$.

More explicitly, if we write $\varphi_\alpha = c_\alpha^U x_\alpha^U$ locally for some one form c_α^U, then by the decomposition of $\Phi_{\hat{\mathfrak{g}}}^+$ in Sect. 2.3, we have (here we omit the local chart U for simplicity):

$$\overline{\partial}_{(\varphi,\Phi)} := \overline{\partial}_0 + \sum_{n\in\mathbb{Z}_{\geq 0}} \sum_{\alpha\in\Phi^+} \left(c_{\alpha+nF}\,\mathrm{ad}(x_\alpha e_{nF}) + c_{-\alpha+(n+1)F}\,\mathrm{ad}\left(x_{-\alpha} e_{(n+1)F}\right)\right)$$
$$+ \sum_{n\in\mathbb{Z}_{\geq 0}} \sum_{i=1}^{r} c^i_{(n+1)F}\,\mathrm{ad}\left(h_i e_{(n+1)F}\right).$$

Proposition 16 $\overline{\partial}_{(\varphi,\Phi)}$ *is compatible with the Lie algebra structure on* $\mathcal{E}_0^{L\mathfrak{g}}$.

Proof $\overline{\partial}_{(\varphi,\Phi)}[,]_{L\mathfrak{g},\Phi} = 0$ follows directly from the Jacobi identity. □

For $\overline{\partial}_{(\varphi,\Phi)}$ to define a holomorphic structure, we need $\overline{\partial}^2_{(\varphi,\Phi)} = 0$, which is equivalent to the following equations:

$$\begin{cases} \overline{\partial}_0 \varphi^i_{nF} &= \sum_{p+q=n} \sum_{\alpha\in\Phi^+} \pm a_i(h_\alpha)\varphi_{\alpha+pF} \wedge \varphi_{-\alpha+qF}, \\ \overline{\partial}_0 \varphi_{\alpha+nF} &= \sum_{p+q=n} \sum_{\alpha_1+\alpha_2=\alpha} \pm\varphi_{\alpha_1+pF} \wedge \varphi_{\alpha_2+qF} \\ & \quad + \sum_{p+q=n} \sum_{i=1}^r \langle \alpha, C_i\rangle\, \varphi_{\alpha+pF} \wedge \varphi^i_{qF}, \\ \overline{\partial}_0 \varphi_{-\alpha+nF} &= \sum_{p+q=n} \sum_{\alpha_2-\alpha_1=\alpha} \pm\varphi_{\alpha_1+pF} \wedge \varphi_{-\alpha_2+qF} \\ & \quad + \sum_{p+q=n} \sum_{i=1}^r \langle -\alpha, C_i\rangle\, \varphi_{-\alpha+pF} \wedge \varphi^i_{qF}, \end{cases}$$

where $a_i(h_\alpha)$ is the coefficient of h_i in h_α.

Proposition 17 *Given any* $(\varphi_{C_i})_{i=0}^r \in \Omega^{0,1}(X, \bigoplus_{i=0}^r \mathcal{O}(C_i))$ *with* $\overline{\partial}\varphi_{C_i} = 0$ *for every* i, *it can be extended to* $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+_{\hat{\mathfrak{g}}}} \in \Omega^{0,1}(X, \bigoplus_{\alpha\in\Phi^+_{\hat{\mathfrak{g}}}} \mathcal{O}(\alpha))$ *satisfying* $\overline{\partial}^2_\varphi = 0$. *Namely we have a holomorphic* $L\mathfrak{g}$*-bundle* $\mathcal{E}_\varphi^{L\mathfrak{g}}$ *over* X.

In order to prove this proposition, we need the following lemma.

Lemma 18 *If* $p_g(X) = 0$, *then for any* $\alpha \in \Phi^+$, $n \in \mathbb{Z}_{\geq 0}$, $H^2(X, \mathcal{O}(nF))$, $H^2(X, \mathcal{O}(\alpha + nF))$ *and* $H^2(X, \mathcal{O}(-\alpha + (n+1)F))$ *are zero.*

Proof Since F is an effective divisor and $H^0(X, K_X) = 0$, we have for any $n \geq 0$, $H^0(X, K_X(-nF)) = 0$. This is equivalent to $H^2(X, \mathcal{O}(nF)) = 0$ by Serre duality. Similarly, $H^2(X, \mathcal{O}(\alpha + nF)) = 0$ follows from $H^0(X, K_X(-\alpha)) \cong H^2(X, \mathcal{O}(\alpha)) = 0$ (Lemma 13). The proof of $H^2(X, \mathcal{O}(-\alpha + (n+1)F)) = 0$ uses the fact that $F - \alpha$ is an effective divisor for any $\alpha \in \Phi^+$. □

Proof of Proposition 17 The equation $\overline{\partial}^2_{(\varphi,\Phi)} = 0$ can be rewritten as follows:

$$\begin{cases} \overline{\partial}_0 \varphi_{C_i} = 0 \quad \text{for } i \in \{1, 2, \ldots, r\}, \\ \overline{\partial}_0 \varphi_\alpha = \sum_{\alpha_1+\alpha_2=\alpha} \left(\pm\varphi_{\alpha_1} \wedge \varphi_{\alpha_2}\right), \\ \overline{\partial}_0 \varphi_{-\alpha_0+F} = \overline{\partial}_0 \varphi_{C_0} = 0, \\ \overline{\partial}_0 \varphi_{-\alpha+F} = \sum_{\alpha_2-\alpha_1=\alpha} \left(\pm\varphi_{\alpha_1} \wedge \varphi_{-\alpha_2+F}\right), \\ \overline{\partial}_0 \varphi^i_F = \sum_{\alpha\in\Phi^+} \left(\pm a_i(h_\alpha)\varphi_\alpha \wedge \varphi_{-\alpha+F}\right), \\ \vdots \end{cases}$$

where $\alpha_0 = F - C_0$ is the longest root in Φ.

Firstly, we can solve for all the $\varphi_\alpha(\alpha \in \Phi^+)$ from $H^2(X, \mathcal{O}(\alpha)) = 0$ (Proposition 12). Secondly, we get all the $\varphi_{-\alpha+F}$ $(\alpha \in \Phi^+)$ from $H^2(X, \mathcal{O}(-\alpha + F)) = 0$. Thirdly, since we have all the φ_α and $\varphi_{-\alpha+F}$, we can solve for all the φ^i_F for $1 \le i \le r$ from $H^2(X, \mathcal{O}(F)) = 0$. Do this process for $\varphi_{\alpha+nF}$, $\varphi_{-\alpha+(n+1)F}$ and $\varphi^i_{(n+1)F}$ inductively on n. □

By Lemma 13, there always exists $\varphi_{C_i} \in \Omega^{0,1}(X, \mathcal{O}(C_i))$ such that $0 \neq [\varphi_{C_i}|_{C_i}] \in H^1(X, \mathcal{O}_{C_i}(C_i)) \cong \mathbb{C}$ for each $i = 0, 1, \ldots, r$.

Theorem 19 *For any given i, the holomorphic $L\mathfrak{g}$-bundle $\mathcal{E}^{L\mathfrak{g}}_\varphi$ over X is trivial on C_i if and only if* $[\varphi_{C_i}|_{C_i}] \neq 0$.

Proof The proof will be given in Sects. 3.4 and 3.5. In Sect. 3.4, we deal with all the loop *ADE* cases except loop E_8 case which will be analyzed in Sect. 3.5. □

3.3 Trivializations in affine *ADE* cases

Follow the notation in Sect. 3.2, we define $\overline{\partial}_{(\varphi,\Phi)} := \overline{\partial}_0 + \mathrm{ad}(\varphi)$ on $\mathcal{E}^{\widehat{\mathfrak{g}}}_0$. Note the adjoint action here is defined using the affine Lie bracket.

Proposition 20 $\overline{\partial}_{(\varphi,\Phi)}$ *is compatible with the Lie algebra structure on* $\mathcal{E}^{\widehat{\mathfrak{g}}}_0$.

Proof $\overline{\partial}_{(\varphi,\Phi)}[,]_{\widehat{\mathfrak{g}},\Phi} = 0$ follows directly from the Jacobi identity and the Killing form being invariant under the adjoint action. □

It is easy to see that $\overline{\partial}^2_{(\varphi,\Phi)} = 0$ in the affine case is equivalent to $\overline{\partial}^2_{(\varphi,\Phi)} = 0$ in the loop case. Hence we have a new holomorphic structure $\overline{\partial}_{(\varphi,\Phi)}$ on $\mathcal{E}^{\widehat{\mathfrak{g}}}_0$.

Theorem 21 *For any given i, the holomorphic $\widehat{\mathfrak{g}}$-bundle $\mathcal{E}^{\widehat{\mathfrak{g}}}_\varphi$ over X is trivial on C_i if and only if* $[\varphi_{C_i}|_{C_i}] \neq 0$.

Proof This follows from Theorem 19, $0 \to \mathcal{O} \to \mathcal{E}^{\widehat{\mathfrak{g}}}_\varphi \to \mathcal{E}^{L\mathfrak{g}}_\varphi \to 0$ and $Ext^1_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}) = H^1(\mathbb{P}^1, \mathcal{O}) = 0$. □

From the construction of $\overline{\partial}_\varphi$ in Sect. 3.1 and $\overline{\partial}_{(\varphi,\Phi)}$ above, we have the following observation: let X be a complex surface with $p_g(X) = 0$. If $\Lambda \subset \mathrm{Pic}(X)$ is isomorphic to the root lattice $\Lambda_\mathfrak{g}$ (resp. $\Lambda_{\widehat{\mathfrak{g}}}$) of *ADE* type (resp. affine *ADE* type) and $C = \bigcup C_i$ is an *ADE* curve of type $\mathfrak{h}$ with each irreducible curve C_i from the corresponding root system $\Phi_\mathfrak{g}$ (resp. $\Phi_{\widehat{\mathfrak{g}}}$), then we can deform the Lie algebra bundle $\mathcal{E}^\mathfrak{g}_0$ (resp. $\mathcal{E}^{\widehat{\mathfrak{g}}}_0$) such that its deformation $\mathcal{E}^\mathfrak{g}_\varphi$ (resp. $\mathcal{E}^{\widehat{\mathfrak{g}}}_\varphi$) is trivial over every C_i. To show this, we will describe the corresponding holomorphic structure $\overline{\partial}_\varphi$ (resp. $\overline{\partial}_{(\varphi,\Phi)}$) in detail. We choose these C_i as basis of $\Phi_\mathfrak{h}$ and extend it to the basis of $\Phi_\mathfrak{g}$ (resp. $\Phi_{\widehat{\mathfrak{g}}}$), then construct $\overline{\partial}_\varphi$ (resp. $\overline{\partial}_{(\varphi,\Phi)}$) as follows:

(1) for $\alpha \in \Phi^+_\mathfrak{g} \backslash \Phi^+_\mathfrak{h}$, take $\varphi_\alpha = 0$;
(2) for $C_i \in \Phi^+_\mathfrak{h}$, take φ_{C_i} such that $[\varphi_{C_i}|_{C_i}] \neq 0$;
(3) for $\alpha \in \Phi^+_\mathfrak{h}$, $\alpha \neq C_i$, take φ_α such that $\overline{\partial}_{(\varphi,\mathfrak{h})} := \overline{\partial}_0 + \sum_{\alpha\in\Phi^+_\mathfrak{h}} \mathrm{ad}(\varphi_\alpha)$ satisfy $\overline{\partial}^2_{(\varphi,\mathfrak{h})} = 0$.

It obviously that such φ_α exist and the corresponding $\overline{\partial}_\varphi$ (resp. $\overline{\partial}_{(\varphi,\Phi)}$) satisfy the integrability condition. And from the above theorem, the new bundle $\mathcal{E}^\mathfrak{g}_\varphi$ (resp. $\mathcal{E}^{\widehat{\mathfrak{g}}}_\varphi$) is trivial over every C_i.

Remark 22 In particular, if the del Pezzo surface X_n (resp. X_9) has a rational double point, then we can construct an E_n-bundle (resp. $\widehat{E}_8$-bundle) on its minimal resolution such that its restriction to each irreducible component of the exceptional locus is trivial, then this E_n-bundle (resp. $\widehat{E}_8$-bundle) can descend to the singular surface X_n (resp. X_9). Therefore for a del Pezzo surface X_n (resp. X_9) with a rational double point, the E_n-bundle (resp. $\widehat{E}_8$-bundle) still exists. The relationship between the deformability of the $\widehat{E}_8$-bundle and the geometry of X_9 is shown in [3].

Remark 23 For a complex surface X with $p_g(X) = 0$ and containing an *ADE* curve (resp. Kodaira curve) C, we have a corresponding type *ADE* bundle (resp. affine *ADE* bundle). If we contract any *ADE* curve C' inside C, then we will get a singular surface with a rational double point. By the above observation, we can deform this bundle such that it can descend to this singular surface.

3.4 Proof (except the loop E_8 case)

In this subsection, we use the symmetry of the affine *ADE* Dynkin diagram (except $\widehat{E}_8$) to show that $\mathcal{E}_\varphi^{L\mathfrak{g}}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$.

Recall that $\mathcal{E}_\varphi^{L\mathfrak{g}}$ and $\mathcal{E}_0^{L\mathfrak{g}}$ have the same underlying C^∞-vector bundle, but with a holomorphic structure $\overline{\partial}_{(\varphi,\Phi)}$ of the following upper triangular block shape:

$$\overline{\partial}_\varphi = \left(\begin{array}{c|c|c|c|c} \ddots & \ddots & \ddots & \ddots & \ddots \\ \hline \ddots & \overline{\partial}_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)} \otimes \mathcal{O}((n+1)F)} & * & * & \ddots \\ \hline \ddots & O & \overline{\partial}_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)} \otimes \mathcal{O}(nF)} & * & \ddots \\ \hline \ddots & O & O & \overline{\partial}_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)} \otimes \mathcal{O}((n-1)F)} & \ddots \\ \hline \ddots & \ddots & \ddots & \ddots & \ddots \end{array}\right).$$

i.e., $\mathcal{E}_\varphi^{L\mathfrak{g}}$ is constructed from successive extensions of these $\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)} \otimes \mathcal{O}(nF)$ $(n \in \mathbb{Z})$.

Note $\overline{\partial}_{(\varphi,\Phi)}|_{\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)}} = \overline{\partial}_0 + \sum_{\alpha\in\Phi^+} \mathrm{ad}(\varphi_\alpha)$. By Theorem 14, for every $i \neq 0$, $\mathcal{E}_\varphi^{(\mathfrak{g},\Phi)}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$. We also know $\mathcal{O}(F)|_{C_i}$ is trivial for every i because $F \cdot C_i = 0$. Thus, when $i \neq 0$, $\mathcal{E}_\varphi^{L\mathfrak{g}}|_{C_i}$ is constructed from successive extensions of trivial vector bundles over $C_i \cong \mathbb{P}^1$. This implies that $\mathcal{E}_\varphi^{L\mathfrak{g}}|_{C_i}$ is trivial if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$ as $Ext^1_{\mathbb{P}^1}(\mathcal{O},\mathcal{O}) = H^1(\mathbb{P}^1,\mathcal{O}) = 0$.

Now we consider $i = 0$. Since $\widehat{\mathfrak{g}} \neq \widehat{E}_8$, the affine Dynkin diagram always admits a diagram automorphism, that means we can write $\mathcal{E}_0^{L\mathfrak{g}}$ as $\bigoplus_{n\in\mathbb{Z}}(\mathcal{E}_0^{(\mathfrak{g},\Psi)} \otimes \mathcal{O}(nF))$ (see Proposition 8). Suppose the extended root corresponding to Ψ is C_k, and the longest root in Ψ is β_0.

We will rewrite the holomorphic structure $\overline{\partial}_{(\varphi,\Phi)}$ in terms of the Ψ root system. Note $\overline{\partial}_{(\varphi,\Phi)}$ is determined by the loop Lie algebra structure which is independent of the choice of the extended root. We choose a local base of $\mathcal{E}_0^{(\mathfrak{g},\Psi)}$ as in Proposition 8 and define $\overline{\partial}_{(\psi,\Psi)}$ to be the same with $\overline{\partial}_{(\varphi,\Phi)}$, then obviously $\psi_D = \varphi_D$ when $D \neq nF$.

Because $(\mathcal{E}_\varphi^{(L\mathfrak{g},\Phi)}, \overline{\partial}_{(\varphi,\Phi)}) = (\mathcal{E}_\psi^{(L\mathfrak{g},\Psi)}, \overline{\partial}_{(\psi,\Psi)})$ as a holomorphic vector bundle, similar to the arguments in $(\mathcal{E}_\varphi^{(L\mathfrak{g},\Phi)}, \overline{\partial}_{(\varphi,\Phi)})$ case, we have when $i \neq k$, $\mathcal{E}_\varphi^{L\mathfrak{g}}$ is trivial on C_i if and only if $[\psi_{C_i}|_{C_i}] \neq 0$. Note $\psi_{C_0} = \varphi_{-\alpha_0+F} = \varphi_{C_0}$. So we have Theorem 19 when $\mathfrak{g} \neq E_8$.

3.5 Proof for the loop E_8 case

Similar to the above subsection, we have when $i \in \{1, 2, \dots, 8\}$, $\mathcal{E}_\varphi^{LE_8}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$. The question is what about C_0?

We recall $\mathcal{E}_0^{E_8} := \mathcal{O}^{\oplus 8} \oplus \bigoplus_{\alpha \in \Phi} \mathcal{O}(\alpha)$. For any $\alpha \in \Phi$, we write $a_1(\alpha)$ as the coefficient of C_1 in α, then $\mathcal{O}(\alpha)|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(a_1(\alpha))$. Among Φ^+, there are 63 roots with $a_1(\alpha) = 0$, corresponding to the positive roots of the Lie sub-algebra E_7; 56 roots with $a_1(\alpha) = 1$, corresponding to weights of the standard representation of E_7; 1 root with $a_1(\alpha) = 2$, which is just the longest root $\alpha_0 = F - C_0$. We denote $\mathcal{E}_0^{E_7} \triangleq \mathcal{O}^{\oplus 7} \oplus \bigoplus_{\alpha \in \Phi, a_1(\alpha)=0} \mathcal{O}(\alpha)$, $V_0^+ \triangleq \bigoplus_{\alpha \in \Phi, a_1(\alpha)=1} \mathcal{O}(\alpha)$ and $V_0^- \triangleq \bigoplus_{\alpha \in \Phi, a_1(\alpha)=-1} \mathcal{O}(\alpha)$, then

$$\mathcal{E}_0^{E_8} = \mathcal{E}_0^{E_7} \oplus \mathcal{O} \oplus V_0^+ \oplus V_0^- \oplus \mathcal{O}(\alpha_0) \oplus \mathcal{O}(-\alpha_0).$$

When $\mathcal{O}(\alpha)$ is a summand of V_0^+, i.e., $\mathcal{O}(\alpha)|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(1)$, we have $\mathcal{O}(\alpha + C_0)|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(-1)$ and $\alpha + C_0 = F - (\alpha_0 - \alpha)$ with $(\alpha_0 - \alpha) \in \Phi^+$, that is $\mathcal{O}(\alpha + C_0)$ is a summand of $V_0^-(F)$. Since $F = \alpha_0 + C_0$ satisfies $F \cdot F = 0$, we have $\mathcal{O}(F)|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}$, $\mathcal{O}(\alpha_0)|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(2)$ and $\mathcal{O}(2F - \alpha_0)|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}(-2)$.

For the loop E_8-bundle, we have

$$\begin{aligned}
\mathcal{E}_0^{LE_8} &= \bigoplus_{n\in\mathbb{Z}} \left(\mathcal{E}_0^{E_8} \otimes \mathcal{O}(nF)\right) \\
&= \bigoplus_{n\in\mathbb{Z}} \left(\left(\mathcal{E}_0^{E_7} \oplus \mathcal{O} \oplus V_0^+ \oplus V_0^- \oplus \mathcal{O}(\alpha_0) \oplus \mathcal{O}(-\alpha_0)\right) \otimes \mathcal{O}(nF)\right) \\
&= \bigoplus_{n\in\mathbb{Z}} \left(\left(\mathcal{E}_0^{E_7} \oplus \mathcal{O} \oplus V_0^+ \oplus V_0^-(F) \oplus \mathcal{O}(\alpha_0 - F) \oplus \mathcal{O}(F - \alpha_0)\right) \otimes \mathcal{O}(nF)\right).
\end{aligned}$$

We denote $L_0^{248} \triangleq \mathcal{E}_0^{E_7} \oplus \mathcal{O} \oplus V_0^+ \oplus V_0^-(F) \oplus \mathcal{O}(\alpha_0 - F) \oplus \mathcal{O}(F - \alpha_0)$. From definition of $\overline{\partial}_\varphi$, $\mathcal{E}_\varphi^{LE_8}$ is built from successive extensions of $L_\varphi^{248} \otimes \mathcal{O}(nF)$, i.e.,

$$\overline{\partial}_\varphi = \left(\begin{array}{c|c|c|c}
\ddots & \ddots & \ddots & \ddots \\ \hline
\ddots & \overline{\partial}_{L_\varphi^{248} \otimes \mathcal{O}((n+1)F)} & * & \ddots \\ \hline
\ddots & O & \overline{\partial}_{L_\varphi^{248} \otimes \mathcal{O}(nF)} & \ddots \\ \hline
\ddots & \ddots & \ddots & \ddots
\end{array}\right).$$

So if we can prove $[\varphi_{C_0}|_{C_0}] \neq 0$ implies $(L_\varphi^{248}, \overline{\partial}_\varphi|_{L_\varphi^{248}})$ is trivial over C_0, then $(\mathcal{E}_\varphi^{LE_8}, \overline{\partial}_\varphi)$ is also trivial over C_0 because of $Ext^1_{\mathbb{P}^1}(\mathcal{O}, \mathcal{O}) = 0$. Note

$$L_0^{248}|_{C_0} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus 133} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus (\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))^{\oplus 56} \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-2).$$

In this decomposition, any of the 56 pairs of $\{\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(1)\}$ is the restriction of $\{\mathcal{O}(\alpha), \mathcal{O}(\alpha + C_0) = \mathcal{O}(F - (\alpha_0 - \alpha))\}$ to C_0 for some α with $a_1(\alpha) = 1$ and the triple $\{\mathcal{O}_{\mathbb{P}^1}(2), \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-2)\}$ is the restriction of $\{\mathcal{O}(-C_0), \mathcal{O}, \mathcal{O}(C_0)\}$ to C_0. We will show that the restriction of $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ to C_0 gives a non-trivial extension for each of these pairs $\{\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(1)\}$ and the triple $\{\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(2)\}$.

In order to write $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ in matrix form, we need to decompose $\mathcal{E}_0^{E_7}$ into positive parts and non-positive parts, i.e., we denote $\mathcal{E}_0^{(E_7,+)} := \bigoplus_{\alpha \in \Phi^+, a_1(\alpha)=0} \mathcal{O}(\alpha)$ and $\mathcal{E}_0^{(E_7,-)} :=$

$\mathcal{O}^{\oplus 7} \oplus \bigoplus_{\alpha \in \Phi^-, a_1(\alpha)=0} \mathcal{O}(\alpha)$. Then $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ can be written as follows: ($\overline{\partial}_\varphi|_{L_\varphi^{248}}$ is a upper triangle matrix since $\overline{\partial}_\varphi|_{L_\varphi^{248}}$ maps any line bundle summand to other more "positive" line bundle summands, i.e., $\overline{\partial}_\varphi : \mathcal{O}(D) \to \mathcal{O}(D')$ is nonzero only if $D' - D \geq 0$)

$$\overline{\partial}_\varphi|_{L_\varphi^{248}} = \left(\begin{array}{c|c|c|c|c|c|c} \overline{\partial}_{V_\varphi^-(F)} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ \hline O & \overline{\partial}_{\mathcal{O}(F-\alpha_0)} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ \hline O & O & \overline{\partial}_{V_\varphi^+} & A_{34} & A_{35} & A_{36} & A_{37} \\ \hline O & O & O & \overline{\partial}_{\mathcal{E}_\varphi^{(E_7,+)}} & A_{45} & A_{46} & A_{47} \\ \hline O & O & O & O & \overline{\partial}_{\mathcal{O}} & A_{56} & A_{57} \\ \hline O & O & O & O & O & \overline{\partial}_{\mathcal{E}_\varphi^{(E_7,-)}} & A_{67} \\ \hline O & O & O & O & O & O & \overline{\partial}_{\mathcal{O}(\alpha_0-F)} \end{array}\right).$$

Now we restrict this to C_0, the 56 pairs $\{\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(1)\}$ are in $V_0^-(F)|_{C_0} \oplus V_0^+|_{C_0}$. Since $A_{23} = (0, 0, \ldots, 0)_{56\times 1}$ and

$$A_{13} = \begin{pmatrix} \pm\varphi_{C_0} & * & \cdots & * \\ 0 & \pm\varphi_{C_0} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm\varphi_{C_0} \end{pmatrix}_{56\times 56},$$

if $[\varphi_{C_0}|_{C_0}] \neq 0$, then we have a trivialization of the 56 pairs $\{\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(1)\}$ over C_0 by Lemma 32 in [2].

For the triple $\{\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(2)\}$, we review the trivialization of A_1 Lie algebra bundle. In A_1 case, we have an A_1-bundle $\mathcal{E}_\varphi^{A_1}$, which topologically is $\mathcal{E}_0^{A_1} = \mathcal{O} \oplus \mathcal{O}(C) \oplus \mathcal{O}(-C)$, but with a holomorphic structure as follows:

$$\overline{\partial}_\varphi = \left(\begin{array}{c|c|c} \overline{\partial}_0 & \pm\varphi_C & 0 \\ \hline 0 & \overline{\partial}_0 & \pm\varphi_C \\ \hline 0 & 0 & \overline{\partial}_0 \end{array}\right),$$

where $\varphi_C \in H^{0,1}(X, \mathcal{O}(C))$. From [2], we know if $[\varphi_C|_C] \neq 0$, then $\mathcal{E}_\varphi^{A_1}$ is trivial on C. Back to our case, the triple $\{\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(2)\}$ has the corresponding submatrices $A_{25} = (\varphi_{C_0})_{1\times 1}$, $A_{57} = (\varphi_{C_0})_{1\times 1}$ and $A_{27} = (0)_{1\times 1}$. Since A_{23}, A_{24}, A_{26}, A_{47} and A_{67} are all zero matrices, from the trivialization of A_1 Lie algebra bundle, we know if $[\varphi_{C_0}|_{C_0}] \neq 0$, then we have a trivialization of the triple $\{\mathcal{O}_{\mathbb{P}^1}(-2), \mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(2)\}$ over C_0.

Hence if $[\varphi_{C_0}|_{C_0}] \neq 0$, then $(L_\varphi^{248}, \overline{\partial}_\varphi|_{L_\varphi^{248}})$ is trivial on C_0, which implies $(\mathcal{E}_\varphi^{LE_8}, \overline{\partial}_\varphi)$ is also trivial on C_0. Hence, we have Theorem 19 for LE_8 case.

Acknowledgments We are grateful to R. Friedman, E. Looijenga and J. J. Zhang for many useful comments and discussions. The first author is supported by the National Natural Science Foundation of China (No. 11501201), the China Postdoctoral Science Foundation (No. 2015M570334) and the Fundamental Research Funds for the Central Universities under Project No. 222201514322. The second author is supported by research grants from the Research Grants Council of the Hong Kong Special Administrative Region, China (Nos. 401411, 14302714).

References

1. Barth, W., Peters, C., Van de Ven, A.: Compact Complex Surfaces. Springer, Berlin (1984)

2. Chen, Y.X., Leung, N.C.: ADE bundles over surfaces with ADE singularities. Int. Math. Res. Not. **15**, 4049–4084 (2014)
3. Chen, Y.X., Leung, N.C.: Deformability of Lie algebra bundles and geometry of rational surfaces. Int. Math. Res. Not. (2015). doi:10.1093/imrn/rnv023
4. Donagi, R.: Principal bundles on elliptic fibrations. Asian J. Math **1**(2), 214–223 (1997)
5. Donagi, R.: Taniguchi Lecture on Principal Bundles on Elliptic Fibrations. arXiv: hep-th/9802094 (1998)
6. Friedman, R., Morgan, J.W.: Exceptional groups and del Pezzo surfaces. In: Contemporary Mathematics 312, Symposium in Honor of C. H. Clemens, pp. 101–116 (2000)
7. Friedman, R., Morgan, J.W., Witten, E.: Vector bundles and F theory. Commun. Math. Phys. **187**, 679–743 (1997)
8. Humphreys, J.E.: Introduction to Lie Algebras and Representation Theory, Graduate Texts in Mathematics 9. Springer-Verlag New York Inc. (1973)
9. Leung, N.C., Xu, M., Zhang, J.J.: Kac–Moody $\widetilde{E_k}$-bundles over elliptic curves and del Pezzo surfaces with singularities of type A. Math. Ann. **352**, 805–828 (2012)
10. Leung, N.C., Zhang, J.J.: Moduli of bundles over rational surfaces and elliptic curves I: simply laced cases. J. Lond. Math. Soc. **80**(3), 750–770 (2009)
11. Leung, N.C., Zhang, J.J.: Moduli of bundles over rational surfaces and elliptic curves II: non-simply laced cases. Int. Math. Res. Not. **24**, 4597–4625 (2009)
12. Looijenga, E.: Rational surfaces with an anti-canonical cycle. Ann. Math. **114**, 267–322 (1981)
13. Malagon, A.: Killing forms of isotropic Lie algebras. J. Pure Appl. Algebra **216**, 2213–2224 (2012)
14. Xu, M., Zhang, J.J.: G-bundles over elliptic curves for non-simply laced Lie groups and configurations of lines in rational surfaces. Pac. J. Math. **261**, 497–510 (2013)