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Instantons in G_2 manifolds from J-holomorphic curves in coassociative submanifolds

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Abstract. In G_2 manifolds, 3-dimensional associative submanifolds (instantons) play a role similar to J-holomorphic curves in symplectic geometry. In [21], instantons in G_2 manifolds were constructed from regular J-holomorphic curves in coassociative submanifolds. In this exposition paper, after reviewing the background of G_2 geometry, we explain the main ingredients in the proofs in [21]. We also construct new examples of instantons.

1. Introduction

A G_2-manifold M is a Riemannian manifold of dimension seven equipped with a nontrivial parallel 2-fold *vector cross product* (VCP) $\times$. If the VCP is 1-fold instead, namely a Hermitian complex structure, then the manifold is a Kähler manifold X.[1] Note that every 2-fold VCP comes from the restriction of the algebra product on the octonions $\mathbb{O}$ or quaternions $\mathbb{H}$ to its imaginary part. Thus the only submanifolds in M preserved by $\times$ are of dimension three and they are called *instantons* or *associative submanifolds*, which are analogs to holomorphic curves in Kähler manifolds.

In Physics, G_2-manifolds are internal spaces for compactification in M-theory in eleven dimensional spacetimes, similar to the role of *Calabi-Yau threefolds* X in string theory in ten dimensional spacetimes. Instantons in string theory are holomorphic curves Σ in X with the natural boundary condition being $\partial\Sigma$ lies inside a Lagrangian submanifold L in X. Very roughly speaking, the Fukaya category $Fuk\,(X)$ is defined by counting holomorphic disks with Lagrangian boundary condition. The homological mirror symmetry (HMS) conjecture of Kontsevich says that $Fuk\,(X)$ is equivalent to the derived category $D^b\,(X^\vee)$ of coherent sheaves of the mirror manifold $X^\vee$. The proof of this conjecture and its generalizations in many specific cases relies on the work of Fukaya-Oh [8] when $X = T^*L$ and the Lagrangian L_t is the graph of an exact one form df scaled by small $t \in \mathbb{R}$. They showed that holomorphic disks with boundary in $L \cup L_t$ one to one correspond to gradient flow lines of f, which are in fact instantons in quantum mechanics according to Witten's Morse theory [31]. (Fukaya-Oh [8] actually proved the case of k

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[1]Throughout this paper M is a G_2-manifold and X is a symplectic manifold possibly with extra structures, like Kähler or CY.

Lagrangians $L_t^j = tdf_j$ $(j = 1, 2, \cdots k)$, where holomorphic polygons bounding on these Lagrangians correspond to gradient flow trees of the Morse functions $\{f_j\}_{j=1}^k$.)

In [21] we proved a corresponding result for instantons in any G_2-manifold with boundary in the coassociative submanifold $C \cup C_t$. Here the family C_t is constructed by a self-dual harmonic two form ω on the four manifold C. We assume that ω is non-degenerate, thus defining an almost complex structure J on C. Our main result, Theorem 13, gives a correspondence between such instantons in M and J-holomorphic curves Σ in C. Thus the number of instantons in M is related to the Seiberg-Witten invariant of C by the celebrated work of Taubes on GW=SW ([26], [28], [27]). We suspect this holds true without the need of the non-degeneracy of ω.

Our result is similar to the $k = 2$ case of Fukaya-Oh [8], however the analysis involved in the proof is essentially different from theirs in the following 3 aspects.

(1) The instanton equation is on 3-*dimensional* domains, and there is no analogous way of finding associative submanifold by constructing *associative maps* as was done in constructing J-holomorphic curves thanks to the *conformality* of Cauchy-Riemann equation. So we have to deform *submanifolds* rather than *maps* as in the Lagrangian Floer theory. A good choice of normal frames of submanifolds turns out to be essential.

(2) The instanton equation is *more nonlinear* than the Cauchy-Riemann equation for J-holomorphic curves. It is a first order PDE system involving cubic terms $\frac{\partial V^i}{\partial x^1}\frac{\partial V^j}{\partial x^2}\frac{\partial V^k}{\partial x^3}$, while the Cauchy-Riemann operator $\frac{\partial u}{\partial \tau}+J(u)\frac{\partial u}{\partial t}$ has no product of derivative terms. Consequently, the needed *quadratic estimate* appears to be unavailable in the $W^{1,p}$ setting, as for $V^i, V^j, V^k \in W^{1,p}$, $\frac{\partial V^i}{\partial x^1}\frac{\partial V^j}{\partial x^2}\frac{\partial V^k}{\partial x^3} \notin L^p$ in general. So instead we use the *Schauder* ($C^{1,\alpha}$) *setting*, which creates new complications (Subsection 3.2.4).

(3) The linearized instanton equation is *more weakly coupled* than the Cauchy-Riemann equation. It is a Dirac type equation for spinors $(u, v) \in \mathbb{S}^+ \oplus \mathbb{S}^-$ where u and v play the role of the real and imaginary parts in Cauchy-Riemann equations, but the interrelation between ∇u and ∇v becomes weaker. This causes several difficulties in the $C^{1,\alpha}$ estimates (See comments below (14)).

Besides above difficulties, our domains are $[0, \varepsilon] \times \Sigma$ for compact Riemannian surfaces Σ, and they *collapse* to Σ as $\varepsilon \to 0$, causing *lack of uniform ellipticity,* which in turn creates difficulty to obtain a uniform right inverse bound needed in gluing arguments. This also occurs in Proposition 6.1 of Fukaya-Oh [8] in the $W^{1,p}$ setting, but in our $C^{1,\alpha}$ setting the boundary estimates become more subtle.

To deal with these difficulties, our paper [21] becomes rather technical. Therefore in this article we give an outline of the main arguments. The organization is as follows. In Section 2, we review the background of G_2 geometry, instantons, coassociative boundary condition and give the motivations of counting instantons. In Section 3 we state the main theorem in [21] and explain the main ingredients in the proof. In Section 4 we apply the theorem to construct new examples of instantons and discuss possible generalizations.

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2. Review on G_2-geometry

2.1. G_2 manifolds

G_2 manifolds are 7-dimensional Riemannian manifolds (M, g) with a parallel *vector cross product* (VCP) $\times$, i.e., for $u, v \in T_pM$ there is a VCP $\times$ such that

(i) $u \times v$ is perpendicular to both u and v.
(ii) $|u \times v| = |u \wedge v| =$ Area of the parallelogram spanned by u and v,

and $\times$ is invariant under the parallel transport of the Levi-Civita connection ∇.

In the octonion algebra $\mathbb{O}$, we can construct a VCP $\times$ on Im $\mathbb{O}$ as follows:

$$u \times v = \operatorname{Im}(\overline{v}u), \tag{1}$$

where $\overline{v}$ is the conjugate of v. The same formula for $\mathbb{H}$ gives another VCP on $\operatorname{Im}\mathbb{H}$, which is indeed the useful vector product on $\mathbb{R}^3$. Together, they form the complete list of VCP because the normed algebra structures can be recovered from the VCP structures. In particular,

$$Aut\left(\mathbb{R}^7, \times\right) = Aut_{alg}(\mathbb{O}).$$

Hence a G_2 manifold is simply a 7-dimensional Riemannian manifold (M, g) with holonomy group inside the exceptional Lie group $G_2 = Aut_{alg}(\mathbb{O})$.

Equivalently, G_2 manifolds are 7-dimensional Riemannian manifolds (M, g) with a *nondegenerate* 3*-form* Ω such that $\nabla\Omega = 0$. The relation between Ω and $\times$ is

$$\Omega(u, v, w) = g(u \times v, w), \quad \text{for } u, v, w \in T_pM.$$

Example 1 (Linear case)**.** *The* G_2 *manifold* $\operatorname{Im}\mathbb{O} \simeq \mathbb{R}^7$*: Let*

$$\mathbb{R}^7 \simeq \operatorname{Im}\mathbb{O} \simeq \operatorname{Im}\mathbb{H} \oplus \mathbb{H} = \{(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, x_4 + x_5\mathbf{i} + x_6\mathbf{j} + x_7\mathbf{k})\},$$

and the standard basis $e_i = \frac{\partial}{\partial x_i}$ $(i = 1, 2, \cdots, 7)$*. The vector cross product* $\times$ *for* u, v *in* $\operatorname{Im}\mathbb{O}$ *is defined by* (1)*. The standard* G_2 3*-form* $\Omega = \Omega_0$ *is*

$$\Omega_0 = \omega^{123} - \omega^{167} - \omega^{527} - \omega^{563} - \omega^{154} - \omega^{264} - \omega^{374},$$

where $\omega^{ijk} = dx^i \wedge dx^j \wedge dx^k$*.*

Example 2 (Product case). *$M = X \times S^1$ is a G_2 manifold if and only if X is a Calabi-Yau threefold. The G_2 3-form Ω of X is related to the holomorphic volume form Ω_X and the Kähler form ω_X of X as follow*

$$\Omega = \operatorname{Re}\Omega_X + \omega_X \wedge d\theta,$$

where $d\theta$ is the standard angular-form on S^1.

So far all compact irreducible G_2 manifolds are constructed by solving the nonlinear PDE of the G_2 metric using the implicit function theorem, including (i) resolving orbifold singularities by Joyce [17], [18] and (ii) twisted connected sum by Kovalev [7] and Corti-Haskins-Nordstrom-Pancini [19].

We remark a useful construction of local G_2 frames.

Remark 3 (Cayley-Dickson construction). *A convenient basis in $\operatorname{Im}\mathbb{O}$ can be constructed inductively (Lemma A.15 in [13]): Given orthogonal unit vectors v_1 and v_2, we define $v_3 := v_1 \times v_2$. Then we take any unit vector $v_4 \perp span\{v_1, v_2, v_3\}$, and let $v_{i+4} := v_i \times v_4$ for $i = 1, 2, 3$. Then $\{v_i\}_{i=1}^7$ is an orthonormal basis of $\operatorname{Im}\mathbb{O}$ satisfying the same vector cross product relation as the standard basis $\{e_i\}_{i=1}^7$ in $\operatorname{Im}\mathbb{O}$.*

This basis $\{v_i\}_{i=1}^7$ is useful in many local calculations in [21] (see also [3]), where $\{v_1, v_2\}$ span the tangent spaces of a J-holomorphic curve.

2.2. Instantons (associative submanifolds)

A 3-dimensional submanifold A in a G_2 manifold M is called an **associative submanifold** (or an **instanton**) if its tangent space TA is closed under $\times$. This notion was introduced by Harvey-Lawson ([13], see also [20]). The following are two other equivalent conditions for A to be an associative submanifold: (1) A is calibrated by Ω, namely $\Omega|_A = dv_A$; (2) $\tau|_A = 0 \in \Omega^3(A, T_M|_A)$ where $\tau \in \Omega^3(M, TM)$ is defined by the following equation

$$g(\tau(u, v, w), z) = (*\Omega)(u, w, w, z) \tag{2}$$

for any u, v, w and $z \in T_pM$, where $*\Omega \in \Omega^4(M)$ is the Hodge-$*$ of Ω. This measurement $\tau|_A$ of associativity is important for perturbing almost instantons to an instanton.

Example 4. *In $\operatorname{Im}\mathbb{O} \simeq \operatorname{Im}\mathbb{H} \oplus \mathbb{H}$, the subspace $\operatorname{Im}\mathbb{H} \oplus \{0\}$ is an instanton. Explicitly τ is ((5.4) in [24])*

$$\begin{aligned}\tau = &\left(\omega^{256} - \omega^{247} + \omega^{346} - \omega^{357}\right)\partial_1 + \left(\omega^{156} - \omega^{147} - \omega^{345} + \omega^{367}\right)\partial_2 \\ &+ \left(\omega^{245} - \omega^{267} - \omega^{146} - \omega^{157}\right)\partial_3 + \left(\omega^{567} - \omega^{127} + \omega^{136} - \omega^{235}\right)\partial_4 \\ &+ \left(\omega^{126} - \omega^{467} + \omega^{137} + \omega^{234}\right)\partial_5 + \left(\omega^{457} - \omega^{125} - \omega^{134} + \omega^{237}\right)\partial_6 \\ &+ \left(\omega^{124} - \omega^{456} - \omega^{135} - \omega^{236}\right)\partial_7,\end{aligned} \tag{3}$$

where $\partial_i = \frac{\partial}{\partial x^i}$. For a map $V = \left(V^4, V^5, V^6, V^7\right) : \operatorname{Im}\mathbb{H} \to \mathbb{H}$, we let

$$A = graph(V) = \{(x_1, x_2, x_3, V(x_1, x_2, x_3)) \,|\, (x_1, x_2, x_3) \in \operatorname{Im}\mathbb{H}\}.$$

By (3)*, the condition* $\tau|_A = 0$ *becomes* $V^*\tau = 0$ *in coordinates* (x_1, x_2, x_3)*, and we see the instanton equation for* A *is a* first order cubic PDE *system involving product terms* $\frac{\partial V^i}{\partial x^1}\frac{\partial V^j}{\partial x^2}\frac{\partial V^k}{\partial x^3}$.

Example 5. *For any holomorphic curve* Σ *in a Calabi-Yau manifold* X, $\Sigma \times S^1$ *is an instanton in the* G_2 *manifold* $X \times S^1$.

The *deformations* of an instanton A in a G_2 manifold M are governed by a *twisted Dirac operator* on its normal bundle $N_{A/M}$, regarded as a Clifford module over TA with $\times$ as the Clifford multiplication.

Theorem 6 ([24])**.** *Infinitesimal deformations of instantons at* A *are parameterized by the space of harmonic twisted spinors on* A*, i.e., kernel of the twisted Dirac operator.*

Example 7. *For the instanton* $\operatorname{Im}\mathbb{H}\oplus\{0\} \subset \operatorname{Im}\mathbb{O}$ *in Example 4, regarding sections* $V = V^4 + \mathbf{i}V^5 + \mathbf{j}V^6 + \mathbf{k}V^7$ *in its normal bundle as* $\mathbb{H}$*-valued functions, then the twisted Dirac operator is* $\mathcal{D} = \mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3$*, and*

$$\begin{aligned}\mathcal{D}V = &-\left(V_1^5 + V_2^6 + V_3^7\right) + \mathbf{i}\left(V_1^4 + V_3^6 - V_2^7\right)\\ &+ \mathbf{j}\left(V_2^4 - V_3^5 + V_1^7\right) + \mathbf{k}\left(V_3^4 + V_2^5 - V_1^6\right).\end{aligned} \tag{4}$$

Remark 8. *Constructing instantons in* G_2 *manifolds in general is difficult, partly because the deformation theory can be obstructed. Our main theorem in* [21] *provides a construction of instantons from* J*-holomorphic curves in coassociative submanifolds. Other constructions are in* [7].

2.3. Boundary value problem on coassociative submanifolds

Coassociative submanifolds are analogues of Lagrangian submanifolds in symplectic geometry.

Definition 9. *A* 4*-dimensional submanifold* C *in a* G_2 *manifold* (M, g, Ω) *is called coassociative if* $\Omega|_C = 0$*. Equivalently,* C *is calibrated by* $*\Omega$.

Example 10. *When* $M = X \times S^1$ *is a product with* X *a Calabi-Yau threefold, then (i)* $C = L \times S^1$ *is coassociative if and only if* $L \subset X$ *is a special Lagrangian submanifold* L *of phase* $\pi/2$ *and (ii)* $C = S \times \{\theta\}$ *is coassociative if and only if* $S \subset X$ *is a complex surface.*

Deformations of a coassociative submanifold C is studied by McLean [24]: Infinitesimally, they are parameterized by $H_+^2(C)$, the space of self-dual harmonic 2-forms on C, and they are always unobstructed.

Finding instantons with boundaries on coassociative submanifolds is an elliptic problem, similar to finding J-holomorphic curves with boundaries on Lagrangian submanifolds in symplectic geometry.

Theorem 11 (Theorem 4.2, [10]). *The linearization of the instanton equation on an instanton A with boundaries Σ lying on a coassociative submanifold C is an elliptic Fredholm operator, with the Fredholm index given by*

$$\int_{\Sigma} c_1\left(N_{\Sigma/C}\right) + 1 - g\left(\Sigma\right).$$

We have the following orthogonal decomposition of TM along Σ,

$$TM|_{\Sigma} = \mathbb{R}\left\langle n\right\rangle \oplus T\Sigma \oplus N_{\Sigma/C} \oplus N_{\langle C,n\rangle/M}|_{\Sigma},$$

where n be the unit inward normal vector field of Σ in A. We call $N_{\Sigma/C}$ the *"intrinsic"* normal bundle of Σ in C and $N_{\langle C,n\rangle/M}$ the *"extrinsic"* normal bundle defined as the orthogonal complement of $TC\oplus\langle n\rangle$ in TM. In [10], Gayet-Witt showed that (1) $T\Sigma$, $N_{\Sigma/C}$ and $N_{\langle C,n\rangle/M}|_{\Sigma}$ are complex line bundles with respect to the almost complex structure $J_n := n\times$ and (2) $N_{\Sigma/C} \otimes_{\mathbb{C}} \wedge^{0,1}_{\mathbb{C}}\left(T^*\Sigma\right) \simeq N^*_{\langle C,n\rangle/M}$ as complex line bundles over Σ which is obtained by changing the tensor product $\otimes$ to the vector cross product $\times$, and used the metric to identify conjugate bundle with dual bundle. This relationship between intrinsic and extrinsic geometry of Σ in C is needed ([21]) in order to show two natural Dirac operators on Σ agree which is needed in subsection 3.2.5 for the proof of our main theorem.

The following table gives an interesting comparison between G_2-geometry and symplectic geometry.

G_2 **manifold**	**Symplectic manifold**
nondegenerate 3-form Ω, $d\Omega = 0$	nondegenerate 2-form ω, $d\omega = 0$
vector cross product $\times$	almost complex structure J
$\Omega\left(u,v,w\right) = g\left(u\times v, w\right)$	$\omega\left(u,v\right) = g\left(Ju, v\right)$
instanton A TA preserved by$\times$ A calibrated by Ω	J-holomorphic curve Σ $T\Sigma$ preserved by J Σ calibrated by ω
coassociative submanifold C $\Omega\vert_C = 0$ and $\dim C = 4$	Lagrangian submanifold L $\omega\vert_L = 0$ and $\dim L = \frac{1}{2}\dim X$

2.4. Intersection theory?

Intersection theory of Lagrangian submanifolds is an essential part of symplectic geometry. By counting the number of holomorphic disks bounding intersecting Lagrangian submanifolds, Floer and others defined the celebrated Floer homology theory. It plays an important role in mirror symmetry for Calabi-Yau manifolds and string theory in physics.

In M-theory, Calabi-Yau threefolds are replaced by G_2-manifolds. The analogs of holomorphic disks (resp. Lagrangian submanifolds) are instantons or associative submanifolds (resp. coassociative submanifolds or branes) in M [20]. The problem of counting instantons has been discussed by many physicists. For example Harvey and Moore [14] discussed the mirror symmetry aspects of it; Aganagic and Vafa [2] related it to the open Gromov-Witten invariants for local Calabi-Yau threefolds; Beasley and Witten [4] argued that one should count instantons using the Euler characteristic of their moduli spaces.

On the mathematical side, the compactness issues of the moduli of instantons is a very challenging problem because the bubbling-off phenomena of (3-dimensional) instantons is not well understood. This makes it very difficult to define an honest invariant by counting instantons. Note that the Fredholm theory for instantons with coassociative boundary conditions has been set up [10].

In symplectic geometry, Fukaya and Oh [8] considered two *nearby* Lagrangian submanifolds L and L_t, where L_t is the graph of a closed 1-form α on L scaled by a small t, then holomorphic disks bounding them with *small volume* is closely related to gradient flow lines of α. They actually deal with J-holomorphic polygons bounding k Lagrangians and need to smooth vertex singularities on gradient flow trees when $k \geq 3$. For simplicity we only state their result for the $k = 2$ case here.

Given a closed 1-form α on L, L_t :=graph($t\alpha$) is a Lagrangian submanifold in T^*L near the zero section L. By Weinstein's neighborhood theorem, a small tubular neighborhood of a Lagrangian L in a symplectic manifold can be identified as a tubular neighborhood of L in T^*L.

Theorem 12. *(*[8]*) For any compact Riemannian manifold (L, g), T^*L has a natural almost Kähler structure (ω, J_g). Let L_t :=graph$(t\alpha)$ be a Lagrangian submanifold in T^*L for a closed* 1*-form α on L. There is a constant $\varepsilon_0 > 0$, such that for any $t \in (0, \varepsilon_0]$, there is a* 1-1 *correspondence between J_g-holomorphic curves bounding $L \cup L_t$ and gradient flow lines of α on L.*

We want to build the following analogue: instantons A bounding $C \cup C_t$ are in 1-1 correspondence to J_n-holomorphic curves Σ on C.

Symplectic manifold M	G_2 **manifold** M
Lagrangian submanifolds L and L_t	coassociative submanifolds C and C_t
J-holo. curve bounding $L \cup L_t$	instanton A bounding $C \cup C_t$
gradient flow line of α on L	J_n-holomorphic curve Σ on C

The meaning of J_n, C_t and the precise statment of our result will be explained in the next section.

3. Main Theorem: instantons from J-holomorphic curves

3.1. Statement of the main theorem

Let $\mathcal{C} = \cup_{0 \leq t \leq \varepsilon} C_t$ be a family of coassociative manifolds C_t in a G_2-manifold M, regarded as a deformation of $C = C_0$ along the normal vector field $n := \frac{dC_t}{dt}|_{t=0}$. Then $\iota_n \Omega$

is a self-dual harmonic 2-form[2] on C by McLean's Theorem on deformations of coassociative submanifolds (Section 2.3). In particular $\omega_n := \iota_n \Omega$ defines a *symplectic structure* on $C \backslash n^{-1}(0)$ as $\omega_n \wedge \omega_n = *\omega_n \wedge \omega_n = |\omega_n|^2 \, dv_C$ is nonzero outside $\{n = 0\}$. Furthermore $J_n := \frac{n}{|n|} \times$ defines a compatible *almost complex structure* on $\left(C \backslash n^{-1}(0), \omega_n\right)$. When n has no zeros, we have the following main theorem in [21]

Theorem 13. *Suppose that (M, Ω) is a G_2-manifold and $\{C_t\}$ is an one-parameter smooth family of coassociative submanifolds in M. When $\iota_n \Omega \in \Omega_+^2(C_0)$ is nonvanishing, then*

(1) *(Proposition 6) If $\{\mathtt{A}_t\}$ is any one-parameter family of associative submanifolds (i.e. instantons) in M satisfying*
$$\partial \mathtt{A}_t \subset C_t \cup C_0, \ \lim_{t \to 0} \mathtt{A}_t \cap C_0 = \Sigma_0 \text{ in the } C^1\text{-topology,}$$
then Σ_0 is a J_n-holomorphic curve in C_0.

(2) *(Theorem 24) Conversely, every regular J_n-holomorphic curve Σ_0 (namely those for which the linearization of $\overline{\partial}_{J_n}$ on Σ_0 is surjective) in C_0 is the limit of a family of associative submanifolds $\mathtt{A}_t$'s as described above.*

Our results are similar to those in Fukaya-Oh [8] and the proofs also share some similarities: relating the Fredholm regular property of higher dimensional linearized instanton equations to lower dimensional ones; necessity to deal with the lack of uniform ellipticity as the domain collapses when $\varepsilon \to 0$; using the periodic reflection technique to "thicken" the collapsing domain to achieve a uniform right inverse estimate in the $W^{1,p}$ setting.

However the proof in our case has more difficulties than those needed in the $k = 2$ case of [8], as explained in the introduction. For $k \geq 3$, [8] contains difficulties we have not encountered here: to find the local models of the singularities of degenerating J-holomorphic polygons and resolve them.

Remark 14. *Given any Riemann surface $\Sigma \subset M$, it can always be thickened to an instanton by the Cartan-Kähler theory (*[13],[10]*). However its boundary may not lie inside any coassociative submanifold (see* [10]*). In our case, we produce an instanton A_ε with boundary in the coassociative submanifold $C \cup C_\varepsilon$, but $\partial A_\varepsilon \cap C$ is only close but not equal to Σ.*

3.2. Main ingredients of the proof

3.2.1. Formulating the instanton equation near an almost instanton

We first produce an almost instanton with boundaries on $C_0 \cup C_\varepsilon$. Let $\varphi : [0, \varepsilon] \times C \to M$ be a parametrization of the family of coassociative submanifolds $\{C_t\}_{0 \leq t \leq \varepsilon}$. Under the assumptions that $n = \frac{dC_t}{dt}|_{t=0}$ is nonvanishing and Σ is a J_n-holomorphic curve in $C = C_0$, we define
$$A_\varepsilon := [0, \varepsilon] \times \Sigma, \text{ and } A'_\varepsilon = \varphi(A_\varepsilon) \subset M,$$

[2]Here ι_n is the contraction of a form by the vector field n.

then A'_ε is an *almost instanton* with $\partial A'_\varepsilon \subset C_0 \cup C_\varepsilon$ in the sense that $\left|\tau|_{A'_\varepsilon}\right|_{C^0}$ is small. Recall that a 3-dimensional submanifold $A \subset M$ is an instanton if and only if $\tau|_A = 0$, where $\tau \in \Omega^3(M, TM)$ is defined in (2). The reason for the smallness is that, at the point $p \in \varphi(\{0\} \times \Sigma) \subset A'_\varepsilon$, $T_p A'_\varepsilon$ is associative by the J_n-holomorphic property of Σ so $\tau|_{T_p A'_\varepsilon} = 0$, and any point q on A'_ε has ε-order distance to $\varphi(\{0\} \times \Sigma)$ while $\tau|_{T_q A'_\varepsilon}$ smoothly depends on q.

Next we formulate $\tau|_A$ as a nonlinear map F_ε on the space $\Gamma\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)$ of sections of the *normal bundle* $N_{A'_\varepsilon/M}$ *of* A'_ε, in particular $F_\varepsilon(V) = 0$ if and only if

$$\mathtt{A}_\varepsilon(V) := (\exp V)(\mathtt{A}'_\varepsilon) \tag{5}$$

is an instanton, where $\exp V : \mathtt{A}'_\varepsilon \to M$.

To do this, we let

$$\begin{aligned} C^\alpha\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right) &= \left\{V \in \Gamma\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right) | V \ \in C^\alpha\right\}, \\ C^{1,\alpha}_-\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right) &= \left\{V \in \Gamma\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right) | V \ \in C^{1,\alpha},\, V|_{\partial \mathtt{A}'_\varepsilon} \subset TC_0 \cup TC_\varepsilon\right\}, \end{aligned}$$

(the "$-$" in $C^{1,\alpha}_-$ is for the coassociative boundary condition), and

$$F_\varepsilon : C^{1,\alpha}_-\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right) \to C^\alpha\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right),$$

$$F_\varepsilon(V) = *_{\mathtt{A}'_\varepsilon} \circ \perp_{\mathtt{A}'_\varepsilon} \circ \left(T_V \circ (\exp V)^* \tau\right),$$

where

(1) $(\exp V)^*$ pulls back the *differential form part* of the tensor τ
(2) $T_V : T_{\exp_p(tV)} M \to T_p M$ pulls back the *vector part* of τ by parallel transport along the geodesic $\exp_p(tV)$
(3) $\perp_{\mathtt{A}'_\varepsilon} : TM|_{\mathtt{A}'_\varepsilon} \to N_{\mathtt{A}'_\varepsilon/M}$ is the orthogonal projection with respect to g
(4) $*_{\mathtt{A}'_\varepsilon} : \Omega^3(\mathtt{A}'_\varepsilon) \to \Omega^0(\mathtt{A}'_\varepsilon)$ is the quotient by the volume form $dvol_{\mathtt{A}'_\varepsilon}$

If $\mathtt{A}'_\varepsilon$ is an almost instanton, then a G_2-linear algebra argument shows that when $\|V\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}$ is small we have

$$F_\varepsilon(V) = 0 \iff \mathtt{A}_\varepsilon(V) \text{ is an instanton.}$$

To ensure that $\mathtt{A}_\varepsilon(V)$ satisfies the boundary condition, in the definition of $\exp V$ we actually need to modify the metric g near $C_0 \cup C_\varepsilon$ to make them totally geodesic, but we will keep the original metric in T_V, $\perp_{\mathtt{A}'_\varepsilon}$ and $*_{\mathtt{A}'_\varepsilon}$. This modification will not change the expression of $F'_\varepsilon(0)$ in (7) (see Remark 10 (1) in [21]). So our estimate for $\left\|F'_\varepsilon(0)^{-1}\right\|$ is still valid in the new metric.

3.2.2. Linearizing the instanton equation using a good frame

We can make $F_\varepsilon(V)$ more explicit by using a good local frame field $\{W_\alpha\}_{\alpha=1}^7$ near $p\in A'_\varepsilon=\mathtt{A}_\varepsilon(0)$. Since τ is a vector-valued 3-form, following Einstein's summation convention, we can write

$$\tau=\omega^\alpha\otimes W_\alpha\in\Omega^3(M,TM),$$

where local 3-forms $\{\omega^\alpha\}_{\alpha=1}^7$ are determined by $\{W_\alpha\}_{\alpha=1}^7$. For any $q=\exp_p V\in\mathtt{A}_\varepsilon(V)$, we let

$$F(V)(p)=(\exp V)^*\,\omega^\alpha(p)\otimes T_V W_\alpha(q).$$

$F(V)$ is the essential part of $F_\varepsilon(V)$ as $*_{\mathtt{A}'_\varepsilon}\circ\perp_{\mathtt{A}'_\varepsilon}$ is only a projection. It is *independent* of the choice of frame $\{W_\alpha\}_{\alpha=1}^7$. To compute $F'(0)$, we have the following *generalized Cartan formula* (6):

$$\begin{aligned}
&F'(0)\,V(p)\\
&=L_V\omega^\alpha\otimes W_\alpha+\omega^\alpha\otimes\nabla_V W_\alpha\\
&=\underbrace{d(i_V\omega^\alpha)\otimes W_\alpha}_{\text{symbol part}}+\underbrace{i_Vd\omega^\alpha(p)\otimes W_\alpha+\omega^\alpha\otimes\nabla_V W_\alpha(p)}_{\text{0-th order part}}. \qquad (6)\\
&=d(i_V\omega^\alpha)\otimes W_\alpha(p)+B_\alpha(p)\,V\otimes W_\alpha+\omega^\alpha\otimes C_\alpha(p)\,V\\
&=\mathcal{D}V\otimes vol_{A'_\varepsilon}+E(p)(V) \qquad (7)
\end{aligned}$$

where B_α, C_α and E are certain matrix-valued functions.

We require a *"good"* frame field $\{W_\alpha\}_{\alpha=1}^7$ to satisfy the following conditions in any small *ε-ball* around *p:*

(1) B_α, C_α and E are of ε-order in C^1 norm,
(2) ω^a's are ε-close to ω^{ijk}'s in (3) for $\mathbb{R}^7$ in C^1 norm, in the sense that ω^{ijk} are replaced by $W_i^*\wedge W_j^*\wedge W_k^*$, where W_i^* is the dual vector of W_i,
(3) $\mathcal{D}V(p)$ is ε-close to the twisted Dirac operator (4) for $\mathbb{R}^7$ in C^1 norm, in the sense that $\frac{\partial}{\partial x^i}$ in (4) are replaced by $\nabla^\perp_{W_i}$.

Condition 1 holds when $\{W_\alpha\}_{\alpha=1}^7$ is parallel along the normal bundle directions. Conditions 2 holds if whenever $e_\gamma=e_\alpha\times e_\beta$ in $\operatorname{Im}\mathbb{O}$ we have $\|W_\gamma-W_\alpha\times W_\beta\|_{C^1}=O(\varepsilon)$. Condition 3 holds if we further have the *normal covariant derivatives* $\left\|\nabla^\perp_{W_i}W_k\right\|_{C^1}=O(\varepsilon)$ assuming that W_i's span TA'_ε and W_k's span $N_{A'_\varepsilon/M}$.

Such a *good* frame $\{W_\alpha\}_{\alpha=1}^7$ can be constructed by the Cayley-Dickson construction as explained in Remark 3. The principal part of the linearized instanton equation $F'(0)\,V$ on A'_ε is the term $\mathcal{D}V$ in (7), which is a first order differential operator with a nice geometric meaning (see the next subsection).

3.2.3. A simplified model: Dirac operators on thin manifolds

We temporarily leave G_2 geometry and consider a (*"thin"* when ε is small) 3-manifold

$$\mathtt{A}_\varepsilon := [0,\varepsilon] \times \Sigma = \{(x_1, z := x_2 + ix_3)\},$$

with the *warped product metric*

$$g_{\mathtt{A}_\varepsilon,h} = h(z)\, dx_1^2 + g_\Sigma,$$

where $h(z) = |n|^2 > 0$ and $n = \frac{dC_t}{dt}|_{t=0}$ is the nonvanishing normal vector field on C_0. This is a first order approximation of the induced metric on $\mathtt{A}'_\varepsilon \subset M$.

We first consider the geometry of a J_n-holomorphic curve Σ in C. Let $L = N_{\Sigma/C}$ be the normal bundle of Σ in C, then L is a Hermitian J_n-complex line bundle over Σ (see Proposition 24). Let $\bar{\partial} = \left(\overline{\partial}, \overline{\partial}^*\right)$ be the Dirac operator on the Dolbeault complex $\Omega^0_{\mathbb{C}}(L) \oplus \Omega^{0,1}_{\mathbb{C}}(L)$ of the spinor bundle of Σ

$$\mathbb{S}_\Sigma = \mathbb{S}^+_\Sigma \oplus \mathbb{S}^-_\Sigma = L \oplus \wedge^{0,1}_{\mathbb{C}}(L)$$

such that

$$\Omega^0_{\mathbb{C}}(L) \oplus \Omega^{0,1}_{\mathbb{C}}(L) \overset{(\overline{\partial},\overline{\partial}^*)}{\longrightarrow} \Omega^{0,1}_{\mathbb{C}}(L) \oplus \Omega^0_{\mathbb{C}}(L), \tag{8}$$

where $\overline{\partial} : \Omega^0_{\mathbb{C}}(L) \to \Omega^{0,1}_{\mathbb{C}}(L)$ is the *normal Cauchy-Riemann operator* of J_n-holomorphic curve Σ in C, and $\overline{\partial}^* : \Omega^{0,1}_{\mathbb{C}}(L) \to \Omega^0_{\mathbb{C}}(L)$ is its adjoint. (In [21], we use the notation $\partial^+ = -\,\mathbf{i}\overline{\partial}^*$ and $\partial^- = \mathbf{i}\overline{\partial}$).

To describe the spinor bundle $\mathbb{S}$ over the 3-manifold $\mathtt{A}_\varepsilon$, we pullback $\mathbb{S}_\Sigma = \mathbb{S}^+_\Sigma \oplus \mathbb{S}^-_\Sigma$ to $\mathtt{A}_\varepsilon = [0,\varepsilon] \times \Sigma$, and denote it as $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$. Let e_1 be the unit tangent vector field on $\mathtt{A}_\varepsilon$ along x_1-direction. For $\mathbb{S}$ to be the spinor bundle of $\mathtt{A}_\varepsilon$, the Clifford multiplication with e_1 should be $\pm i$ on $\mathbb{S}^\pm$. To describe the Dirac operator on $\mathbb{S}$, we define

$$\mathcal{D} = e_1 \cdot h^{-\frac{1}{2}}(z) \frac{\partial}{\partial x_1} + \bar{\partial},$$

where $\bar{\partial} = \left(\overline{\partial}, \overline{\partial}^*\right)$ is the Dirac Dolbeault operator in equation (8). $\mathcal{D}$ acts on the sections $V = (u,v)$ of the spinor bundle $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ over $\mathtt{A}_\varepsilon$ with *local elliptic boundary condition* (in the sense of [5]):

$$v|_{\partial \mathtt{A}_\varepsilon} = 0.$$

We can write a local expression for $\mathcal{D}$. Consider the section $V = (u,v)$ of $\mathbb{S}$ with $u = V^4 + \mathbf{i}V^5 \in \mathbb{S}^+$ and $v = V^6 + \mathbf{i}V^7 \in \mathbb{S}^-$ (for $\mathbb{S}^\pm$ are complex line bundles). Then

$e_1\cdot = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix}$ and

$$\begin{aligned}\mathcal{D}V &= \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \left(h^{-\frac{1}{2}}(z) \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & \mathbf{i}\partial_z \\ \mathbf{i}\bar{\partial}_z & 0 \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix}, \\ &= \left(h^{-\frac{1}{2}}(z) \frac{\partial u}{\partial x_1}\mathbf{i} - \partial_z v \right) + \left(-h^{-\frac{1}{2}}(z) \frac{\partial v}{\partial x_1}\mathbf{i} + \bar{\partial}_z u \right) \cdot \mathbf{j} \end{aligned} \tag{9}$$

where $\bar{\partial}_z := \nabla_2 + \mathbf{i}\nabla_3$ and $\partial_z := \nabla_2 - \mathbf{i}\nabla_3$.

One can check that $\mathcal{D}$ agrees with the linearized instanton equation on $\{0\} \times \Sigma$, and on A_ε they are very close (Subsection 3.2.6). This is why we can use $\mathcal{D}$ of the linear model to study deformations of instantons in G_2 manifolds. The precise comparison is in Subsection 3.2.5 and Subsection 3.2.6.

3.2.4. Key estimates of $\mathcal{D}^{-1}$ of the linear model

The most difficult part of [21] is to derive an explicit ε-dependent bound of the operator norm of $\mathcal{D}^{-1} : C^\alpha(\mathtt{A}_\varepsilon, \mathbb{S}) \to C_-^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})$. This is nontrivial because we lose uniform ellipticity as $\mathtt{A}_\varepsilon = [0, \varepsilon] \times \Sigma$ collapses to Σ and consequently $\|\mathcal{D}^{-1}\|$ may blow up. A good control of $\|\mathcal{D}^{-1}\|$ is crucial for singular perturbation problems in general. We could achieve this in our case roughly because of

1. The Fredholm regularity property of the J_n-holomorphic curve Σ, which supplies the transversality for $\mathcal{D}$,
2. The coassociative boundary condition on $\partial\mathtt{A}_\varepsilon$, which enables us to periodically reflect $\mathtt{A}_\varepsilon$ to a bigger domain $\mathtt{A}_{k(\varepsilon)\varepsilon}$ with integer $k(\varepsilon)$ such that $1/2 \le k(\varepsilon)\varepsilon \le 3/2$, thus restoring the uniform ellipticity.

This is an over-simplified description as sections on $\mathtt{A}_\varepsilon$ may become *discontinuous* after periodical reflection, so condition 2 only helps in the $W^{1,p}$ setting to get a uniform bound of $\|\mathcal{D}^{-1}\|$ as in [8]. More effort is needed to estimate $\|\mathcal{D}^{-1}\|$ in the Schauder $C^{1,\alpha}$ setting.

Recall in Subsection 3.2.1 we have formulated instantons nearby A'_ε as solutions of the nonlinear equation $F_\varepsilon(V) = 0$, and it turns out $\|F_\varepsilon(0)\|_{C^\alpha} \le C\varepsilon^{1-\alpha}$. In Subsection 3.2.2 we have computed the linearization $F'_\varepsilon(0)$. To apply the implicit function theorem to perturb A'_ε to a true instanton, we need to estimate $\left\|F'_\varepsilon(0)^{-1}\right\|$, and we will see in Subsection 3.2.6 it is comparable to $\|\mathcal{D}^{-1}\|$ (Proposition 25). So a key estimate that we need is the following

Theorem 15. *(ε-dependent bound) Suppose that the first eigenvalues for $\overline{\partial\partial^*}$ and $\overline{\partial}^*\overline{\partial}$ are bounded below by $\lambda > 0$. Then for any $0 < \alpha < 1$ and $p > 3$ there is a positive constant $C = C(\alpha, p, \lambda, h)$ such that for any $V \in C_-^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S})$, we have*

$$\|V\|_{C_-^{1,\alpha}} \le C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} \|\mathcal{D}V\|_{C^\alpha}. \tag{10}$$

Here the notations $C^{1,\alpha}_{\pm}(\mathtt{A}_\varepsilon, \mathbb{S})$ are chosen to indicate the *local elliptic boundary conditions* for sections $V = (u, v)$ of $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$:

$$C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon, \mathbb{S}) = \left\{V \in C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S}) : v|_{\partial \mathtt{A}_\varepsilon} = 0\right\}$$
$$C^{1,\alpha}_{+}(\mathtt{A}_\varepsilon, \mathbb{S}) = \left\{V \in C^{1,\alpha}(\mathtt{A}_\varepsilon, \mathbb{S}) : u|_{\partial \mathtt{A}_\varepsilon} = 0\right\}.$$

Similarly $W^{k,p}_{\pm}(\mathtt{A}_\varepsilon, \mathbb{S})$ will be used to indicate these boundary conditions.

To prove this theorem we first recall (for simplicity, we assume $h \equiv 1$),

$$\mathcal{D}V = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \left(\frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & -\mathbf{i}\overline{\partial}^* \\ \mathbf{i}\overline{\partial} & 0 \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix}. \tag{11}$$

Along Σ we have the following

Lemma 16. *Assume the J_n-holomorphic curve $\Sigma \subset C$ is Fredholm regular, then*

$$\lambda_{\overline{\partial}^*} > 0$$

where $\lambda_{\overline{\partial}^*}$ is the first eigenvalue of $\Delta_\Sigma = \overline{\partial}\overline{\partial}^*$ on $W^{1,2}(\Sigma, \mathbb{S}^+)$.

Proof. This follows from the fact that $\overline{\partial}$ is the normal Cauchy-Riemann operator on $N_{\Sigma/C}$, and its adjoint is $\overline{\partial}^*$. So the Fredholm regular property of Σ is equivalent to $\ker \overline{\partial}^* = \{0\}$, i.e., $\lambda_{\overline{\partial}^*} > 0$. ☐

L^2 estimate

The operator $\mathcal{D} = \mathcal{D}_\pm : W^{1,2}_{\pm}(\mathrm{A}_\varepsilon, \mathbb{S}) \to L^2(\mathrm{A}_\varepsilon, \mathbb{S})$ is self-adjoint by the boundary condition, thus $\mathrm{coker}\mathcal{D}_- = \ker \mathcal{D}_+$. By the Rayleigh quotient method, we know

$$\lambda_{\mathcal{D}_\pm} := \inf_{0 \neq V \in W^{1,2}_{\pm}(\mathtt{A}_\varepsilon)} \frac{\|\mathcal{D}V\|^2_{L^2(\mathtt{A}_\varepsilon)}}{\|V\|^2_{L^2(\mathtt{A}_\varepsilon)}} \tag{12}$$

is the first eigenvalue of the Laplacian $\mathcal{D}_\mp \mathcal{D}_\pm$.

Theorem 17 (First eigenvalue estimate)**.** *For $\mathcal{D}_\pm : W^{1,2}_{\pm}(\mathtt{A}_\varepsilon, \mathbb{S}) \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$, we have*

$$\lambda_{\mathcal{D}_-} \geq \min\left\{\lambda_{\overline{\partial}}, \frac{2}{\varepsilon^2}\right\} \text{ and } \lambda_{\mathcal{D}_+} \geq \min\left\{\lambda_{\overline{\partial}^*}, \frac{2}{\varepsilon^2}\right\}.$$

Remark 18. *Theorem 17 enables us to control the first eigenvalue of $\mathcal{D}_\pm$ on 3-dimensional $\mathtt{A}_\varepsilon$ by that of $\bar{\partial}$ on 2-dimensional Σ, when ε is small. The control is due to the boundary condition of $\mathcal{D}_\pm$, as will be clear from the following proof.*

Proof. We prove the estimate for $\lambda_{\mathcal{D}_-}$ ($\lambda_{\mathcal{D}_+}$ is similar). By the boundary condition $v|_{\partial \mathtt{A}_\varepsilon} = 0$ we have

$$\begin{aligned}\langle \mathcal{D}V, \mathcal{D}V \rangle_{L^2} &= \int_{[0,\varepsilon]\times\Sigma} \left(\left| \frac{\partial V}{\partial x_1} \right|^2 + \left\| \overline{\partial}^* v \right\|^2 + \left\| \overline{\partial} u \right\|^2 \right) \\ &\geq \int_{[0,\varepsilon]\times\Sigma} \left(\left\| \overline{\partial} u \right\|^2 + \left\| v_{x_1} \right\|^2 \right).\end{aligned}$$

Then use the Rayleigh quotient for $\overline{\partial}^* \overline{\partial}$ and notice that $v|_{\partial \mathtt{A}_\varepsilon} = 0$. □

Corollary 19. *If $\Sigma \subset C$ is Fredholm regular, then for small enough $\varepsilon > 0$, the map $\mathcal{D} : W^{1,2}_{-}(\mathtt{A}_\varepsilon, \mathbb{S}) \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$ is surjective.*

Proof. By Lemma 16, the Fredholm regular property of Σ implies that $\lambda_{\overline{\partial}^*} > 0$. So by Theorem 17, $\lambda_{\mathcal{D}_+} > 0$, i.e. $\ker \mathcal{D}_+ = \{0\}$. By the self-adjoint property of $\mathcal{D}$, we get $\operatorname{coker}\mathcal{D}_- = \ker \mathcal{D}_+ = \{0\}$. □

Now, from the definition of $\lambda_{\mathcal{D}_-}$ we obtain the L^2 estimate

Corollary 20 (L^2-estimate)**.** *For any $V \in W^{1,2}_{-}(\mathtt{A}_\varepsilon, \mathbb{S})$ and small enough ε, we have*

$$\|V\|_{W^{1,2}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})} \leq C(\lambda) \|\mathcal{D}V\|_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}, \tag{13}$$

where the constant $C(\lambda)$ only depends on $\lambda_{\overline{\partial}}$.

In the following we will derive a C^0-estimate of V. For this purpose let us examine the equation $\mathcal{D}_- V = W$ more closely.

For $V = (u, v)$, $W = (w_1, w_2)$, the equation $\mathcal{D}_- V = W$ (assuming $h = 1$) is equivalent to the following system

$$\begin{cases} u_{x_1} - \mathbf{i}\overline{\partial}^* v &= w_1 \\ v_{x_1} + \mathbf{i}\overline{\partial} u &= w_2 \end{cases} \quad \text{and} \quad v|_{\partial \mathtt{A}_\varepsilon} = 0. \tag{14}$$

This is similar to the Cauchy-Riemann equation, but the relation between u and v is *weaker*, since ∇u can only control $\overline{\partial}^* v$ and v_{x_1}, which are only *half* of partial derivatives of v; the same applies to ∇v. This issue makes it more difficult to obtain the C^0-estimate of u than in the Cauchy-Riemann type equations.

C^0 estimate

The C^0-estimate of V is derived from a $W^{1,p}$-estimate of V and Sobolev embedding. To obtain a uniform estimate for $\left\| \mathcal{D}^{-1} \right\|$ in the L^p setting, we use the periodic reflection technique.

Theorem 21 (L^p-estimate, $p>3$)**.** *For any $V \in W_-^{1,p}(\mathtt{A}_\varepsilon, \mathbb{S})$, we have*

$$\|V\|_{W_-^{1,p}(\mathtt{A}_\varepsilon,\mathbb{S})} \le C_p(\lambda) \|\mathcal{D}V\|_{L^p(\mathtt{A}_\varepsilon,\mathbb{S})}, \tag{15}$$

where the constant $C_p(\lambda)$ only depends on $\lambda_{\overline{\partial}}$, $Vol(\Sigma)$, and p.

Proof. Given any $\varepsilon>0$, we choose an integer $k(\varepsilon)$ so that $1/2 \le k(\varepsilon)\varepsilon \le 3/2$. In the following we will simply write $k(\varepsilon)$ as k. We reflect A_ε to $\mathrm{A}_{k\varepsilon}$ periodically and extend (u,v) along the boundaries $\varepsilon\mathbb{Z}\times\Sigma$ such that

$$v(x,z) = \begin{cases} -v((2j+2)\varepsilon - x, z) & x\in[(2j+1)\varepsilon,(2j+2)\varepsilon] \\ v(x-2j\varepsilon, z) & x\in[2j\varepsilon,(2j+1)\varepsilon] \end{cases},$$

$$u(x,z) = \begin{cases} u((2j+2)\varepsilon - x, z) & x\in[(2j+1)\varepsilon,(2j+2)\varepsilon] \\ u(x-2j\varepsilon, z) & x\in[2j\varepsilon,(2j+1)\varepsilon] \end{cases},$$

(Notice that a $W^{k,p}$ section will remain so after reflections), i.e., we do *odd extensions* of v and *even extensions* of u along the boundaries. By (14), this induces *odd extensions* of w_1 and *even extensions* of w_2 along the boundaries. Since the shape of $\mathrm{A}_{k\varepsilon}$ is uniformly bounded, we have the elliptic estimate

$$\tilde{C}(p)\|V\|_{W_-^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \le \|\mathcal{D}V\|_{L^p(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})} + \|V\|_{L_-^p(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})},$$

with a uniform constant $\tilde{C}(p)$ for all ε. Then we use the interpolation inequality

$$\|V\|_{L_-^p(\mathtt{A}_{k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})} \le C\left(\delta^{\frac{p}{p-1}}\|V\|_{W^{1,p}(\mathtt{A}_{k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})} + \delta^{-p}\|V\|_{L_-^2(\mathtt{A}_{k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})}\right) \tag{16}$$

to pass from the L^2 estimate to the L^p estimate

$$\|V\|_{W_-^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \le C_p(\lambda)\|\mathcal{D}V\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})}. \tag{17}$$

This is because the boundary condition of $V\in W_-^{1,p}(\mathtt{A}_\varepsilon,\mathbb{S})$ and (13) yields

$$\|V\|_{L_-^2(\mathtt{A}_{k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})} \le 2C(\lambda)\|\mathcal{D}V\|_{L^2(\mathtt{A}_{k\varepsilon},\mathbb{S})} \le C_p(\lambda)\|\mathcal{D}V\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})},$$

where $C_p(\lambda) = 2C(\lambda)\left(\frac{3}{2}Vol(\Sigma)\right)^{\frac{1}{2}-\frac{1}{p}}$. Last we obtain the inequality (15) on $\mathtt{A}_\varepsilon$ from (17) by the periodicity of the L^p integrals on the reflected domains. □

Corollary 22 (C^0 estimate)**.** *If for each $z\in\Sigma$ there exist $x,x'\in[0,\varepsilon]$ such that both $u(x,z)=0$ and $v(x',z)=0$, then*

$$\|V\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \le C\varepsilon^{1-\frac{3}{p}}\|\mathcal{D}V\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})}. \tag{18}$$

Proof. We have

$$\begin{aligned} \|V\|_{C^0(\mathtt{A}_{k\varepsilon},\mathbb{S})} &\overset{\text{Schauder}}{\le} C\varepsilon^{1-\frac{3}{p}}\|V\|_{C^{1-\frac{3}{p}}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \overset{\text{Sobolev}}{\le} C\varepsilon^{1-\frac{3}{p}}\|V\|_{W^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\ &\le C\varepsilon^{1-\frac{3}{p}}\|\mathcal{D}V\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} \text{ (by (15))} \le C\varepsilon^{1-\frac{3}{p}}\|\mathcal{D}V\|_{C^0(\mathtt{A}_{k\varepsilon},\mathbb{S})}, \end{aligned}$$

where in the first inequality we have used the condition $u(x,z)=0$, and $v(x',z)=0$ and the definition of the Schauder $C^{1-\frac{3}{p}}$ norm. Then we notice the periodicity on $\mathtt{A}_{k\varepsilon}$ so we get (18). □

$C^{1,\alpha}$-estimate

We can change $\mathcal{D}_{-}V=W$ (14) into a system of second order elliptic equations

$$\begin{cases} u_{x_1x_1}-\overline{\partial}^*\overline{\partial}u=\mathbf{i}\overline{\partial}^*w_2+\partial_{x_1}w_1 \\ v_{x_1x_1}-\overline{\partial}\,\overline{\partial}^*v=-\mathbf{i}\overline{\partial}w_1+\partial_{x_1}w_2 \end{cases} \quad \text{and} \quad v|_{\partial\mathtt{A}_\varepsilon}=0, \tag{19}$$

where $V=(u,v)$ and $W=(w_1,w_2)$.

To get the uniform estimate of $\left\|\mathcal{D}^{-1}\right\|$ in the $C^{1,\alpha}$ setting, we can not just rely on periodic reflections, because if $w_1\neq 0$ then the extension of $\mathcal{D}V$ is no longer continuous on $\mathtt{A}_{k\varepsilon}$, not to mention in $C^{\alpha}(\mathtt{A}_{k\varepsilon},\mathbb{S})$.

Theorem 23 ($C^{1,\alpha}$ estimate)**.** *For any $V\in C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})$, we have*

$$C\varepsilon^{\frac{3}{p}+2\alpha}\|V\|_{C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})}\leq\|\mathcal{D}V\|_{C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}.$$

Proof. We can decouple $W=(w_1,w_2)$ into dealing with $(0,w_2)$ and $(w_1,0)$ cases separately because $\mathcal{D}$ is linear and *surjective* from Corollary 19.

When $w_1=0$, along $\partial\mathrm{A}_\varepsilon$ we have $u_{x_1}=\overline{\partial}^*v=0$ (for $v|_{\partial\mathtt{A}_\varepsilon}=0$), so we can reflect u by even extension and it is still in $C^{1,\alpha}$. We can extend W in $C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$ as well, for $w_1=0$. In this case we can *restore uniform ellipticity* in the Schauder setting by periodic reflection.

We deal with the more difficult case when $w_1\neq 0$ but $w_2=0$. The v component is easier, for it satisfies the Dirichlet boundary condition $v|_{\partial\mathrm{A}_\varepsilon}=0$. Standard Schauder estimate on half balls ([11]) implies

$$C\varepsilon^{1+\alpha}\|v\|_{C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})}\leq\varepsilon\|w_1\|_{C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S}^+)}+\|v\|_{C^0_{-}(\mathtt{A}_\varepsilon,\mathbb{S})}. \tag{20}$$

Plugging the C^0 estimate (18) in above inequality we get

$$\|v\|_{C^{1,\alpha}_{-}(\mathtt{A}_\varepsilon,\mathbb{S})}\leq C\varepsilon^{-\left(\frac{3}{p}+\alpha\right)}\|W\|_{C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}.$$

The u component is much harder, since $u|_{\partial\mathrm{A}_\varepsilon}\neq 0$ in general. We carry out the following steps:

(a) We *homogenize* u by introducing

$$\widetilde{u}=u-\rho\left(\frac{x_1}{\varepsilon}\right)u(\varepsilon,z)-\left(1-\rho\left(\frac{x_1}{\varepsilon}\right)\right)u(0,z),$$

where $\rho:[0,1]\to[0,1],\rho(0)=0,\rho(1)=1$ is a smooth cut-off function such that $\|\rho\|_{C^{2,\alpha}[0,1]}\leq C$. Then we get

$$\widetilde{u}_{x_1x_1}-\partial^+\partial^-\widetilde{u}=\partial_{x_1}\widetilde{w_1}+g \quad \text{and} \quad \widetilde{u}|_{\partial\mathtt{A}_\varepsilon}=0,$$

where the functions $\widetilde{w_1}$ and g involve w_1 and the boundary values of u. Applying elliptic regularity for this Dirichlet problem of $\widetilde{u}$ and then writing back in u, we have

$$\varepsilon \left\| w_1 \right\|_{C^\alpha} + \varepsilon^{1+\alpha} \left\| u \right\|_{C^0_-} + \left\| v \right\|_{C^0_-} \geq C\varepsilon^{1+2\alpha} \left\| u \right\|_{C^{1,\alpha}_-}. \tag{21}$$

This is not quite easy, since we can not use ∇v to completely control ∇u from (14). This was done by carefully analyzing the ε-orders behaviors of the derivatives of the cut-off function $\rho\left(\frac{x_1}{\varepsilon}\right)$ and elliptic estimates of $\overline{\partial}$ and $\overline{\partial}^*$ on slices $\{x\} \times \Sigma$.

(b) We show on each segment $[0, \varepsilon] \times \{z\}$ *there exists x with $u(x, z) = 0$*. Consider the *average* of u along each segment $[0, \varepsilon] \times \{z\}$:

$$\overline{u}(z) = \int_0^\varepsilon u(x_1, z)\, dx_1.$$

From (14) we have (note that $w_2 = 0$)

$$\mathbf{i}\overline{\partial}\overline{u}(z) = -\int_0^\varepsilon \partial_{x_1} v(x_1, z)\, dx_1 = v(0, z) - v(\varepsilon, z) = 0.$$

But from the assumption that $\overline{\partial}^*\overline{\partial}$ has trivial kernel on Σ we get that

$$\overline{u}(z) \equiv 0.$$

Then by the intermediate value theorem there must exist some $x \in [0, \varepsilon]$ such that $u(x, z) = 0$. So by Corollary 22 we get the desired C^0 estimate of u. Similarly we get an estimate for v.

(c) Putting the C^0 estimate of $V = (u, v)$ back to (21), we finally get

$$\left\| V \right\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon, \mathbb{S})} \leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} \left\| DV \right\|_{C^\alpha(\mathtt{A}_\varepsilon, \mathbb{S})}.$$

Theorem 15 is proved. □

3.2.5. Geometry of J_n-holomorphic curves in G_2 manifolds: agreement of two Dirac operators

Recall given a family $\mathcal{C} = \cup_{0 \leq t \leq \varepsilon} C_t$ of coassociative manifolds C_t in M, the (nonvanishing) deformation vector field $n := \frac{dC_t}{dt}|_{t=0}$ defines an almost complex structure J_n on $C = C_0$. For any J_n-holomorphic curve $\Sigma \subset C$, the G_2-structure on M gives a natural identification between $N_{\Sigma/C}$ and $N_{\mathcal{C}/M}|_\Sigma$ ([10]). Further calculations [21] established the following close relation between the "intrinsic" and "extrinsic" Dirac operators.

Proposition 24 (Proposition 16,17 in [21])**.** *Let $\Sigma \subset C \subset \mathcal{C} \subset M$ be as above. We have an orthogonal decomposition*

$$TM|_\Sigma = T\Sigma \oplus N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma \oplus N_{C/\mathcal{C}}|_\Sigma,$$

(1) *For $L = T\Sigma$, $N_{\Sigma/C}$ or $N_{\mathcal{C}/M}|_\Sigma$, the induced connection ∇^L from the Levi-Civita connection on M is Hermitian, i.e., $\nabla^L J_n = 0$.*

(2) *The spinor bundle* $\mathbb{S}_\Sigma$ *over* Σ *is identified with* $N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma$ *in such a way that the Clifford multiplication is given by the* G_2 *multiplication* $\times$ *and the spinor connection equals* $\nabla^{N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma}$.

(3) *The Dirac operator on* $\mathbb{S}_\Sigma = N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma$ *agrees with the Dolbeault operator on* $N_{\Sigma/C} \oplus \wedge^{0,1}_{\mathbb{C}} \left(N_{\Sigma/C}\right)$.

The agreement of the two Dirac type operators on Σ is the geometric reason that the Fredholm regularity property of the "intrinsic" Cauchy-Riemann operator on $N_{\Sigma/C}$ gives control of the "extrinsic" linearized instanton operator.

3.2.6. Comparison of $\mathcal{D}$ and linearized instanton equation

When we move from Σ to the interior of the almost instanton A'_ε, the nice agreement of the two Dirac operators no longer holds. We need to control the difference between the Dirac operator $\mathcal{D}$ on $\mathbb{S}$ and the linearized instanton operator $F'_\varepsilon(0)$ on $N_{\mathtt{A}'_\varepsilon/M}$. In order to compare them, we need a *good* identification between $\mathbb{S}$ and $N_{\mathtt{A}'_\varepsilon/M}$.

For this purpose we defined an *exponential-like map* $\widetilde{\exp} : \mathbb{S} \to M$ which satisfies $\widetilde{\exp}(\mathtt{A}_\varepsilon) = \mathtt{A}'_\varepsilon$ and its differential $d\widetilde{\exp}|_{\mathtt{A}_\varepsilon}$ has the following properties on $\{0\} \times \Sigma$ (see Appendix of [21]):

(1) $d\widetilde{\exp}|_{\{0\}\times\Sigma} = (id, f) : N_{\Sigma/C} \oplus \wedge^{0,1}_{\mathbb{C}} \left(N_{\Sigma/C}\right) \to N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma$ on fiber directions of $\mathbb{S}$, where f is the isomorphism between the second summands given in Lemma 3.2 of [10];

(2) $d\widetilde{\exp}|_{\{0\}\times\Sigma} = id : T\Sigma \to T\Sigma$ and $d\widetilde{\exp}|_{\{0\}\times\Sigma} : \frac{\partial}{\partial x_1} \to n(z)$ on base directions of $\mathbb{S}$;

(3) $\widetilde{\exp}|_{\{0\}\times\Sigma}(\mathbb{S}^+ \oplus 0) \subset C_0$, $\widetilde{\exp}|_{\{\varepsilon\}\times\Sigma}(\mathbb{S}^+ \oplus 0) \subset C_\varepsilon$ on the boundary $\{0, \varepsilon\} \times \Sigma$ of $\mathtt{A}_\varepsilon$.

Using the map $d\widetilde{\exp}|_{\mathtt{A}_\varepsilon}$, we can relate the *spin bundle* $\mathbb{S} \to \mathtt{A}_\varepsilon$ to the *normal bundle* $N_{\mathtt{A}'_\varepsilon/M} \to \mathtt{A}'_\varepsilon$. Because of the compatibilities given in 1-3 above, using Proposition 24 we have the following comparison result.

Proposition 25 (Proposition 18 in [21]). *For any* $V_1 \in C^{1,\alpha}(A_\varepsilon, \mathbb{S})$, *we have*

$$\left\|F'_\varepsilon(0)\left(d\widetilde{\exp} \cdot V_1\right) - \left(d\widetilde{\exp}\right) \circ \mathcal{D}V_1\right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} \le C\varepsilon^{1-\alpha} \left\|V_1\right\|_{C^{1,\alpha}(A_\varepsilon,\mathbb{S})}.$$

Then we get the uniform inverse estimate of $F'_\varepsilon(0)$ from $\left\|\mathcal{D}^{-1}\right\|$ by the following diagram, using that $d\widetilde{\exp}$ and $\left(d\widetilde{\exp}\right)^{-1}$ are both smooth with a uniform C^2 bound for all ε:

$$\begin{array}{ccc} C^{1,\alpha}_-(\mathtt{A}_\varepsilon, \mathbb{S}) & \xrightarrow{\mathcal{D}} & C^\alpha(\mathtt{A}_\varepsilon, \mathbb{S}) \\ {\scriptstyle d\widetilde{\exp}}\big\downarrow & & \big\downarrow{\scriptstyle d\widetilde{\exp}} \\ C^{1,\alpha}_-\left(N_{\mathtt{A}'_\varepsilon/M}\right) & \xrightarrow{F'_\varepsilon(0)} & C^\alpha\left(N_{\mathtt{A}'_\varepsilon/M}\right) \end{array}$$

3.2.7. Quadratic estimate

Using any local frame field $\{W_\alpha\}_{\alpha=1}^{7}$, we could compare the linearizations of $F(V)$ at two different almost instantons, up to a curvature term B as follows:

$$\begin{aligned} &F'(V_0)\,|_{A_{\varepsilon(0)}} V(p) \\ &= \left((\exp V_0)^* \otimes T_{V_0}\right) \circ \left[F'(0)\,|_{A_\varepsilon(V_0)} V_1(q)\right] + B^\alpha(V_0, V)\, W_\alpha(p)\,, \end{aligned} \tag{22}$$

where $V_1 = (d\exp V_0)\,|_{A_\varepsilon(0)} V$. This means the derivative $F'(V_0)$ on $A_\varepsilon(0)$ can be expressed by the derivative $F'(0)$ on $A_\varepsilon(V_0)$ via the transform $(\exp V_0)^* \otimes T_{V_0}$, up to the curvature term $B^\alpha(V_0, V)\, W_\alpha(p)$. Using (22) we get

$$F'(V_0)\, V(p) - F'(0)\, V(p) = (I) + (II)$$

with

$$\begin{aligned} (I) &= \left[(\exp V_0)^* d\left(i_{V_1}\omega^\alpha\right) - d\left(i_V \omega^\alpha\right)\right]|_{A_{\varepsilon(0)}} \otimes W_\alpha(p)\,, \\ (II) &= \left((\exp V_0)^* \otimes T_{V_0}\right) \circ \left[i_{V_1} d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_{V_1} W_\alpha\right](p) + B^\alpha(V_0, V)\, W_\alpha(p)\,. \end{aligned}$$

Here (I) consists of 1^{st} order terms whose C^α-norm are bounded by $\|V_0\|_{C^{1,\alpha}} \|V\|_{C^{1,\alpha}}$ because d, the pull back operator, the parallel transport and various exponential maps are Frechet smooth with respect to variations of V. (II) consists of 0^{th} order terms and it is easier to bound. Thus we get

Proposition 26 (Quadratic estimate)**.** *For any $V_0, V \in C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)$ with small $\|V_0\|_{C^{1,\alpha}}$, we have*

$$\|F'_\varepsilon(V_0)\, V - F'_\varepsilon(0)\, V\|_{C^\alpha} \le C\,\|V_0\|_{C^{1,\alpha}} \|V\|_{C^{1,\alpha}}, \tag{23}$$

where the constant C is independent of ε.

Remark 27. *We also have some pointwise estimates tied to the feature that τ is a cubic-form. In* [21] *we derived that*

$$\begin{aligned} &|F'(V_0)\, V - F'(0)\, V|\,(p) \\ &\le \Big[C_7\left(|d\varphi| + |\nabla V_0|\right)^3\left(|V_0|\,|V| + |\nabla V_0|\,|V| + |V_0|\,|\nabla V|\right) + C_5\,|V_0|\,|V|\Big](p)\,, \end{aligned}$$

where $\varphi : A_\varepsilon \to M$ is the embedding we used to define A'_ε. Because of the cubic terms, the following quadratic estimate needed in the implicit function theorem in $W^{1,p}$ setting

$$\|F'(V_0)\, V - F'(0)\, V\|_{L^p} \le C\,\|V_0\|_{W^{1,p}} \|V\|_{W^{1,p}}.$$

appears unavailable, for $\left(|\nabla V_0|^3\right)^p \notin L^p$ in general. In contrast, the Cauchy-Riemann operator of J-holomorphic curves is more linear in the L^p setting: it has the quadratic estimate (Proposition 3.5.3 in [23]*)*

$$\|F'(V_0)\, V - F'(0)\, V\|_{L^p(\Sigma)} \le C\,\|V_0\|_{W^{1,p}(\Sigma)} \|V\|_{W^{1,p}(\Sigma)}.$$

This is one of the key reasons that we use the Schauder setting.

By our construction of the almost instanton $A'_\varepsilon = \varphi(A_\varepsilon)$ and the smoothness of φ, it is not hard to get the error estimate

$$\|F_\varepsilon(0)\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} \leq C\varepsilon^{1-\alpha}.$$

Combining (10), (23), by the implicit function theorem (e.g., Proposition A.3.4 in [23]) we can solve $F_\varepsilon(V) = 0$, thus our main result, Theorem 13, is proved. QED.

4. Applications and further discussions

4.1. New examples of instantons

Our main theorem can be used to construct new examples of instantons.

Let X be a Calabi-Yau threefold containing a complex surface $S \subset X$ which contains a smooth curve $\Sigma \subset S$ that satisfies (i) $H^0(S, K_S) \neq 0$ (i.e. $p_g(S) \neq 0$) and (ii) $H^0(\Sigma, K_S|_\Sigma) = 0$. Condition (i) implies that S can be deformed inside X and condition (ii) is equivalent to $H^1\left(\Sigma, N_{\Sigma/S}\right) = 0$, namely $\Sigma \subset S$ is Fredholm regular.

Let $\{S_t\}_{0\leq t\leq\varepsilon}$ be a smooth family of deformations of S inside X and let $v = \frac{dS_t}{dt}|_{t=0}$ be the normal vector field on $S = S_0$. By our assumption, v is nontrivial, and after possible rescaling of the parameter t for the family S_t, we may assume v is very small. Let $M = X \times S^1$ be the G_2 manifold as in Example 2, and $C_t := S_t \times \{t\} \subset M$ $(0 \leq t \leq \varepsilon)$ be the family of coassociative submanifolds. They are disjoint since their second components t are different. Let $n_0 := \left(0, \frac{\partial}{\partial\theta}\right)$ be the normal vector field on $C_0 = S \times \{0\}$, then the original complex structure J_0 on S is induced from n_0, i.e., $J_0 = \frac{n_0}{|n_0|}\times$.

Let $n = \frac{dC_t}{dt}|_{t=0} = \left(v, \frac{\partial}{\partial\theta}\right)$ be the other normal vector field on C_0. Then n is nonvanishing since $\frac{\partial}{\partial\theta}$ is nonvanishing on S^1. The almost complex structure $J_n = \frac{n}{|n|}\times$ is close to but not equal to the original complex structure J_0 on S, because n is close to but not equal to n_0.

There must exist a Fredholm regular J_n-holomorphic curve $\Sigma_n \subset S$ near the original J_0-holomorphic curve Σ, because $\Sigma \subset S$ is Fredholm regular and will persist after small perturbations of J_0 on S.

Applying our main theorem to $\Sigma_n \subset C_0$, we get an instanton $A \subset M$ with boundaries on $C_0 \cup C_\varepsilon$. It is *not* the trivial instanton $\Sigma \times [0, \varepsilon]$, which has upper boundary lying on $S \times \{\varepsilon\}$, not on $C_\varepsilon = S(\varepsilon) \times \{\varepsilon\}$.

4.2. Further remarks

A few remarks on our main theorem are in order: First, counting such thin instantons is basically a problem in four manifold theory because of Bryant's result [6] which says that the zero section C in $\Lambda^2_+(C)$ is always a coassociative submanifold for an incomplete G_2-metric on its neighborhood provided that the bundle $\Lambda^2_+(C)$ is topologically trivial.

Second, when the normal vector field $n = \frac{dC_t}{dt}|_{t=0}$ has zeros, our main Theorem 13 should still hold true (work in progress [22]). However it would require a possible change of the Fredholm set-up of the current method and a good understanding of the Seiberg-Witten theory on any four manifold with a *degenerated symplectic form* as in Taubes program ([29], [30]). In the special case when Σ is disjoint from $\{n = 0\}$, Theorem 13 is obviously true as our analysis only involves the local geometry of Σ in M.

Third, if we do not restrict to instantons of small volume, then we have to take into account *bubbling* phenomena as in the pseudo-holomorphic curves case, and gluing of instantons of big and small volumes similar to [25] in Floer trajectory case. Nevertheless, one does not expect bubbling can occur when volumes of instantons are small, thus they would converge to a J_n-holomorphic curve in C as $\varepsilon \to 0$.

Last, we expect our result still holds in the *almost* G_2 setting, namely Ω is only a closed form rather than a parallel form. This is because our gluing analysis relies mainly on the Fredholm regularity property of the linearized instanton equation. Such a flexibility could be useful for finding regular holomorphic curves $\Sigma \subset C$ in order to apply our theorem.

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