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ADE Bundles over Surfaces with *ADE* Singularities

Yunxia Chen and Naichung Conan Leung

Department of Mathematics, The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, NT, Hong Kong

Correspondence to be sent to: yxchen76@gmail.com

Given a compact complex surface X with an ADE singularity and $p_g = 0$, we construct ADE bundles over X and its minimal resolution Y. Furthermore, we describe their minuscule representation bundles in terms of configurations of (reducible) (-1)-curves in Y.

1 Introduction

It has long been known that there are deep connections between Lie theory and the geometry of surfaces. A famous example is an amazing connection between Lie groups of type E_n and del Pezzo surfaces X of degree $9-n$ for $1 \leq n \leq 8$. The root lattice of E_n can be identified with $K_X^{\perp}$, the orthogonal complement to K_X in $\mathrm{Pic}(X)$. Furthermore, all the lines in X form a representation of E_n. Using the configuration of these lines, we can construct an E_n Lie algebra bundle over X [15]. If we restrict it to the anti-canonical curve in X, which is an elliptic curve Σ, then we obtain an isomorphism between the moduli space of degree $9-n$ del Pezzo surfaces which contain Σ and the moduli space of E_n-bundles over Σ. This work is motivated from string/F-theory duality, and it has been studied extensively by Friedman–Morgan–Witten [8–10], Donagi [3–5, 7], Leung–Zhang [14–16], and others [6, 13, 17, 18].

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In this paper, we study the relationships between simply laced, or ADE, Lie theory and rational double points of surfaces. Suppose

$$\pi : Y \to X$$

is the minimal resolution of a compact complex surface X with a rational double point. Then the dual graph of the exceptional divisor $\sum_{i=1}^{n} C_i$ in Y is an ADE Dynkin diagram. From this, we have an ADE root system $\Phi := \{\alpha = \sum a_i[C_i] | \alpha^2 = -2\}$ and we can construct an ADE Lie algebra bundle over Y:

$$\mathcal{E}_0^{\mathfrak{g}} := O_Y^{\oplus n} \oplus \bigoplus_{\alpha \in \Phi} O_Y(\alpha).$$

Even though this bundle cannot descend to X, we show that it can be deformed to one that can descend to X, provided that $p_g(X) = 0$.

Theorem 1 (Propositions 6, 7, Theorem 9, and Lemma 10)**.** Assume that Y is the minimal resolution of a surface X with a rational double point at p of type $\mathfrak{g}$ and $C = \Sigma_{i=1}^{n} C_i$ is the exceptional divisor. If $p_g(X) = 0$, then

(1) given any $(\varphi_{C_i})_{i=1}^{n} \in \Omega^{0,1}(Y, \bigoplus_{i=1}^{n} O(C_i))$ with $\bar{\partial}\varphi_{C_i} = 0$ for every i, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha))$ such that $\bar{\partial}_\varphi := \bar{\partial} + \mathrm{ad}(\varphi)$ is a holomorphic structure on $\mathcal{E}_0^{\mathfrak{g}}$. We denote this new holomorphic bundle as $\mathcal{E}_\varphi^{\mathfrak{g}}$;

(2) such a $\bar{\partial}_\varphi$ is compatible with the Lie algebra structure;

(3) $\mathcal{E}_\varphi^{\mathfrak{g}}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(C_i, O_{C_i}(C_i)) \cong \mathbb{C}$;

(4) there exists $[\varphi_{C_i}] \in H^1(Y, O(C_i))$ such that $[\varphi_{C_i}|_{C_i}] \neq 0$;

(5) such a $\mathcal{E}_\varphi^{\mathfrak{g}}$ can descend to X if and only if $[\varphi_{C_i}|_{C_i}] \neq 0$ for every i. □

Remark 2. Infinitesimal deformations of holomorphic bundle structures on $\mathcal{E}_0^{\mathfrak{g}}$ are parameterized by $H^1(Y, \mathrm{End}(\mathcal{E}_0^{\mathfrak{g}}))$, and those that also preserve the Lie algebra structure are parameterized by $H^1(Y, \mathrm{ad}(\mathcal{E}_0^{\mathfrak{g}})) = H^1(Y, \mathcal{E}_0^{\mathfrak{g}})$, since $\mathfrak{g}$ is semi-simple. If $p_g(X) = q(X) = 0$, for example, rational surface, then for any $\alpha \in \Phi^-$, $H^1(Y, O(\alpha)) = 0$. Hence, $H^1(Y, \mathcal{E}_0^{\mathfrak{g}}) = H^1(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha))$. □

This generalizes the work of Friedman–Morgan [8], in which they considered E_n bundles over generalized del Pezzo surfaces. In this paper, we will also describe the minuscule representation bundles of these Lie algebra bundles in terms of (-1)-curves in Y.

Here is an outline of our results. We first study (-1)-curves in Y which are (possibly reducible) rational curves with self-intersection -1. If there exists a (-1)-curve C_0 in X passing through p with minuscule multiplicity C_k (Definition 15), then (-1)-curves l's in Y with $\pi(l) = C_0$ form the minuscule representation V of $\mathfrak{g}$ corresponding to C_k (Proposition 21). (Here V is the lowest weight representation with lowest weight dual to $-C_k$, that is, V is dual to the highest weight representation with highest weight dual to C_k.) When V is the standard representation of $\mathfrak{g}$, the configuration of these (-1)-curves determines a symmetric tensor f on V such that $\mathfrak{g}$ is the space of infinitesimal symmetries of (V, f). We consider the bundle

$$\mathfrak{L}_0^{(\mathfrak{g},V)} := \bigoplus_{\substack{l:(-1)-\text{curve}\\ \pi(l)=C_0}} O_Y(l),$$

over Y constructed from these (-1)-curves l's. This bundle cannot descend to X as it is not trivial over each C_i. (Unless specified otherwise, C_i always refers to an irreducible component of C, that is, $i \neq 0$.)

Theorem 3 (Theorems 23 and 24)**.** For the bundle $\mathfrak{L}_0^{(\mathfrak{g},V)}$ with the corresponding minuscule representation $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$,

(1) there exists $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$ such that $\bar{\partial}_\varphi := \bar{\partial}_0 + \rho(\varphi)$ is a holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)}$. We denote this new holomorphic bundle as $\mathfrak{L}_\varphi^{(\mathfrak{g},V)}$;

(2) $\mathfrak{L}_\varphi^{(\mathfrak{g},V)}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i))$;

(3) when V is the standard representation of $\mathfrak{g}$, there exists a holomorphic fiberwise symmetric multi-linear form

$$f : \bigotimes^r \mathfrak{L}_\varphi^{(\mathfrak{g},V)} \longrightarrow O_Y(D),$$

with $r = 0, 2, 3, 4$ when $\mathfrak{g} = A_n, D_n, E_6, E_7$, respectively, such that $\mathcal{E}_\varphi^{\mathfrak{g}} \cong \mathrm{aut}_0(\mathfrak{L}_\varphi^{(\mathfrak{g},V)}, f)$. □

When V is a minuscule representation of $\mathfrak{g}$, there exists a unique holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)} := \bigoplus_l O(l)$ such that the action of $\mathcal{E}_\varphi^{\mathfrak{g}}$ on this bundle is holomorphic and it can descend to X as well.

Example 4. When we blowup two distinct points, we have a surface Y with two (-1)-curves l_1 and l_2 as exceptional curves. $\mathfrak{L}_0 := O_Y(l_1) \oplus O_Y(l_2)$ is a $\mathbb{C}^2$-bundle and the bundle $\zeta_0^{A_1}$ of its symmetries is an sl(2)- or A_1-bundle over Y.

When the two points become infinitesimally close, then $C_1 = l_2 - l_1$ is effective, namely a (-2)-curve in Y. If we blow down C_1 in Y, we get a surface X with an A_1 singularity. $\mathfrak{L}_0$ cannot descend to X as $\mathfrak{L}_0|_{C_1} \cong O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(1)$. Using the Euler sequence $0 \to O_{\mathbb{P}^1}(-1) \to O_{\mathbb{P}^1}^{\oplus 2} \to O_{\mathbb{P}^1}(1) \to 0$, we deform $\mathfrak{L}_0|_{C_1}$ to become trivial and using $p_g = 0$ to lift this deformation to Y. The resulting bundles $\mathfrak{L}_\varphi$ and $\zeta_\varphi^{A_1}$ do descend to X. $\square$

For every ADE case with V the standard representation, we have $\mathfrak{L}_0^{(\mathfrak{g},V)}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus m} + (O_{\mathbb{P}^1}(1) + O_{\mathbb{P}^1}(-1))^{\oplus n}$. For A_n cases, our arguments are similar to the above A_1 case. For D_n cases, further arguments are needed as the pairs of $O_{\mathbb{P}^1}(\pm 1)$ in $\mathfrak{L}_0^{(D_n,\mathbb{C}^{2n})}|_{C_i}$ are in different locations comparing with the A_n cases, and we also need to check that the holomorphic structure $\bar{\partial}_\varphi$ on $\mathfrak{L}_0^{(D_n,\mathbb{C}^{2n})}$ preserves the natural quadratic form q. For the E_6 (respectively, E_7) case, since the cubic form c (respectively, quartic form t) is more complicated than the quadratic form q in D_n cases, the calculations are more involved. The E_8 case is rather different and we handle it by reductions to A_7 and D_7 cases.

The organization of this paper is as follows. Section 2 gives the construction of ADE Lie algebra bundles over Y directly. In Section 3, we review the definition of minuscule representations and construct all minuscule representations using (-1)-curves in Y. Using these, we construct the Lie algebra bundles and minuscule representation bundles which can descend to X in A_n, D_n, and E_n $(n \neq 8)$ cases separately in Sections 4–6. The proofs of the main theorems in this paper are given in Section 7.

Notation: For a holomorphic bundle $(E_0, \bar{\partial}_0)$, if we construct a new holomorhic structure $\bar{\partial}_\varphi$ on E_0, then we denote the resulting bundle as E_φ.

2 *ADE* Lie Algebra Bundles

2.1 *ADE* singularities

A rational double point p in a surface X can be described locally as a quotient singularity $\mathbb{C}^2/\Gamma$ with Γ a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. It is also called a Kleinian singularity or ADE singularity [2]. We can write $\mathbb{C}^2/\Gamma$ as zeros of a polynomial $F(X,Y,Z)$ in $\mathbb{C}^3$, where $F(X,Y,Z)$ is $X^n + YZ$, $X^{n+1} + XY^2 + Z^2$, $X^4 + Y^3 + Z^2$, $X^3Y + Y^3 + Z^2$, or $X^5 + Y^3 + Z^2$ and the corresponding singularity is called of type A_n, D_n, E_6, E_7, or E_8, respectively. The reason is if we consider the minimal resolution Y of X, then every irreducible component of the exceptional divisor $C = \sum_{i=1}^n C_i$ is a smooth rational curve with normal bundle

$O_{\mathbb{P}^1}(-2)$, that is, a (-2)-curve, and the dual graph of the exceptional divisor is an *ADE* Dynkin diagram.

There is a natural decomposition

$$H^2(Y,\mathbb{Z}) = H^2(X,\mathbb{Z}) \oplus \Lambda,$$

where $\Lambda = \{\sum a_i[C_i] | a_i \in \mathbb{Z}\}$. The set $\Phi := \{\alpha \in \Lambda | \alpha^2 = -2\}$ is a simply laced (i.e., *ADE*) root system of a simple Lie algebra $\mathfrak{g}$ and $\Delta = \{[C_i]\}$ is a base of Φ. For any $\alpha \in \Phi$, there exists a unique divisor $D = \sum a_i C_i$ with $\alpha = [D]$, and we define a line bundle $O(\alpha) := O(D)$ over Y.

2.2 Lie algebra bundles

We define a Lie algebra bundle of type $\mathfrak{g}$ over Y as follows:

$$\mathcal{E}_0^{\mathfrak{g}} := O^{\oplus n} \oplus \bigoplus_{\alpha\in\Phi} O(\alpha).$$

For every open chart U of Y, we take x_α^U to be a nonvanishing holomorphic section of $O_U(\alpha)$ and h_i^U $(i = 1, \dots, n)$ nonvanishing holomorphic sections of $O_U^{\oplus n}$. Define a Lie algebra structure $[,]$ on $\mathcal{E}_0^{\mathfrak{g}}$ such that $\{x_\alpha^U\text{'s}, h_i^U\text{'s}\}$ is the Chevalley basis [12], that is,

(1) $[h_i^U, h_j^U] = 0$, $1 \le i, j \le n$;
(2) $[h_i^U, x_\alpha^U] = \langle \alpha, C_i\rangle x_\alpha^U$, $1 \le i \le n$, $\alpha \in \Phi$;
(3) $[x_\alpha^U, x_{-\alpha}^U] = h_\alpha^U$ is a $\mathbb{Z}$-linear combination of h_i^U;
(4) if α, β are independent roots, and $\beta - r\alpha, \dots, \beta + q\alpha$ is the α-string through β, then $[x_\alpha^U, x_\beta^U] = 0$ if $q = 0$; otherwise $[x_\alpha^U, x_\beta^U] = \pm(r+1)x_{\alpha+\beta}^U$.

Since $\mathfrak{g}$ is simply laced, all its roots have the same length, we have that any α-string through β is of length at most 2. So (4) can be written as $[x_\alpha^U, x_\beta^U] = n_{\alpha,\beta} x_{\alpha+\beta}^U$, where $n_{\alpha,\beta} = \pm 1$ if $\alpha+\beta \in \Phi$, otherwise $n_{\alpha,\beta} = 0$. From the Jacobi identity, we have for any $\alpha, \beta, \gamma \in \Phi$, $n_{\alpha,\beta}n_{\alpha+\beta,\gamma} + n_{\beta,\gamma}n_{\beta+\gamma,\alpha} + n_{\gamma,\alpha}n_{\gamma+\alpha,\beta} = 0$. This Lie algebra structure is compatible with different trivializations of $\mathcal{E}_0^{\mathfrak{g}}$ [15].

By Friedman–Morgan [8], a bundle over Y can descend to X if and only if its restriction to each irreducible component C_i of the exceptional divisor is trivial. But $\mathcal{E}_0^{\mathfrak{g}}|_{C_i}$ is not trivial as $O([C_i])|_{C_i} \cong O_{\mathbb{P}^1}(-2)$. We will construct a new holomorphic structure on $\mathcal{E}_0^{\mathfrak{g}}$, which preserves the Lie algebra structure, and therefore, the resulting bundle $\mathcal{E}_\varphi^{\mathfrak{g}}$ can descend to X.

As we have fixed a base Δ of Φ, we have a decomposition $\Phi = \Phi^+ \cup \Phi^-$ into positive and negative roots.

Definition 5. Given any $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$, we define $\bar\partial_\varphi : \Omega^{0,0}(Y, \mathcal{E}_0^{\mathfrak{g}}) \longrightarrow \Omega^{0,1}(Y, \mathcal{E}_0^{\mathfrak{g}})$ by

$$\bar\partial_\varphi := \bar\partial_0 + \mathrm{ad}(\varphi) := \bar\partial_0 + \sum_{\alpha\in\Phi^+} \mathrm{ad}(\varphi_\alpha),$$

where $\bar\partial_0$ is the standard holomorphic structure of $\mathcal{E}_0^{\mathfrak{g}}$. More explicitly, if we write $\varphi_\alpha = c_\alpha^U x_\alpha^U$ locally for some one form c_α^U, then $\mathrm{ad}(\varphi_\alpha) = c_\alpha^U \mathrm{ad}(x_\alpha^U)$. □

Proposition 6. $\bar\partial_\varphi$ is compatible with the Lie algebra structure, that is, $\bar\partial_\varphi[,] = 0$. □

Proof. This follows directly from the Jacobi identity. ■

For $\bar\partial_\varphi$ to define a holomorphic structure, we need

$$0 = \bar\partial_\varphi^2 = \sum_{\alpha\in\Phi^+} \left(\bar\partial_0 c_\alpha^U + \sum_{\beta+\gamma=\alpha} (n_{\beta,\gamma} c_\beta^U c_\gamma^U) \right) \mathrm{ad}(x_\alpha^U),$$

that is, $\bar\partial_0\varphi_\alpha + \sum_{\beta+\gamma=\alpha}(n_{\beta,\gamma}\varphi_\beta\varphi_\gamma) = 0$ for any $\alpha \in \Phi^+$. Explicitly:

$$\begin{cases} \bar\partial_0\varphi_{C_i} = 0, & i = 1, 2 \ldots, n, \\ \bar\partial_0\varphi_{C_i+C_j} = n_{C_i,C_j}\varphi_{C_i}\varphi_{C_j}, & \text{if } C_i + C_j \in \Phi^+, \\ \vdots & \end{cases}$$

Proposition 7. Given any $(\varphi_{C_i})_{i=1}^n \in \Omega^{0,1}(Y, \bigoplus_{i=1}^n O(C_i))$ with $\bar\partial_0\varphi_{C_i} = 0$ for every i, it can be extended to $\varphi = (\varphi_\alpha)_{\alpha\in\Phi^+} \in \Omega^{0,1}(Y, \bigoplus_{\alpha\in\Phi^+} O(\alpha))$ such that $\bar\partial_\varphi^2 = 0$. Namely, we have a holomorphic vector bundle $\mathcal{E}_\varphi^{\mathfrak{g}}$ over Y. □

To prove this proposition, we need the following lemma. For any $\alpha = \sum_{i=1}^n a_i C_i \in \Phi^+$, we define $\mathrm{ht}(\alpha) := \sum_{i=1}^n a_i$.

Lemma 8. For any $\alpha \in \Phi^+$, $H^2(Y, O(\alpha)) = 0$. □

Proof. If $\mathrm{ht}(\alpha) = 1$, that is, $\alpha = C_i$, $H^2(Y, O(C_i)) = 0$ follows from the long exact sequence associated to $0 \to O_Y \to O_Y(C_i) \to O_{C_i}(C_i) \to 0$ and $p_g = 0$.

By induction, suppose that the lemma is true for every β with $\mathrm{ht}(\beta) = m$. Given any α with $\mathrm{ht}(\alpha) = m + 1$, by Humphreys [12, Section 10.2, Lemma A], there exists some C_i such that $\alpha \cdot C_i = -1$, that is, $\beta := \alpha - C_i \in \Phi^+$ with $\mathrm{ht}(\beta) = m$. Using the long exact sequence associated to $0 \to O_Y(\beta) \to O_Y(\alpha) \to O_{C_i}(\alpha) \to 0$, $O_{C_i}(\alpha) \cong O_{\mathbb{P}^1}(-1)$, and $H^2(Y, O(\beta)) = 0$ by induction, we have $H^2(Y, O(\alpha)) = 0$. ■

Proof of Proposition 7. We solve the equations $\bar{\partial}_0 \varphi_\alpha = \sum_{\beta+\gamma=\alpha} n_{\beta,\gamma} \varphi_\beta \varphi_\gamma$ for $\varphi_\alpha \in \Omega^{0,1}(Y, O(\alpha))$ inductively on $\mathrm{ht}(\alpha)$.

For $\mathrm{ht}(\alpha) = 2$, that is, $\alpha = C_i + C_j$ with $C_i \cdot C_j = 1$, since $[\varphi_{C_i} \varphi_{C_j}] \in H^2(Y, O(C_i + C_j)) = 0$, we can find $\varphi_{C_i+C_j}$ satisfying $\bar{\partial}_0 \varphi_{C_i+C_j} = \pm \varphi_{C_i} \varphi_{C_j}$.

Suppose that we have solved the equations for all φ_β's with $\mathrm{ht}(\beta) \leq m$. For

$$\bar{\partial}_0 \varphi_\alpha = \sum_{\beta+\gamma=\alpha} n_{\beta,\gamma} \varphi_\beta \varphi_\gamma,$$

with $\mathrm{ht}(\alpha) = m+1$, we have $\mathrm{ht}(\beta), \mathrm{ht}(\gamma) \leq m$. Using $\bar{\partial}_0 (\sum_{\beta+\gamma=\alpha} n_{\beta,\gamma} \varphi_\beta \varphi_\gamma) = \sum_{\delta+\lambda+\mu=\alpha} (n_{\delta,\lambda} n_{\delta+\lambda,\mu} + n_{\lambda,\mu} n_{\lambda+\mu,\delta} + n_{\mu,\delta} n_{\mu+\delta,\lambda}) \varphi_\delta \varphi_\lambda \varphi_\mu = 0$, $[\sum_{\beta+\gamma=\alpha} n_{\beta,\gamma} \varphi_\beta \varphi_\gamma] \in H^2(Y, O(\alpha)) = 0$, we can solve for φ_α. ■

Denote

$$\Psi_Y \triangleq \left\{ \varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Omega^{0,1}\left(Y, \bigoplus_{\alpha \in \Phi^+} O(\alpha)\right) | \bar{\partial}_\varphi^2 = 0 \right\}$$

and

$$\Psi_X \triangleq \{\varphi \in \Psi_Y | [\varphi_{C_i}|_{C_i}] \neq 0 \text{ for } i = 1, 2, \ldots, n\}.$$

Theorem 9. $\mathcal{E}_\varphi^{\mathfrak{g}}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i))$. □

Proof. We will discuss the *ADE* cases separately in Sections 4–6 and the proof will be completed in Section 7. ■

The next lemma says that given any C_i, there always exists $\varphi_{C_i} \in \Omega^{0,1}(Y, O(C_i))$ such that $0 \neq [\varphi_{C_i}|_{C_i}] \in H^1(Y, O_{C_i}(C_i)) \cong \mathbb{C}$.

Lemma 10. For any C_i in Y, the restriction homomorphism $H^1(Y, O_Y(C_i)) \to H^1(Y, O_{C_i}(C_i))$ is surjective. □

Proof. The above restriction homomorphism is part of a long exact sequence induced by $0 \to O_Y \to O_Y(C_i) \to O_{C_i}(C_i) \to 0$. The lemma follows directly from $p_g(Y) = 0$. ■

3 Minuscule Representations and (−1)-Curves

3.1 Standard representations

For *ADE* Lie algebras, $A_n = \mathrm{sl}(n+1)$ is the space of tracefree endomorphisms of $\mathbb{C}^{n+1}$ and $D_n = o(2n)$ is the space of infinitesimal automorphisms of $\mathbb{C}^{2n}$ which preserve a nondegenerate quadratic form q on $\mathbb{C}^{2n}$. In fact, E_6 (respectively, E_7) is the space of infinitesimal automorphisms of $\mathbb{C}^{27}$ (respectively, $\mathbb{C}^{56}$) which preserve a particular cubic form c on $\mathbb{C}^{27}$ (respectively, quartic form t on $\mathbb{C}^{56}$) [1]. We call the above representation the *standard representation* of $\mathfrak{g}$, that is,

$\mathfrak{g}$	Standard representation
$A_n = \mathrm{sl}(n+1)$	$\mathbb{C}^{n+1}$
$D_n = o(2n)$	$\mathbb{C}^{2n}$
E_6	$\mathbb{C}^{27}$
E_7	$\mathbb{C}^{56}$

Note that all these standard representations are the fundamental representations corresponding to the left nodes (i.e. C_1) in the corresponding Dynkin diagrams (Figures 1–3) and they are minuscule representations.

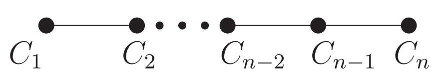

Fig. 1. The Dynkin diagram of A_n.

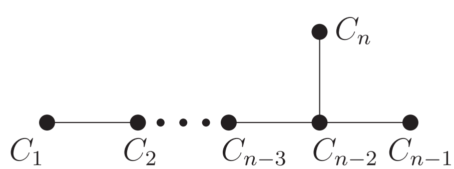

Fig. 2. The Dynkin diagram of D_n.

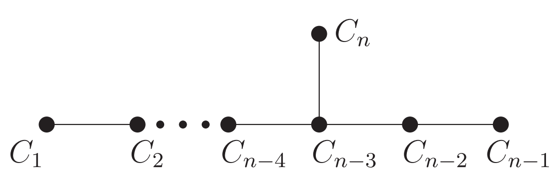

Fig. 3. The Dynkin diagram of E_n.

3.2 Minuscule representations

Definition 11. A minuscule (respectively, quasi-minuscule) representation of a semi-simple Lie algebra is an irreducible representation such that the Weyl group acts transitively on all the weights (respectively, nonzero weights). □

Minuscule representations are always fundamental representations and quasi-minuscule representations are either minuscule or adjoint representations.

$\mathfrak{g}$	Miniscule representations
$A_n = \mathrm{sl}(n+1)$	$\wedge^k \mathbb{C}^{n+1}$ for $k = 1, 2, \ldots, n$
$D_n = o(2n)$	$\mathbb{C}^{2n}$, $\mathcal{S}^+$, $\mathcal{S}^-$
E_6	$\mathbb{C}^{27}$, $\bar{\mathbb{C}}^{27}$
E_7	$\mathbb{C}^{56}$

Note that E_8 has no minuscule representation.

3.3 Configurations of (-1)-curves

In this subsection, we describe (-1)-curves in X and Y.

Definition 12. A (-1)-curve in a surface Y is a genus-zero (possibly reducible) curve l in Y with $l \cdot l = -1$. □

Remark 13. The genus-zero condition can be replaced by $l \cdot K_Y = -1$ by the genus formula, where K_Y is the canonical divisor of Y. □

Let C_0 be a curve in X passing through p.

Definition 14. (1) C_0 is called a (-1)-curve in X if there exists a (-1)-curve l in Y such that $\pi(l) = C_0$, or equivalently, the strict transform of C_0 is a (-1)-curve $\tilde{C}_0$ in Y. (2) The multiplicity of C_0 at p is defined to be $\sum_{i=1}^{n} a_i [C_i] \in \Lambda$, where $a_i = \tilde{C}_0 \cdot C_i$. □

Recall from Lie theory that any irreducible representation of a simple Lie algebra is determined by its lowest weight. The fundamental representations are those irreducible representations whose lowest weight is dual to the negative of some base root. (The usual definition for fundamental representations uses highest weight. But in this paper, we will use lowest weight for simplicity of notations.) If $C_0 \subset X$ has multiplicity C_k at p whose dual weight determines a minuscule representation V, then we use C_0^k to denote $\tilde{C}_0$. The construction of such X's and C_0's can be found in Appendix.

Definition 15. (1) We call C_0 has minuscule multiplicity $C_k \in \Lambda$ at p if C_0 has multiplicity C_k and the dual weight of $-C_k$ determines a minuscule representation V. (2) In this case, we denote $I^{(\mathfrak{g},V)} = \{l : (-1)\text{-curve in } Y | \pi(l) = C_0\}$. □

If there is no ambiguity, we will simply write $I^{(\mathfrak{g},V)}$ as I. Note that $I \subset C_0^k + \Lambda_{\geq 0}$, where $\Lambda_{\geq 0} = \{\sum a_i[C_i] : a_i \geq 0\}$.

Lemma 16. In the above situation, the cardinality of I is given by $|I| = \dim V$. □

Proof. By the genus formula and every $C_i \cong \mathbb{P}^1$ being a (-2)-curve, we have $C_i \cdot K_Y = 0$. Since $C_0^k \cdot K_Y = -1$, each (-1)-curve has the form $l = C_0^k + \sum a_i C_i$ with a_i's nonnegative integers. From $l \cdot l = -1$, we can determine $\{a_i\}$'s for l to be a (-1)-curve by direct computations. ■

Remark 17. The intersection product is negative definite on the sublattice of $\mathrm{Pic}(X)$ generated by $C_0^k, C_1, \ldots, C_n$ and we use its negative as an inner product. □

Lemma 18. In the above situation, for any $l \in I$, $\alpha \in \Phi$, we have $|l \cdot \alpha| \leq 1$. □

Proof. We claim that for any $v \in C_0^k + \Lambda$, we have $v \cdot v \leq -1$. We prove the claim by direct computations. In $(A_n, \wedge^k \mathbb{C}^{n+1})$ case:

$$\begin{aligned}\left(C_0^k + \sum a_i C_i\right)^2 &= -1 + 2a_k - (a_1^2 + (a_1 - a_2)^2 + \cdots + (a_{k-1} - a_k)^2) - ((a_k - a_{k+1})^2 + \cdots + a_n^2) \\ &\leq -1.\end{aligned}$$

The other cases can be proved similarly.

Since $l, l + \alpha$, $l - \alpha \in C_0^k + \Lambda$ by assumptions, we have $l \cdot l = -1 \geq (l + \alpha) \cdot (l + \alpha)$, and hence $l \cdot \alpha \leq 1$. Also $l \cdot l = -1 \geq (l - \alpha) \cdot (l - \alpha)$, and hence $l \cdot \alpha \geq -1$. ■

Lemma 19. In the above situation, for any $l \in I$ that is not C_0^k, there exists C_i such that $l \cdot C_i = -1$. □

Proof. From $l = C_0^k + \sum a_i C_i \neq C_0^k$ $(a_i \geq 0)$, we have $a_k \geq 1$. From $l \cdot l = -1$, we have $(\sum a_i C_i)^2 = -2a_k$. If there does not exist such an i with $l \cdot C_i = -1$, then by Lemma 18, $l \cdot C_i \geq 0$ for every i, $l \cdot (\sum a_i C_i) \geq 0$. But $l \cdot (\sum a_i C_i) = a_k + (\sum a_i C_i)^2 = -a_k \leq -1$ leads to a contradiction. ■

Lemma 20. In the above situation, for any $l, l' \in I$, $H^2(Y, O(l-l')) = 0$. □

Proof. Firstly, we prove $H^2(Y, O(C_0^k - l)) = 0$ for any $l = C_0^k + \sum a_i C_i \in I$ inductively on $\mathrm{ht}(l) := \sum a_i$. If $\mathrm{ht}(l) = 0$, that is, l is C_0^k, the claim follows from $p_g = 0$. Suppose that the claim is true for any $l' \in I$ with $\mathrm{ht}(l') \leq m-1$. Then for any $l \in I$ with $\mathrm{ht}(l) = m$, by Lemma 19, there exists i such that $l \cdot C_i = -1$. This implies $(l - C_i) \in I$ with $\mathrm{ht}(l - C_i) = m-1$, and therefore, $H^2(Y, O(C_0^k - (l - C_i))) = 0$ by induction hypothesis. Using the long exact sequence induced from

$$0 \to O_Y(C_0^k - l) \to O_Y(C_0^k - (l - C_i)) \to O_{C_i}(C_0^k - (l - C_i)) \to 0,$$

and $O_{C_i}(C_0^k - (l - C_i)) \cong O_{\mathbb{P}^1}(-1)$ or $O_{\mathbb{P}^1}$, we have the claim.

If $H^2(Y, O(l - l')) \neq 0$, then there exists a section $s \in H^0(Y, K_Y(l' - l))$ by Serre duality. Since there exists a nonzero section $t \in H^0(Y, O(l - C_0^k))$, we have $st \in H^0(Y, K_Y(l' - C_0^k)) \cong H^2(Y, O(C_0^k - l')) = 0$, which is a contradiction. ■

3.4 Minuscule representations from (-1)-curves

Recall from the *ADE* root system Φ that we can recover the corresponding Lie algebra $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. As before, we use $\{x_\alpha$'s, h_i's$\}$ to denote its Chevalley basis. If C_0 has minuscule multiplicity C_k, we denote

$$V_0 := \mathbb{C}^I = \bigoplus_{l \in I} \mathbb{C}\langle v_l \rangle,$$

where v_l is the base vector of V_0 generated by l. Then we define a bilinear map $[,]: \mathfrak{g} \otimes V_0 \to V_0$ (possibly up to $\pm$ signs) as follows:

$$[x, v_l] = \begin{cases} \langle \alpha, l \rangle & \text{if } x = h_\alpha \\ \pm v_{l+\alpha} & \text{if } x = x_\alpha,\ l + \alpha \in I, \\ 0 & \text{if } x = x_\alpha,\ l + \alpha \notin I. \end{cases}$$

Proposition 21. The signs in the above bilinear map $\mathfrak{g} \otimes V_0 \to V_0$ can be chosen so that it defines an action of $\mathfrak{g}$ on V_0. Moreover, V_0 is isomorphic to the minuscule representation V. □

Proof. For the first part, similar to [19], we use Lemma 18 to show $[[x, y], v_l] = [x, [y, v_l]] - [y, [x, v_l]]$.

For the second part, since $[x_\alpha, v_{C_0^k}] = 0$ for any $\alpha \in \Phi^-$, $v_{C_0^k}$ is the lowest weight vector of V_0 with weight corresponding to $-C_k$. Also we know the fundamental representation V corresponding to $-C_k$ has the same dimension with V_0 by Lemma 16. Hence, V_0 is isomorphic to the minuscule representation V. ■

Here, we show how to determine the signs. Take any $l \in I$, v_l is a weight vector of the above action. For $x = x_\alpha$ and v_l with weight w, we define $[x, v_l] = n_{\alpha,w} v_{l+\alpha}$, where $n_{\alpha,w} = \pm 1$ if $l + \alpha \in I$, otherwise $n_{\alpha,w} = 0$. By $[[x, y], v_l] = [x, [y, v_l]] - [y, [x, v_l]]$, we have $n_{\alpha,\beta} n_{\alpha+\beta,w} - n_{\beta,w} n_{\alpha,\beta+w} + n_{\alpha,w} n_{\beta,\alpha+w} = 0$.

Remark 22. Recall that for any $l = C_0^k + \sum a_i C_i \in I$, we define $\mathrm{ht}(l) := \sum a_i$. Using this, we can define a filtered structure for $I : I = I_0 \supset I_1 \supset \cdots \supset I_m$, where $m = \max_{l \in I} \mathrm{ht}(l)$, $I_i = \{l \in I | \mathrm{ht}(l) \leq m - i\}$ and $I_i \backslash I_{i+1} = \{l \in I | \mathrm{ht}(l) = m - i\}$. This $\mathrm{ht}(l)$ also enables us to define a partial order of I. Say $|I| = N$, we denote $l_N := C_0^k$ since it is the only element with $\mathrm{ht} = 0$. Similarly, $l_{N-1} := C_0^k + C_k$. Of course, there are some ambiguity of this ordering, and if so, we will just make a choice to order these (-1)-curves. □

3.5 Bundles from (-1)-curves

The geometry of (-1)-curves in Y can be used to construct representation bundles of $\mathcal{E}_\varphi^{\mathfrak{g}}$ for every minuscule representation of $\mathfrak{g}$. The proofs of theorems in this subsection will be given in Section 7.

When $C_0 \subset X$ has minuscule multiplicity C_k at p with the corresponding minuscule representation V, we define (When X is a del Pezzo surface, we use lines in X to construct bundles [8]. So here we use (-1)-curves in X to construct bundles.)

$$\mathfrak{L}_0^{(\mathfrak{g},V)} := \bigoplus_{l \in I^{(\mathfrak{g},V)}} O(l).$$

$\mathfrak{L}_0^{(\mathfrak{g},V)}$ has a natural filtration $F^\bullet$: $\mathfrak{L}_0^{(\mathfrak{g},V)} = F^0\mathfrak{L} \supset F^1\mathfrak{L} \supset \cdots \supset F^m\mathfrak{L}$, induced from the flittered structure on I, namely $F^i\mathfrak{L}_0^{(\mathfrak{g},V)} = \bigoplus_{l \in I_i} O(l)$.

$\mathfrak{L}_0^{(\mathfrak{g},V)}$ cannot descend to X as $O_{C_k}(C_0^k) \cong O_{\mathbb{P}^1}(1)$ (because $C_k \cdot C_0^k = 1$ by the definition of the minuscule multiplicity). For any C_i and any $l \in I$, we have $O_{C_i}(l) \cong O_{\mathbb{P}^1}(\pm 1)$ or $O_{\mathbb{P}^1}$ by Lemma 18. For every fixed C_i, if there is a $l \in I$ such that $O_{C_i}(l) \cong O_{\mathbb{P}^1}(1)$, then $(l + C_i)^2 = -1 = (l + C_i) \cdot K_Y$, that is, $l + C_i \in I$, also $O_{C_i}(l + C_i) \cong O_{\mathbb{P}^1}(-1)$. That means that among the direct summands of $\mathfrak{L}_0^{(\mathfrak{g},V)}|_{C_i}$, $O_{\mathbb{P}^1}(1)$ and $O_{\mathbb{P}^1}(-1)$ occur in pairs, and each pair is given by two (-1)-curves in I whose difference is C_i. This gives us a chance to deform $\mathfrak{L}_0^{(\mathfrak{g},V)}$ to get another bundle that can descend to X.

Theorem 23. If there exists a (-1)-curve C_0 in X with minuscule multiplicity C_k at p and $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is the corresponding representation, then

$$\left(\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)} := \bigoplus_{l \in I} O(l),\ \bar{\partial}_{\varphi} := \bar{\partial}_0 + \rho(\varphi)\right),$$

with $\varphi \in \Psi_Y$ is a holomorphic bundle over Y which preserves the filtration on $\mathfrak{L}_0^{(\mathfrak{g},V)}$ and it is a holomorphic representation bundle of $\mathcal{E}_{\varphi}^{\mathfrak{g}}$. Moreover, $\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)}$ is trivial on C_i if and only if $[\varphi_{C_i}|_{C_i}] \neq 0 \in H^1(Y, O_{C_i}(C_i))$. □

For C_k with $k = 1$, the corresponding minuscule representation V is the standard representation of $\mathfrak{g}$. When $\mathfrak{g} = A_n$, it is simply $\mathrm{sl}(n+1) = \mathrm{aut}_0(V)$. When $\mathfrak{g} = D_n$ (respectively, E_6 and E_7), there exists a quadratic (respectively, cubic and quartic) form f on V such that $\mathfrak{g} = \mathrm{aut}(V, f)$. The next theorem tells us that we can globalize this construction over Y to recover the Lie algebra bundle $\mathcal{E}_{\varphi}^{\mathfrak{g}}$ over Y. But this does not work for $\mathcal{E}_{\varphi}^{E_8}$ as E_8 has no standard representation.

Theorem 24. Under the same assumptions as in Theorem 23 with $k = 1$, there exists a holomorphic fiberwise symmetric multi-linear form

$$f : \bigotimes^{r} \mathfrak{L}_{\varphi}^{(\mathfrak{g},V)} \longrightarrow O_Y(D),$$

with $r = 0, 2, 3, 4$ when $\mathfrak{g} = A_n, D_n, E_6, E_7$, respectively, such that $\mathcal{E}_{\varphi}^{\mathfrak{g}} \cong \mathrm{aut}_0(\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)}, f)$. □

It is obvious that $\mathcal{E}_{\varphi}^{\mathfrak{g}}$ does not depend on the existence of the (-1)-curve C_0, for the minuscule representation bundles, we have the following results.

Theorem 25. There exists a divisor B in Y and an integer k, such that the bundle $\mathbb{L}_{\varphi}^{(\mathfrak{g},V)} := S^k\mathfrak{L}_{\varphi}^{(\mathfrak{g},V)} \otimes O(-B)$ with $\varphi \in \Psi_X$ can descend to X and does not depend on the existence of C_0. □

3.6 Outline of proofs for $\mathfrak{g} \neq E_8$

When $\mathfrak{g} \neq E_8$, there exists a natural symmetric tensor f on its standard representation V such that $\mathfrak{g} = \mathrm{aut}_0(V, f)$. The set $I^{(\mathfrak{g},V)}$ of (-1)-curves has cardinality $N = \dim V$. Given $\eta := (\eta_{i,j})_{N\times N}$ with $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for every $l_i \neq l_j \in I^{(\mathfrak{g},V)}$, we consider the operator $\bar{\partial}_{\eta} := \bar{\partial}_0 + \eta$ on $\mathfrak{L}_0^{(\mathfrak{g},V)} := \bigoplus_{l \in I^{(\mathfrak{g},V)}} O_Y(l)$. We will look for η that satisfies:

(1) (filtration) $\eta_{i,j} = 0$ for $i > j$ for the partial ordering introduced in Section 3.4;
(2) (holomorphic structure) $(\bar{\partial}_0 + \eta)^2 = 0$;

(3) (Lie algebra structure) $\bar{\partial}_\eta f = 0$;

(4) (descendent) for every C_k, if $l_i - l_j = C_k$, then $0 \neq [\eta_{i,j}|_{C_k}] \in H^1(Y, O_{C_k}(C_k))$.

Remark 26. Property (2) implies that we can define a new holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)}$. Properties (1) and (3) require that for any $\eta_{i,j} \neq 0$, $\eta_{i,j} \in \Omega^{0,1}(Y, O(\alpha))$ for some $\alpha \in \Phi^+$. We will show that if η satisfies (1)–(3), then (4) is equivalent to $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ being trivial on every C_k, that is, $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ can descend to X. □

Denote

$$\Xi_Y^{\mathfrak{g}} \triangleq \{\eta = (\eta_{i,j})_{N\times N} | \eta \text{ satisfies (1)–(3)}\}$$

and

$$\Xi_X^{\mathfrak{g}} \triangleq \{\eta \in \Xi_Y^{\mathfrak{g}} | \eta \text{ satisfies (4)}\};$$

then each η in $\Xi_Y^{\mathfrak{g}}$ determines a filtered holomorphic bundle $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ over Y together with a holomorphic tensor f on it. It can descend to X if $\eta \in \Xi_X^{\mathfrak{g}}$.

Since $\mathfrak{g} = \mathrm{aut}(V, f)$, for any $\eta \in \Xi_Y^{\mathfrak{g}}$, we have a holomorphic Lie algebra bundle $\zeta_\eta^{\mathfrak{g}} := \mathrm{aut}(\mathfrak{L}_\eta^{(\mathfrak{g},V)}, f)$ over Y of type $\mathfrak{g}$, and $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ is automatically a representation bundle of $\zeta_\eta^{\mathfrak{g}}$. Furthermore, if $\eta \in \Xi_X^{\mathfrak{g}}$, then $\zeta_\eta^{\mathfrak{g}}$ can descend to X.

For a general minuscule representation of $\mathfrak{g}$, given any $\eta \in \Xi_Y^{\mathfrak{g}}$, we show that there exists a unique holomorphic structure on $\mathfrak{L}_0^{(\mathfrak{g},V)}$, such that the action of $\zeta_\eta^{\mathfrak{g}}$ on the new holomorphic bundle $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ is holomorphic. Furthermore, if $\eta \in \Xi_X^{\mathfrak{g}}$, then $\mathfrak{L}_\eta^{(\mathfrak{g},V)}$ can descend to X.

4 A_n Case

We recall that $A_n = \mathrm{sl}(n+1, \mathbb{C}) = \mathrm{aut}_0(\mathbb{C}^{n+1})$ (where aut_0 means tracefree endomorphisms). The standard representation of A_n is $\mathbb{C}^{n+1}$ and minuscule representations of A_n are $\wedge^k \mathbb{C}^{n+1}$, $k = 1, 2, \ldots, n$.

4.1 A_n standard representation bundle $\mathfrak{L}_\eta^{(A_n,\mathbb{C}^{n+1})}$

We consider a surface X with an A_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1; then $I^{(A_n,\mathbb{C}^{n+1})} = \{C_0^1 + \sum_{i=1}^k C_i |\ 0 \leq k \leq n\}$ has cardinality $n+1$. We order these (-1)-curves: $l_k = C_0^1 + \sum_{i=1}^{n+1-k} C_i$ for $1 \leq k \leq n+1$. For any $l_i \neq l_j \in I$, $l_i \cdot l_j = 0$.

Fix any C_i, we have

$$l_k \cdot C_i = \begin{cases} 1, & k = n+2-i, \\ -1, & k = n+1-i, \\ 0, & \text{otherwise}. \end{cases}$$

Define $\mathfrak{L}_0^{(A_n,\mathbb{C}^{n+1})} := \bigoplus_{l \in I} O(l)$ over Y, for simplicity, we write it as $\mathfrak{L}_0^{A_n}$. $\mathfrak{L}_0^{A_n}$ cannot descend to X, since for any C_i,

$$\mathfrak{L}_0^{A_n}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus(n-1)} \oplus O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1).$$

Our aim is to find a new holomorphic structure on $\mathfrak{L}_0^{A_n}$ such that the resulting bundle can descend to X. First, we define $\bar{\partial}_\eta : \Omega^{0,0}(Y, \mathfrak{L}_0^{A_n}) \longrightarrow \Omega^{0,1}(Y, \mathfrak{L}_0^{A_n})$ on $\mathfrak{L}_0^{A_n} = \bigoplus_{k=1}^{n+1} O(l_k)$ as follows:

$$\bar{\partial}_\eta = \begin{pmatrix} \bar{\partial} & \eta_{1,2} & \cdots & \eta_{1,n+1} \\ 0 & \bar{\partial} & \cdots & \eta_{2,n+1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{\partial} \end{pmatrix},$$

where $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for any $j > i$. When $j > i$, $l_i - l_j \in \Lambda$ is a positive root because of $l_i \cdot l_j = 0$ and our ordering of l_k's.

The integrability condition $\bar{\partial}_\eta^2 = 0$ is equivalent to, for $i = 1, 2, \ldots, n$,

$$\begin{cases} \bar{\partial}\eta_{i,i+1} = 0, \\ \bar{\partial}\eta_{i,j} = -\displaystyle\sum_{m=i+1}^{j-1} \eta_{i,m} \cdot \eta_{m,j}, & j \geq i+2. \end{cases}$$

Note that $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j)) = \Omega^{0,1}(Y, O(\alpha))$ for some $\alpha \in \Phi^+$. From

$$\sum_{m=i+1}^{j-1} [\eta_{i,m} \cdot \eta_{m,j}] \in H^2(Y, O(l_i - l_j)) = 0,$$

we can find $\eta_{i,j}$, such that $\bar{\partial}\eta_{i,j} = -\sum_{m=i+1}^{j-1} \eta_{i,m} \cdot \eta_{m,j}$. That is,

Proposition 27. Given any $\eta_{i,i+1} \in \Omega^{0,1}(Y, O(l_i - l_{i+1}))$ with $\bar{\partial}\eta_{i,i+1} = 0$ for $i = 1, 2, \ldots n$, there exists $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for every $j > i$ such that $\bar{\partial}_\eta$ defines a holomorphic structure on $\mathfrak{L}_0^{A_n}$, that is, $\bar{\partial}_\eta^2 = 0$. $\square$

We want to prove that there exists $\eta \in \Xi_Y^{A_n}$ such that $\mathfrak{L}_\eta^{A_n}$ can descend to X, that is, $\mathfrak{L}_\eta^{A_n}|_{C_i}$ is trivial for every C_i. To prove this, we will construct $n+1$ holomorphic sections of $\mathfrak{L}_\eta^{A_n}|_{C_i}$ which are linearly independent everywhere on C_i. The following lemma will be needed for all the ADE cases.

Lemma 28. Consider a vector bundle $(\mathfrak{L} := \bigoplus_{i=1}^N O(l_i),\ \bar\partial_{\mathfrak{L}} = \bar\partial_0 + (\eta_{i,j})_{N\times N})$ over Y with $\eta_{i,j} = 0$ whenever $i \geq j$. Suppose that C is a smooth (-2)-curve in Y with $H^1(C, O_C(l_i)) = 0$ for every $i = 1, 2, \ldots, N$; then for any fixed i and any $s_i \in H^0(C, O_C(l_i))$, the following equation for $s_1, s_2, \ldots, s_{i-1}$ has a solution,

$$\begin{pmatrix} \bar\partial & \eta_{1,2}|_C & \eta_{1,3}|_C & \cdots & \cdots & \eta_{1,N}|_C \\ 0 & \bar\partial & \eta_{2,3}|_C & \cdots & \cdots & \eta_{2,N}|_C \\ 0 & 0 & \bar\partial & \cdots & \cdots & \eta_{3,N}|_C \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & \bar\partial \end{pmatrix} \begin{pmatrix} s_1 \\ \vdots \\ s_i \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.$$

□

Proof. The above equation is equivalent to:

$$\bar\partial s_i = 0, \tag{1}$$

$$\eta_{i-1,i} s_i + \bar\partial s_{i-1} = 0, \tag{2}$$

$$\vdots$$

$$\eta_{1,i} s_i + \cdots + \eta_{1,2} s_2 + \bar\partial s_1 = 0. \tag{i}$$

Equation (1) is automatic as $s_i \in H^0(C, O_C(l_i))$. For Equation (2), since $\bar\partial \eta_{i-1,i} = 0$ and $\bar\partial s_i = 0$, we have $[\eta_{i-1,i} s_i] \in H^1(C, O_C(l_{i-1})) = 0$, and hence, we can find s_{i-1} satisfying $\bar\partial s_{i-1} = -\eta_{i-1,i} s_i$.

Inductively, suppose that we have found $s_i, \ldots, s_{j-1}$ for the first $(i-j)$ equations; then for the $(i-j+1)$th equation: $\eta_{j,i} s_i + \cdots + \eta_{j,j+1} s_{j+1} + \bar\partial s_j = 0$, we have

$$\eta_{j,i} s_i + \cdots + \eta_{j,j+1} s_{j+1} \in \Omega^{0,1}(C, O_C(l_j)).$$

From $\bar\partial_{\mathfrak{L}}^2 = 0$, we have

$$\bar\partial \eta_{k,m} = -(\eta_{k,k+1} \cdot \eta_{k+1,m} + \eta_{k,k+2} \cdot \eta_{k+2,m} + \cdots + \eta_{k,m-1} \cdot \eta_{m-1,m}).$$

Then

$$\bar{\partial}(s_m) = -(\eta_{m,m+1}s_{m+1} + \cdots + \eta_{m,i}s_i)$$

implies

$$\bar{\partial}(\eta_{j,i}s_i + \cdots + \eta_{j,j+1}s_{j+1}) = 0.$$

Therefore, $[\eta_{j,i}s_i + \cdots + \eta_{j,j+1}s_{j+1}] \in H^1(C, O_C(l_j)) = 0$, and hence, we can find s_j such that $\bar{\partial}s_j = -(\eta_{j,i}s_i + \cdots + \eta_{j,j+1}s_{j+1})$. ■

Let us recall a standard result that says that the only nontrivial extension of $O_{\mathbb{P}^1}(1)$ by $O_{\mathbb{P}^1}(-1)$ is the trivial bundle. We will give an explicit construction of this trivialization as we will need a generalization of it later.

Lemma 29. For an exact sequence over $\mathbb{P}^1 : 0 \to O_{\mathbb{P}^1}(-1) \to E \to O_{\mathbb{P}^1}(1) \to 0$, the bundle E is determined by the extension class $[\varphi] \in \mathrm{Ext}^1_{\mathbb{P}^1}(O(1), O(-1)) \cong \mathbb{C}$ up to a scalar multiple. If $[\varphi] \neq 0$, E is trivial, that is there exists two holomorphic sections for E that are linearly independent at every point in $\mathbb{P}^1$. □

Proof. With respect to the (topological) splitting $E = O_{\mathbb{P}^1}(-1) \oplus O_{\mathbb{P}^1}(1)$, the holomorphic structure on E is given by

$$\bar{\partial}_E = \begin{pmatrix} \bar{\partial} & \varphi \\ 0 & \bar{\partial} \end{pmatrix},$$

with $\varphi \in \mathrm{Ext}^1_{\mathbb{P}^1}(O(1), O(-1))$. Let t_1, t_2 be a base of $H^0(\mathbb{P}^1, O(1)) \cong \mathbb{C}^2$. Since $[\varphi t_i] \in H^1(\mathbb{P}^1, O(-1)) = 0$, we can find $u_1, u_2 \in \Omega^0(\mathbb{P}^1, O(-1))$, such that

$$\begin{pmatrix} \bar{\partial} & \varphi \\ 0 & \bar{\partial} \end{pmatrix} \cdot \begin{pmatrix} u_i \\ t_i \end{pmatrix} = 0,$$

that is, $s_1 = (u_1, t_1)^t$ and $s_2 = (u_2, t_2)^t$ are two holomorphic sections of E. Explicitly, we can take $s_1 = (\frac{1}{1+|z|^2}, z)^t$, $s_2 = (\frac{-\bar{z}}{1+|z|^2}, 1)^t$ in the coordinate chart $\mathbb{C} \subset \mathbb{P}^1$. It can be checked that s_1 and s_2 are linearly independent over $\mathbb{P}^1$. ■

From the above lemma, we have the following result.

Lemma 30. Under the same assumption as in Lemma 28. Suppose that $\mathfrak{L}|_C \cong O_{\mathbb{P}^1}^{\oplus m} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus n}$ with each pair of $O_{\mathbb{P}^1}(\pm 1)$ corresponding to two (-1)-curves l_i and l_{i+1} with $l_i - l_{i+1} = C$. Then $\mathfrak{L}|_C$ is trivial if and only if $[\eta_{i,i+1}|_C] \neq 0$ for every $\eta_{i,i+1} \in \Omega^{0,1}(Y, O(C))$. □

Proof. For simplicity, we assume $m=n=1$ and $O_C(l_1)\cong O_{\mathbb{P}^1}$, $O_C(l_2)\cong O_{\mathbb{P}^1}(-1)$, $O_C(l_3)\cong O_{\mathbb{P}^1}(1)$ with $l_2-l_3=C$. If $[\eta_{2,3}|_C]\neq 0$, by Lemmas 28 and 29, there exists two holomorphic sections for $\mathfrak{L}|_C$ that are linearly independent at every point in C: $s_1=(x_1,u_1,t_1)^t$ and $s_2=(x_2,u_2,t_2)^t$ with u_1,t_1,u_2,t_2 given in the proof of Lemma 29. By $H^0(Y,O_C(l_1))\cong H^0(\mathbb{P}^1,O)\cong\mathbb{C}$, there exists one holomorphic section for $\mathfrak{L}|_C$ that is nowhere zero on C: $s_3=(x_3,0,0)^t$. These s_1,s_2,s_3 give a trivialization of $\mathfrak{L}|_C$. If $[\eta_{2,3}|_C]=0$, then $\mathfrak{L}_N|_C$ is an extension of $O_{\mathbb{P}^1}(1)\oplus O_{\mathbb{P}^1}(-1)$ by $O_{\mathbb{P}^1}$ and there is no such nontrivial extension. ■

Proposition 31. The bundle $\mathfrak{L}_\eta^{A_n}$ over Y with $\eta\in\Xi_Y^{A_n}$ can descend to X if and only if $0\neq[\eta_{n+1-i,n+2-i}|_{C_i}]\in H^1(Y,O_{C_i}(C_i))$ for every i, that is, $\eta\in\Xi_X^{A_n}$. □

Proof. Restricting $\mathfrak{L}_0^{A_n}$ to C_i, the corresponding line bundle summands are

$$O_{C_i}(l_k)\cong\begin{cases} O_{\mathbb{P}^1}(1), & k=n+2-i,\\ O_{\mathbb{P}^1}(-1), & k=n+1-i,\\ O_{\mathbb{P}^1}, & \text{otherwise.}\end{cases}$$

By Lemma 30 and our assumption, we have the proposition. ■

4.2 A_n Lie algebra bundle $\zeta_\eta^{A_n}$

As $A_n=\mathrm{sl}(n+1,\mathbb{C})=\mathrm{aut}_0(\mathbb{C}^{n+1})$, $\zeta_\eta^{A_n}:=\mathrm{aut}_0(\mathfrak{L}_\eta^{A_n})$ ($\eta\in\Xi_X^{A_n}$) is an A_n Lie algebra bundle over Y that can descend to X. This $\zeta_\eta^{A_n}$ does not depend on the existence of C_0. And $\mathfrak{L}_\eta^{A_n}$ is automatically a representation bundle of $\zeta_\eta^{A_n}$.

4.3 A_n minuscule representation bundle $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})}$

Consider a surface X with an A_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_k. By Proposition 16, $I^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ has cardinality $\binom{k}{n+1}$. Define $\mathfrak{L}_0^{(A_n,\wedge^k\mathbb{C}^{n+1})}:=\bigoplus_{l\in I}O(l)$ over Y.

Lemma 32. $\mathfrak{L}_0^{(A_n,\wedge^k\mathbb{C}^{n+1})}=(\wedge^k\mathfrak{L}_0^{A_n})(C_0^k-kC_0^1-\sum_{j=1}^{k-1}(k-j)C_j)$. □

Proof. The bundles on both sides have the same rank, so we only need to check that every line bundle summand in the right-hand side is $O_Y(l)$ for l a (-1)-curve in $I^{(A_n,\wedge^k\mathbb{C}^{n+1})}$. For any k distinct elements l_{i_j} in $I^{(A_n,\mathbb{C}^{n+1})}$, we denote $l=l_{i_1}+l_{i_2}+\cdots+l_{i_k}+C_0^k-(l_1+l_2+\cdots l_k)$. Then $O_Y(l)$ is a summand in the right-hand side. Since the intersection number of any two distinct (-1)-curves in $I^{(A_n,\mathbb{C}^{n+1})}$ is zero, we have $l^2 = l \cdot K_Y = -1$, that is, $l \in I^{(A_n,\wedge^k\mathbb{C}^{n+1})}$. ■

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{(A_n,\wedge^k\mathbb{C}^{n+1})}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus(\binom{k}{n-1}+\binom{k-2}{n-1})} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus\binom{k-1}{n-1}}.$$

Proposition 33. Fix any $\eta \in \Xi_Y^{A_n}$. There exists a unique holomorphic structure on $\mathfrak{L}_0^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ such that the action of $\zeta_\eta^{A_n}$ on the resulting bundle $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ is holomorphic. Furthermore, if $\eta \in \Xi_X^{A_n}$, then $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})}$ can descend to X. □

Proof. As the action of $\zeta_\eta^{A_n}$ on $\mathfrak{L}_\eta^{A_n}$ is holomorphic, $\zeta_\eta^{A_n}$ acts on $\mathfrak{L}_\eta^{(A_n,\wedge^k\mathbb{C}^{n+1})} := (\wedge^k\mathfrak{L}_\eta^{A_n})(C_0^k - kC_0^1 - \sum_{j=1}^{k-1}(k-j)C_j)$ holomorphically. The last assertion follows from Proposition 31 and the fact that $O(C_0^k - kC_0^1 - \sum_{j=1}^{k-1}(k-j)C_j)|_{C_i}$ is trivial for every C_i. ■

5 D_n Case

We recall that $D_n = o(2n,\mathbb{C}) = \mathrm{aut}(\mathbb{C}^{2n}, q)$ for a nondegenerate quadratic form q on the standard representation $\mathbb{C}^{2n}$. The other minuscule representations are $\mathcal{S}^+$ and $\mathcal{S}^-$ and the adjoint representation is $\wedge^2\mathbb{C}^{2n}$.

5.1 D_n standard representation bundle $\mathfrak{L}_\eta^{(D_n,\mathbb{C}^{2n})}$

We consider a surface X with a D_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1. Then $I^{(D_n,\mathbb{C}^{2n})} = I_1 \cup I_2$ with $I_1 = \{C_0^1 + \sum_{i=1}^k C_i | 0 \le k \le n-1\}$ and $I_2 = \{F - l | l \in I_1\}$, where $F = 2C_0^1 + 2C_1 + \cdots + 2C_{n-2} + C_{n-1} + C_n$. We order these (-1)-curves: $l_k = F - C_0^1 - \sum_{i=1}^{k-1} C_i$ and $l_{2n-k+1} = C_0^1 + \sum_{i=1}^{k-1} C_i$ for $1 \le k \le n$.

For any $l_i \ne l_j \in I$, we have $l_i \cdot l_j = 0$ or 1. Given any $l_i \in I$, there exists a unique $l_j \in I$ such that $l_i \cdot l_j = 1$. In this case, $l_i + l_j = F$.

Define $\mathfrak{L}_0^{(D_n,\mathbb{C}^{2n})} := \bigoplus_{l\in I} O(l)$ over Y. For simplicity, we write it as $\mathfrak{L}_0^{D_n}$. If we ignore C_n, then we recover the A_{n-1} case as in the last section. They are related by the following.

Lemma 34. $\mathfrak{L}_0^{D_n} = \mathfrak{L}_0^{A_{n-1}} \oplus (\mathfrak{L}_0^{A_{n-1}})^*(F)$. □

Proof. Since A_{n-1} is a Lie subalgebra of D_n, we can decompose the representation of D_n as sum of irreducible representations of A_{n-1}. By the branching rule, we have $2n = n + n$, that is, $\mathbb{C}^{2n} = \mathbb{C}^n \oplus (\mathbb{C}^n)^*$ with $\mathbb{C}^{2n}$ and $\mathbb{C}^n$ the standard representations of D_n and A_{n-1}, respectively. For $I^{(D_n,\mathbb{C}^{2n})} = I_1 \cup I_2$, I_1 forms the standard representation $\mathbb{C}^n$ of A_{n-1}, and I_2 forms the $(\mathbb{C}^n)^*$. ■

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{D_n}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus(2n-4)} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 2}.$$

Similar to the $(A_n, \mathbb{C}^{n+1})$ case, we define $\bar{\partial}_\eta : \Omega^{0,0}(Y, \mathfrak{L}_0^{D_n}) \longrightarrow \Omega^{0,1}(Y, \mathfrak{L}_0^{D_n})$ on $\mathfrak{L}_0^{D_n} = \bigoplus_{k=1}^{2n} O(l_k)$ by $\bar{\partial}_\eta := \bar{\partial}_0 + (\eta_{i,j})_{2n\times 2n}$, where $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for any $j > i$, otherwise $\eta_{i,j} = 0$.

By Lemma 20 and arguments similar to those in the proof of Proposition 27 for the A_n case, given any $\eta_{i,i+1}$ with $\bar{\partial}\eta_{i,i+1} = 0$ for every i, there exists $\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j))$ for every $j > i$ such that $\bar{\partial}_\eta^2 = 0$.

From the configuration of these $2n$ (-1)-curves, we can define a quadratic form q on the vector space $V_0 = \mathbb{C}^I = \bigoplus_{l\in I} \mathbb{C}\langle v_l\rangle$ spanned by these (-1)-curves,

$$q : V_0 \otimes V_0 \longrightarrow \mathbb{C}, \qquad q(v_{l_i}, v_{l_j}) = l_i \cdot l_j.$$

The D_n Lie algebra is the space of infinitesimal automorphism of q, that is, $D_n = \mathrm{aut}(V_0, q)$.

Correspondingly, we have a fiberwise quadratic form q on the bundle $\mathfrak{L}_\eta^{D_n}$:

$$q : \mathfrak{L}_\eta^{D_n} \otimes \mathfrak{L}_\eta^{D_n} \longrightarrow O(F).$$

Proposition 35. There exists η with $\bar{\partial}_\eta^2 = 0$ such that $\bar{\partial}_\eta q = 0$. □

Proof. $\bar{\partial}_\eta q = 0$ if and only if $q(\bar{\partial}_\eta s_i, s_j) + q(s_i, \bar{\partial}_\eta s_j) = 0$ for any $s_i \in H^0(Y, O(l_i))$ and $s_j \in H^0(Y, O(l_j))$. From the definition of q, this is equivalent to $\eta_{2n+1-j,i} + \eta_{2n+1-i,j} = 0$, that is, $\eta_{i,j} = -\eta_{2n+1-j,2n+1-i}$ for any $j > i$. From $l_i + l_{2n+1-i} = l_j + l_{2n+1-j} = F$, we have

$$\eta_{i,j} \in \Omega^{0,1}(Y, O(l_i - l_j)) = \Omega^{0,1}(Y, O(l_{2n+1-j} - l_{2n+1-i})) \ni \eta_{2n+1-j,2n+1-i}.$$

We construct η that satisfies $\bar{\partial}_\eta^2 = 0$ with $\eta_{i,j} = -\eta_{2n+1-j,2n+1-i}$ inductively on $j - i$. For $j - i = 1$, we can always take $\eta_{i,i+1} = -\eta_{2n-i,2n+1-i}$. Note we have $\eta_{n,n+1} = 0$. For $j - i = 2$, we have

$$\bar{\partial}\eta_{i,i+2} = -\eta_{i,i+1}\eta_{i+1,i+2},$$

$$\bar{\partial}\eta_{2n-i-1,2n-i+1} = -\eta_{2n-i-1,2n-i}\eta_{2n-i,2n-i+1} = -\eta_{i+1,i+2}\eta_{i,i+1} = -\bar{\partial}\eta_{i,i+2},$$

so we can take $\eta_{i,i+2} = -\eta_{2n-i-1,2n-i+1}$.

Repeat this process inductively on $j-i$. We can take $\eta_{i,j} = -\eta_{2n+1-j,2n+1-i}$ for any $j > i$. So there exists η satisfying $\bar{\partial}_\eta q = 0$. ■

Until now, we have proved that $\Xi_Y^{D_n}$ is not empty.

Restricting $\mathfrak{L}_0^{D_n}$ to C_n, the corresponding line bundle summands are:

$$O_{C_n}(l_j) \cong \begin{cases} O_{\mathbb{P}^1}(1), & j = n+1 \text{ or } n+2, \\ O_{\mathbb{P}^1}(-1), & j = n-1 \text{ or } n, \\ O_{\mathbb{P}^1}, & \text{otherwise.} \end{cases}$$

The pairs of $O_{\mathbb{P}^1}(\pm 1)$ in $\mathfrak{L}_0^{D_n}|_{C_n}$ are given by $\{l_{n-1}, l_{n+1}\}$ and $\{l_n, l_{n+2}\}$. To construct a trivialization of $\mathfrak{L}_\eta^{D_n}|_{C_n}$, we need the following generalizations of Lemmas 29 and 30.

Lemma 36. Under the same assumption as in Lemma 28. Assume that $l_{i+1}, l_{i+2}, \ldots, l_{i+2k}$ satisfy $l_{i+j} \cdot C = -1$ and $l_{i+k+j} = l_{i+j} - C$ for $j = 1, 2, \ldots, k$. If $\eta_{i+p,i+q} = 0$ for $2 \le p \le k$, $k+1 \le q \le 2k-1$ and $q - p \le k-1$, that is, the corresponding submatrix of $\bar{\partial}_{\mathfrak{L}}$ given by $l_{i+1}, l_{i+2}, \ldots, l_{i+2k}$ looks like

$$\left(\begin{array}{c|cccc} & \eta_{i+1,i+k+1} & \eta_{i+1,i+k+2} & \cdots & \eta_{i+1,i+2k} \\ & 0 & \eta_{i+2,i+k+2} & \cdots & \eta_{i+2,i+2k} \\ * & \vdots & \vdots & \ddots & \vdots \\ & 0 & 0 & \cdots & \eta_{i+k,i+2k} \\ \hline 0_{k\times k} & & * & & \end{array}\right),$$

with $\eta_{i+1,i+k+1}, \eta_{i+2,i+k+2}, \ldots, \eta_{i+k,i+2k}$ in $\Omega^{0,1}(Y, O(C))$. Suppose that $[\eta_{i+1,i+k+1}|_C], [\eta_{i+2,i+k+2}|_C], \ldots, [\eta_{i+k,i+2k}|_C]$ are nonzero, we can construct $2k$ holomorphic sections of $\mathfrak{L}|_C$ that are linearly independent at every point in C. □

Proof. In order to keep our notation simpler, we assume $k = 2$. The above matrix given by $l_{i+1}, l_{i+2}, l_{i+3}, l_{i+4}$ has the form

$$\left(\begin{array}{cc|cc} \bar{\partial} & \eta_{i+1,i+2} & \eta_{i+1,i+3} & * \\ 0 & \bar{\partial} & 0 & \eta_{i+2,i+4} \\ \hline \multicolumn{2}{c|}{\multirow{2}{*}{$0_{2\times 2}$}} & \bar{\partial} & \eta_{i+3,i+4} \\ & & 0 & \bar{\partial} \end{array}\right).$$

From $H^0(Y, O_C(l_{i+4})) \cong H^0(\mathbb{P}^1, O(1)) \cong \mathbb{C}^2$ and $[\eta_{i+2,i+4}|_C] \ne 0$, there exist two holomorphic sections of $\mathfrak{L}|_C$ that are linearly independent at every point in C: $s_1 = (y_1, u_1, x_1, t_1)^t$ and

$s_2 = (y_2, u_2, x_2, t_2)^t$ with u_1, t_1, u_2, t_2 given in Lemma 29. Similarly, from $H^0(Y, O_C(l_{i+3})) \cong \mathbb{C}^2$ and $[\eta_{i+1,i+3}|_C] \neq 0$, we also have two holomorphic sections of $\mathfrak{L}|_C$ that are linearly independent at every point in C: $s_3 = (y_3, 0, x_3, 0)^t$ and $s_4 = (y_4, 0, x_4, 0)^t$. If there exist a_1, a_2, a_3, a_4 such that $a_1 s_1 + a_2 s_2 + a_3 s_3 + a_4 s_4 = 0$ at some point in C, then we have $a_1 t_1 + a_2 t_2 = 0$ and $a_1 u_1 + a_2 u_2 = 0$ at some point, which is impossible by the explicit formulas for u_1, t_1, u_2, t_2 in Lemma 29. Hence, we have the lemma. ■

Lemma 37. Under the same assumption as in Lemma 28, we assume $\mathfrak{L}|_C \cong O_{\mathbb{P}^1}^{\oplus m} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus n}$ with each pair of $O_{\mathbb{P}^1}(\pm 1)$ and the corresponding holomorphic structure as in Lemma 36. Then $\mathfrak{L}|_C$ is trivial if and only if $[\eta_{i,j}|_C] \neq 0$ for any $\eta_{i,j} \in \Omega^{0,1}(Y, O(C))$. □

Proof. The same arguments as in the proof of Lemmas 30 and 36. ■

Proposition 38. The bundle $\mathfrak{L}_\eta^{D_n}$ over Y with $\eta \in \Xi_Y^{D_n}$ can descend to X if and only if for every C_k and $\eta_{i,j} \in \Omega^{0,1}(Y, O(C_k))$, $[\eta_{i,j}|_{C_k}] \neq 0$, that is, $\eta \in \Xi_X^{D_n}$. □

Proof. Restricting $\mathfrak{L}_0^{D_n}$ to C_i $(1 \leq i \leq n-1)$, the line bundle summands are

$$O_{C_i}(l_j) \cong \begin{cases} O_{\mathbb{P}^1}(1), & j = i+1 \text{ or } 2n-i, \\ O_{\mathbb{P}^1}(-1), & j = i \text{ or } 2n-i+1, \\ O_{\mathbb{P}^1}, & \text{otherwise.} \end{cases}$$

By Lemma 30, $\mathfrak{L}_\eta^{D_n}|_{C_i}$ is trivial if and only if $[\eta_{i,i+1}|_{C_i}]$, $[\eta_{2n-i,2n+1-i}|_{C_i}]$ are not zeros. For C_n, the pairs of $O_{\mathbb{P}^1}(\pm 1)$ in $\mathfrak{L}_0^{D_n}|_{C_n}$ are given by $\{l_{n-1}, l_{n+1}\}$ and $\{l_n, l_{n+2}\}$. By Lemma 37 and $\eta_{n,n+1} = 0$ (Proposition 35), $\mathfrak{L}_\eta^{D_n}|_{C_n}$ is trivial if and only if $[\eta_{n-1,n+1}|_{C_n}]$, $[\eta_{n,n+2}|_{C_n}]$ are not zeros. In fact, this $\mathfrak{L}_\eta^{D_n}$ is just an extension of $\mathfrak{L}_{\eta'}^{A_{n-1}}$ by $(\mathfrak{L}_{\eta'}^{A_{n-1}})^*(F)$ for some $\eta' \in \Xi_X^{A_{n-1}}$ with $\eta' \subset \eta$. ■

5.2 D_n Lie algebra bundle $\zeta_\eta^{D_n}$

Note that $\zeta_\eta^{D_n} = \mathrm{aut}(\mathfrak{L}_\eta^{D_n}, q)$ is a D_n Lie algebra bundle over Y. In order for $\zeta_\eta^{D_n}$ to descend to X as a Lie algebra bundle, we need to show that $q|_{C_i} : \mathfrak{L}_\eta^{D_n}|_{C_i} \otimes \mathfrak{L}_\eta^{D_n}|_{C_i} \longrightarrow O_{C_i}(F)$ is a constant map for every C_i. This follows from the fact that both $\mathfrak{L}_\eta^{D_n}$ and $O(F)$ are trivial on all C_i's and $\bar{\partial}_\eta q = 0$. From the construction, $\mathfrak{L}_\eta^{D_n}$ is a representation bundle of $\zeta_\eta^{D_n}$.

5.3 D_n spinor representation bundles $\mathfrak{L}_\eta^{(D_n,\mathcal{S}^\pm)}$

We will only deal with $\mathcal{S}^+$, as the $\mathcal{S}^-$ case is analogous. Consider a surface X with a D_n singularity p and a (-1)-curve C_0 passing through p with multiplicity C_n. By Proposition 16, $|I^{(D_n,\mathcal{S}^+)}| = 2^{n-1}$. Define $\mathfrak{L}_0^{(D_n,\mathcal{S}^+)} := \bigoplus_{l\in I} O(l)$ over Y.

Lemma 39. $\mathfrak{L}_0^{(D_n,\mathcal{S}^+)} = \bigoplus_{m=0}^{[\frac{n}{2}]} \wedge^{2m}(\mathfrak{L}_0^{A_{n-1}})^*(mF + C_0^n)$. □

Proof. First we check that every line bundle summand in the right-hand side is $O_Y(l)$ for a (-1)-curve l in $I^{(D_n,\mathcal{S}^+)}$. For any $l_i \in I^{(A_{n-1},\mathbb{C}^n)}$, we have $l_i \cdot C_0^n = 0$, $l_i \cdot F = 0$ and $F \cdot F = 0$, $F \cdot C_0^n = 1$. For any $2m$ distinct elements l_{i_j}'s in $I^{(A_{n-1},\mathbb{C}^n)}$, we denote $l = -(l_{i_1} + \cdots + l_{i_{2m}}) + mF + C_0^n$. Then $O_Y(l)$ is a summand in the right-hand side. Since $l^2 = -1$ and $l \cdot K_Y = -1$, $l \in I^{(D_n,\mathcal{S}^+)}$. Also the rank of these two bundles are the same, which is $2^{n-1} = \binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2[\frac{n}{2}]}$. Hence, we have the lemma. ■

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{(D_n,\mathcal{S}^+)}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus 2^{n-2}} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 2^{n-3}}.$$

The D_n Lie algebra bundle $\zeta_0^{D_n}$ has a natural fiberwise action on $\mathfrak{L}_0^{(D_n,\mathcal{S}^+)}$,

$$\rho : \zeta_0^{D_n} \otimes \mathfrak{L}_0^{(D_n,\mathcal{S}^+)} \longrightarrow \mathfrak{L}_0^{(D_n,\mathcal{S}^+)},$$

which can be described easily using the reduction to A_{n-1} (with the node C_n being removed): recall

$$\zeta_0^{D_n} = (\wedge^2 \mathfrak{L}_0^{A_{n-1}}(-F)) \oplus ((\mathfrak{L}_0^{A_{n-1}})^* \otimes \mathfrak{L}_0^{A_{n-1}}) \oplus ((\wedge^2 \mathfrak{L}_0^{A_{n-1}})^*(F)),$$

$$\mathfrak{L}_0^{(D_n,\mathcal{S}^+)} = \bigoplus_{m=0}^{[\frac{n}{2}]} \wedge^{2m}(\mathfrak{L}_0^{A_{n-1}})^*(mF),$$

and ρ is given by interior and exterior multiplications for $\wedge^{\cdot}\mathfrak{L}_0^{A_{n-1}}$. (For simplicity, we omit the C_0^n factor.)

Proposition 40. Fix any $\eta \in \Xi_Y^{D_n}$, there exists a unique holomorphic structure on $\mathfrak{L}_0^{(D_n,\mathcal{S}^+)}$ such that the action of $\zeta_\eta^{D_n}$ on the resulting bundle $\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)}$ is holomorphic. Furthermore, if $\eta \in \Xi_X^{D_n}$, then $\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)}$ can descend to X. □

Proof. First, we recall the holomorphic structure on $\zeta_\eta^{D_n}$. In $I^{(D_n,\mathbb{C}^{2n})} = I_1 \cup I_2$ with $I_1 = \{l_i = C_0^1 + \sum_{m=1}^{2n-i} C_m | n+1 \le i \le 2n\}$ and $I_2 = \{F - l_i | l_i \in I_1\}$, let s_i, s_i^*, and f be local holomorphic sections of $O(l_i)$, $O(F-l_i)$, and $O(-F)$, respectively. By Proposition 35, we have

$$\bar{\partial}_{\mathfrak{L}_\eta^{D_n}} s_i^* = \sum_{p=1}^{i-1} \eta_{p,i} s_p^*$$

and

$$\bar{\partial}_{\mathfrak{L}_\eta^{D_n}} s_i = \sum_{p=1}^{n} \eta_{p,2n+1-i} s_p^* - \sum_{p=i+1}^{n} \eta_{i,p} s_p.$$

Back to $(\mathfrak{L}_0^{\mathcal{S}^+})^*$, we define $s_{i_1\cdots i_{2m}} := s_{i_1} \wedge \cdots \wedge s_{i_{2m}} \otimes f^m \in \Gamma(\wedge^{2m}\mathfrak{L}_0^{A_{n-1}}(-mF))$, where $i_j \in \{1, 2, \ldots, n\}$ and define $\bar{\partial}_{(\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)})^*}$ as follows:

$$\bar{\partial}_{\mathfrak{L}} s_{i_1\cdots i_{2m}} = \sum_{p,q} (-1)^{p+q} \eta_{i_p,2n+1-i_q} s_{i_1\cdots \hat{i}_p \cdots \hat{i}_q \cdots i_{2m}} - \sum_{p} \sum_{k \ne i_p} \eta_{i_p,k} s_{i_1\cdots i_{p-1} k i_{p+1} \cdots i_{2m}},$$

where $\hat{i}_j$ means deleting the i_j component. We verify $\bar{\partial}_{\mathfrak{L}}^2 = 0$ by direct computations.

We claim that $\bar{\partial}_{\mathfrak{L}}$ is the unique holomorphic structure such that the action of $\zeta_\eta^{D_n}$ on $(\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)})^*$ is holomorphic, that is,

$$\bar{\partial}_{\zeta_\eta^{D_n}}(g) \cdot x + g \cdot (\bar{\partial}_{\mathfrak{L}} x) = \bar{\partial}_{\mathfrak{L}}(g \cdot x), \qquad (*)$$

for any $g \in \Gamma(\zeta_\eta^{D_n})$ and $x \in \Gamma((\mathfrak{L}_0^{\mathcal{S}^+})^*)$.

We prove the above claim by induction on m. When $m = 0$, $x = s_0 \in \Gamma(\wedge^0 \mathfrak{L}_0^{A_{n-1}})$, by direct computations, $(*)$ holds for any $g \in \Gamma(\zeta_\eta^{D_n})$ if and only if $\bar{\partial}_{\mathfrak{L}} s_0 = 0$ and $\bar{\partial}_{\mathfrak{L}} s_{ij} = -\eta_{i,2n+1-j} s_0 - \sum_{p=i+1}^{n} \eta_{i,p} s_{pj} - \sum_{p=j+1}^{n} \eta_{j,p} s_{ip}$ for any $s_{ij} \in \Gamma(\wedge^2 \mathfrak{L}_0^{A_{n-1}})$. When $m = 2$, from the above formula for $\bar{\partial}_{\mathfrak{L}} s_{ij}$, we can get the formula for $\bar{\partial}_{\mathfrak{L}} s_{ijkl}$. Repeat this process inductively, and we can get the above formula for $\bar{\partial}_{\mathfrak{L}} s_{i_1\cdots i_{2m}}$. Hence, we have the first part of this proposition.

For the second part, we will rewrite $\bar{\partial}_{\mathfrak{L}}$ in matrix form. Firstly, we have

$$\bar{\partial}_{\mathfrak{L}_\eta^{(D_n,\mathbb{C}^{2n})}} = \left(\begin{array}{c|c} \bar{\partial}_{(\mathfrak{L}_{\eta'}^{A_{n-1}})^*(F)} & B \\ \hline 0 & \bar{\partial}_{\mathfrak{L}_{\eta'}^{A_{n-1}}} \end{array}\right),$$

with $\eta' \subset \eta$ and the upper right block B has the following shape:

$$B = \begin{pmatrix} \vdots & \vdots & \ddots \\ \beta & * & \cdots \\ 0 & -\beta & \cdots \end{pmatrix},$$

for $[\beta] \in H^1(Y, O(C_n))$.

In particular, we have an exact sequence of holomorphic bundles:

$$0 \to (\mathfrak{L}_{\eta'}^{A_{n-1}})^*(F) \to \mathfrak{L}_{\eta}^{D_n} \to \mathfrak{L}_{\eta'}^{A_{n-1}} \to 0. \qquad (\Delta)$$

By tensoring (Δ) with $\mathfrak{L}_{\eta'}^{A_{n-1}}(-F)$, we obtain a bundle S_1 as follows:

$$0 \to O_Y \to S_1 \to \wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F) \to 0,$$

with the induced holomorphic structure given by

$$\bar{\partial}_{S_1} = \left(\begin{array}{c|c} \bar{\partial}_{\wedge^0 \mathfrak{L}_{\eta'}^{A_{n-1}}} & B_1 \\ \hline 0 & \bar{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} \end{array}\right) = \left(\begin{array}{c|c} \bar{\partial}_{\wedge^0 \mathfrak{L}_{\eta'}^{A_{n-1}}} & \pm\beta \cdots \\ \hline 0 & \bar{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} \end{array}\right).$$

The occurrence of $\pm\beta$ in that location is because $l_{n+1} + l_{n+2}$ with $l_{n+1}, l_{n+2} \in I^{(D_n, \mathbb{C}^{2n})}$ is the largest element in $I^{(A_{n-1}, \wedge^2 \mathbb{C}^n)}$ and $F - l_{n+1} - l_{n+2} = C_n$ because $F = 2C_0^1 + 2C_1 + \cdots + 2C_{n-2} + C_{n-1} + C_n$.

Similarly, we have an extension bundle

$$0 \to \wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F) \to S_2 \to \wedge^4 \mathfrak{L}_{\eta'}^{A_{n-1}}(-2F) \to 0,$$

with

$$\bar{\partial}_{S_2} = \left(\begin{array}{c|c} \bar{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} & B_2 \\ \hline 0 & \bar{\partial}_{\wedge^4 \mathfrak{L}_{\eta'}^{A_{n-1}}(-2F)} \end{array}\right),$$

where

$$B_2 = \begin{pmatrix} \pm\beta & & & \\ 0 & \pm\beta & & \\ \vdots & \vdots & \ddots & \\ 0 & 0 & \cdots & \pm\beta \end{pmatrix},$$

for $[\beta] \in H^1(Y, \mathrm{Hom}(O(l_i + l_j + l_{n+1} + l_{n+2} - 2F), O(l_i + l_j - F))) = H^1(Y, O(C_n))$ with $i, j \in \{n+3, n+4, \ldots, 2n\}$. And the number of $\pm\beta$'s is $\binom{n-2}{2}$.

Inductively, we obtain $\bar{\partial}_{(\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)})^*}$ as above which has the shape that satisfies Lemma 37:

$$\bar{\partial}_{(\mathfrak{L}_\eta^{(D_n,\mathcal{S}^+)})^*} = \left(\begin{array}{c|c|c|c} \bar{\partial}_{\wedge^0 \mathfrak{L}_{\eta'}^{A_{n-1}}} & B_1 & \cdots & \cdots \\ \hline 0 & \bar{\partial}_{\wedge^2 \mathfrak{L}_{\eta'}^{A_{n-1}}(-F)} & B_2 & \ddots \\ \hline 0 & 0 & \bar{\partial}_{\wedge^4 \mathfrak{L}_{\eta'}^{A_{n-1}}(-2F)} & \ddots \\ \hline \vdots & \ddots & \ddots & \ddots \end{array}\right).$$

The number of $\pm\beta \in \Omega^1(Y, O(C_n))$ in $\bar{\partial}_{\mathfrak{L}}$ is $\binom{n-2}{0} + \binom{n-2}{2} + \cdots + \binom{n-2}{2[\frac{n-2}{2}]} = 2^{n-3}$.

To prove that $(\mathfrak{L}_\eta^{\mathcal{S}^+})^*$ can descend to X when $\eta \in \Xi_X^{D_n}$, we need to show that $(\mathfrak{L}_\eta^{\mathcal{S}^+})^*|_{C_i}$ is trivial for every C_i. When $i \neq n$, this follows from the fact that $\mathfrak{L}_{\eta'}^{A_{n-1}}$ is trivial (Proposition 31) and $\mathrm{Ext}^1_{\mathbb{P}^1}(O, O) \cong 0$. When $i = n$, this follows from Lemma 37 and $\beta = \eta_{n-1,n+1} \in \Omega^{0,1}(Y, O(C_n))$ with $[\eta_{n-1,n+1}|_{C_n}] \neq 0$. ■

6 E_n Case

6.1 E_6 case

We recall that [1] E_6=aut$(\mathbb{C}^{27}, c)$ for a nondegenerate cubic form c on the standard representation $\mathbb{C}^{27}$. The other minuscule representation is $\bar{\mathbb{C}}^{27}$.

We consider a surface X with an E_6 singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1. By Proposition 16, $I^{(E_6,\mathbb{C}^{27})}$ has cardinality 27. For any two distinct (-1)-curves l_i and l_j in I, we have $l_i \cdot l_j = 0$ or 1.

Define $\mathfrak{L}_0^{(E_6,\mathbb{C}^{27})} := \bigoplus_{l\in I} O(l)$ over Y, for simplicity, we write it as $\mathfrak{L}_0^{E_6}$. If we ignore C_6, then we recover the A_5 case as in Section 4.1.

Lemma 41. $\mathfrak{L}_0^{E_6} = \mathfrak{L}_0^{A_5} \oplus (\wedge^2 \mathfrak{L}_0^{A_5})^*(H) \oplus (\wedge^5 \mathfrak{L}_0^{A_5})^*(2H)$, where $H = 3C_0^1 + 3C_1 + 3C_2 + 3C_3 + 2C_4 + C_5 + C_6$. □

Proof. E_6 has A_5 as a Lie subalgebra. The branching rule is $27 = 6 + 15 + 6$, that is, $\mathbb{C}^{27} = \mathbb{C}^6 \oplus \wedge^2(\mathbb{C}^6)^* \oplus \wedge^5(\mathbb{C}^6)^*$. The first six (-1)-curves in $I: l_1 = C_0^1$, $l_2 = C_0^1 + C_1, \ldots, l_6 = C_0^1 + C_1 + C_2 + C_3 + C_4 + C_5$ form the standard representation $\mathbb{C}^6$ of A_5. The next 15 (-1)-curves are given by $H - l_i - l_j$ with $i \neq j \in \{1, 2, \ldots, 6\}$. The remaining six (-1)-curves are given by $2H - l_1 - l_2 - \cdots - \hat{l}_i - \cdots - l_6$. ■

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{E_6}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus 15} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 6}.$$

Since the Weyl group of E_6 acts transitively on these (-1)-curves, we can easily determine the configuration of these 27 (-1)-curves [20]: Fix any (-1)-curve, there are exactly 10 (-1)-curves intersect it, together with the fixed (-1)-curve, they form 5 triangles. A triple l_i, l_j, l_k is called a triangle if $l_i + l_j + l_k = K'$, where $K' = 3C_0^1 + 4C_1 + 5C_2 + 6C_3 + 4C_4 + 2C_5 + 3C_6$.

From the configuration of these 27 (-1)-curves in Y, we can define a cubic form c on the vector space $V_0 = \mathbb{C}^I = \bigoplus_{l\in I}\mathbb{C}\langle v_l\rangle$ spanned by (-1)-curves,

$$c: V_0 \otimes V_0 \otimes V_0 \longrightarrow \mathbb{C}, \quad (v_{l_i}, v_{l_j}, v_{l_k}) \mapsto \begin{cases} \pm 1 & \text{if } l_i + l_j + l_k = K', \\ 0 & \text{otherwise.} \end{cases}$$

The signs above can be determined explicitly [1, 11] such that $E_6 = \mathrm{aut}(V_0, c)$.

Correspondingly, we have a fiberwise cubic form c on the bundle $\mathfrak{L}_\eta^{E_6}$,

$$c: \mathfrak{L}_\eta^{E_6} \otimes \mathfrak{L}_\eta^{E_6} \otimes \mathfrak{L}_\eta^{E_6} \longrightarrow O(K').$$

Proposition 42. There exists η with $\bar\partial_\eta^2 = 0$ such that $\bar\partial_\eta c = 0$. □

Proof. Note $\bar\partial_\eta c = 0$ if and only if

$$c(\bar\partial_\eta s_i, s_j, s_k) + c(s_i, \bar\partial_\eta s_j, s_k) + c(s_i, s_j, \bar\partial_\eta s_k) = 0, \tag{*}$$

for any $s_i \in H^0(Y, O(l_i))$, $s_j \in H^0(Y, O(l_j))$, and $s_k \in H^0(Y, O(l_k))$. From the definition of c, if $l_i + l_j + l_k = K'$, then the above equation $(*)$ holds automatically. If $l_i + l_j + l_k \neq K'$, without loss of generality, we assume $l_i \cdot l_j = 0$, then we have the following four cases.

Case (i), if $l_i \cdot l_k = 0$ and $l_j \cdot l_k = 0$, then $(*)$ holds automatically.

Case (ii), if $l_i \cdot l_k = 0$ and $l_j \cdot l_k = 1$, then $(*)$ holds if $\eta_{l_i, K'-l_j-l_k} = 0$.

Case (iii), if $l_i \cdot l_k = 1$ and $l_j \cdot l_k = 0$, then $(*)$ holds if $\eta_{l_j, K'-l_i-l_k} = 0$.

Case (iv), if $l_i \cdot l_k = 1$ and $l_j \cdot l_k = 1$, then $(*)$ holds if $\eta_{l_i, K'-l_j-l_k} \pm \eta_{l_j, K'-l_i-l_k} = 0$, here the sign is determined by the signs of cubic form.

In conclusion, for any $l_i, l_j \in I^{(E_6, \mathbb{C}^{27})}$, if $l_i \cdot l_j \neq 0$, then $\eta_{i,j} = 0$. If $l_i \cdot l_j = 0$, then $l_i - l_j = \alpha$ $(j > i)$ for $\alpha \in \Phi^+$, that is, $\eta_{i,j} \in \Omega^{0,1}(Y, O(\alpha))$. And for any other $\eta_{p,q} \in \Omega^{0,1}(Y, O(\alpha))$, we have $\eta_{i,j} \pm \eta_{p,q} = 0$. From the signs of the cubic form c, we know that given any positive root α, there exists six $\eta_{i,j}$'s in $\Omega^{0,1}(Y, O(\alpha))$, where three of them are the same and the other three different from the first three by a sign. We use a computer to prove that we can find such $\eta_{i,j}$'s satisfying $\bar\partial_\eta^2 = 0$. ■

Until now, we have proved that $\Xi_Y^{E_6}$ is not empty.

Proposition 43. The bundle $\mathfrak{L}_{\eta}^{E_6}$ over Y with $\eta \in \Xi_Y^{E_6}$ can descend to X if and only if for every C_k and $\eta_{i,j} \in \Omega^{0,1}(Y, O(C_k))$, $[\eta_{i,j}|_{C_k}] \neq 0$, that is, $\eta \in \Xi_X^{E_6}$. □

Proof. From Lemma 41, Proposition 42 and the order of $I^{(E_6,\mathbb{C}^{27})}$, for $\eta \in \Xi_Y^{E_6}$, $\mathfrak{L}_{\eta}^{E_6}$ can be constructed from $\mathfrak{L}_{\eta'}^{A_5}$ for some $\eta' \in \Xi_Y^{A_5}$ with $\eta' \subset \eta$. Under the (nonholomorphic) direct sum decomposition $\mathfrak{L}_0^{E_6} = \mathfrak{L}_0^{A_5} \oplus (\wedge^2 \mathfrak{L}_0^{A_5})^*(H) \oplus (\wedge^5 \mathfrak{L}_0^{A_5})^*(2H)$, $\bar\partial_\eta$ for $\mathfrak{L}_{\eta}^{E_6}$ has the following block decomposition:

$$\left(\begin{array}{c|c|c}
\bar\partial_{(\wedge^5 \mathfrak{L}_{\eta'}^{A_5})^*(2H)} & \begin{matrix} \vdots & \vdots & \vdots & \ddots \\ \pm\beta & * & * & \cdots \\ 0 & \pm\beta & * & \cdots \\ 0 & 0 & \pm\beta & \cdots \end{matrix} & * \\
\hline
0 & \bar\partial_{(\wedge^2 \mathfrak{L}_{\eta'}^{A_5})^*(H)} & \begin{matrix} \vdots & \vdots & \vdots & \ddots \\ \pm\beta & * & * & \cdots \\ 0 & \pm\beta & * & \cdots \\ 0 & 0 & \pm\beta & \cdots \end{matrix} \\
\hline
0 & 0 & \bar\partial_{\mathfrak{L}_{\eta'}^{A_5}}
\end{array}\right).$$

Here $\pm\beta \in \Omega^{0,1}(Y, O(C_6))$, it is because of the corresponding two (-1)-curves l and l' satisfying $l - l' = C_6$. The signs of β can be determined by $\bar\partial_\eta c = 0$.

From the above, we know that $\mathfrak{L}_{\eta}^{E_6}|_{C_k}$ $(k \neq 6)$ is trivial if and only if $\mathfrak{L}_{\eta'}^{A_5}|_{C_k}$ $(k \neq 6)$ is trivial. From Proposition 31, we have the theorem for $k \neq 6$. For C_6, from Lemma 37, $\mathfrak{L}_{\eta}^{E_6}|_{C_6}$ is trivial if and only if these $\pm\beta$'s satisfy $[\beta|_{C_6}] \neq 0$. ■

Note that $\zeta_\eta^{E_6} = \mathrm{aut}(\mathfrak{L}_{\eta}^{E_6}, c)$ is an E_6 Lie algebra bundle over Y. In order for $\zeta_\eta^{E_6}$ to descend to X as a Lie algebra bundle, we need to show that $c|_{C_i} : \mathfrak{L}_{\eta}^{E_6}|_{C_i} \otimes \mathfrak{L}_{\eta}^{E_6}|_{C_i} \otimes \mathfrak{L}_{\eta}^{E_6}|_{C_i} \longrightarrow O_{C_i}(K')$ is a constant map for every C_i. This follows from the fact that both $\mathfrak{L}_{\eta}^{E_6}$ and $O(K')$ are trivial on all C_i's and $\bar\partial_\eta c = 0$. From the construction, $\mathfrak{L}_{\eta}^{E_6}$ is a representation bundle of $\zeta_\eta^{E_6}$.

The only other minuscule representation $\bar{\mathbb{C}}^{27}$ of E_6 is the dual of the standard representation $\mathbb{C}^{27}$. Therefore $\mathfrak{L}_{\eta}^{(E_6,\bar{\mathbb{C}}^{27})} = (\mathfrak{L}_{\eta}^{(E_6,\mathbb{C}^{27})})^*$.

6.2 E_7 case

We recall that [1] $E_7 = \mathrm{aut}(\mathbb{C}^{56}, t)$ for a nondegenerate quartic form t on the standard representation $\mathbb{C}^{56}$. There is no other minuscule representation of E_7.

We consider a surface X with an E_7 singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1. By Proposition 16, $I^{(E_7,\mathbb{C}^{56})}$ has cardinality 56. For any two distinct (-1)-curves l_i and l_j in I, we have $l_i \cdot l_j = 0, 1,$ or 2.

Define $\mathfrak{L}_0^{(E_7,\mathbb{C}^{56})} := \bigoplus_{l\in I} O(l)$ over Y, for simplicity, we write it as $\mathfrak{L}_0^{E_7}$. If we ignore C_7, we recover the A_6 case as in Section 4.1.

Lemma 44. $\mathfrak{L}_0^{(E_7,\mathbb{C}^{56})} = \mathfrak{L}_0^{A_6} \oplus (\wedge^2 \mathfrak{L}_0^{A_6})^*(H) \oplus (\wedge^5 \mathfrak{L}_0^{A_6})^*(2H) \oplus (\wedge^6 \mathfrak{L}_0^{A_6})^*(3H)$, where $H = 3C_0^1 + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 2C_5 + C_6 + C_7$. □

Proof. Similar to the E_6 case. ■

From the above lemma and direct computations, for any C_i,

$$\mathfrak{L}_0^{E_7}|_{C_i} \cong O_{\mathbb{P}^1}^{\oplus 32} \oplus (O_{\mathbb{P}^1}(1) \oplus O_{\mathbb{P}^1}(-1))^{\oplus 12}.$$

Since the Weyl group of E_7 acts transitively on these 56 (-1)-curves, the configuration of these 56 (-1)-curves is as follows: Fix any (-1)-curve, there are exactly 27 (-1)-curves intersect it once, 1 (-1)-curve intersects it twice. If $l_i + l_j + l_p + l_q = 2K'$ with $K' = 2C_0^1 + 3C_1 + 4C_2 + 5C_3 + 6C_4 + 4C_5 + 2C_6 + 3C_7$, the four (-1)-curves l_i, l_j, l_p, and l_q will form a quadrangle.

From this configuration, we can define a quartic form t on the vector space $V_0 = \mathbb{C}^I = \bigoplus_{l\in I} \mathbb{C}\langle v_l \rangle$ spanned by all the (-1)-curves,

$$t : V_0 \otimes V_0 \otimes V_0 \otimes V_0 \longrightarrow \mathbb{C}, \quad (v_{l_i}, v_{l_j}, v_{l_p}, v_{l_q}) \mapsto \begin{cases} \pm 1 & \text{if } l_i + l_j + l_p + l_q = 2K', \\ 0 & \text{otherwise.} \end{cases}$$

The signs above can be determined explicitly [1] such that $E_7 = \mathrm{aut}(V_0, t)$.

Correspondingly, we have a fiberwise quartic form t on the bundle $\mathfrak{L}_\eta^{E_7}$,

$$t : \mathfrak{L}_\eta^{E_7} \otimes \mathfrak{L}_\eta^{E_7} \otimes \mathfrak{L}_\eta^{E_7} \otimes \mathfrak{L}_\eta^{E_7} \longrightarrow O(2K').$$

Proposition 45. There exists η with $\bar{\partial}_\eta^2 = 0$ such that $\bar{\partial}_\eta t = 0$. □

Proof. Similar to the E_6 case, but even more calculations involved. We will omit the calculations here and only list the conditions for $\bar{\partial}_\eta t = 0$. From $\bar{\partial}_\eta t = 0$, we have when $l_i \cdot l_j \neq 0$, $\eta_{i,j} = 0$. That means all the nonzero $\eta_{i,j}$'s correspond to $l_i \cdot l_j = 0$, and then $l_i - l_j = \alpha$ for some root α, that is, $\eta_{i,j} \in \Omega^{0,1}(Y, O(\alpha))$. Conversely, given any positive root α, there exists 12 $\eta_{i,j}$'s in $\Omega^{0,1}(Y, O(\alpha))$, where 6 of them are the same and the other 6 different from the first 6 by a sign. We use a computer to prove that we can find such $\eta_{i,j}$'s satisfying $\bar{\partial}_\eta^2 = 0$. ■

Until now, we have proved that $\Xi_Y^{E_7}$ is not empty.

Proposition 46. The bundle $\mathfrak{L}_\eta^{E_7}$ over Y with $\eta \in \Xi_Y^{E_7}$ can descend to X if and only if for every C_k and $\eta_{i,j} \in \Omega^{0,1}(Y, O(C_k))$, $[\eta_{i,j}|_{C_k}] \neq 0$, that is, $\eta \in \Xi_X^{E_7}$. □

Proof. Similar to the E_6 case (Proposition 43). ■

Note that $\zeta_\eta^{E_7} = \mathrm{aut}(\mathfrak{L}_\eta^{E_7}, t)$ is an E_7 Lie algebra bundle over Y. In order for $\zeta_\eta^{E_7}$ to descend to X as a Lie algebra bundle, we need to show that $t|_{C_i} : \mathfrak{L}_\eta^{E_7}|_{C_i} \otimes \mathfrak{L}_\eta^{E_7}|_{C_i} \otimes \mathfrak{L}_\eta^{E_7}|_{C_i} \otimes \mathfrak{L}_\eta^{E_7}|_{C_i} \longrightarrow O_{C_i}(2K')$ is a constant map for every C_i. This follows from the fact that both $\mathfrak{L}_\eta^{E_7}$ and $O(2K')$ are trivial on all C_i's and $\bar{\partial}_\eta t = 0$. It is obvious that $\mathfrak{L}_\eta^{E_7}$ is a representation bundle of $\zeta_\eta^{E_7}$.

6.3 E_8 case

Although E_8 has no minuscule representation, the fundamental representation corresponding to C_1 is the adjoint representation of E_8.

We consider a surface X with an E_8 singularity p and a (-1)-curve C_0 passing through p with multiplicity C_1. By direct computations, $|I| = 240$. In this case, $l \in I$ if and only if $l - K' \in \Phi$, where $K' = C_0^1 + 2C_1 + 3C_2 + 4C_3 + 5C_4 + 6C_5 + 4C_6 + 2C_7 + 3C_8$. So $\mathcal{E}_0^{E_8}$ defined in Section 2 can be written as follows:

$$\mathcal{E}_0^{E_8} := O^{\oplus 8} \oplus \bigoplus_{\alpha \in \Phi} O(\alpha) = \left(O(K')^{\oplus 8} \oplus \bigoplus_{l \in I} O(l) \right)(-K').$$

We will prove that $(\mathcal{E}_\varphi^{E_8}, \bar{\partial}_\varphi)$ with $\varphi = (\varphi_\alpha)_{\alpha \in \Phi^+} \in \Psi_X$ descends to X in Section 7.

7 Proof of Main Results

In the above three sections, we have constructed and studied the Lie algebra bundles and minuscule representation bundles in A_n, D_n, and E_n $(n \neq 8)$ cases separately. We will prove that the holomorphic structures on these bundles can be expressed by forms in the positive root classes and the representation actions.

Proof of Theorems 23 and 24. Recall that when $\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ is the standard representation, $\mathfrak{L}_\eta^{(\mathfrak{g},V)} (\eta \in \Xi_Y^{\mathfrak{g}})$ admits of a holomorphic fiberwise symmetric multi-linear form f. And $\bar\partial_\eta f=0$ implies that $\eta_{i,j}=0$ unless $l_i-l_j=\alpha$ $(j>i)$ for some $\alpha\in\Phi^+$. Thus, $\eta_{i,j}=\varphi_\alpha\in\Omega^{0,1}(Y,O(\alpha))$. Furthermore, if $\eta_{i,j}$ and $\eta_{i',j'}$ are in $\Omega^{0,1}(Y,O(\alpha))$, then they are the same up to sign. Thus, we can write $\eta_{i,j}=n_{\alpha,w_i}\varphi_\alpha$, where n_{α,w_i}'s are as in Section 3, since ρ preserves f. Namely, $\bar\partial_\eta=\bar\partial_0+\sum_{\alpha\in\Phi^+}c_\alpha\rho(x_\alpha)=\bar\partial_0+\sum_{\alpha\in\Phi^+}\rho(\varphi_\alpha)$ with $\varphi_\alpha=c_\alpha x_\alpha$.

The holomorphic structure on the bundle $\zeta_\eta^{\mathfrak g}:=\mathrm{aut}(\mathfrak{L}_\eta^{(\mathfrak g,V)},f)$ is $\bar\partial_\eta=\bar\partial_0+\sum_{\alpha\in\Phi^+}c_\alpha\mathrm{ad}(x_\alpha)$, which is the same as $\bar\partial_\varphi$ for $\mathcal{E}_\varphi^{\mathfrak g}$ in Section 2, that is, $\zeta_\eta^{\mathfrak g}=\mathcal{E}_\varphi^{\mathfrak g}$.

The only minuscule representations $(\mathfrak g,V)$ besides standard representations are $(A_n,\wedge^k C^{n+1})$, $(D_n,S^\pm)$, and $(E_6,\bar{\mathbb{C}}^{27})$. We denote corresponding actions as ρ as usual. In each case, for $\mathcal{E}_\varphi^{\mathfrak g}$ to act holomorphically on the corresponding vector bundle, the holomorphic structure on $\mathfrak{L}_0^{(\mathfrak g,V)}$ can only be $\bar\partial_\varphi$.

The filtration of $\mathfrak{L}_0^{(\mathfrak g,V)}$ gives one on $\mathfrak{L}_\eta^{(\mathfrak g,V)}$, since it is constructed from extensions using elements in $I_i\backslash I_{i+1}$ (Section 3.3).

We note that all the above Lie algebra bundles and representation bundles over Y can descend to X if and only if $0\neq[\varphi_{C_i}|_{C_i}]\in H^1(Y,O_{C_i}(C_i))$ for all C_i's, that is, $\varphi=(\varphi_\alpha)_{\alpha\in\Phi^+}\in\Psi_X$. ■

From the above arguments, Theorem 9 holds true for *ADE* except E_8 case.

Proof of Theorem 9. It remains to prove the E_8 case.

$$\mathcal{E}_0^{E_8}:=O^{\oplus 8}\oplus\bigoplus_{\alpha\in\Phi}O(\alpha)=\left(O(K')^{\oplus 8}\oplus\bigoplus_{l\in I}O(l)\right)(-K').$$

We want to show that the bundle $(\mathcal{E}_\varphi^{E_8},\bar\partial_\varphi)$ with $\varphi=(\varphi_\alpha)_{\alpha\in\Phi^+}\in\Psi_X$ can descend to X, that is, $\mathcal{E}_\varphi^{E_8}|_{C_i}$ is trivial for $i=1,2,\dots,8$. Note that $O(K')|_{C_i}$ is trivial for every i, but $O(l)|_{C_i}$ can be $O_{\mathbb{P}^1}(\pm 2)$, and hence, Lemma 28 is not sufficient. However, if we ignore C_8 (respectively, C_7) in Y, then we recover the A_7 case (respectively, D_7 case). Our approach is to reduce the problem of trivializing $\mathcal{E}_\varphi^{E_8}|_{C_i}$ to one for a representation bundle of A_7 (respectively, D_7).

Step 1: As A_7 is a Lie subalgebra of E_8, the adjoint representation of E_8 decomposes as a sum of irreducible representations of A_7. The branching rule is $248=8+28+56+64+56+28+8$, and correspondingly, we have the following decomposition of $\mathcal{E}_0^{E_8}$ over Y:

$$\begin{aligned}\mathcal{E}_0^{E_8}&=\mathfrak{L}_0^{A_7}(-K')\oplus\wedge^2(\mathfrak{L}_0^{A_7})^*(H-K')\oplus\wedge^5(\mathfrak{L}_0^{A_7})^*(2H-K')\\&\quad\oplus\mathfrak{L}_0^{A_7}\otimes(\mathfrak{L}_0^{A_7})^*\oplus\wedge^3(\mathfrak{L}_0^{A_7})^*(H)\oplus\wedge^6(\mathfrak{L}_0^{A_7})^*(2H)\oplus(\mathfrak{L}_0^{A_7})^*(K'),\end{aligned}$$

where $H = 3C_0^1 + 3C_1 + 3C_2 + 3C_3 + 3C_4 + 3C_5 + 2C_6 + C_7 + C_8$ and $K' = C_0^1 + 2C_1 + 3C_2 + 4C_3 + 5C_4 + 6C_5 + 4C_6 + 2C_7 + 3C_8$.

Step 2: Instead of $\mathfrak{L}_0^{A_7}$, we use $\mathfrak{L}_\varphi^{A_7}$, which is trivial on C_i for $i \neq 8$. We consider the bundle

$$\mathcal{E}'^{E_8} = \mathfrak{L}_\varphi^{A_7}(-K') \oplus \wedge^2(\mathfrak{L}_\varphi^{A_7})^*(H-K') \oplus \wedge^5(\mathfrak{L}_\varphi^{A_7})^*(2H-K')$$
$$\oplus\, \mathfrak{L}_\varphi^{A_7} \otimes (\mathfrak{L}_\varphi^{A_7})^* \oplus \wedge^3(\mathfrak{L}_\varphi^{A_7})^*(H) \oplus \wedge^6(\mathfrak{L}_\varphi^{A_7})^*(2H) \oplus (\mathfrak{L}_\varphi^{A_7})^*(K').$$

We have $\bar\partial_{\mathcal{E}'^{E_8}} = \bar\partial_0 + \sum_{\alpha\in\Phi_{A_7}^+} \mathrm{ad}(\varphi_\alpha)$. Since $O(K')$ and $O(H)$ are both trivial on C_i for $i \neq 8$, $\mathcal{E}'^{E_8}$ is trivial on C_i for $i \neq 8$.

Step 3: We compare $\mathcal{E}'^{E_8}$ with $\mathcal{E}_\varphi^{E_8}$. Topologically they are the same. Holomorphically,

$$\bar\partial_{\mathcal{E}_\varphi^{E_8}} = \bar\partial_0 + \sum_{\alpha\in\Phi_{E_8}^+} \mathrm{ad}(\varphi_\alpha) = \bar\partial_{\mathcal{E}'^{E_8}} + \sum_{\alpha\in\Phi_{E_8}^+\backslash\Phi_{A_7}^+} \mathrm{ad}(\varphi_\alpha).$$

If we write the holomorphic structure of $\mathcal{E}_\varphi^{E_8}$ as a 248×248 matrix, then φ_α with $\alpha\in\Phi_{E_8}^+\backslash\Phi_{A_7}^+$ must appear at those positions (β,γ) with $\beta-\gamma=\alpha$, where β has at least one more C_8 than γ. That means that after taking extensions between the summands of $\mathcal{E}'^{E_8}$, we can get $\mathcal{E}_\varphi^{E_8}$. Since $\mathcal{E}'^{E_8}$ is trivial on C_i for $i\neq 8$ and $\mathrm{Ext}^1_{\mathbb{P}^1}(O,O)\cong 0$, we have $\mathcal{E}_\varphi^{E_8}$ trivial on C_i for $i \neq 8$.

Similarly, if we consider the reduction of E_8 to D_7, from the branching rule $248 = 14+64+1+91+64+14$, we have the following decomposition of $\mathcal{E}_0^{E_8}$:

$$\mathcal{E}_0^{E_8} = \mathfrak{L}_0^{D_7}(-K') \oplus \mathfrak{L}_0^{(D_7,\mathcal{S}^+)}(C_7 - C_0^6) \oplus O \oplus \mathcal{E}_0^{D_7} \oplus (\mathfrak{L}_0^{(D_7,\mathcal{S}^+)})^*(C_0^6 - C_7) \oplus (\mathfrak{L}_0^{D_7})^*(K').$$

Instead of $\mathfrak{L}_0^{D_7}$, we consider $\mathfrak{L}_\varphi^{D_7}$. Similar to the reduction to the A_7 case as above, we will get for $(\mathcal{E}_\varphi^{E_8}, \bar\partial_\varphi)$, if we take $[\varphi_{C_i}|_{C_i}]\neq 0$, then $\mathcal{E}_\varphi^{E_8}$ is trivial on C_i for $i\neq 7$. Hence, we have proved Theorem 9 for type E_8. ■

Proof of Theorem 25. We only need to find a divisor B in Y such that (1) B is a combination of C_i's and $\tilde C_0$ with the coefficient of $\tilde C_0$ not zero and (2) $O(B)$ can descend to X. Then if we take k to be the coefficient of $\tilde C_0$ in B, $\mathbb{L}_\varphi^{(\mathfrak{g},V)} := S^k\mathfrak{L}_\varphi^{(\mathfrak{g},V)} \otimes O(-B)$ with $\varphi\in\Psi_X$ can descend to X and does not depend on the existence of C_0.

(A_n, C^{n+1}) case, $B = (n+1)\tilde C_0 + nC_1 + (n-1)C_2 + \cdots + C_n$,

$(A_n, \wedge^k C^{n+1})$ case, $B = (n+1)\tilde C_0 + (n-k+1)C_1 + \cdots + (k-1)(n-k-1)C_{k-1} + k(n-k)C_{k+1} + \cdots + kC_n$,

(D_n, C^{2n}) case, $B = F = 2\tilde C_0 + 2C_1 + \cdots + 2C_{n-2} + C_{n-1} + C_n$,

(D_n, S^+) case, $B = 4\tilde{C}_0 + 2C_1 + 4C_2 + \cdots + 2(n-2)C_{n-2} + (n-2)C_{n-1} + nC_n$,
(E_6, C^{27}) case, $B = 3\tilde{C}_0 + 4C_1 + 5C_2 + 6C_3 + 4C_4 + 2C_5 + 3C_6$,
(E_7, C^{56}) case, $B = 2\tilde{C}_0 + 3C_1 + 4C_2 + 5C_3 + 6C_4 + 4C_5 + 2C_6 + 3C_7$. ■

Remark 47. We can determine Chern classes of the Lie algebra bundles and minuscule representation bundles. For any minuscule representation bundle $\mathfrak{L}_\varphi^{(\mathfrak{g},V)}$,

$$c_1(\mathfrak{L}_\varphi^{(\mathfrak{g},V)}) = \sum_{l \in I^{(\mathfrak{g},V)}} [l] \in H^2(Y, \mathbb{Z}).$$

For any Lie algebra bundle $\mathcal{E}_\varphi^{\mathfrak{g}}$, we have

$$c_1(\mathcal{E}_\varphi^{\mathfrak{g}}) = 0$$

and

$$c_2(\mathcal{E}_\varphi^{\mathfrak{g}}) = \sum_{\alpha \neq \beta \in \Phi} c_1(O(\alpha))c_1(O(\beta)) = \sum_{\alpha \in \Phi^+} c_1(O(\alpha))c_1(O(-\alpha)) = \dim(\mathfrak{g}) - \operatorname{rank}(\mathfrak{g}).$$

In particular, the bundles we defined above are not trivial. □

Remark 48. There are choices in the construction of our Lie algebra bundles and minuscule representation bundles. We will see that these bundles are not unique. Take $\mathfrak{L}_\varphi^{A_2}$ ($\varphi = (\varphi_\alpha)_{\alpha \in \Phi_{A_2}^+} \in \Psi_X$) as an example. The holomorphic structure on $\mathfrak{L}_\varphi^{A_2}$ is as follows:

$$\bar{\partial}_\varphi = \begin{pmatrix} \bar{\partial} & \varphi_{C_2} & \varphi_{C_1+C_2} \\ 0 & \bar{\partial} & \varphi_{C_1} \\ 0 & 0 & \bar{\partial} \end{pmatrix},$$

with $[\varphi_{C_1}|_{C_1}] \neq 0$ and $[\varphi_{C_2}|_{C_2}] \neq 0$. We replace $\varphi_{C_1+C_2}$ by $\varphi_{C_1+C_2} + \psi$, where $\psi \in H^1(Y, O(C_1 + C_2)) \neq 0$. If $[\psi] \neq 0$, then $\bar{\partial}_{\varphi+\psi}$ is not isomorphic to $\bar{\partial}_\varphi$. □

Remark 49. Our $\mathfrak{g}$-bundle $\mathcal{E}_\eta^{\mathfrak{g}}$ over Y is given by $\operatorname{aut}(\mathfrak{L}_\eta^{(\mathfrak{g},V)}, f)$ with $f : \bigotimes^r \mathfrak{L}_\eta^{(\mathfrak{g},V)} \longrightarrow O_Y(D)$. If $O(D) = O(rD')$ for some divisor D', then

$$f : \bigotimes^r \mathfrak{L}_\eta^{(\mathfrak{g},V)}(-D') \longrightarrow O_Y.$$

And $\operatorname{Aut}(\mathfrak{L}_\eta^{(\mathfrak{g},V)}(-D'), f)$ is a Lie group bundle over Y lifting $\mathcal{E}_\eta^{\mathfrak{g}}$. In general, we only have a $G \times \mathbb{Z}_r$-bundle, or so-called conformal G-bundle in [8]. □

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Appendix

We now construct examples of surface with an ADE singularity p of type $\mathfrak{g}$ and a (-1)-curve C_0 passing through p with minuscule multiplicity C_k. We call its minimal resolution a surface with minuscule configuration of type $(\mathfrak{g}, V)$, where V is the fundamental representation corresponding to $-C_k$.

First we consider the standard representation $V \simeq \mathbb{C}^{n+1}$ of $A_n = \mathrm{sl}(n+1)$. When we blowup a point on any surface, the exceptional curve is a (-1)-curve E. If we blowup a point on E, the strict transform of E becomes a (-2)-curve. By repeating this process $n+1$ times, we obtain a chain of (-2)-curves with a (-1)-curve attached to the last one. Namely, we have a surface with a minuscule configuration of type $(A_n, \mathbb{C}^{n+1})$.

Suppose that D is a smooth rational curve on a surface with $D^2 = 0$. By blowing up a point on D, we obtain a surface with a chain of two (-1)-curves. If we blowup their intersection point and iterative blowing up points in exceptional curves, then we obtain a surface with minuscule configuration of type $(D_n, \mathbb{C}^{2n})$.

Given a surface together with a smooth rational curve C with $C^2 = 1$ on it, we could obtain every minuscule configuration by the following process. If we blowup three points on C, then the strict transform of C is a (-2)-curve. By the previous construction of iterated blowups of points in these three exceptional curves E_i's, we could obtain many minuscule configurations. Let us denote the number of iterated blowups of the exceptional curve E_i as m_i with $i \in \{1, 2, 3\}$. Then we can obtain minuscule configuration of type $(\mathfrak{g}, V)$ by taking suitable m_i's as follows.

Minuscule configuration of type $(\mathfrak{g}, V)$	(m_1, m_2, m_3)
$(A_n, \Lambda^k\mathbb{C}^{n+1})$ for any k	$(k-1, 0, n-k)$
$(D_n, \mathbb{C}^{2n})$, (D_n, S^+), and (D_n, S^-)	$(n-3, 1, 1)$
$(E_6, 27)$, $(E_6, \bar{27})$	$(2, 1, 2)$
$(E_7, 56)$	$(3, 1, 2)$

Note that we could obtain such a configuration for every adjoint representation of E_n this way. We remark that surfaces in this last construction are necessarily rational surfaces because of the existence of C with $C^2 = 1$.

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