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THIN INSTANTONS IN G_2-MANIFOLDS AND SEIBERG-WITTEN INVARIANTS

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Abstract

For two nearby disjoint coassociative submanifolds C and C' in a G_2-manifold, we construct thin instantons with boundaries lying on C and C' from regular J-holomorphic curves in C. We explain their relationship with the Seiberg-Witten invariants for C.

1. Introduction

Intersection theory of Lagrangian submanifolds is an essential part of symplectic geometry. By counting the number of holomorphic disks bounding intersecting Lagrangian submanifolds, Floer and others defined the celebrated Floer homology theory. It plays an important role in mirror symmetry for Calabi-Yau manifolds and string theory in physics. In M-theory, Calabi-Yau threefolds are replaced by seven dimensional G_2-manifolds M (i.e. oriented Octonion manifolds [**26**]). The analogs of holomorphic disks (resp. Lagrangian submanifolds) are instantons or associative submanifolds (resp. coassociative submanifolds or branes) in M [**25**]. In [**17**], the Fredholm theory for instantons with coassociative boundary conditions has been set up. However, existence of instantons is still a difficult problem. As a first step, we want to give a construction modeled on the work of Fukaya and Oh [**10**] in symplectic geometry. As it was shown in [**10**], if we choose two nearby Lagrangian submanifolds in such a way that one is the graph of a closed one form on the other, then the holomorphic disks bounding two are closely related to gradient flow lines of the one form. Searching for the analog in G_2-geometry leads us to study the following problem.

Problem: Given two nearby coassociative submanifolds C and C' in a (almost) G_2-manifold M, relate the number of instantons in M bounding $C \cup C'$ to the Seiberg-Witten invariants of C.

The basic idea is as follows: When the coassociative submanifold C' is sufficiently close to C, then it is the graph of a self-dual two form on C. This two form is essentially a symplectic form on C away from the intersection $C \cap C'$. Instantons bounding $C \cup C'$ would become *holomorphic*

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curves on C when C' collapses onto C modulo the possible *bubblings*. By the seminal work of Taubes [**36**] on the equivalence of Gromov-Witten and Seiberg-Witten invariants, we expect that the number counted with algebraic weights of such instantons is given by the Seiberg-Witten invariants of C.

Settling the above problem completely is very difficult at the current stage. We treat the special case when C and C' are disjoint, i.e. C is a symplectic four manifold in this paper. The basic tool is the gluing technique. But even in this simpler case, the set up is quite different from the Lagrangian Floer theory (cf. Fukaya and Oh [**10**]). Our domains are three dimensional instead of two dimensional, and we have to deform the submanifolds rather than deforming the maps as was done in Floer theory. This is because we do not have the luxury of applying the conformal geometry in dimension two to transform the problem of finding holomorphic curves into the one of finding holomorphic maps. Furthermore, the linear theory is more difficult in this case since we have to deal with a problem that lacks uniform ellipticity, as we will explain in Section 3.3. Since the G_2 form is cubic, the needed quadratic estimate of the 3-dimensional instanton equation appears unavailable in the L^p setting (see end of Section 4.3). So we set up the problem in the Schauder setting, and in linear theory we need to go further from L^p estimates to Schauder estimates. This is different from [**10**].

As C' should be sufficiently close to C, we assume that they arise in a one-parameter smooth family of coassociative submanifolds C_t with small t. Contracting with the normal vector field $n := dC_t/dt|_{t=0}$ for the infinitesimal deformation with the G_2-form Ω, we obtain a closed self-dual two form $\iota_n\Omega \in \Omega^2_+(C_0)$. Using the induced metric, from n one can define an *almost complex structure* $J = J_n$ on C_0 away from the zero set of $\iota_n\Omega$ (see (2) for details).

Theorem 1. *Suppose that* (M, Ω) *is a* G_2*-manifold and* $\{C_t\}$ *is a one-parameter smooth family of coassociative submanifolds in* M*. When* $\iota_n\Omega \in \Omega^2_+(C_0)$ *is nonvanishing, then:*

1) *(Proposition* 6*) If* $\{A_t\}$ *is any one-parameter family of associative submanifolds (i.e. instantons) in* M *satisfying*
$$\partial A_t \subset C_t \cup C_0,\ \lim_{t\to 0} A_t \cap C_0 = \Sigma_0 \text{ in the } C^1\text{-topology,}$$
then Σ_0 *is a* J_n*-holomorphic curve in* C_0*.*
2) *(Theorem* 27*) Conversely, every regular* J_n*-holomorphic curve* Σ_0 *(namely those for which the linearization of* $\overline{\partial}_{J_n}$ *on* Σ_0 *is surjective) in* C_0 *is the limit of a family of associative submanifolds* A_t *as described above.*

Notice that in the product situation, where
$$(M, \Omega) := \left(X \times S^1, \operatorname{Re}\Omega_X + \omega_X \wedge d\theta\right)$$

(cf. Section 2.2) with (X, Ω_X) being a Calabi-Yau threefold, then our theorem would follow from the work of [**10**].

The paper is organized as follows: In Section 2, we first recall some basics of Floer theory; next we describe their G_2-counterparts; then we explain the connection between instantons and Seiberg-Witten invariants; and last we study the deformation of instantons with the aim to generalize to almost instantons. In Section 3, we first study the linear differential operator $\mathcal{D}$ (defined in (19)) on a type of thin 3-manifold, which is a linear approximation of the instanton equation; then we give the L^2 and Schauder estimates of its inverse $\mathcal{D}^{-1}$. In Section 4, we first compare the linearized instanton equation on almost instantons with the operator $\mathcal{D}$ on linear models, then we use the implicit function theorem to perturb almost instantons to true instantons, thus proving our main theorem.

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2. Instantons of dimension 2 and 3

2.1. Review of symplectic geometry. Given any symplectic manifold (X, ω) of dimension $2n$, there exists a compatible metric g so that the equation

$$\omega(u, v) = g(Ju, v)$$

defines a Hermitian almost complex structure

$$J : T_X \to T_X,$$

that is, $J^2 = -id$ and $g(Ju, Jv) = g(u, v)$.

A **holomorphic curve**, or **instanton** of dimension 2, is a two dimensional submanifold Σ in X whose tangent bundle is preserved by J.

Equivalently, Σ is calibrated by ω, i.e. $\omega|_\Sigma = vol_\Sigma$. By **algebraic counting** of the number of instantons in X, one can define a highly nontrivial invariant for the symplectic structure on X, called the Gromov-Witten invariant.

When the instanton Σ has nontrivial boundary, then the corresponding boundary value problem would require $\partial\Sigma$ to lie on a **Lagrangian submanifold** L in X, i.e. $\dim L = n$ and $\omega|_L = 0$. Floer studied the intersection theory of Lagrangian submanifolds and defined the Floer homology group $HF(L, L')$ under certain assumptions.

Suppose that X is a compact Calabi-Yau manifold, i.e. the holonomy group of the Levi-Civita connection is inside $SU(n)$; equivalently, J is an integrable complex structure on X and there exists a holomorphic volume form $\Omega_X \in \Omega^{n,0}(X)$ on X satisfying

$$(-1)^{\frac{n(n-1)}{2}} (i/2)^n \, \Omega_X \wedge \bar{\Omega}_X = \omega^n/n!.$$

Under the mirror symmetry transformation, $HF(L, L')$ is expected to correspond to the Dolbeault cohomology group of coherent sheaves in the mirror Calabi-Yau manifold.

A Lagrangian submanifold L in X is called a **special Lagrangian submanifold** with phase zero (resp. $\pi/2$) if $\operatorname{Im}\Omega_X|_L = 0$ (resp. $\operatorname{Re}\Omega_X|_L = 0$). With suitable choice of orientation of L, L is calibrated by $\operatorname{Re}\Omega_X|_L$ (resp. $\operatorname{Im}\Omega_X|_L$), that is, $\operatorname{Re}\Omega_X|_L$ is the volume form of L. They play important roles in the Strominger-Yau-Zaslow mirror conjecture for Calabi-Yau manifolds [**35**].

When X is a Calabi-Yau threefold, there are conjectures of Vafa and others (e.g. [**3**][**14**]) that relate the (partially defined) open Gromov-Witten invariant of the number of instantons with Lagrangian boundary condition to the large N Chern-Simons invariants of knots in three manifolds.

2.2. Counting instantons in (almost) G_2-manifolds. Notice that a real linear homomorphism $J : \mathbb{R}^m \to \mathbb{R}^m$ being a Hermitian complex structure on $\mathbb{R}^m$ is equivalent to the following conditions: for any vector $v \in \mathbb{R}^m$, we have:

1) Jv is perpendicular to v.
2) $|Jv| = |v|$.

We can generalize J to the case involving more than one vector. We call a skew symmetric bilinear map

$$\times : \mathbb{R}^m \otimes \mathbb{R}^m \to \mathbb{R}^m$$

a (2-fold) **vector cross product** if it satisfies:

(i) $(u \times v)$ is perpendicular to both u and v.
(ii) $|u \times v| =$ Area of parallelogram spanned by u and $v = |u \wedge v|$.

The obvious example of this is the standard vector product on $\mathbb{R}^3$. By identifying $\mathbb{R}^3$ with $\operatorname{Im}\mathbb{H}$, the imaginary part of the quaternion numbers, we have

$$u \times v = \operatorname{Im} \bar{v}u.$$

The same formula defines a vector cross product on $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$, the imaginary part of the octonion numbers. Brown and Gray [**15**] showed that these two are the only possible vector cross product structures on $\mathbb{R}^m$ up to the automorphism of $\mathbb{R}^m$.

Suppose that M is a seven dimensional Riemannian manifold with a vector cross product $\times$ on each of its tangent spaces. The analog of the symplectic form is a degree three differential form Ω on M defined as follows:

$$\Omega(u, v, w) = g(u \times v, w).$$

Definition 2. Suppose that (M, g) is a Riemannian manifold of dimension seven with a vector cross product $\times$ on its tangent bundle. Then

1) M is called an **almost G_2-manifold** if $d\Omega = 0$.
2) M is called a **G_2-manifold** if $\nabla\Omega = 0$ with ∇ being the Levi-Civita connection.

Remark 3. M is a G_2-manifold if and only if its holonomy group is inside the exceptional Lie group $G_2 = Aut(\mathbb{O})$. The geometry of G_2-manifolds can be interpreted as the symplectic geometry on its knot space (see e.g. [**25**], [**29**]).

A typical family of examples of G_2-manifolds can be obtained via the product manifold $M := X \times S^1$ with (X, ω_X) being a Calabi-Yau threefold with a holomorphic volume form Ω_X, and the G_2-form is given by

$$\Omega = \operatorname{Re}\Omega_X + \omega_X \wedge d\theta.$$

Next we define the analogs of holomorphic curves and Lagrangian submanifolds in the G_2 setting.

Definition 4. Suppose that A is a three dimensional submanifold of an almost G_2-manifold M. We call A an **instanton** or **associative submanifold** if TA is preserved by the vector cross product $\times$.

Harvey and Lawson [**18**] showed that $A \subset M$ is an instanton if and only if A is calibrated by Ω, i.e. $\Omega|_A = vol_A$. This is in turn equivalent to $\tau|_{TA} = 0$ in our Lemma 7 for τ defined in (3) (i.e. Corollary 1.7 in Section IV.1.A. of [**18**] and Corollary 14 in [**25**]).

In M-theory, associative submanifolds are also called *M2-branes*. In the case when $M = X \times S^1$ with X a Calabi-Yau threefold, $\Sigma \times S^1$ (resp. $L \times \{p\}$) is an instanton in M if and only if Σ (resp. L) is a holomorphic curve (resp. special Lagrangian submanifold with zero phase) in X.

A natural interesting question is to count the number of instantons in M. In the special case of $M = X \times S^1$, these numbers are related to the conjectural invariants proposed by Joyce [**22**] by counting special Lagrangian submanifolds in Calabi-Yau threefolds, and the product of holomorphic curves with S^1. This problem has been discussed by many physicists. For example, Harvey and Moore discussed in [**19**] the mirror symmetry aspects of these invariants; Aganagic and Vafa in [**3**] related these invariants to the open Gromov-Witten invariants for local Calabi-Yau threefolds; Beasley and Witten argued in [**4**] that when there is a moduli of instantons, then one should count them using the Euler characteristic of the moduli space. The compactness issues of the moduli of instantons are a very challenging problem because the bubbling-off phenomena of (3-dimensional) instantons have not been well understood. This makes it very difficult to define an honest invariant by counting instantons.

Analogous to the Floer homology for Lagrangian intersections, when an instanton A has a nontrivial boundary, $\partial A \neq \phi$, one should require it to lie inside a **brane** or a **coassociative submanifold** to make it a well-posed elliptic problem (see [**17**] for Fredholmness and index computation), i.e. submanifolds of maximal dimension in M where the restriction of Ω is zero. Branes are the analog of Lagrangian submanifolds in symplectic geometry.

Definition 5. Suppose that C is a submanifold of an almost G_2-manifold M. We call C a **coassociative submanifold** if

$$\Omega|_C = 0 \text{ and } \dim C = 4.$$

For example, when $M = X \times S^1$ with X a Calabi-Yau threefold, $H \times S^1$ (resp. $C \times \{p\}$) is a coassociative submanifold in M if and only if H (resp. C) is a special Lagrangian submanifold with phase $\pi/2$ (resp. complex surface) in X. In [**25**], J.H. Lee and the first author showed that the isotropic knot space $\hat{\mathcal{K}}_{S^1}X$ of X admits a natural holomorphic symplectic structure. Moreover, $\hat{\mathcal{K}}_{S^1}H$ (resp. $\hat{\mathcal{K}}_{S^1}C$) is a complex Lagrangian submanifold in $\hat{\mathcal{K}}_{S^1}X$ with respect to the complex structure J (resp. K).

Constructing special Lagrangian submanifolds with zero phase in X with boundaries lying on H (resp. C) corresponds to the Dirichlet (resp. Neumann) boundary value problem for minimizing volume among Lagrangian submanifolds as studied by Schoen, Wolfson ([**33**], [**34**]), and Butcher [**5**]. For a general G_2-manifold M, the natural boundary value for an instanton is a coassociative submanifold. Similar to the intersection theory of Lagrangian submanifolds in symplectic manifolds, we propose to study the following problem: Count the number of instantons in G_2-manifolds bounding two coassociative submanifolds.

The product of a coassociative submanifold with a two dimensional plane inside the eleven dimension spacetime $M \times \mathbb{R}^{3,1}$ is called a *D5-brane* in M-theory. Counting the number of M2-branes between two D5-branes has also been studied in the physics literature.

In general, this is a very difficult problem. For instance, counting S^1-invariant instantons in $M = X \times S^1$ is the open Gromov-Witten invariant. However, when the two coassociative submanifolds C and C' are *close* to each other, we can relate the number of thin instantons between them to the number of J-holomorphic curves in C (Theorem 1), and hence, by Taubes' work, to the Seiberg-Witten invariant of C.

2.3. Relationships to Seiberg-Witten invariants. To determine the number of instantons between nearby coassociative submanifolds, we first recall the deformation theory of compact coassociative submanifolds C inside any G_2-manifold M, as developed by McLean [**28**]. Given any normal vector field $n \in \Gamma\left(N_{C/M}\right)$, the interior product $\iota_n \Omega$ is naturally a self-dual two form on C because of $\Omega|_C = 0$. This gives a natural identification,

$$\begin{aligned} \Gamma\left(N_{C/M}\right) &\overset{\simeq}{\to} \Lambda^2_+\left(C\right) \\ n &\to \eta_0 = \iota_n \Omega. \end{aligned} \tag{1}$$

Furthermore, infinitesimal deformations of coassociative submanifolds are parameterized by *self-dual harmonic two forms* $\eta_0 \in H^2_+\left(C\right)$, and they are always unobstructed, i.e. any such forms with sufficiently small norm give actual deformations to nearby coassociative submanifolds (see section 4 of [**28**]). Notice that the zero set of η_0 is the intersection of C with a infinitesimally nearby coassociative submanifold C_t, that is,

$$\{\eta_0 = 0\} = \lim_{t \to 0}\left(C \cap C_t\right),$$

where $C = C_0$ and $\eta_0 = dC_t/dt|_{t=0}$.

Since

$$\eta_0 \wedge \eta_0 = \eta_0 \wedge *\eta_0 = \left|\eta_0\right|^2 * 1,$$

η_0 defines a natural symplectic structure on $C^{reg} := C \backslash \{\eta_0 = 0\}$. If we normalize η_0 to η,

$$\eta = \eta_0 / \left|\eta_0\right|,$$

then the equation

$$\eta\left(u, v\right) = g\left(Ju, v\right)$$

defines a Hermitian almost complex structure J on C^{reg}. The J is determined by η, which in turn is determined by n, so we denote it by J_n. More explicitly, for $u \in TC^{reg}$,

$$J_n\left(u\right) = \left|n\right|^{-1} n \times u. \tag{2}$$

The next proposition says that when two coassociative submanifolds C and C' come together, not necessarily disjoint, then the limit of instantons bounding them will be a J_n-holomorphic curve Σ in C^{reg} with boundary $C \cap C'$.

Proposition 6. *Let M be a G_2-manifold. Suppose that for some $\varepsilon_0 > 0$, there is a smooth map*

$$\psi : C \times [0, \varepsilon_0] \longrightarrow M$$

such that for each $t \in [0, \varepsilon_0]$, $\psi_t(\cdot) := \psi(\cdot, t)$ is a smooth immersion of C into M as a coassociative submanifold $C_t := \psi(C \times \{t\})$. Suppose that $n = dC_t/dt|_{t=0} \in \Gamma\left(N_{C/M}\right)$ is nowhere vanishing, and

$$\phi_t : \Sigma \times [0, t] \longrightarrow M$$

is a smooth family of instantons in M such that for each $t \in (0, \varepsilon_0]$, ϕ_t is an associative immersion with boundary condition

$$\phi_t(\Sigma \times \{0\}) \subset C_0 := \psi(C \times \{0\}),\ \phi_t(\Sigma \times \{t\}) \subset C_t := \psi(C \times \{t\}).$$

Suppose that the C^1-limit of $\phi_t(\Sigma \times \{0\})$ exists as $t \to 0$. Then $\Sigma_0 := \lim_{t\to 0} \phi_t(\Sigma \times \{0\})$ is a J_n-holomorphic curve in C_0.

Proof. Let us denote the boundary component of $A_t = \text{Image}(\phi_t)$ in C_0 by Σ_t, i.e. $\Sigma_t := \phi_t(\Sigma \times \{0\})$. Let w_t be a unit normal vector field for Σ_t in A_t. We claim that w_t is *perpendicular to* C_0. To see this, note that TA_t being preserved by the vector cross product implies that

$$w_t = u \times v$$

for some tangent vectors u and v in Σ_t. In fact, for any unit vector $u \in T\Sigma_t$, $v := w_t \times u \in T\Sigma_t$ by associativity condition of A_t and $v \perp w_t$, and

$$\begin{aligned} u \times v &= -u \times (u \times w_t) \\ &= \tau(u, u, w_t) + g(u, w_t)\, u + g(u, u)\, w_t \\ &= 0 + 0 + w_t = w_t, \end{aligned}$$

where we have used in the second line the definition of τ in (3) and τ is a (vector-valued) form. Therefore, given any tangent vector w along C_0, we have

$$g(w_t, w) = g(u \times v, w) = \Omega(u, v, w) = 0,$$

where the last equality follows from C_0 being coassociative and $\Sigma_t \subset C_0$. Reparameterize $\phi_t : \Sigma \times [0, \varepsilon] \to M$ when necessary, then for small t we can assume that $\frac{d}{ds}\phi_t(z, s)\big|_{s=0}$ is parallel to $w_t(z)$ for any $z \in \Sigma$. Noting that

$$\phi_t(z, 0) \subset C_0 \text{ and } \phi_t(z, t) \subset C_t,$$

we see $\lim_{t\to 0} w_t(z)$ is parallel to

$$\left(\left.\frac{dC_t}{dt}\right|_{t=0}\right)\Big|_{\Sigma_0} = n \in \Gamma\left(\Sigma_0, N_{C_0/M}\right).$$

Therefore, along Σ_0,

$$\lim_{t\to 0} w_t(z) = |n(z)|^{-1} n(z).$$

For any $u_t \in T\Sigma_t$, $w_t \times u_t \perp w_t$ in associative A_t, so

$$w_t \times u_t \in T\Sigma_t.$$

Since $\Sigma_t \to \Sigma_0$ in C^1 topology, for any $u \in T\Sigma_0$, u can be realized as the limit of u_t. Therefore

$$J_n(u) = |n|^{-1} n \times u = \lim_{t\to 0}(w_t \times u_t) \in \lim_{t\to 0} T\Sigma_t = T\Sigma_0,$$

i.e. Σ_0 is a J_n-holomorphic curve in C_0 with respect to the almost complex structure J_n defined in (2). q.e.d.

The reverse of the above proposition is also true (Theorem 27). The Lagrangian analog of it was proven by Fukaya and Oh in [**10**]. On the other hand, by the celebrated work of Taubes, we expect that the number (counted with algebraic weights) of such holomorphic curves in C_0 equals the Seiberg-Witten invariant of C_0. We conjecture the following statement.

Conjecture: Suppose that C and C' are nearby coassociative submanifolds in a G_2-manifold M. Then the number of instantons counted with algebraic weights in M with small volume and with boundary lying on $C \cup C'$ is given by the Seiberg-Witten invariants of C.

The main result of our paper is to solve a special case of the above conjecture; namely, we will concentrate on the case that C and C' are both **compact** and they do NOT intersect. (In the remainder of the paper, we will always assume C and C' are compact and they do NOT intersect.) The basic ideas are (i) the limit of such instantons is a J-holomorphic curve for almost complex structure J compatible to the (degenerated) symplectic form η on C coming from its deformations as coassociative submanifolds and this process can be reversed; (ii) the number of J-holomorphic curves in the four manifold C should be related to the Seiberg-Witten invariant of C by the work of Taubes ([**37**], [**38**]). Note that one only gets one symplectic form η (and hence one almost complex structure J) from a given coassociative deformation of C, though of course one can get more (from different coassociative deformations).

Suppose that η is a self-dual two form on C with constant length $\sqrt{2}$; in particular, it is a (non-degenerate) symplectic form, and Σ is a smooth holomorphic curve in C, possibly disconnected. If Σ is **regular** in the sense that the linearized Cauchy-Riemann operator $D_\Sigma \bar{\partial}_J$ has trivial cokernel [**36**], then Taubes showed that the perturbed Seiberg-Witten equations,

$$\begin{aligned} F_a^+ &= q(\psi) - r\sqrt{-1}\eta, \\ D_{A(a)}\psi &= 0, \end{aligned}$$

have solutions for all sufficiently large r. Here a is a connection on the complex line bundle E over C whose first Chern class equals the Poincaré dual of Σ, $PD[\Sigma]$, F_a is the curvature 2-form of E and F_a^+ is the projection of F_a to $\wedge_+^2(C)$, ψ is a section of the twisted spinor bundle $S_+ = E \oplus (K^{-1} \otimes E)$ and $D_{A(a)}$ is the twisted Dirac operator, and $q(\cdot)$ is a certain canonical quadratic map from S_+ to $i \cdot \wedge_+^2(C)$. The number of such solutions (counted with algebraic weights) is the Seiberg-Witten invariant $SW_C(\Sigma)$ of C. Furthermore, the converse is also true; namely, the Seiberg-Witten invariant $SW_C(\Sigma)$ is equal to the Gromov-Witten invariants counting holomorphic curves Σ. Thus Taubes established an equivalence between Seiberg-Witten theory and Gromov-Witten theory for symplectic four manifolds. This result has far reaching applications in four dimensional symplectic geometry.

For a general four manifold C with nonzero $b^+(C)$, using a generic metric, any self-dual two form η on C defines a degenerate symplectic form on C, i.e. η is a symplectic form on the complement of $\{\eta = 0\}$, which is a finite union of circles (see [**12**][**21**]). Therefore, one might expect to have a relationship between the Seiberg-Witten invariants of C and the number of holomorphic curves with boundaries $\{\eta = 0\}$ in C. Part of this Taubes' program has been verified in [**37**], [**38**].

Suppose that η is a nowhere vanishing self-dual harmonic two form on a coassociative submanifold C in a G_2-manifold M. For any holomorphic curve Σ in C, we want to construct an instanton in M bounding C and C', where C' is a small deformation of the coassociative submanifold C along the normal direction given by η. Notice that C and C' do not intersect. We will construct such an instanton using a perturbation argument which requires a lower bound on the first eigenvalue for the appropriate elliptic operator. Recall that the deformation of an instanton is governed by a twisted Dirac operator. We will reinterpret it as a complexified version of the Cauchy-Riemann operator in Section 3.1.

2.4. Deformation of instantons. To construct an instanton A in M from a holomorphic curve Σ in C, we need to perturb an almost instanton A$'$ to a honest one using a quantitative version of the implicit function theorem. Let us first recall the deformation theory of instantons A ([**18**] and [**25**]) in a Riemannian manifold (M, g) with a parallel (or closed) r-fold vector cross product

$$\times : \Lambda^r TM \to TM.$$

In our situation, we have $r = 2$. By taking the wedge product with TM, we obtain a homomorphism τ,

$$\tau : \Lambda^{r+1}TM \to \Lambda^2 TM \cong \Lambda^2 T^*M,$$

where the last isomorphism is induced from the Riemannian metric. As a matter of fact, the image of τ lies inside the subbundle $\mathfrak{g}_M^{\perp}$ which is the orthogonal complement of $\mathfrak{g}_M \subset \mathfrak{so}\,(TM) \cong \Lambda^2 T^*M$, the bundle of infinitesimal isometries of TM preserving $\times$. That is,

$$\tau \in \Omega^{r+1}\left(M, \mathfrak{g}_M^{\perp}\right).$$

Lemma 7. ([**18**], [**25**]) *An $r+1$ dimensional submanifold $A \subset M$ is an instanton, i.e. TA is preserved by $\times$, if and only if*

$$\tau|_A = 0 \in \Omega^{r+1}\left(A, \mathfrak{g}_M^{\perp}\right).$$

This lemma is important in describing deformations of an instanton. Mclean [**28**] used this to show that the normal bundle to an instanton A is a twisted spinor bundle over A and infinitesimal deformations of A are parameterized by twisted harmonic spinors.

In our present situation, (M, g) is a G_2-manifold. Using the cross product, we can identify $\mathfrak{g}_M^{\perp} \subset \Lambda^2 T^*M \cong \Lambda^2 TM$ with the tangent bundle TM, i.e. for $u \wedge v \in \Lambda^2 TM$, we identify it with $w \in TM$ that $w = u \times v$. Then we can also characterize $\tau \in \Omega^3(M, TM)$ by the following formula:

$$(*\Omega)(u, v, w, z) = g(\tau(u, v, w), z).$$

More explicitly,

$$\tau(u, v, w) = -u \times (v \times w) - g(u, v)\, w + g(u, w)\, v. \tag{3}$$

Therefore $\mathrm{A} \subset M$ is an instanton if and only if $*_{\mathrm{A}}(\tau|_{\mathrm{A}}) = 0 \in T_M|_{\mathrm{A}}$.

Example. The G_2 manifold $\mathbb{R}^7$:

$$\mathbb{R}^7 \simeq \operatorname{Im}\mathbb{O} \simeq \operatorname{Im}\mathbb{H} \oplus \mathbb{H} = \{(x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, x_4 + x_5\mathbf{i} + x_6\mathbf{j} + x_7\mathbf{k})\},$$

the standard basis consists of $e_i = \frac{\partial}{\partial x_i}$ $(i = 1, 2, \ldots 7)$, and the multiplication $\times$ for $(a, b), (c, d) \in \operatorname{Im}\mathbb{H} \oplus \mathbb{H} \simeq \operatorname{Im}\mathbb{O}$ is

$$(a, b) \times (c, d) = (ac - d^*b, da + bc^*) \tag{4}$$

(Cayley–Dickson construction), where z^* denotes the conjugate of the quaternion z. The G_2 form Ω is

$$\Omega = \omega^{123} - \omega^{167} - \omega^{527} - \omega^{563} - \omega^{154} - \omega^{264} - \omega^{374},$$

and the form τ is the following ((5.4) in [**28**]):

$$\tau = \left(\omega^{256} - \omega^{247} + \omega^{346} - \omega^{357}\right)\frac{\partial}{\partial x_1} + \left(\omega^{156} - \omega^{147} - \omega^{345} + \omega^{367}\right)\frac{\partial}{\partial x_2}$$
$$+ \left(\omega^{245} - \omega^{267} - \omega^{146} - \omega^{157}\right)\frac{\partial}{\partial x_3} + \left(\omega^{567} - \omega^{127} + \omega^{136} - \omega^{235}\right)\frac{\partial}{\partial x_4}$$
$$+ \left(\omega^{126} - \omega^{467} + \omega^{137} + \omega^{234}\right)\frac{\partial}{\partial x_5} + \left(\omega^{457} - \omega^{125} - \omega^{134} + \omega^{237}\right)\frac{\partial}{\partial x_6}$$
$$+ \left(\omega^{124} - \omega^{456} - \omega^{135} - \omega^{236}\right)\frac{\partial}{\partial x_7}, \tag{5}$$

where $\omega^{ijk} = dx_i \wedge dx_j \wedge dx_k$. $\operatorname{Im}\mathbb{H} \oplus \{0\}$ is associative (i.e. an instanton), and $\{0\} \oplus \mathbb{H}$ is coassociative.

As a matter of fact, if A is already close to being an instanton, then we only need the normal components of $*_{\mathtt{A}}(\tau|_{\mathtt{A}})$ to vanish.

Proposition 8. *There is a positive constant δ such that for any* 3-*plane A in $\left(\mathbb{R}^7, \Omega\right)$ with $|\tau|_A| < \delta$, A is an instanton if and only if $*_A(\tau|_A) \in T_A$.*

Proof. McLean observed (from formula (5.6) in [**28**]) that if A_t is a family of linear subspaces in $M \cong \mathbb{R}^7$ with A_0 an instanton, then

$$*_{\mathrm{A}_t}\left(\frac{d\tau|_{\mathrm{A}_t}}{dt}\right)\Bigg|_{t=0} \in N_{\mathrm{A}_0/M} \subset T_M|_{\mathrm{A}_0}.$$

We may assume that A is spanned by e_1, e_2 and $\tilde{e}_3 = e_3 + \sum_{a=4}^{7} t_a e_a$ for some small t_a's where e_i's are a standard basis for $\mathbb{R}^7$, in particular $e_1 \times e_2 = e_3$. This is because the natural action of G_2 on the Grassmannian $Gr(2,7)$ is transitive. An easy computation (cf. equation (5.4) in [**28**]) shows that the normal component of $*(\tau|_{\mathrm{A}})$ in $N_{\mathrm{A}/M}$ is given by

$$*(\tau|_{\mathrm{A}})^{\perp} = -t_5(e_4)^{\perp} + t_4(e_5)^{\perp} + t_7(e_6)^{\perp} - t_6(e_7)^{\perp},$$

where $(\cdot)^{\perp}$ denote the orthogonal projection onto $N_{\mathrm{A}/M}$. When t_a's are all zero, we have $(e_a)^{\perp} = e_a$ for $4 \le a \le 7$. In particular, they are linearly independent when t_a's are small. In that case, $*(\tau|_{\mathrm{A}})^{\perp} = 0$ will actually imply that $t_a = 0$ for all a, i.e. A is an instanton in M. Hence the proposition. q.e.d.

This proposition will be needed later when we perturb an almost instanton to an honest one. We also need to identify the normal bundle $N_{\mathrm{A}/M}$ to an instanton A with a twisted spinor bundle over A as following [**28**]: We denote P to be the $SO(4)$-frame bundle of $N_{\mathrm{A}/M}$. Using the identification

$$a : SO(4) = (Sp(1) \times Sp(1)) / \pm(1,1) \to SO(\mathbb{H}),$$
$$(p,q)\cdot y = py\bar{q}, \text{ with } p,q \in Sp(1) \text{ and } y \in \mathbb{H}.$$

Mclean [**28**] showed that the normal bundle $N_{\mathtt{A}/M}$ can be identified as an associated bundle to P for the representation $SO(4) \to SO(\mathbb{H})$ given by $(p,q) \cdot y = py\bar{q}$. The spinor bundle $\mathbb{S}$ of $\mathtt{A}$ is associated to P for the representation $s : SO(4) \to SO(\mathbb{H})$ given by $(p,q) \cdot y = y\bar{q}$. Let E be the associated bundle to P for the representation $e : SO(4) \to SO(\mathbb{H})$ given by $(p,q) \cdot y = py$. Then because the 3 representations a, s, and e have the relation

$$a = s \circ e,$$

we obtain

$$N_{\mathrm{A}/M} \cong \mathbb{S} \otimes_{\mathbb{H}} E.$$

An alternative proof of $N_{\mathtt{A}/M} \cong \mathbb{S} \otimes_{\mathbb{H}} E$ using explicit frame identification is contained in the proof of the next theorem.

We re-derive McLean's theorem on deformation of associative submanifolds A in a G_2 manifold M. The original proof (Theorem 5.2 in [**28**]) is not quite precise: the associative form $\tau \in \Omega^3(M, TM)$ is *vector-valued* rather than a usual differential form, so the pull back operation and Cartan formula need to be clarified. The key is to define a suitable notion of pull back for vector-valued forms. Our calculation is flexible and can be extended to almost associative submanifolds in later sections. Other proofs were given in [**2**] and [**16**].

Theorem 9 (McLean)**.** *Under the correspondence of normal vector fields with twisted spinors, the Zariski tangent space to associative sub-manifolds at an associative sub-manifold A is the space of harmonic twisted spinors on A, that is, the kernel of the twisted Dirac operator.*

Proof. For any section V of $N_{A/M}$ with C^0 norm smaller than the injectivity radius δ_0 of M, we define a nonlinear map

$$F : \Gamma\left(N_{A/M}\right) \to \Omega^3\left(A, i^*TM\right),$$

$$F(V) = T_V \circ (\exp V)^* \tau, \tag{6}$$

where for the embedding $\exp V : A \to M$, $(\exp V)^*$ pulls back the differential form part of the tensor τ, and $T_V : T_{\exp_p(tV)}M \to T_pM$ pulls back the vector part of the tensor τ by parallel transport along the geodesic $\exp_p(tV)$. There is an ambiguity of the form part and vector part of tensor τ up to a scalar function-valued matrix transform Θ and Θ^{-1} respectively, but by the linearity of T_V and $(\exp V)^*$ on scalar function factors, one can easily show the definition of F is independent on such Θ, so F is well-defined. We make F more explicit by using a good frame. At a point $p \in A$, we pick two orthonormal vectors $\{W_1, W_2\}$ in T_pA, then *with respect to the induced connection* ∇^A on A, we parallel transport $\{W_1, W_2, W_3 = W_1 \times W_2\}$ from p to a neighborhood B in A along geodesic rays from p. From the construction we see

$$\nabla^A_{W_i} W_j(p) = 0, \text{ for } 1 \le i, j \le 3. \tag{7}$$

Then at any $q \in B \subset A$, we have orthonormal basis

$$\{W_1(q), W_2(q), W_3(q) = W_1(q) \times W_2(q)\}$$

spanning T_qA. (This uses that $W_1(q) \times W_2(q)$ and the parallel transported $W_3(p)$ are both orthogonal to $W_1(q)$ and $W_2(q)$ in 3 dimensional A). We further choose a smooth normal unit vector field W_4 on B in M as follows: we choose a $W_4 \in N_{A/M}(p)$, then use the parallel transport of $N_{A/M}$ *with respect to the induced connection* $\nabla^{\perp}$ in $N_{A/M}$ defined as $\nabla^{\perp} = \perp\nabla$ from the metric on M, where $\perp : TM \to N_{A/M}$ is the natural projection. Thus on B,

$$W_4 \perp \{W_1, W_2, W_1 \times W_2\}.$$

Using Lemma A.15 in [**18**] (Cayley–Dickson construction), we can uniquely extend $\{W_\alpha(q)\}_{1,2,3,4}$ to basis $\{W_\alpha(q)\}_{\alpha=1,2,\ldots,7}$ of T_qM, such that

$$W_{i+3}(q) = W_i(q) \times W_4(q)$$

for $i = 1, 2, 3$, and the correspondence

$$T_qM \ni W_\alpha \overset{i}{\leftrightarrow} e_\alpha \in \operatorname{Im}\mathbb{O}, \quad \alpha = 1, 2, \ldots, 7 \tag{8}$$

preserves the inner product $\cdot$ and cross product $\times$, where $\{e_\alpha\}_{\alpha=1,2,\ldots,7}$ is the standard basis of $\mathbb{R}^7 \simeq \operatorname{Im}\mathbb{O}$ defined in the previous example. So locally $N_{A/M}$ is trivialized as $B \times \mathbb{H}$, and at any $q \in B$, by the above basis W_α we have algebra isomorphism

$$T_qM = T_qA \oplus N_{A/M}(q) \overset{i}{\simeq} \operatorname{Im}\mathbb{H} \oplus \mathbb{H} \simeq \operatorname{Im}\mathbb{O}$$

that smoothly depends on $q \in B$. Hence $N_{A/M}$ is a quaternion valued bundle over A isomorphic to $\mathbb{S} \otimes_{\mathbb{H}} E$. From our construction we also have

$$\nabla^{\perp}_{W_i} W_k(p) = 0 \text{ for } i = 1, 2, 3 \text{ and } k = 4, 5, 6, 7, \tag{9}$$

because $\nabla^{\perp}_{W_i} W_4(p) = 0$ by construction of W_4, and for $k = 5, 6, 7$, say $k = 5$,

$$\begin{aligned}
\nabla^{\perp}_{W_i} W_5(p) &= \perp(\nabla_{W_i}(W_1 \times W_4))(p) \\
&= \perp(\nabla_{W_i} W_1 \times W_4 + W_1 \times \nabla_{W_i} W_4)(p) \\
&= \nabla^{A}_{W_i} W_1(p) \times W_4 + W_1 \times \nabla^{\perp}_{W_i} W_4(p) = 0,
\end{aligned} \tag{10}$$

where the second row is because $N_{A/M}(p) \times W_4 \subset T_pA$ (for $\operatorname{Im}\mathbb{O} \simeq \operatorname{Im}\mathbb{H} \oplus \mathbb{H}$, $(0, \mathbb{H}) \times (0, 1) \subset (\operatorname{Im}\mathbb{H}, 0)$ by (4)) and $W_1 \times T_pA \subset T_pA$ by associative condition. We remark that the parallel transport in $N_{A/M}$ w.r.t $\nabla^{\perp}$ is an *isometry*, for if $\nabla^{\perp}_T W = 0$ for section W in $N_{A/M}$, then

$$\nabla_T \langle W, W \rangle = 2 \langle \nabla_T W, W \rangle = 2 \left\langle \nabla^{\perp}_T W, W \right\rangle = 0. \tag{11}$$

Next, for each $q \in B$, we *parallel transport* the frame $\{W_\alpha (q)\}_{\alpha=1,2,\dots,7}$ along geodesical rays emanating from q in M in $N_{A/M}(q)$ directions up to length δ_0. This extends the frame to a tubular neighborhood of A in M. Then $\nabla_V W_\alpha (q) = 0$ for any $V \in \Gamma\left(N_{A/M}\right)$. If we write

$$\tau = \omega^\alpha \otimes W_\alpha \qquad (\alpha = 1, 2, \dots, 7)$$

following Einstein's summation convention, then

$$F(V)(q) = (\exp V)^* \omega^\alpha (q) \otimes T_V W_\alpha \left(\exp_q V\right).$$

We have

$$\begin{aligned} F'(0)V &= \left.\frac{d}{dt}\right|_{t=0} F(tV) \\ &= \left.\frac{d}{dt}\right|_{t=0} \left[(\exp tV)^* \omega^\alpha \otimes T_{tV} W_\alpha\right] \\ &= L_V \omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha \\ &= d\left(i_V \omega^\alpha\right) \otimes W_\alpha + i_V d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha. \end{aligned} \tag{12}$$

Since $\nabla\tau = 0$ and $\nabla_V W_\alpha (q) = 0$ by the parallel property of τ and W_α, we have

$$0 = \nabla_V \tau (q) = \nabla_V \omega^\alpha \otimes W_\alpha (q) + \omega^\alpha \otimes \nabla_V W_\alpha (q) = \nabla_V \omega^\alpha (q) \otimes W_\alpha (q),$$

and so $\nabla_V \omega^\alpha (q) = 0$. Since $\nabla = d + A$ and in normal coordinates the connection 1-form A vanishes at q along the fiber direction of $N_{A/M}$, we have $i_V d\omega^\alpha (q) = 0$. Therefore, by (12), at q we have

$$F'(0)V(q) = d\left(i_V \omega^\alpha\right) \otimes W_\alpha (q). \tag{13}$$

By our choice of W_a, the $\tau = \omega^a \otimes W_\alpha$ at q is the standard form (5), and by the parallel property of the cross product $\times$ and τ, in the neighborhood of q in M the τ is also of standard form as (5), in the sense that the coordinate vector $\frac{\partial}{\partial \omega^i}$ is replaced by the frame W_i, and $\omega^{ijk} = dx^i \wedge dx^j \wedge dx^k$ is replaced by $W_i^* \wedge W_j^* \wedge W_k^*$ where W_α^* is the dual vector of W_α. Namely,

$$\tau = \left(W_2^* \wedge W_5^* \wedge W_6^* - W_2^* \wedge W_4^* \wedge W_7^* + \cdots\right) \otimes W_1 + \text{similar terms}.$$

This is because $\nabla_V \left(\tau\left(W_i, W_j, W_k\right)\right) = 0$ for $V \in \Gamma\left(N_{A/M}\right)$, by $\nabla\tau = 0$ and the parallel property of $\{W_\alpha\}_{\alpha=1,2,\dots,7}$ in $N_{A/M}$ directions. Therefore from (13), similar to the way of deriving (5.6) in [**28**], and noting $dW_i^*(p) = 0$ $(i = 1, 2, \dots 7)$ from (7) and (9), we get

$$\begin{aligned} F'(0)V(p) &= d\left(i_V \omega^\alpha\right) \otimes W_\alpha (p) \\ &= \mathcal{D}V(p) \otimes W_1^* \wedge W_2^* \wedge W_3^* (p) \\ &= \mathcal{D}V(p) \otimes dvol_A (p), \end{aligned} \tag{14}$$

where for $V = V^4W_4 + V^5W_5 + V^6W_6 + V^7W_7$,

$$\mathcal{D}V\,(p) = -\left(V_1^5 + V_2^6 + V_3^7\right)W_4 + \left(V_1^4 + V_3^6 - V_2^7\right)W_5 \\ + \left(V_2^4 - V_3^5 + V_1^7\right)W_6 + \left(V_3^4 + V_2^5 - V_1^6\right)W_7$$

is the twisted Dirac operator (5.2) in [**28**] and also (17) below, with $V_i^k = dV^k\,(W_i)$, and $dvol_A = W_1^* \wedge W_2^* \wedge W_3^*$ is the induced volume form on $A \subset M$ since $\{W_\alpha\}_{\alpha=1,2,3}$ is the orthonormal basis of TA. Since $p \in A$ and the section V are arbitrary, we have

$$F'\,(0) = \mathcal{D} \otimes dvol_A. \tag{15}$$

Note that both sides of the above identity are independent on the choice of the frame $\{W_\alpha\}_{\alpha=1,2,\ldots,7}$. If the normal vector field V is induced from deformation of associative submanifolds $\{A_t\}_{0\leq t\leq\varepsilon_0}$, i.e.

$$A_t = \exp U\,(t)\cdot A \quad \text{for } U\,(t) \in \Gamma\left(N_{A/M}\right) \text{ and } U'\,(0) = V,$$

then differentiating $F\,(U\,(t)) = 0$ at $t = 0$ and using (15) we get $\mathcal{D}V = 0$ on A, namely V is a harmonic twisted spinor. q.e.d.

Remark 10. 1) The vector field V is only defined on $A \subset M$, but along the geodesic rays from A in the fiber directions of $N_{A/M}$, one can extend V to an open neighborhood of A by parallel transport; therefore the Lie derivative $L_V\omega$ for any 3-form ω on M makes sense in this neighborhood. However, the Lie derivative of ω restricted on A, namely $i_A^*\,(L_V\omega)$ for the inclusion $i_A : A \hookrightarrow M$, is actually independent of the extension of V, as mentioned in [**28**]. One can see this from the Cartan formula

$$L_V\omega = d\,(i_V\omega) + i_V d\omega$$

as follows: On A, the second term $i_V d\omega$ is independent on the extension of V. For the first term $d\,(i_V\omega)$, since we restrict it on $A \subset M$, one can directly check that this term only involves the derivatives of the section V and ω in tangent directions of A, for any derivative in normal direction will contribute a covector not in T^*A and make the corresponding summand in $i_A^*\,(L_V\omega)$ vanish. So $i_A^*\,(L_V\omega)$ is independent on the extension of V and ω to M.

2) In the preceding proof, the normal vector field V is used only to ensure that $\exp V : A \to M$ is an embedding and give $d\,(i_V\omega^\alpha) \otimes W_\alpha$ a twisted Dirac operator interpretation on the normal bundle $N_{A/M}$. Actually, to write down the linearization of F, one only needs the normal bundle of A in M in differential topology sense (namely at any $p \in A$, $N_{A/M}\,(p) + T_pA = T_pM$, but not necessarily $N_{A/M}\,(p) \perp T_pA$). If we denote the differential topological normal bundle by $N_{A/M}^{top}$, then for section $V \in \Gamma\left(N_{A/M}^{top}\right)$ with small C^0 norm, $\exp V : A \to M$ is an embedding so we can define the linearization of F as before.

3) The linearization formula

$$F'(0)\,V = d\,(i_V\omega^\alpha)\otimes W_\alpha + i_V d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha \tag{16}$$

holds for any section $V \in \Gamma\left(N^{top}_{A/M}\right)$, and any tangent frame $\{W_a\}_{i=1,2,\ldots 7}$ along any submanifold $A \subset M$ (not necessarily associative). The term $d\,(i_V\omega^\alpha)\otimes W_\alpha$ is the principal symbol term of the first order linear differential operator $F'(0)$ and behaves functorially under the diffeomorphism between two manifolds. To get $F'(0)\,V\,(p) = d\,(i_V\omega^\alpha)\otimes W_\alpha\,(p)$, one only needs to choose the frame $\{W_\alpha\}_{\alpha=1,\ldots,7}$ that is parallel along the curves $\exp_p(tv)$ for all $p\in A$ and $v \in N^{top}_{A/M}(p)$.

4) On an almost instanton A, its normal bundle is not closed under $\times$ by TA in general, so the linearized instanton equation $F'(0)\,V$ can NOT be interpreted as a twisted Dirac operator. For this reason, we will seldom use the twisted Dirac operator, but mainly (16) to do computations and estimates of $F'(0)\,V$, with the aid of good local frames $\{W_a\}_{i=1,2,\ldots 7}$.

5) At a point p in a general 3-manifold $A \subset M$ with volume form $dvol_A$, we can write the linearization

$$F'(0)\,V\,(p) = G\circ V\,(p)\otimes dvol_A\,(p)\,,$$

where G is a first order linear differential operator, whose principal symbol depends on the algebraic relation of the frame $\{W_a\}$ under the products $\times$ and $\cdot$, and the volume form on A. Besides the associative $\{W_a\}_{i=1,2,3}$+coassociative $\{W_a\}_{i=4,5,6,7}$ type frame, there may be some other type of frames leading to a meaningful differential operator G.

By McLean's theorem, the normal bundle to any instanton A is a twisted spinor bundle $\mathbb{S}$ over A corresponding to the representation $SO(4)\to SO(\mathbb{H})$ given by $(p,q)\cdot y = py\bar{q}$. Let $\mathcal{D}$ be the twisted Dirac operator on A. We want to write it down explicitly in local coordinates near p. We let $\{W_\alpha\}_{\alpha=1}^{7}$ be a local orthonormal frame constructed as above. Suppose

$$V = V^4W_4 + V^5W_5 + V^6W_6 + V^7W_7$$

is a normal vector field to A and we write the covariant differentiation of V as $\nabla(V) := V_i^\alpha W_\alpha \otimes \omega^i$ with $\{\omega^i\}$ being the co-frame dual to $\{W_i\}$; then the twisted Dirac operator is

$$\mathcal{D} = W_1\times\nabla_1 + W_2\times\nabla_2 + W_3\times\nabla_3, \tag{17}$$

where $\nabla_i := \nabla^\perp_{W_i}$ $(i=1,2,3)$. For instance, when the G_2 manifold is $\operatorname{Im}\mathbb{O}\simeq \operatorname{Im}\mathbb{H}\oplus\mathbb{H}$ and the instanton is $\operatorname{Im}\mathbb{H}$, then by viewing V as a $\mathbb{H}$ valued function, $V = V^4 + \mathbf{i}V^5 + \mathbf{j}V^6 + \mathbf{k}V^7$, the twisted Dirac operator is $\mathcal{D} = \mathbf{i}\nabla_1 + \mathbf{j}\nabla_2 + \mathbf{k}\nabla_3$. Expression (17) can be easily shown to be independent on the choice of orthonormal basis $\{W_a\}_{i=1,2,3}$ of TA; thus $\mathcal{D}$ is globally defined on A. From (17) and (9) we have

$$\begin{aligned}\mathcal{D}V(p) &= (W_1 \times \nabla_1 + W_2 \times \nabla_2 + W_3 \times \nabla_3) \\ &\quad \left(V^4 W_4 + V^5 W_5 + V^6 W_6 + V^7 W_7\right) \\ &= -\left(V_1^5 + V_2^6 + V_3^7\right) W_4 + \left(V_1^4 + V_3^6 - V_2^7\right) W_5 \\ &\quad + \left(V_2^4 - V_3^5 + V_1^7\right) W_6 + \left(V_3^4 + V_2^5 - V_1^6\right) W_7. \end{aligned} \tag{18}$$

We remark that away from p the expression of $\mathcal{D}V$ may have more 0-th order terms in general.

3. Dirac Equation on Thin 3-manifolds

3.1. A simplified model. To motivate our analytical estimates for later sections, let us first consider a simplified model. Suppose that $\mathtt{A}_\varepsilon = [0, \varepsilon] \times \Sigma$ is a three manifold and (x_1, z) are coordinates on $\mathtt{A}_\varepsilon$. On $\mathtt{A}_\varepsilon$ we put a warped product metric $g_{\mathtt{A}_\varepsilon, h} = h(z)\, dx_1^2 + g_\Sigma$, where Σ is a Riemann surface with a background metric g_Σ and $h(z) > 0$ is a C^∞ function on Σ. Let e_1 be the unit tangent vector field on $\mathtt{A}_\varepsilon$ normal to Σ, namely along the x_1-direction and $e_1 = h^{-1/2}(z) \frac{\partial}{\partial x_1}$. We introduce a first order linear differential operator $\mathcal{D}$ that

$$\mathcal{D} = e_1 \cdot \nabla_1 + \bar{\partial} = e_1 \cdot h^{-1/2}(z) \frac{\partial}{\partial x_1} + \bar{\partial}, \tag{19}$$

where $\nabla_1 = \nabla_{e_1} = h^{-1/2}(z) \frac{\partial}{\partial x_1}$, and $\bar{\partial} = \left(\overline{\partial}, \overline{\partial}^*\right)$ is the Dirac operator on the Dolbeault complex $\Omega_{\mathbb{C}}^{0,*}(L)$ of Hermitian line bundle L over the Riemann surface Σ with a connection ∇ (cf. Proposition 3.67 of [**6**] or Proposition 1.4.25 of [**31**], by taking the Kahler manifold to be Σ and the Hermitian line bundle to be L). Here $\bar{\partial}$ acts on the spinor bundle $\mathbb{S}_\Sigma := \mathbb{S}^+ \oplus \mathbb{S}^-$ via the following identification:

$$\begin{array}{ccc} \Omega_{\mathbb{C}}^{0}(L) \oplus \Omega_{\mathbb{C}}^{0,1}(L) & \xrightarrow{(\overline{\partial}, \overline{\partial}^*)} & \Omega_{\mathbb{C}}^{0,1}(L) \oplus \Omega_{\mathbb{C}}^{0}(L) \\ \downarrow & & \downarrow \\ \mathbb{S}^+ \oplus \mathbb{S}^- & \xrightarrow{\bar{\partial}} & \mathbb{S}^- \oplus \mathbb{S}^+ \end{array} \tag{20}$$

where on the left $\mathbb{S}^+$ and $\mathbb{S}^-$ are identified to complex line bundles L and $L \otimes \Lambda_{\mathbb{C}}^{0,1}(\Sigma)$ respectively, $\overline{\partial} : \Omega_{\mathbb{C}}^{0}(L) \to \Omega_{\mathbb{C}}^{0,1}(L)$ is the Dolbeault operator, and $\overline{\partial}^* : \Omega_{\mathbb{C}}^{0,1}(L) \to \Omega_{\mathbb{C}}^{0}(L)$ is its formal adjoint operator. The Dolbeault operator $\overline{\partial}$ depends on the complex structures j on Σ, J on L, and the connection ∇ of L.

When $h(z)$ is a constant, from the spinor bundle $\mathbb{S}_\Sigma = \mathbb{S}^+ \oplus \mathbb{S}^-$ over Σ one can construct a spinor bundle $\mathbb{S}$ over the odd dimensional manifold $\mathrm{A}_\varepsilon := [0, \varepsilon] \times \Sigma$ by taking the Cartesian product of $\mathbb{S}_\Sigma$ with $[0, \varepsilon]$ (see Chapter 22 in [**8**]). When there is no confusion, we also write

$\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ where the $\mathbb{S}^\pm$ are the Cartesian products of the $\mathbb{S}^\pm$ of $\mathbb{S}_\Sigma$ with $[0,\varepsilon]$. [**8**] also constructs a Dirac operator on the spinor bundle $\mathbb{S} \to \mathrm{A}_\varepsilon$ right from the Dirac operator $\bar\partial$ on the spinor bundle $\mathbb{S}_\Sigma \to \Sigma$, and it is $\mathcal{D} = e_1 \cdot \nabla_1 + \bar\partial$ as in (19). For general $h(z)$, our $\mathcal{D}$ is a Dirac type operator but not necessarily a genuine Dirac operator.

Let's recall the Clifford multiplication of $T\mathrm{A}_\varepsilon$ on $\mathbb{S}$. Since $\mathbb{S}$ is the Cartesian product of $\mathbb{S}_\Sigma$ with $[0,\varepsilon]$, it is enough to define the Clifford multiplication $\cdot$ of e_1 on $\mathbb{S}_\Sigma$. Let σ be the volume element of the Clifford bundle $Cl(\Sigma)$, $\sigma^2 = 1$. Then $\mathbb{S}^+$ and $\mathbb{S}^-$ are the ± 1 eigenbundles of σ and $\mathbb{S}_\Sigma = \mathbb{S}^+ \oplus \mathbb{S}^-$. Since the Clifford multiplication $\cdot$ of e_1 on $\mathbb{S}_\Sigma$ satisfies $e_1^2 = -1$, the natural choice of the action $e_1 \cdot$ on $\mathbb{S}_\Sigma$ is to let $\mathbb{S}^+$ and $\mathbb{S}^-$ be the $\pm i$ eigenbundles; namely, for $(u,v) \in \mathbb{S}^+ \oplus \mathbb{S}^-$, $e_1 \cdot (u,v) = (iu, -iv)$. The connection of $\mathbb{S}$ along the x_1 direction is trivial.

Given $z_0 \in \Sigma$, in its neighborhood in Σ we can choose a complex coordinate $z = x_2 + ix_3 \in \mathbb{C}$ with $\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}$ orthonormal at z_0. We locally trivialize the spinor bundle $\mathbb{S}_\Sigma = \mathbb{S}^+ \oplus \mathbb{S}^- \to \Sigma$ as

$$\mathbb{H} = \mathbb{C} + \mathbb{C}\mathbf{j} = \mathbb{C}^{\mathbf{2}} \to \mathbb{C}.$$

We may choose the trivialization such that the Dirac operator

$$\left(\overline{\partial}, \overline{\partial}^*\right)\Big|_{z_0} = \left(\overline{\partial}_z, -\partial_z\right).$$

We may write the section V of $\mathbb{S} \to \mathrm{A}_\varepsilon$ as $V = (u,v) := u + v\mathbf{j} \in \mathbb{S}^+ \oplus \mathbb{S}^-$ with

$$u = V^4 + \mathbf{i}V^5 \in \mathbb{S}^+ \simeq \mathbb{C},$$
$$v = V^6 + \mathbf{i}V^7 \in \mathbb{S}^+ \simeq \mathbb{C}, \text{ and } \ v\mathbf{j} \in \mathbb{S}^+\mathbf{j} = \mathbb{S}^-.$$

With these understood, we may write (19) as

$$\mathcal{D}V = \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \left(h^{-1/2}(z) \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & -\mathbf{i}\overline{\partial}^* \\ \mathbf{i}\overline{\partial} & 0 \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix} \tag{21}$$

$$\overset{\text{at } z_0}{=} \begin{bmatrix} \mathbf{i} & 0 \\ 0 & -\mathbf{i} \end{bmatrix} \left(h^{-1/2}(z) \frac{\partial}{\partial x_1} + \begin{bmatrix} 0 & \mathbf{i}\partial_z \\ \mathbf{i}\overline{\partial}_z & 0 \end{bmatrix} \right) \begin{bmatrix} u \\ v \end{bmatrix}$$
$$= \left((\nabla_1 u + \mathbf{i}\partial_z v) + \left(\nabla_1 v + \mathbf{i}\overline{\partial}_z u\right) \cdot \mathbf{j} \right) \cdot \mathbf{i}$$
$$= (\mathbf{i}\nabla_1 u - \partial_z v) + \left(-\mathbf{i}\nabla_1 v + \overline{\partial}_z u\right) \cdot \mathbf{j}, \tag{22}$$

where at z_0 we have the identification

$$\overline{\partial} = \overline{\partial}_z = \frac{1}{2}(\nabla_2 + \mathbf{i}\nabla_3), \quad -\overline{\partial}^* = \partial_z = \frac{1}{2}(\nabla_2 - \mathbf{i}\nabla_3)$$

with $\nabla_i = \nabla_{\frac{\partial}{\partial x_i}}$ for $i = 2,3$ and $\nabla_1 = h^{-1/2}(z) \frac{\partial}{\partial x_1}$. We will also denote

$$-\mathbf{i}\overline{\partial}^* = \partial^+ \text{ and } \mathbf{i}\overline{\partial} = \partial^-$$

respectively. They satisfy $\partial^+ = (\partial^-)^*$.

This implies that the equation $\mathcal{D}V = 0$ is equivalent to the following equations, an analog of the Cauchy-Riemann equation,

$$\begin{aligned} \partial^- u + h^{-1/2}(z) \frac{\partial v}{\partial x_1} &= 0, \\ \partial^+ v + h^{-1/2}(z) \frac{\partial u}{\partial x_1} &= 0. \end{aligned} \tag{23}$$

We put the boundary condition

$$v|_{\partial \mathtt{A}_\varepsilon} = 0. \tag{24}$$

(The more precise formulation will be given in (25).) It is similar to the totally real boundary condition for J-holomorphic maps. When $h(z)$ is a constant, it is a special case of the theory of boundary value problems for Dirac operators developed in [**8**]. For general $h(z)$, the fact that the boundary value problem is Fredholm follows from our elliptic estimates in Subsection 3.3. Also see [**17**] for relevant discussion.

Later in Section 4, we will apply the above $\mathcal{D}$ to the case when $h(z) = |n(z)|^2$, where $n(z) = \frac{d}{dt} C_t\big|_{t=0}$ is the normal vector field on C_0 coming from the deformation of coassociative submanifolds C_t, and the Hermitian line bundle L is the normal bundle N_{Σ_0/C_0} of the J_n-holomorphic curve Σ_0 in a coassociative manifold C_0, whose connection ∇ is the normal connection from the induced metric of $\Sigma_0 \subset C_0$.

3.2. First eigenvalue estimates. In this subsection we will establish a quantitative estimate of the eigenvalue of the linearized operator for the simplified model $\mathtt{A}_\varepsilon = [0, \varepsilon] \times \Sigma$ with warped product metric $g_{\mathtt{A}_\varepsilon, h} = h(z)\, dx_1^2 + g_\Sigma$, where Σ is a **compact** Riemann surface, and $h(z)$ is a smooth function on Σ with $\frac{1}{K} \leq h \leq K$ for some constant $K > 0$. The volume form of this metric on $\mathtt{A}_\varepsilon$ is

$$dvol_{\mathtt{A}_\varepsilon} = h^{\frac{1}{2}}(z)\, dvol_\Sigma dx_1.$$

We introduce the following function spaces for spinors $V = (u, v)$ over $\mathtt{A}_\varepsilon$.

Definition 11. Let $\mathbb{S}$ be the spinor bundle over $(\mathtt{A}_\varepsilon, g_{\mathtt{A}_\varepsilon})$ and V be a smooth section of $\mathbb{S}$.

1) We define the norm

$$\|V\|_{L^{m,p}(\mathtt{A}_\varepsilon, \mathbb{S})} := \left(\sum_{\alpha+\beta \leq m} \int_0^\varepsilon \int_\Sigma \left| (\nabla_{x_1})^\alpha (\nabla_\Sigma)^\beta V \right|^p h^{\frac{1}{2}}(z)\, dvol_\Sigma dx_1 \right)^{1/p}$$

and

$$\|V\|_{C^m(\mathtt{A}_\varepsilon, \mathbb{S})} := \sum_{\alpha+\beta \leq m} \sup \left| (\nabla_{x_1})^\alpha (\nabla_\Sigma)^\beta V \right|$$

where ∇_{x_1} and ∇_Σ are the covariant differentiation along x_1-direction and Σ-directions (i.e. two tangent directions on Σ) respectively with respect to the metric $g_{\mathtt{A}_\varepsilon}$, and the L^p-norm is with respect to $g_{\mathtt{A}_\varepsilon}$ too. By the standard Sobolev embedding theorem, we have an ε independent constant C such that for any smooth section V,

$$\|V\|_{L^p(\mathtt{A}_\varepsilon,\mathbb{S})} \leq C\|V\|_{L^{m,q}(\mathtt{A}_\varepsilon,\mathbb{S})}, \text{ for } p \leq \frac{3q}{3-mq}$$
$$\|V\|_{C^m(\mathtt{A}_\varepsilon,\mathbb{S})} \leq C\|V\|_{L^{l,p}(\mathtt{A}_\varepsilon,\mathbb{S})}, \text{ for } p \geq \frac{3}{l-m}$$

as long as ε is bounded below and above by some universal constant. For a later purpose, let us fix $\varepsilon \in [1/2, 3/2]$.

2) We define the function spaces

$$L^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S}) := \left\{ V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S}) \mid \|V\|_{L^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S})} < +\infty \right\}$$

and $L_-^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S})$ (resp.$L_+^{m,p}(\mathtt{A}_\varepsilon,\mathbb{S})$) to be the closure (with respect to the norm $\|\cdot\|_{L^{m,p}(\mathtt{A}_\varepsilon)}$) of the subspace of smooth sections $V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S})$ such that $v \in C_0^\infty(\mathtt{A}_\varepsilon \backslash \partial \mathtt{A}_\varepsilon)$ (resp. $u \in C_0^\infty(\mathtt{A}_\varepsilon \backslash \partial \mathtt{A}_\varepsilon)$), where $C_0^\infty(\mathtt{A}_\varepsilon \backslash \partial \mathtt{A}_\varepsilon)$ denotes the space of smooth functions with compact support inside $\mathtt{A}_\varepsilon \backslash \partial \mathtt{A}_\varepsilon$. Let us also introduce the space

$$C^m(\mathtt{A}_\varepsilon,\mathbb{S}) := \left\{ V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S}) \mid \|V\|_{C^m(\mathtt{A}_\varepsilon,\mathbb{S})} < +\infty \right\},$$
$$C_-^m(\mathtt{A}_\varepsilon,\mathbb{S}) := \left\{ V = (u,v) \in \Gamma(\mathtt{A}_\varepsilon,\mathbb{S}) \mid \|V\|_{C^m(\mathtt{A}_\varepsilon,\mathbb{S})} < +\infty, v|_{\partial \mathtt{A}_\varepsilon} = 0 \right\}.$$

Let $\mathcal{D}$ be the operator (19) in our linear model. We will impose boundary conditions for sections V that $\mathcal{D}$ acts on. It is known (cf. [**8**] Theorem 21.5) that the Dirac operators

$$\mathcal{D}_\pm := \mathcal{D}|_{L_\pm^{1,2}} : L_\pm^{1,2}(\mathtt{A}_\varepsilon,\mathbb{S}) \to L^2(\mathtt{A}_\varepsilon,\mathbb{S}) \tag{25}$$

give well-posed *local elliptic boundary problems* and their formal adjoint operators are $\mathcal{D}_\pm^* = \mathcal{D}_\mp$. This boundary condition restricted on smooth sections $V = (u,v)$ in $L_-^{1,2}(\mathtt{A}_\varepsilon,\mathbb{S})$ means

$$v|_{\partial \mathrm{A}_\varepsilon} = 0, \text{ i.e. } v(0,\Sigma) = v(\varepsilon,\Sigma) = 0. \tag{26}$$

The following theorem compares the first eigenvalue for Dirac operator $\bar\partial$ on the Riemann surface Σ with the operator $\mathcal{D}$ on product three manifold $\mathtt{A}_\varepsilon$. The theorem refers to the L^2 metric which is defined as follows: Let U, V be two sections in $L^2(\mathtt{A}_\varepsilon,\mathbb{S})$. We let the inner product $\langle\cdot,\cdot\rangle_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}$ be

$$\langle U,V\rangle_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} := \int_{[0,\varepsilon]}\int_\Sigma \langle U,V\rangle(x_1,z)\, h^{\frac{1}{2}}(z)\, dvol_\Sigma dx_1,$$

where in the integral the $\langle\cdot,\cdot\rangle$ is the inner product in fibers of the spinor bundle $\mathbb{S}$. We let

$$\begin{aligned}|V(x_1,z)|^2 &:= \langle V(x_1,z), V(x_1,z)\rangle\\ \|V\|^2_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} &:= \langle V,V\rangle_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}.\end{aligned}$$

Theorem 12. *Suppose λ_{∂^-} is the first eigenvalue of the Laplacian $\Delta_\Sigma = \partial^+\partial^-$ acting on the space $L^{1,2}_-(\Sigma,\mathbb{S}^+)$ and λ_{∂^+} is the first eigenvalue of the Laplacian $\Delta_\Sigma = \partial^-\partial^+$ acting on the space $L^{1,2}_+(\Sigma,\mathbb{S}^-)$. Let*

$$\lambda_{\mathcal{D}_\pm} := \inf_{0\neq V\in L^{1,2}_\pm(\mathtt{A}_\varepsilon,\mathbb{S})} \frac{\|\mathcal{D}V\|^2_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}}{\|V\|^2_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}}.$$

(Notice the boundary conditions (25) *and* (26) *for V.) Then for the operator $\mathcal{D}_\pm : L^{1,2}_\pm(\mathtt{A}_\varepsilon,\mathbb{S}) \to L^2(\mathtt{A}_\varepsilon,\mathbb{S})$, we have*

$$\lambda_{\mathcal{D}_\pm} \geq \min \frac{1}{K}\left\{\lambda_{\partial^\pm}, \frac{2}{K\varepsilon^2} - K\left\|h^{-\frac{1}{2}}\right\|^2_{C^1(\Sigma)}\right\},$$

where $K>0$ is some constant such that $\frac{1}{K}\leq h(z)\leq K$ for all $z\in\Sigma$.

Proof. In the following we will assume the volume form $dvol_{\mathtt{A}_\varepsilon} = dvol_\Sigma dx_1$ to simplify calculation. General cases can be reduced to this case by observing

(27)
$$\frac{\|\mathcal{D}V\|^2_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}}{\|V\|^2_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}} \geq \frac{\int_{\mathtt{A}_\varepsilon} |DV|^2 K^{-1/2} dvol_\Sigma dx_1}{\int_{\mathtt{A}_\varepsilon} |V|^2 K^{1/2} dvol_\Sigma dx_1} = \frac{1}{K}\frac{\int_{\mathtt{A}_\varepsilon} |DV|^2 dvol_\Sigma dx_1}{\int_{\mathtt{A}_\varepsilon} |V|^2 dvol_\Sigma dx_1}$$

from the condition $\frac{1}{K}\leq h(z)\leq K$. It is enough to consider the case $\mathcal{D}_- : L^{1,2}_-(\mathtt{A}_\varepsilon,\mathbb{S}) \to L^2(\mathtt{A}_\varepsilon,\mathbb{S})$. The $\mathcal{D}_+$ case is similar. For any $V=(u,v)\in L^{1,2}_-(\mathtt{A}_\varepsilon)$, we have

$$\begin{aligned}\langle \mathcal{D}V,\mathcal{D}V\rangle_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} = \int_{\mathtt{A}_\varepsilon}\Bigg(& h^{-1}(z)\left|\frac{\partial V}{\partial x_1}\right|^2\\ &+ 2\left\langle h^{-\frac{1}{2}}(z)\frac{\partial V}{\partial x_1}, \begin{bmatrix} 0 & \partial^+\\ \partial^- & 0\end{bmatrix} V\right\rangle\\ &+ \left|\partial^+ v\right|^2 + \left|\partial^- u\right|^2\Bigg) dvol_{\mathtt{A}_\varepsilon}\end{aligned}$$

Using the formula $\partial^- = (\partial^+)^*$, we have

$$
\begin{aligned}
&\int_{\mathtt{A}_\varepsilon} \left\langle h^{-\frac{1}{2}}(z)\frac{\partial V}{\partial x_1}, \begin{bmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{bmatrix} V \right\rangle dvol_{\mathtt{A}_\varepsilon} \\
&= \int_{\mathtt{A}_\varepsilon} \left(\left\langle h^{-\frac{1}{2}}\frac{\partial u}{\partial x_1}, \partial^+ v \right\rangle + \left\langle h^{-\frac{1}{2}}\frac{\partial v}{\partial x_1}, \partial^- u \right\rangle \right) dvol_{\mathtt{A}_\varepsilon} \\
&(\because \mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- \text{ orthogonal decomposition}) \\
&= \int_{\mathtt{A}_\varepsilon} \left(\left\langle \partial^- \left(h^{-\frac{1}{2}}\frac{\partial u}{\partial x_1}\right), v \right\rangle - \left\langle h^{-\frac{1}{2}} v, \partial^- \left(\frac{\partial u}{\partial x_1}\right) \right\rangle \right) dvol_{\mathtt{A}_\varepsilon} \\
&+ \int_{\{\varepsilon\}\times\Sigma} \left\langle h^{-\frac{1}{2}} v, \partial^- u \right\rangle dvol_\Sigma - \int_{\{0\}\times\Sigma} \left\langle h^{-\frac{1}{2}} v, \partial^- u \right\rangle dvol_\Sigma \\
&= \int_{\mathtt{A}_\varepsilon} \left(\left\langle \partial^- \left(h^{-\frac{1}{2}}\right) \cdot \frac{\partial u}{\partial x_1}, v \right\rangle + \left\langle h^{-\frac{1}{2}} \partial^- \left(\frac{\partial u}{\partial x_1}\right), v \right\rangle \right. \\
&\left. - \left\langle h^{-\frac{1}{2}} v, \partial^- \left(\frac{\partial u}{\partial x_1}\right) \right\rangle \right) dvol_{\mathtt{A}_\varepsilon} \text{ (since } v \text{ vanishes on } \partial \mathtt{A}_\varepsilon) \\
&= \int_{\mathtt{A}_\varepsilon} \left\langle \partial^- \left(h^{-\frac{1}{2}}\right) \cdot \frac{\partial u}{\partial x_1}, v \right\rangle \; dvol_{\mathtt{A}_\varepsilon} \\
&\qquad (\because \text{the above 2nd and 3rd terms cancel}).
\end{aligned} \tag{28}
$$

Therefore

$$
\begin{aligned}
&2\left| \int_{\mathtt{A}_\varepsilon} \left\langle h^{-\frac{1}{2}}(z)\frac{\partial V}{\partial x_1}, \begin{bmatrix} 0 & \partial^+ \\ \partial^- & 0 \end{bmatrix} V \right\rangle dvol_{\mathtt{A}_\varepsilon} \right| \\
&\leq 2 \int_{\mathtt{A}_\varepsilon} \left| \left\langle \partial^- \left(h^{-\frac{1}{2}}\right) \cdot \frac{\partial u}{\partial x_1}, v \right\rangle dvol_{\mathtt{A}_\varepsilon} \right| \\
&\leq \left\| h^{-\frac{1}{2}} \right\|_{C^1(\Sigma)} \int_{\mathtt{A}_\varepsilon} \left| 2 \left\langle \frac{\partial u}{\partial x_1}, v \right\rangle dvol_{\mathtt{A}_\varepsilon} \right| \\
&\leq \int_{\mathtt{A}_\varepsilon} \left(\frac{1}{K} \left| u_{x_1} \right|^2 + K \left\| h^{-\frac{1}{2}} \right\|^2_{C^1(\Sigma)} |v|^2 \right) dvol_{\mathtt{A}_\varepsilon}.
\end{aligned} \tag{29}
$$

In order to estimate $\int_{\mathtt{A}_\varepsilon} h^{-1}(z) \left|V_{x_1}\right|^2 dvol_{\mathtt{A}_\varepsilon}$, we notice that, for any fixed point $p \in \Sigma$, $v|_{[0,\varepsilon]\times\{p\}}$ can be treated as a $\mathbb{C}$-valued function over the interval $[0,\varepsilon]$ with boundary value $v(0,z)=0$. Hence

$$
\begin{aligned}
\int_0^\varepsilon |v|^2\,dx_1 &= \int_0^\varepsilon \left|\int_0^{x_1} \frac{\partial v}{\partial x_1}(t)\,dt\right|^2 dx_1 \\
&\le \int_0^\varepsilon \left(\int_0^{x_1} ds\right)\left(\int_0^{x_1}\left|\frac{\partial v}{\partial x_1}(t)\right|^2 dt\right) dx_1 \\
&\le \int_0^\varepsilon x_1 dx_1 \int_0^\varepsilon \left|\frac{\partial v}{\partial x_1}(t)\right|^2 dt \\
&\le K\frac{\varepsilon^2}{2}\int_0^\varepsilon h^{-1}(z)\left|\frac{\partial v}{\partial x_1}(t)\right|^2 dt, \qquad (30)
\end{aligned}
$$

where the last inequality is by $\frac{1}{K}\le h(z)\le K$ for all $z\in\Sigma$. Putting $(29),(30)$ in $\langle \mathcal{D}V,\mathcal{D}V\rangle_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}$, we have

$$
\begin{aligned}
&\langle \mathcal{D}V,\mathcal{D}V\rangle_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} \\
&\ge \int_{\mathtt{A}_\varepsilon}\Bigg(h^{-1}(z)\left|u_{x_1}\right|^2+\left|\partial^- u\right|^2+h^{-1}(z)\left|v_{x_1}\right|^2+\left|\partial^+ v\right|^2 \\
&-2\left|\left\langle h^{-\frac{1}{2}}(z)\frac{\partial V}{\partial x_1},\begin{bmatrix} 0 & \partial^+ \\ \partial^- & 0\end{bmatrix}V\right\rangle\right|\Bigg)dvol_{\mathtt{A}_\varepsilon} \\
&\ge \int_{\mathtt{A}_\varepsilon}\Bigg(\left(h^{-1}(z)-\frac{1}{K}\right)\left|u_{x_1}\right|^2+\left|\partial^- u\right|^2+h^{-1}(z)\left|v_{x_1}\right|^2 \\
&-K\left\|h^{-\frac{1}{2}}\right\|^2_{C^1(\Sigma)}|v|^2\Bigg)dvol_{\mathtt{A}_\varepsilon}(\text{by } (29)) \\
&\ge 0+\lambda_{\partial^-}\int_{\mathtt{A}_\varepsilon}|u|^2\,dvol_{\mathtt{A}_\varepsilon}+\left(\frac{2}{K\varepsilon^2}-K\left\|h^{-\frac{1}{2}}\right\|^2_{C^1(\Sigma)}\right)\int_{\mathtt{A}_\varepsilon}|v|^2\ dvol_{\mathtt{A}_\varepsilon} \\
&(\text{by definition of } \lambda_{\partial^-} \text{ and } (30)) \\
&\ge \min\left\{\lambda_{\partial^-},\frac{2}{K\varepsilon^2}-K\left\|h^{-\frac{1}{2}}\right\|^2_{C^1(\Sigma)}\right\}\left(\int_{\mathtt{A}_\varepsilon}\left(|u|^2+|v|^2\right)dvol_{\mathtt{A}_\varepsilon}\right) \\
&= \min\left\{\lambda_{\partial^-},\frac{2}{K\varepsilon^2}-K\left\|h^{-\frac{1}{2}}\right\|^2_{C^1(\Sigma)}\right\}\|V\|^2_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})}.
\end{aligned}
$$

For general volume form $dvol_{\mathtt{A}_\varepsilon}=h^{\frac{1}{2}}(z)\,dvol_\Sigma dx_1$, by (27) there is an extra factor $\frac{1}{K}$ for the lower bound of the L^2 eigenvalue $\lambda_{\mathcal{D}_\pm}$. Hence the result. q.e.d.

Suppose $V=(u,v)\in C^\infty(\mathtt{A}_\varepsilon,\mathbb{S})\cap L^{1,2}_-(\mathtt{A}_\varepsilon,\mathbb{S})$ and $W=(f,g)\in C^\infty(\mathtt{A}_\varepsilon,\mathbb{S})\cap L^2(\mathtt{A}_\varepsilon,\mathbb{S})$ and $\mathcal{D}$ is the operator (19). Then with respect to the induced volume form $dvol_{\mathtt{A}_\varepsilon} = h^{1/2}(z)\, d\Sigma dx_1$ on $(\mathtt{A}_\varepsilon, g_{\mathtt{A}_\varepsilon,h})$, we have

$$\begin{aligned}
&\int_{\mathtt{A}_\varepsilon} \langle \mathcal{D}V, W\rangle\, dvol_{\mathtt{A}_\varepsilon} \\
&= \int_0^\varepsilon \int_\Sigma \left[\left\langle \mathbf{i}\left(h^{-1/2}u_{x_1} + \partial^+ v\right), f\right\rangle - \left\langle \mathbf{i}\left(h^{-1/2}v_{x_1} + \partial^- u\right), g\right\rangle\right] h^{1/2} d\Sigma dx_1 \\
&= \mathbf{i}\int_0^\varepsilon \int_\Sigma \left[-\langle u, f_{x_1}\rangle + \left\langle h^{1/2}v, \partial^- f\right\rangle - \left\langle h^{1/2}u, \partial^+ g\right\rangle + \langle v, g_{x_1}\rangle\right] d\Sigma dx_1 \\
&+ \mathbf{i}\int_\Sigma \left(\langle u, f\rangle|_0^\varepsilon - \langle v, g\rangle|_0^\varepsilon\right) d\Sigma \\
&= \mathbf{i}\int_0^\varepsilon \int_\Sigma \left[\left\langle u, -f_{x_1} - h^{1/2}\partial^+ g\right\rangle + \left\langle v, g_{x_1} + h^{1/2}\partial^- f\right\rangle\right] d\Sigma dx_1 \\
&+ \mathbf{i}\int_\Sigma \left(\langle u, f\rangle|_0^\varepsilon - \langle v, g\rangle|_0^\varepsilon\right) d\Sigma \\
&= \int_0^\varepsilon \int_\Sigma \left\langle u, \mathbf{i}\left(h^{-1/2}f_{x_1} + \partial^+ g\right)\right\rangle - \left\langle v, \mathbf{i}\left(h^{-1/2}g_{x_1} + \partial^- f\right)\right\rangle \left(h^{1/2} d\Sigma dx_1\right) \\
&+ \mathbf{i}\int_\Sigma \left(\langle u, f\rangle|_0^\varepsilon - \langle v, g\rangle|_0^\varepsilon\right) d\Sigma
\end{aligned}$$

(31)

$$= \int_{\mathtt{A}_\varepsilon} \langle V, \mathcal{D}W\rangle\, dvol_{\mathtt{A}_\varepsilon} + \mathbf{i}\int_\Sigma \left(\langle u, f\rangle|_0^\varepsilon - \langle v, g\rangle|_0^\varepsilon\right) d\Sigma$$

since h is independent of x_1. This is the Green's formula.

When $V \in L_-^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})$ and $W \in L_+^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})$, the above boundary terms are zero, so we have

$$\int_{\mathtt{A}_\varepsilon} \langle \mathcal{D}V, W\rangle\, dvol_{\mathtt{A}_\varepsilon} = \int_{\mathtt{A}_\varepsilon} \langle V, \mathcal{D}W\rangle\, dvol_{\mathtt{A}_\varepsilon}.$$

This implies that $\mathcal{D}$ is a *self-adjoint* operator from $L_\pm^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})$ to $L_\mp^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})$ in the sense of [**8**]. Since $L_+^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})$ is dense in $L^2(\mathtt{A}_\varepsilon, \mathbb{S})$, this implies that $\mathcal{D} : L_-^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S}) \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$ is surjective if and only if $\ker \mathcal{D}|_{L_+^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})} = 0 \subset L_+^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})$. By Theorem 12, for small enough ε, we have

$$\ker \mathcal{D}|_{L_+^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})} = 0 \Leftarrow \ker \partial^-\partial^+ = 0 \Leftrightarrow \ker \partial^+ = 0,$$

where the last "$\Leftrightarrow$" is because $(\partial^+)^* = \partial^-$. Hence we have obtained the following result.

Theorem 13. *Let λ_{∂^-} and λ_{∂^+} be the first eigenvalue for $\partial^+\partial^-$ and $\partial^-\partial^+$ respectively. For $\mathcal{D} : L_-^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S}) \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$, if ε is sufficiently small, then we have*

$$\ker \partial^- = \{0\} \Leftrightarrow \lambda_{\partial^-} > 0 \Rightarrow \mathcal{D} \text{ injective},$$
$$\ker \partial^+ = \{0\} \Leftrightarrow \lambda_{\partial^+} > 0 \Rightarrow \mathcal{D} \text{ surjective}.$$

Especially if both $\lambda_{\partial^-}, \lambda_{\partial^+} > 0$, then $\mathcal{D}$ is one-to-one and onto.

3.3. Schauder estimates for the linear model. In this section we will develop the necessary linear theory for the equation

$$\mathcal{D}V = W \text{ on } \mathtt{A}_\varepsilon = \Sigma \times [0, \varepsilon]$$

with a warped product metric $g_{\mathtt{A}_\varepsilon, h} := h(z)\, dx_1^2 + g_\Sigma$, where $\mathcal{D}$ is the operator (19) in our linear model. The key issue is to estimate the operator norm of the inverse operator of $\mathcal{D}$ with explicit dependence on ε, as ε goes to zero. When ε is further away from zero, say $\varepsilon \in [1/2, 3/2]$, we have ε-free Schauder estimates. For ε small, we overcome the difficulty coming from ε by choosing an appropriate integer k so that $k\varepsilon \in [1/2, 3/2]$ and we extend any solution $V = (u, v)$ on $\mathtt{A}_\varepsilon$ to $\mathtt{A}_{k\varepsilon}$ in an L^p sense by suitable reflection. However, much care will be needed to obtain the C^α-estimate, because after the reflection of W across the boundary of $\mathtt{A}_\varepsilon$, it will no longer be continuous in general. This problem will be resolved in the case (ii) part of the proof of the following theorem.

Estimating the operator norm of $\mathcal{D}^{-1}$ in the Schauder setting rather than in the L^p setting is crucial. Since our goal is to construct instantons A governed by the equation $\tau|_A = 0$, which involves the associative 3-*form* τ, for deformations of an approximate solution A by normal vector fields V in M that are in $W^{1,p}$ class, the nonlinear equation will have cubic terms of ∇V that are outside the L^p space. The cubic terms also cause difficulty for obtaining desired quadratic estimates for the implicit function theorem in the L^p setting (see Remark 25).

The Schauder estimates are harder to obtain than for the Cauchy-Riemann type equations, partly because in our equation (23), the derivative of v only controls the ∂_{x_1} and $\bar{\partial}_z$ derivatives of u, not the full derivatives.

We recall the definition of Hölder norms for functions f on a domain Ω in $\mathbb{R}^n$:

$$[f]_{\alpha;\Omega} := \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|f(x) - f(y)|}{|x-y|^\alpha}, \tag{32}$$

$$[f]_{C^\alpha(\Omega)} = \|f\|_{C^0(\Omega)} + [f]_{\alpha;\Omega}\,,$$

$$[f]_{C^{1,\alpha}(\Omega)} = \|f\|_{C^1(\Omega)} + [\nabla f]_{\alpha;\Omega}\,.$$

Using trivialization of the bundle $\mathbb{S}$ over $\mathtt{A}_\varepsilon$, the Hölder norms for sections V of the bundle $\mathbb{S}$ are defined by patching the (finitely many) $C^{1,\alpha}$-coordinate charts on $\mathtt{A}_\varepsilon$.

Theorem 14. *Let $\mathcal{D} : L^{1,2}_-(\mathtt{A}_\varepsilon, \mathbb{S}) \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$ be the operator (23) defined on $\mathtt{A}_\varepsilon = \Sigma \times [0, \varepsilon]$ with warped product metric $g_{\mathtt{A}_\varepsilon, h} := h(z)\, dx_1^2 + g_\Sigma$. Suppose that the first eigenvalues for $\partial^-\partial^+$ and $\partial^+\partial^-$ are bounded below by $\lambda > 0$. Then for any $0 < \alpha < 1$ and $p > 3$ there is a positive constant $C = C(\alpha, p, \lambda, h)$ independent of ε such that for any $V \in$*

$C_{-}^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$ *and* $W\in C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$ *satisfying*

$$\mathcal{D}V=W,$$

we have

$$C\left\|V\right\|_{C_{-}^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}\leq\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}\left\|W\right\|_{C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}.$$

In other words, there exists a right inverse

$$Q_\varepsilon:C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})\to C_{-}^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$$

of $\mathcal{D}:C_{-}^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})\to C^{\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$ *such that* $\left\|Q_\varepsilon\right\|\leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}$.

Proof. For the operator (23), it is known (cf. [**8**] Theorem 21.5) that the Dirac operators

$$\mathcal{D}_{\pm}:=\mathcal{D}|_{L_{\pm}^{1,2}}:L_{\pm}^{1,2}(\mathtt{A}_\varepsilon,\mathbb{S})\to L^{2}(\mathtt{A}_\varepsilon,\mathbb{S})$$

give well-posed local elliptic boundary problems. Using the orthogonal decomposition $\mathbb{S}=\mathbb{S}^{+}\oplus\mathbb{S}^{-}$, we write sections $V=(u,v)\in C_{-}^{\infty}(\mathtt{A}_\varepsilon,\mathbb{S})$ and $W=(w_1,w_2)\in C^{\infty}(\mathtt{A}_\varepsilon,\mathbb{S})$, where u,w_1 are sections of $\mathbb{S}^{+}$ and v,w_2 are sections of $\mathbb{S}^{-}$. So the equation $\mathcal{D}V=W$ may be explicitly written as (cf. equation (23))

$$\left\{\begin{array}{lcl}h^{-1/2}(z)\,u_{x_1}+\partial^{+}v&=&w_1\\ h^{-1/2}(z)\,v_{x_1}+\partial^{-}u&=&w_2\end{array}\right.\quad\text{with } v|_{\partial\mathtt{A}_\varepsilon}=0.$$

To make the exposition more transparent, we will assume that $h\equiv1$, and it is clear from the proof below that the argument works equally well for any $h(z)\in C^{\infty}(\Sigma)$ such that $\frac{1}{K}\leq h(z)\leq K$ for some constant $K>0$. For any $0<\varepsilon<3/2$, we take an integer $k=k(\varepsilon)>0$ (which *depends* on ε) such that

$$k(\varepsilon)\,\varepsilon\in[1/2,3/2].$$

For notation brevity, we will write k for $k(\varepsilon)$ in the remainder of our paper. Then $\mathrm{A}_{k\varepsilon}=\Sigma\times[0,k\varepsilon]$ is a product region whose second component has length *uniformly bounded below and above* for any ε. We divide the estimates into two cases: *Case (i)*: Suppose that $w_1=0$. Then we will have along the boundary $\partial\mathtt{A}_\varepsilon$, $u_{x_1}=0$ since $v=0$. Since $v|_{\partial\mathtt{A}_\varepsilon}=0$, we can extend v from $\mathtt{A}_\varepsilon$ to $\mathtt{A}_{k\varepsilon}$ by odd reflection along the walls $\Sigma\times\{j\varepsilon\}$ with $0\leq j\leq k-1$, while still keeping v in $C^{1,\alpha}(A_{k\varepsilon})$. Similarly, we consider an even extension of u to $\mathtt{A}_{k\varepsilon}$; then u is still in $C^{1,\alpha}(A_{k\varepsilon})$ since $u_{x_1}|_{\partial\mathtt{A}_\varepsilon}=0$. The extension formula is

$$v(x,z)=\left\{\begin{array}{cc}-v((2j+2)\varepsilon-x,z)&\text{for } x\in[(2j+1)\varepsilon,(2j+2)\varepsilon]\\ v(x-2j\varepsilon,z)&\text{for } x\in[2j\varepsilon,(2j+1)\varepsilon]\end{array}\right.,$$

$$u(x,z)=\left\{\begin{array}{cc}u((2j+2)\varepsilon-x,z)&\text{for } x\in[(2j+1)\varepsilon,(2j+2)\varepsilon]\\ u(x-2j\varepsilon,z)&\text{for } x\in[2j\varepsilon,(2j+1)\varepsilon]\end{array}\right..$$

This will induce an even extension of w_2 so that the equation $\mathcal{D}V=W$ is satisfied in the C^{α} sense on $\mathtt{A}_{k\varepsilon}$. The motivation of even and odd extension is to have an ε-independent L^p and Schauder estimate. By differentiating both sides of the equation $v_{x_1} + \partial^- u = w_2$ with respect to x_1, we obtain an equation which is equivalent to the Dirichlet problem of the second order elliptic equation

$$\begin{aligned} v_{x_1x_1} - \partial^-\partial^+ v &= \frac{\partial w_2}{\partial x_1} \text{ and } v|_{\partial \mathtt{A}_{k\varepsilon}} = 0, \\ u_{x_1x_1} - \partial^+\partial^- u &= -\partial^+ w_2, \end{aligned}$$

noting that $\partial^-\partial^+$ is a positive operator. (Since u, v are only in $C^{1,\alpha}$, they should be understood as weak solutions of the above equations.) Since the C^α-norm is preserved under the above extension, Schauder and L^p interior estimate for the second order elliptic equation on the region

$$\mathtt{A}_{k\varepsilon} \subset \mathtt{A}_{2k\varepsilon} \cup \mathtt{A}_{-2k\varepsilon}$$

would then imply that there are constants $C(\alpha)$ and $\tilde{C}(p)$ independent of ε (because $k\varepsilon \in [1/2, 3/2]$) such that

$$\begin{aligned} \|w_2\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^-)} + \|V\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} &= \|w_2\|_{C^\alpha(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S}^-)} + \|V\|_{C^0(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})} \\ &\geq C(\alpha)\|V\|_{C^{1,\alpha}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\ &= C(\alpha)\|V\|_{C_-^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}, \end{aligned} \tag{33}$$

and

$$\|w_2\|_{L^p(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S}^-)} + \|V\|_{L^p(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})} \geq 4\tilde{C}(p)\|V\|_{L_-^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})}$$

(cf. [**13**] section 8.11 Theorem 8.32 and [**30**] Theorem B.3.2 respectively), so

$$\|w_2\|_{L^p(\mathtt{A}_\varepsilon,\mathbb{S}^-)} + \|V\|_{L^p(\mathtt{A}_\varepsilon,\mathbb{S})} \geq \tilde{C}(p)\|V\|_{L_-^{1,p}(\mathtt{A}_\varepsilon,\mathbb{S})}$$

by periodicity of reflection. We also have

$$\|V\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \|V\|_{C_-^{1-3/p}(\mathtt{A}_\varepsilon,\mathbb{S})} \leq C(p,\lambda)\|W\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})},$$

whose proof is identical to that of (38) in the following case (ii). Plugging this into (33), we have

$$C(\alpha)\|V\|_{C_-^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})} \leq (C(p,\lambda)+1)\|w_2\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^-)}. \tag{34}$$

Case (ii): Suppose that $w_2 = 0$ and $w_1 \in C^\alpha(\mathtt{A}_\varepsilon, \mathbb{S}^+)$. Since $v = 0$, this implies that if we consider the odd extension of v and the even extension of u to $\mathtt{A}_{2k\varepsilon} \cup \mathtt{A}_{-2k\varepsilon}$ as in the previous case, then they induce an odd extension of w_1 so that the equation

$$\begin{cases} u_{x_1} + \partial^+ v &= w_1 \\ v_{x_1} + \partial^- u &= 0 \end{cases} \tag{35}$$

is satisfied in the weak sense on $\mathtt{A}_{2k\varepsilon} \cup \mathtt{A}_{-2k\varepsilon}$. Notice that w_1 does not vanish on $\partial\mathtt{A}_\varepsilon$ in general, so after the odd extension, w_1 is no longer continuous, but still we have $w_1 \in L^p(\mathtt{A}_{2k\varepsilon} \cup \mathtt{A}_{-2k\varepsilon}, \mathbb{S})$ for $\forall p$. Since $\mathrm{A}_{k\varepsilon}$

is a proper subdomain in $\mathtt{A}_{2k\varepsilon} \cup \mathtt{A}_{-2k\varepsilon}$ and $k\varepsilon \in [1/2, 3/2]$, the interior L^p-estimate then implies that there is a constant $\tilde{C}(p)$ independent of ε such that

$$\|w_1\|_{L^p(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S}^-)} + \|V\|_{L^p(\mathtt{A}_{2k\varepsilon}\cup\mathtt{A}_{-2k\varepsilon},\mathbb{S})} \geq 4\tilde{C}(p)\|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})}.$$

By the periodicity of w_1 and V, that is

$$\|w_1\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S}^-)} + \|V\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} \geq \tilde{C}(p)\|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})}. \tag{36}$$

Then we proceed to the L^p estimates of V purely in terms of W. It follows from (36) and Theorem 12 that for $\varepsilon < 3/2$, we have

$$\|W\|_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} \geq C(\lambda)\|V\|_{L^2(\mathtt{A}_\varepsilon,\mathbb{S})} \text{ for } V \in C^\infty_-(\mathtt{A}_\varepsilon,\mathbb{S})$$

with the constant $C(\lambda)$ dependent on $\lambda_{\partial+}$ but independent of ε. To go from L^2 to L^p, note the following interpolation inequality:

$$\begin{aligned}
\|V\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} &= \left(\int_{\mathtt{A}_{k\varepsilon}} |V|^p\right)^{1/p} \leq \|V\|^{\frac{p-1}{p}}_{C^0(\mathtt{A}_{k\varepsilon},\mathbb{S})} \cdot \left(\int_{\mathtt{A}_{k\varepsilon}} |V|\right)^{1/p} \\
&\leq \frac{p-1}{p}\delta^{\frac{p}{p-1}}\|V\|^{\frac{p-1}{p}\cdot\frac{p}{p-1}}_{C^0(\mathtt{A}_{k\varepsilon},\mathbb{S})} + \frac{1}{p}\frac{1}{\delta^p}\left(\int_{\mathtt{A}_{k\varepsilon}} |V|\right)^{\frac{p}{p}} \\
&\quad \text{(by Young's Inequality)} \\
&= \frac{p-1}{p}\delta^{\frac{p}{p-1}}\|V\|_{C^0(\mathtt{A}_{k\varepsilon},\mathbb{S})} + \frac{1}{p}\frac{1}{\delta^p}\|V\|_{L^1(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq C\left(\delta^{\frac{p}{p-1}}\|V\|_{L^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} + \delta^{-p}\|V\|_{L^2(\mathtt{A}_{k\varepsilon},\mathbb{S})}\right) \\
&\quad \text{(by Sobolev embedding),}
\end{aligned}$$

where C is independent of ε in the last inequality because $k\varepsilon \in [1/2, 3/2]$. Putting this into (36), we have

$$\begin{aligned}
\left(\tilde{C}(p) - C\delta^{\frac{p}{p-1}}\right)\|V\|_{L^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} &\leq \|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} + C\delta^{-p}\|V\|_{L^2(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq \|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})} + \frac{C\delta^{-p}}{C(\lambda)}\|W\|_{L^2(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq \|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})}\left(1 + \frac{C\delta^{-p}}{C(\lambda)}\right).
\end{aligned}$$

So

$$\|V\|_{L^{1,p}_-(\mathtt{A}_{k\varepsilon},\mathbb{S})} \leq \left(\tilde{C}(p) - C\delta^{\frac{p}{p-1}}\right)^{-1}\left(1 + \frac{C\delta^{-p}}{C(\lambda)}\right)\|W\|_{L^p(\mathtt{A}_{k\varepsilon},\mathbb{S})}. \tag{37}$$

Fix a small δ such that $\tilde{C}(p) - C\delta^{\frac{p}{p-1}} > 0$ and let the constant

$$\tilde{C}(p,\lambda) := \left(\tilde{C}(p) - C\delta^{\frac{p}{p-1}}\right)^{-1}\left(1 + \frac{C\delta^{-p}}{C(\lambda)}\right)$$

which is independent of ε. Then for $p > 3$, by Sobolev embedding $C_{-}^{1-3/p}(\mathtt{A}_{k\varepsilon}, \mathbb{S}) \hookrightarrow L_{-}^{1,p}(\mathtt{A}_{k\varepsilon}, \mathbb{S})$ and (37), we have

$$
\begin{aligned}
\|V\|_{C_{-}^{1-3/p}(\mathtt{A}_{\varepsilon},\mathbb{S})} &= \|V\|_{C_{-}^{1-3/p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq C\|V\|_{L_{-}^{1,p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \leq C \cdot \tilde{C}(p,\lambda)\|W\|_{L^{p}(\mathtt{A}_{k\varepsilon},\mathbb{S})} \\
&\leq C(p,\lambda)\|W\|_{C^{0}(\mathtt{A}_{k\varepsilon},\mathbb{S})} = C(p,\lambda)\|W\|_{C^{0}(\mathtt{A}_{\varepsilon},\mathbb{S})}, \qquad (38)
\end{aligned}
$$

where the constant $C(p,\lambda) = C \cdot \tilde{C}(p,\lambda)\left(\frac{3}{2}vol(\Sigma)\right)^{\frac{1}{p}}$ is independent of ε. We remark that the above argument also works in case (i), so (38) holds in all cases. In the following, we give the Schauder estimate of $V = (u, v)$. Differentiating the second equation in (35) with respect to x_1, we get the Dirichlet problem of the second order elliptic equation for (weak solutions) v and u:

$$
\begin{aligned}
&v_{x_1x_1} - \partial^{-}\partial^{+}v = -\partial^{-}w_1 \quad \text{and} \quad v|_{\partial\mathtt{A}_{\varepsilon}} = 0, \qquad (39)\\
&u_{x_1x_1} - \partial^{+}\partial^{-}u = \partial_{x_1}w_1.
\end{aligned}
$$

Recall that in [**13**] chapter 4, the *non-dimensional* Schauder norm $\|f\|'_{C^{k,\alpha}(\overline{\Omega})}$ is defined as

$$
\|f\|'_{C^{k,\alpha}(\overline{\Omega})} = \Sigma_{j=0}^{k} d^{j}\left|D^{j}u\right|_{0;\Omega} + d^{k+\alpha}\left[D^{k}u\right]_{\alpha;\Omega}, \qquad (40)
$$

where d is the diameter of Ω. For component v, since $v|_{\partial\mathtt{A}_{\varepsilon}} = 0$, the standard Schauder estimate for second order elliptic equation on radius ε half balls B_{ε} with centers on $\partial\mathtt{A}_{\varepsilon}$ (cf. [**13**] Section 8.11 for regularity of weak solutions and Section 4.4 equation (4.43)) gives

$$
\varepsilon\|w_1\|'_{C^{\alpha}(B_{\varepsilon},\mathbb{S}^{+})} + \|v\|_{C_{-}^{0}(B_{\varepsilon},\mathbb{S})} \geq C(\alpha)\|v\|'_{C_{-}^{1,\alpha}(B_{\varepsilon/2},\mathbb{S})},
$$

or equivalently, by the definition of $\|f\|'_{C^{k,\alpha}(\overline{\Omega})}$,

$$
\begin{aligned}
&\varepsilon\|w_1\|_{C^{\alpha}(B_{\varepsilon},\mathbb{S}^{+})'} + \|v\|_{C_{-}^{0}(B_{\varepsilon},\mathbb{S})} \qquad (41)\\
&\geq C(\alpha)\left[\|v\|_{C_{-}^{0}(B_{\varepsilon/2},\mathbb{S})} + \varepsilon\|\nabla v\|_{C_{-}^{0}(B_{\varepsilon/2},\mathbb{S})} + \varepsilon^{1+\alpha}[\nabla v]_{\alpha;(B_{\varepsilon/2},\mathbb{S})}\right],
\end{aligned}
$$

and hence

$$
\varepsilon\|w_1\|_{C^{\alpha}(B_{\varepsilon},\mathbb{S}^{+})} + \|v\|_{C_{-}^{0}(B_{\varepsilon},\mathbb{S})} \geq C(\alpha)\varepsilon^{1+\alpha}\|v\|_{C_{-}^{1,\alpha}(B_{\varepsilon/2},\mathbb{S})}, \qquad (42)
$$

where the constant $C(\alpha)$ is independent of ε. Covering $\mathtt{A}_{\varepsilon}$ by such radius-ε half balls and radius-$\varepsilon/2$ full balls centered on meridian $\Sigma \times \{\varepsilon/2\}$ and taking supremum on $\mathtt{A}_{\varepsilon}$, we have

$$
\varepsilon\|w_1\|_{C^{\alpha}(\mathtt{A}_{\varepsilon},\mathbb{S}^{+})} + \|v\|_{C_{-}^{0}(\mathtt{A}_{\varepsilon},\mathbb{S})} \geq C(\alpha)\varepsilon^{1+\alpha}\|v\|_{C_{-}^{1,\alpha}(\mathtt{A}_{\varepsilon},\mathbb{S})}. \qquad (43)
$$

By combining this with the fact that $v|_{\partial \mathtt{A}_\varepsilon = 0}$, and the definition of the Schauder norm, we have

$$\|v\|_{C^0_-(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \|v\|_{C^{1-3/p}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-3/p} \leq C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-\frac{3}{p}}. \tag{44}$$

Plugging these into (43), we obtain

$$C(\alpha)\|v\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \leq \varepsilon^{-\alpha}\|w_1\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^+)} + C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{-\left(\frac{3}{p}+\alpha\right)}. \tag{45}$$

For the component u, its boundary value is nonzero, so we cannot directly apply the above inequalities as v. First we derive the C^0 estimate of u, using the assumption that $\partial^+\partial^-$ has *trivial kernel* on Σ. Consider the section $\overline{u}$ of the bundle $\mathbb{S}^+ \to \Sigma$ defined as

$$\overline{u}(z) = \int_0^\varepsilon u(x_1,z)\,dx_1.$$

From equation (35), we have

$$\begin{aligned}\partial^-\overline{u}(z) &= -\int_0^\varepsilon h^{-1/2}(z)\,\partial_{x_1} v(x_1,z)\,dx_1\\ &= h^{-1/2}(z)\,(v(0,z) - v(\varepsilon,z)) = 0.\end{aligned}$$

But from the assumption that $\partial^+\partial^-$ has trivial kernel on Σ, we get that

$$\overline{u}(z) \equiv 0.$$

Let $\operatorname{Re}\overline{u}$ and $\operatorname{Im}\overline{u}$ be the real and imaginary parts of the section $\overline{u}$ (notice that $\mathbb{S}^+$ is a complex line bundle). Then for any fixed $z \in \Sigma$,

$$\int_0^\varepsilon \operatorname{Re} u(x_1,z)\,dx_1 = 0 = \int_0^\varepsilon \operatorname{Im} u(x_1,z)\,dx_1.$$

By the mean value theorem for the $\mathbb{R}$-valued function $\operatorname{Re} u(x_1,z)$, there exists $s \in [0,\varepsilon]$ depending on z, such that $\operatorname{Re} u(s,z) = 0$. Therefore, for $x_1 \neq s$, by the definition of the Schauder norm, we have

$$\begin{aligned}|\operatorname{Re} u(x_1,z)| &= \frac{|\operatorname{Re} u(x_1,z) - \operatorname{Re} u(s,z)|}{|x_1 - s|^{1-\frac{3}{p}}} \cdot |x_1 - s|^{1-\frac{3}{p}}\\ &\leq \|\operatorname{Re} u\|_{C^{1-3/p}(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-\frac{3}{p}}.\end{aligned}$$

Then we have

$$\begin{aligned}\|\operatorname{Re} u\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} &\leq \|\operatorname{Re} u\|_{C^{1-3/p}(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-\frac{3}{p}}\\ &\leq \|V\|_{C^{1-3/p}_-(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-\frac{3}{p}}\\ &\leq C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-\frac{3}{p}}\end{aligned}$$

by (38). Similarly,

$$\|\operatorname{Im} u\|_{C^0_-(\mathtt{A}_\varepsilon,\mathbb{S}^+)} \leq C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})} \varepsilon^{1-\frac{3}{p}}.$$

Combining these, we get

$$\|u\|_{C^0_-(\mathtt{A}_\varepsilon,\mathbb{S}^+)} \le C(p,\lambda)\,\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})}\,\varepsilon^{1-\frac{3}{p}}. \tag{46}$$

Now we are ready to derive the Schauder estimate of u. Using the equation

$$\begin{cases} u_{x_1} + \partial^+ v &= w_1 \\ v_{x_1} + \partial^- u &= 0 \end{cases} \quad \text{with } v|_{\partial \mathtt{A}_\varepsilon} = 0,$$

the Schauder estimate of u_{x_1} and $\partial^- u$ can be reduced to that of v. For the full covariant derivative $\nabla_\Sigma u$ on Σ, we observe that

$$[\nabla_\Sigma u]_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})} \le [\nabla_\Sigma u]^z_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})} + [\nabla_\Sigma u]^{x_1}_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})}$$

where $[\cdot]^z_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})}$ and $[\cdot]^{x_1}_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})}$ as defined in (32) are the Schauder α-components for the Σ and $[0,\varepsilon]$ directions, respectively. For $[\nabla_\Sigma u]^z_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})}$, on each slice $\Sigma_s := \Sigma \times \{s\}$, we use the elliptic estimate on compact closed surface Σ_s to control $\nabla_\Sigma u$ by $\partial^- u$,

$$[\nabla_\Sigma u]^z_{\alpha;(\Sigma_s,\mathbb{S})} \le |u|_{C^{1,\alpha}(\Sigma_s,\mathbb{S})} \le C\left(\left|\partial^- u\right|_{C^\alpha(\Sigma_s,\mathbb{S})} + |u|_{C^0(\Sigma_s,\mathbb{S})}\right),$$

and then take the sup for $0 \le s \le \varepsilon$ to get

$$\begin{aligned}[\nabla_\Sigma u]^z_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})} &\le C\left(\left|\partial^- u\right|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S})} + |u|_{C^0(\mathrm{A}_\varepsilon,\mathbb{S})}\right) \\ &= C\left(|v_{x_1}|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S})} + |u|_{C^0(\mathrm{A}_\varepsilon,\mathbb{S})}\right).\end{aligned} \tag{47}$$

Since the estimate for v_{x_1} is known in (45), the only term left to estimate is $[\nabla_\Sigma u]^{x_1}_{\alpha;(\mathtt{A}_\varepsilon,\mathbb{S})}$. For this we reduce u to the zero boundary value case by introducing

$$\widetilde{u} = u - \rho\left(\frac{x_1}{\varepsilon}\right)u(\varepsilon,z) - \left(1 - \rho\left(\frac{x_1}{\varepsilon}\right)\right)u(0,z),$$

where $\rho : [0,1] \to [0,1], \rho(0) = 0, \rho(1) = 1$ is a smooth cut-off function such that $\|\rho\|_{C^{2,\alpha}[0,1]} \le C$. Then $\widetilde{u}$ satisfies

$$\widetilde{u}_{x_1x_1} - \partial^+\partial^-\widetilde{u} = \partial_{x_1}\widetilde{w_1} + g \quad \text{and} \quad \widetilde{u}|_{\partial \mathtt{A}_\varepsilon} = 0,$$

where

$$\widetilde{w_1} = w_1 + \rho\left(\frac{x_1}{\varepsilon}\right)\left[\left(u_{x_1}(\varepsilon,z) - u_{x_1}(0,z)\right) + \left(w_1(0,z) - w_1(\varepsilon,z)\right)\right]$$

and

$$\begin{aligned} g = {}& \frac{1}{\varepsilon^2}\rho''\left(\frac{x_1}{\varepsilon}\right)\left[u(0,z) - u(\varepsilon,z)\right] \\ &+ \frac{1}{\varepsilon}\rho'\left(\frac{x_1}{\varepsilon}\right)\left[\left(u_{x_1}(0,z) - u_{x_1}(\varepsilon,z)\right) + \left(w_1(\varepsilon,z) - w_1(0,z)\right)\right].\end{aligned}$$

In the above derivation of $\widetilde{w_1}$ and g, we have used the equation

$$\partial^+\partial^- u = u_{x_1x_1} - \partial_{x_1} w_1$$

from (39). By [**13**] section 4.4 equation (4.46), we have, on radius-ε half ball B_ε,

$$\|\widetilde{u}\|_{C^0_-(B_\varepsilon,\mathbb{S})} + \varepsilon \|\widetilde{w_1}\|'_{C^\alpha(B_\varepsilon,\mathbb{S}^+)} + \varepsilon^2 |g|_{C^0_-(B_\varepsilon,\mathbb{S})} \geq C(\alpha) \|\widetilde{u}\|'_{C^{1,\alpha}_-\left(B_{\varepsilon/2},\mathbb{S}\right)}.$$

By the definition of the norm $\|f\|'_{C^{k,\alpha}(\overline{\Omega})}$ in (40), this is equivalent to

$$\begin{aligned} &\|\widetilde{u}\|_{C^0_-(B_\varepsilon,\mathbb{S})} + \varepsilon \|\widetilde{w_1}\|_{C^0(B_\varepsilon,\mathbb{S}^+)} + \varepsilon^{1+\alpha} [\widetilde{w_1}]_{\alpha;(B_\varepsilon,\mathbb{S}^+)} + \varepsilon^2 |g|_{C^0_-(B_\varepsilon,\mathbb{S})} \\ &\geq C(\alpha) \left[\|\widetilde{u}\|_{C^0_-\left(B_{\varepsilon/2},\mathbb{S}\right)} + \varepsilon \|\nabla\widetilde{u}\|_{C^0_-\left(B_{\varepsilon/2},\mathbb{S}\right)} + \varepsilon^{1+\alpha} [\nabla\widetilde{u}]_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)}\right]. \end{aligned} \tag{48}$$

Therefore

$$\begin{aligned} &\|\widetilde{u}\|_{C^0_-(B_\varepsilon,\mathbb{S})} + \varepsilon \|\widetilde{w_1}\|_{C^0(B_\varepsilon,\mathbb{S}^+)} + \varepsilon^{1+\alpha} [\nabla\widetilde{w_1}]_{\alpha;(B_\varepsilon,\mathbb{S}^+)} + \varepsilon^2 |g|_{C^0_-(B_\varepsilon,\mathbb{S})} \\ &\geq C(\alpha) \varepsilon^{1+\alpha} [\nabla_\Sigma\widetilde{u}]_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)}. \end{aligned} \tag{49}$$

We are going to get rid of the tilde terms in the above inequality by the following rule: The terms with tilde differ from the original terms by w_1, u, and their derivatives. If the derivative is with respect to x_1 or ∂^-, then we reduce it to the previous estimates of v using (35). If it is with respect to ∇_Σ, the full derivative on Σ, then we use the elliptic estimate on each slice of Riemann surface $\{t\} \times \Sigma$. Notice that all ε powers are coupled with the order of derivatives so all terms are dimensionless, and after the enlargements the estimate of u is of correct ε power. The following is the precise calculation: From the definition of $g, \widetilde{w_1}$, and $\widetilde{u}$ we can easily check that

$$\begin{aligned} \left\|\varepsilon^2 g\right\|_{C^0(B_\varepsilon,\mathbb{S})} &\leq C\left(\|u\|_{C^0(B_\varepsilon,\mathbb{S})} + \varepsilon \|u_{x_1}\|_{C^0(B_\varepsilon,\mathbb{S})} + \varepsilon \|w_1\|_{C^0(B_\varepsilon,\mathbb{S})}\right) \\ |\widetilde{w_1}|_{C^0(B_\varepsilon,\mathbb{S})} &\leq 3\|w_1\|_{C^0(B_\varepsilon,\mathbb{S})} + \|u_{x_1}\|_{C^0(B_\varepsilon,\mathbb{S})} \\ [\widetilde{w_1}]^{x_1}_{\alpha;(B_\varepsilon,\mathbb{S})} &\leq C\left([w_1]^{x_1}_{\alpha;(B_\varepsilon,\mathbb{S})} + \frac{1}{\varepsilon^\alpha}\|u_{x_1}\|_{C^0(B_\varepsilon,\mathbb{S})} + \frac{1}{\varepsilon^\alpha}\|w_1\|_{C^0(B_\varepsilon,\mathbb{S})}\right) \\ [\widetilde{w_1}]^{z}_{\alpha;(B_\varepsilon,\mathbb{S})} &\leq C\left([w_1]^{z}_{\alpha;(B_\varepsilon,\mathbb{S})} + [u_{x_1}]^{z}_{\alpha;(B_\varepsilon,\mathbb{S})}\right) \\ \|\widetilde{u}\|_{C^0(B_\varepsilon,\mathbb{S})} &\leq 2\|u\|_{C^0(B_\varepsilon,\mathbb{S})} \\ \|\nabla_\Sigma\widetilde{u}\|_{C^0(B_\varepsilon,\mathbb{S})} &\leq 2\|\nabla_\Sigma u\|_{C^0(B_\varepsilon,\mathbb{S})}, \\ \|\nabla_{x_1}\widetilde{u}\|_{C^0(B_\varepsilon,\mathbb{S})} &\leq C\left(\|\nabla_{x_1}u\|_{C^0(B_\varepsilon,\mathbb{S})} + \frac{1}{\varepsilon}\|u\|_{C^0(B_\varepsilon,\mathbb{S})}\right) \\ [\nabla_\Sigma u]^{x_1}_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)} &\leq C\left([\nabla_\Sigma\widetilde{u}]^{x_1}_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)} + \frac{1}{\varepsilon^\alpha}\|\nabla_\Sigma u\|_{C^0\left(B_{\varepsilon/2},\mathbb{S}\right)}\right). \end{aligned} \tag{50}$$

Hence, putting these back into (49) and (48), and noticing that from (41) we can control $\|u_{x_1}\|_{C^0(B_\varepsilon,\mathbb{S})}$ and $\|\nabla_\Sigma u\|_{C^0(B_\varepsilon,\mathbb{S})}$ that appeared in the above inequalities by the following:

$$\begin{aligned}
&\varepsilon \left\|u_{x_1}\right\|_{C^0(B_\varepsilon,\mathbb{S})} \leq C\left(\varepsilon\left\|\partial^+ v\right\|_{C^0(B_\varepsilon,\mathbb{S})}+\varepsilon\left\|w_1\right\|_{C^0(B_\varepsilon,\mathbb{S})}\right) \text{ (by (35))}\\
&\text{(by (41))} \qquad \leq C\left[\left(\|v\|_{C^0(\mathrm{A}_\varepsilon,\mathbb{S})}+\varepsilon\left\|w_1\right\|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S})}\right)+\varepsilon\left\|w_1\right\|_{C^0(B_\varepsilon,\mathbb{S})}\right]\\
&\qquad\qquad\qquad \leq C\left(\|v\|_{C^0(\mathrm{A}_\varepsilon,\mathbb{S})}+\varepsilon\left\|w_1\right\|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S})}\right), \qquad (51)
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon\left\|\nabla_\Sigma u\right\|_{C^0(\Sigma_s,\mathbb{S})} &\leq \varepsilon\left\|\nabla_\Sigma u\right\|_{C^\alpha(\Sigma_s,\mathbb{S})}\\
&\leq C\varepsilon\left(\left\|\partial^- u\right\|_{C^\alpha(\Sigma_s,\mathbb{S})}+\|u\|_{C^0(\Sigma_s,\mathbb{S})}\right) \text{ (by ellipticity on } \Sigma_s)\\
\text{(by (35))} \quad &= C\left(\varepsilon\left\|v_{x_1}\right\|_{C^\alpha(\Sigma_s,\mathbb{S})}+\varepsilon\|u\|_{C^0(\Sigma_s,\mathbb{S})}\right)\\
(52)\quad & \\
\text{(by (42))} \quad &\leq C\left(\varepsilon^{-\alpha}\|v\|_{C^0(\mathrm{A}_\varepsilon,\mathbb{S})}+\varepsilon^{1-\alpha}\left\|w_1\right\|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S})}+\varepsilon\|u\|_{C^0(\Sigma_s,\mathbb{S})}\right),
\end{aligned}$$

then from (49) *and* (50), we get the estimate for $[\nabla_\Sigma u]^{x_1}_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)}$:

$$\begin{aligned}
&\varepsilon^{1-\alpha}\left\|w_1\right\|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S}^+)}+\varepsilon\|u\|_{C^0_-(\mathrm{A}_\varepsilon,\mathbb{S})}+\varepsilon^{-\alpha}\|v\|_{C^0_-(\mathrm{A}_\varepsilon,\mathbb{S})}\\
&\geq CC(\alpha)\,\varepsilon^{1+\alpha}\left[\nabla_\Sigma u\right]^{x_1}_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)}.
\end{aligned}$$

Combining (47) about $[\nabla_\Sigma u]^{z}_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)}$, we have the full control of $[\nabla_\Sigma u]_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)}$:

$$\begin{aligned}
&\varepsilon^{1-\alpha}\left\|w_1\right\|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S}^+)}+\varepsilon\|u\|_{C^0_-(\mathrm{A}_\varepsilon,\mathbb{S})}+\varepsilon^{-\alpha}\|v\|_{C^0_-(\mathrm{A}_\varepsilon,\mathbb{S})}\\
&\geq CC(\alpha)\,\varepsilon^{1+\alpha}\left[\nabla_\Sigma u\right]_{\alpha;\left(B_{\varepsilon/2},\mathbb{S}\right)}.
\end{aligned}$$

Combining with (51) and (52) about $\|\nabla u\|_{C^0}$, we get

$$\begin{aligned}
&\varepsilon\left\|w_1\right\|_{C^\alpha(\mathrm{A}_\varepsilon,\mathbb{S}^+)}+\varepsilon^{1+\alpha}\|u\|_{C^0_-(\mathrm{A}_\varepsilon,\mathbb{S})}+\|v\|_{C^0_-(\mathrm{A}_\varepsilon,\mathbb{S})}\\
&\geq CC(\alpha)\,\varepsilon^{1+2\alpha}\|u\|_{C^{1,\alpha}_-\left(B_{\varepsilon/2},\mathbb{S}\right)}. \qquad (53)
\end{aligned}$$

Covering $\mathtt{A}_\varepsilon$ by radius-ε half balls B_ε and radius-$\varepsilon/2$ full balls centered on meridian $\Sigma\times\{\varepsilon/2\}$ and then taking supremum on $\mathtt{A}_\varepsilon$, we derive

$$\begin{aligned}
&\varepsilon\left\|w_1\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^+)}+\varepsilon^{1+\alpha}\|u\|_{C^0_-(\mathtt{A}_\varepsilon,\mathbb{S})}+\|v\|_{C^0_-(\mathtt{A}_\varepsilon,\mathbb{S})}\\
&\geq CC(\alpha)\,\varepsilon^{1+2\alpha}\|u\|_{C^{1,\alpha}_-(\mathtt{A}_\varepsilon,\mathbb{S})}. \qquad (54)
\end{aligned}$$

Notice that here v is involved on the left side to control u, compared to similar inequality (43) about v which needs no u. This is partly because in second order elliptic equations (39) of u and v, the inhomogeneous term $\partial_{x_1}w_1$ is "more discontinuous" than $-\partial^- w_1$, for the (weak) derivative is taken in x_1 direction along which (the extended) w_1 is discontinuous. Plugging these into (44) again, we obtain

$$
\begin{aligned}
& C(\alpha)\|u\|_{C_{-}^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})} \\
& \leq \varepsilon^{-2\alpha}\|w_1\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^+)}+\varepsilon^{-\alpha}\|u\|_{C_{-}^{0}(\mathtt{A}_\varepsilon,\mathbb{S})}+\varepsilon^{-(1+2\alpha)}\|v\|_{C_{-}^{0}(\mathtt{A}_\varepsilon,\mathbb{S})} \\
& \leq \varepsilon^{-2\alpha}\|w_1\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S}^+)}+C(p,\lambda)\|w_1\|_{C^0(\mathtt{A}_\varepsilon,\mathbb{S})}\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}. \qquad (55)
\end{aligned}
$$

The $C^{1,\alpha}$ estimates (55) and (45) also hold when $\mathtt{A}_\varepsilon$ is equipped with a warped product metric $g_{\mathtt{A}_\varepsilon,h}$ with $h(z)\in C^\infty(\Sigma)$ such that $\frac{1}{K}\leq h(z)\leq K$, up to the constant factor $K>0$ on the right hand sides of (55) and (45). This is because they are derived by C^0 estimates (44), (46), and local Schauder estimates (42), (53) for u,v on half balls $B_\varepsilon(p)$ and $B_{\varepsilon/2}(p)$ for $p\in\partial\mathtt{A}_\varepsilon=\{(0,z),(\varepsilon,z)\,|z\in\Sigma\}$, while when $\varepsilon\to 0$, the function $h(z)\,|_{B_\varepsilon(p)}$ converges to constant function $h(p)$ uniformly in C^0 norm in $B_\varepsilon(p)$, and correspondingly the second order equations of u, v converge to those with constant coefficients as above. This is similar to the "frozen coefficient method" in Schauder theory of second order elliptic equations. Finally, we estimate the right inverse bound of $\mathcal{D}$ in the Schauder setting. By Theorem 13, the operator $\mathcal{D}$ is surjective, so for any $W=(w_1,w_2)$ we can write it as the sum of $(w_1,0)$ and $(0,w_2)$, whose preimages $V=\mathcal{D}^{-1}(w_1,0)$ and $\mathcal{D}^{-1}(0,w_2)$ exist. Therefore we can reduce general cases to cases (i) and (ii). Combining the $C^{1,\alpha}$ estimates (34) in case (i) and (55) and (45) for u and v in case (ii), we have

$$C(\alpha,p,\lambda)\|V\|_{C_{-}^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})}\leq\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}\|W\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})},$$

where $C(\alpha,p,\lambda)=C(\alpha)(1+C(p,\lambda))^{-1}$ is independent on ε. Hence the result. q.e.d.

Remark 15. (About the asymmetry of case (i) and case (ii)) In case (i), since $v|_{\partial A_\varepsilon}=0$, after odd reflections of v and even reflections of u and w_2, the $V=(u,v)$ is still a $C^{1,\alpha}$ function on the domain $A_{k(\varepsilon)\varepsilon}$ and w_2 is of class C^α on $A_{k(\varepsilon)\varepsilon}$, so we can use the Schauder estimate of the domain $A_{k(\varepsilon)\varepsilon}$, which has uniform ellipticity; in case (ii), since $w_1|_{\partial A_\varepsilon}\neq 0$ in general, after odd reflections of v and w_1, w_1 is no longer continuous on $A_{k(\varepsilon)\varepsilon}$, not to mention in $C^{1,\alpha}$. So we have to directly work on the thin domain A_ε (which lacks uniform ellipticity as $\varepsilon\to 0$) for the Schauder estimates, with ε-dependent coefficients. The steps there are from L^p, $C^{1-\frac{3}{p}}$, C^0 to $C^{1,\alpha}$.

We remark that when $\lambda_{\partial^+}>0$ but $\lambda_{\partial^-}=0$ (i.e. $\ker\partial^-\neq\{0\}$), there still exists a right inverse $Q_\varepsilon: C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})\to C_{-}^{1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S})$ of $\mathcal{D}$ with the operator norm bound $\|Q_\varepsilon\| \leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}$. To prove this, one needs to consider the restriction

$$\mathcal{D} : L_{-}^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S}) \cap (\ker \mathcal{D})^{\perp} \to L^2(\mathtt{A}_\varepsilon, \mathbb{S})$$

to construct the right inverse Q_ε, where "$\perp$" is the L^2 orthogonal complement, and replace the constants

$$\lambda_{\mathcal{D}_-} = \inf_{0 \neq V \in L_{-}^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S})} \frac{\|\mathcal{D}V\|^2_{L^2(\mathtt{A}_\varepsilon, \mathbb{S})}}{\|V\|^2_{L^2(\mathtt{A}_\varepsilon, \mathbb{S})}},$$

$$\lambda_{\partial^-} = \inf_{0 \neq V \in L_{-}^{1,2}(\Sigma, \mathbb{S}^+)} \frac{\|\partial^- V\|^2_{L^2(\Sigma, \mathbb{S}^+)}}{\|V\|^2_{L^2(\Sigma, \mathbb{S}^+)}}$$

in Theorem 12 by the constants

$$\widetilde{\lambda}_{\mathcal{D}_-} = \inf_{0 \neq V \in L_{-}^{1,2}(\mathtt{A}_\varepsilon, \mathbb{S}) \cap (\ker \mathcal{D})^{\perp}} \frac{\|\mathcal{D}V\|^2_{L^2(\mathtt{A}_\varepsilon, \mathbb{S})}}{\|V\|^2_{L^2(\mathtt{A}_\varepsilon, \mathbb{S})}},$$

$$\widetilde{\lambda}_{\partial^-} = \inf_{0 \neq V \in L_{-}^{1,2}(\Sigma, \mathbb{S}^+) \cap (\ker \partial^-)^{\perp}} \frac{\|\partial^- V\|^2_{L^2(\Sigma, \mathbb{S}^+)}}{\|V\|^2_{L^2(\Sigma, \mathbb{S}^+)}}$$

respectively to get the L^2 estimate of V; and in the C^0 estimate of u in case (ii) of the above theorem, use the fact that

$$u \in (\ker \mathcal{D})^{\perp} \Rightarrow \overline{u} \in \left(\ker \partial^-\right)^{\perp}$$

by integrating in the x_1 direction. The details are left to readers.

From now on, we fix $p > 3$ sufficiently large and $0 < \alpha < 1$ sufficiently small such that

$$0 < \frac{3}{p} + 3\alpha < \frac{1}{2}.$$

This will be necessary for the implicit function theorem in the next section.

4. Proof of the main theorem

Let $C_0 \subset M$ be a **compact** coassociative submanifold. Suppose that n is a normal vector field on C_0 such that its corresponding self-dual two form

$$\eta_0 = \iota_n \Omega \in \wedge^2_+(C_0)$$

is harmonic with respect to the induced metric. So η_0 is actually a symplectic form on the complement of the zero set $Z(\eta_0)$ of η_0 in C_0. Furthermore,

$$J_n(u) := \frac{n}{|n|} \times u \tag{56}$$

defines an almost complex structure J_n on the $C_0 \backslash Z(\eta_0)$. Since deformations of coassociative submanifolds are unobstructed, we may assume that there is a one parameter family of coassociative submanifolds $\varphi : [0, \varepsilon] \times C_0 \longrightarrow M$ such that

$$\left.\frac{\partial \varphi}{\partial t}\right|_{t=0} = n \in \Gamma\left(C_0, N_{C_0/M}\right).$$

For ε small, let

$$\mathtt{C} := [0, \varepsilon] \times C_0, \ \mathcal{C} := \varphi(\mathtt{C}), \text{ and } C_t := \varphi(\{t\} \times C_0).$$

Then $\mathcal{C}$ is diffeomorphic to $\mathtt{C}$, and $\varphi(t, \cdot)$ is an embedding for $\forall t \in [0, \varepsilon]$. We remind the readers about the typefaces of our notations: both $\mathtt{C}$ and $\mathcal{C}$ are 5-dimensional, while each C_t is 4-dimensional. Let

$$g_{\mathtt{C}} := dt^2 \oplus g|_{C_0}$$

be the product metric on $\mathtt{C}$ and $\exp^{\mathtt{C}}$ be the exponential map associated to the metric $g_{\mathtt{C}}$.

In the remaining part of this article, we assume that η_0 is *nowhere vanishing* on C_0, which implies that (C_0, η_0) is a symplectic four manifold, and all coassociative submanifolds C_t's are mutually disjoint. We are going to establish a correspondence between the regular J_n-holomorphic curves Σ in C_0 and the instantons in M with coassociative boundary conditions.

Given such a $\Sigma \subset C_0$, we denote

$$\mathtt{A}_\varepsilon := [0, \varepsilon] \times \Sigma \text{ and } \mathtt{A}'_\varepsilon := \varphi(\mathtt{A}_\varepsilon).$$

Then $\mathtt{A}'_\varepsilon$ is close to being associative in the sense that $\left|\tau|_{\mathtt{A}'_\varepsilon}\right| \leq K\varepsilon$ for some constant K depending on the geometry of the family $\{C_t\}$ and M for small ε. This is due to the smooth dependence of $\tau(\varphi(t, z))$ on $(t, z) \in [0, \varepsilon] \times \Sigma$, and the fact that

$$\tau|_{T\mathtt{A}'_\varepsilon|_{\varphi(0,\Sigma)}} = \tau|_{T\mathtt{A}'_\varepsilon|_\Sigma} = 0,$$

since Σ is a J_n-holomorphic curve, the tangent space $T\Sigma$ is closed under $J_n = \frac{n}{|n|}\times$, and $T\mathtt{A}'_\varepsilon|_\Sigma = T\Sigma \oplus \mathrm{span}\{n\}$.

We want to perturb A'_ε to become an honest associative submanifold in M. In order to apply the implicit function theorem to obtain the desired perturbation for $\mathtt{A}'_\varepsilon$, we need the estimates for the linearized problem to behave well as ε approaches zero. Notice that for small ε, the induced metric on $\mathtt{A}'_\varepsilon$ is close to being a warped product metric

$$g_{\mathtt{A}_\varepsilon, h} := h(z)\, dx_1^2 + g_\Sigma$$

on $\mathtt{A}_\varepsilon$, where $h(z) = |n(z)|^2$ is the length squared of the normal vector n. Namely,

$$(1 - K\varepsilon)\, g_{\mathtt{A}_\varepsilon, h} \leq \varphi^* g_M \leq (1 + K\varepsilon)\, g_{\mathtt{A}_\varepsilon, h}$$

for some uniform constant K. This is because for any vectors $X, Y \in T_{(t,z)}A_{\varepsilon_0}$ and $t \in [0, \varepsilon_0]$, the function $G : [0, \varepsilon_0] \times TA_{\varepsilon_0} \times TA_{\varepsilon_0} \to \mathbb{R}$,

$$G(t, X, Y) := g_M(d\varphi(t, z) X, d\varphi(t, z) Y),$$

is smooth with respect to (t, X, Y), bilinear in (X, Y), and

$$\begin{aligned} & G\left(0, TA_{\varepsilon_0}|_{\{0\}\times\Sigma}, TA_{\varepsilon_0}|_{\{0\}\times\Sigma}\right) \\ & = g_M\left(d\varphi|_{\{0\}\times\Sigma}\left(TA_{\varepsilon_0}|_{\{0\}\times\Sigma}\right), d\varphi|_{\{0\}\times\Sigma}\left(TA_{\varepsilon_0}|_{\{0\}\times\Sigma}\right)\right) \\ & = g_{\mathtt{A}_{\varepsilon_0},h}|_{\{0\}\times\Sigma}, \end{aligned}$$

where in the last identity we have used that $d\varphi|_{\{0\}\times\Sigma} = id : T\Sigma \to T\Sigma$ and $d\varphi|_{\{0\}\times\Sigma} : \frac{\partial}{\partial x_1} \to n$.

For the warped product metric on $\mathtt{A}_\varepsilon$ and the corresponding operator $\mathcal{D}$, the estimates for its inverse have been established in Theorem 14 in the previous section. The above discussion indicates that the linearization of the instanton equation on A'_ε may be compared with $\mathcal{D}$. We will first show they agree on Σ in the next subsection. Then the comparison on A'_ε is a small perturbation from their agreement on Σ.

4.1. Geometry of $\Sigma \subset \mathcal{C} \subset M$. We study the geometry of the J_n-holomorphic curves Σ in a G_2-manifold M. Let $\mathcal{C} = \cup_{0\le t\le\varepsilon} C_t$ be the family of coassociative manifolds in the previous subsection and $\Sigma \subset C = C_0$. We remind the readers that the $\mathcal{C}$ is a 5-dimensional submanifold so its normal bundle in M has real rank 2, while C is a 4-dimensional submanifold. We recall the following results in Lemma 3.2 of [**17**].

Proposition 16. 1) *$N_{\Sigma/C}$ and $N_{\mathcal{C}/M}|_\Sigma$ are complex line bundles with the almost complex structure $J_n = \frac{n}{|n|}\times$ on fibers.*

2) *$\overline{N_{\mathcal{C}/M}|_\Sigma} \simeq \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right)$ as complex line bundles, where $\overline{L}$ is the conjugate of a complex line bundle L, and $\wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right) = N_{\Sigma/C} \otimes_{\mathbb{C}} \wedge^{0,1}_{\mathbb{C}}(T^*\Sigma)$.*

Note that the notations $\partial Y, X, \nu_X$, and μ_X in [**17**] correspond to our $\Sigma, C, N_{\Sigma/C}$, and $N_{\mathcal{C}/M}|_\Sigma$, respectively.

In the following we will use the notation $N_{\mathcal{C}/M}|_\Sigma$ rather than $\overline{N_{\mathcal{C}/M}|_\Sigma}$, but we emphasize that the isomorphism $N_{\mathcal{C}/M}|_\Sigma \simeq \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right)$ *is complex conjugate linear* and should be regarded as an isomorphism between *real vector bundles*. Note that we do NOT complexify $N_{\Sigma/C}$, since it is already a complex line bundle, and more importantly, we want to use its complex structure $J_n = \frac{n}{|n|}\times$ to interplay with the G_2 geometry.

In the following proposition, we will show that $N_{\Sigma/C}$ and $N_{\mathcal{C}/M}|_\Sigma$ are Hermitian line bundles, and $N := N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma \to \Sigma$ is a *Dirac bundle* (in the sense of Definition 5.2 in [**23**]), i.e. N is a left Clifford module over Σ with respect to the G_2 multiplication $\times$, together with a Riemannian metric $\langle , \rangle$ and connection ∇^N on N satisfying: (a) at each $p \in \Sigma$, for any $\sigma_1, \sigma_2 \in N_p$ and any unit vector $e \in T_p\Sigma$, $\langle e\sigma_1, e\sigma_2 \rangle = \langle e\sigma_1, e\sigma_2 \rangle$; (b) for any section φ of $T\Sigma$ and section σ of N, $\nabla^N (\varphi \times \sigma) = \left(\nabla^{T\Sigma}\varphi\right) \times \sigma + \varphi \times \nabla^N \sigma$.

Proposition 17. *Let ∇ be the Levi-Civita connection of (M, g). For any subbundle $L \to \Sigma_0$ of $TM|_{\Sigma_0}$, let the* induced connection *∇^L of L be*

$$\nabla^L := \pi^L \circ \nabla,$$

where π^L is the orthogonal projection of $TM|_\Sigma$ to subbundle L according to the metric g. Then,

1) *For $L = N_{\Sigma/C}$, $N_{\mathcal{C}/M}|_\Sigma$ or $T\Sigma$, the induced connection ∇^L is Hermitian, namely $\nabla^L J_n = 0$.*
2) *Let $N = N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma$. Then with respect to the G_2 multiplication $\times$ as the Clifford multiplication and the induced connection ∇^N, N is a Dirac bundle.*

Proof. For each $p \in \Sigma_0$, the subspace $T_p\Sigma_0 \oplus \text{span}\{n(p)\} \subset T_pM$ is associative since Σ_0 is J_n-holomorphic. By the same Cayley-Dickson construction in Theorem 9, we may choose orthonormal frame $\{W_\alpha\}_{\alpha=1,\dots 7}$ in a neighborhood $B_\varepsilon(p) \subset \Sigma$ of p satisfying standard $\cdot, \times$ relation as the basis of $\operatorname{Im}\mathbb{O}$, such that

$$\begin{aligned} T\Sigma_0 &= \text{span}\{W_2, W_3\}, \ W_1 := W_2 \times W_3, \\ W_4 &\in N_{\Sigma_0/C_0}, W_{i+4} := W_i \times W_4 \text{ for } 1 \le i \le 3, \\ N_{\Sigma_0/C_0} &= \text{span}\{W_4, W_5\}, \text{ and } N_{\mathcal{C}/M}|_{\Sigma_0} = \text{span}\{W_6, W_7\}. \end{aligned}$$

For $i = 2, 3$, we have

$$\nabla^L_{Wi} J_n = \nabla^L_{Wi}(W_1 \times |_L) = \pi^L(\nabla_{Wi}W_1) \times |_L = \langle W_1, \nabla_{Wi}W_1 \rangle W_1 \times |_L = 0,$$

where the third identity is because for $L = T\Sigma = \text{span}\{W_2, W_3\}$, $L = N_{\Sigma_0/C} = \text{span}\{W_4, W_5\}$ or $L = N_{\mathcal{C}/M}|_{\Sigma_0} = \text{span}\{W_6, W_7\}$, under cross product $\times$, only the W_1 component of $\nabla_{Wi}W_1$ can preserve L, i.e.

$$W_1 \times L \subset L, \text{ and } (W_k \times L) \cap L = \{0\} \text{ for } 2 \le k \le 7$$

from octonion multiplication relation, and the last identity is because $\langle W_1, \nabla_{Wi}W_1 \rangle = \frac{1}{2}\nabla_{Wi}\langle W_1, W_1 \rangle = 0$. Thus $\nabla^L_v J = 0$ for any $v \in T\Sigma_0$. Using the standard G_2 multiplication relation of the above "good" frame $\{W_\alpha\}_{\alpha=1,\dots 7}$, it is easy to check that $N := N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma \to \Sigma$ is a Clifford module. To show N is a Dirac bundle, we first note that for any unit vector $e \in T_p\Sigma$, $e\times$ is an isometry on N by the property of $\times$; second, for any section φ of $T\Sigma$ and section σ of $N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma$, with respect to the induced connection ∇^N on N we have

$$\nabla^N(\varphi \times \sigma) = \pi^N \circ (\nabla\varphi \times \sigma + \varphi \times \nabla\sigma) = \left(\nabla^{T\Sigma}\varphi\right) \times \sigma + \varphi \times \nabla^N \sigma,$$

where in the second identity, the first term is because for distinct $i, j \in \{4,5,6,7\}$, $W_i \times W_j \in$span$\{W_1, W_2, W_3\}$, and the second term is because span$\{W_1, W_2, W_3\}$ is closed under $\times$. q.e.d.

It is well known that on a compact Kähler manifold with a Hermitian line bundle L, one can define a Dirac operator on the Dolbeault complex $\Omega_{\mathbb{C}}^{0,*}(L)$ (cf. Proposition 3.67 of [**6**] or Proposition 1.4.25 of [**31**]). Taking the Kähler manifold to be Σ and the Hermitian line bundle to be $N_{\Sigma/C}$, we have the Dolbeault Dirac operator on $N_{\Sigma/C} \oplus \wedge_{\mathbb{C}}^{0,1}(N_{\Sigma/C})$. On the other hand, by the above lemma, $N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_{\Sigma}$ is a Dirac bundle and has a canonically associated Dirac operator. The following proposition compares the two Dirac operators.

Proposition 18. *The Dolbeault Dirac operator on* $N_{\Sigma/C} \oplus \wedge_{\mathbb{C}}^{0,1}(N_{\Sigma/C})$ *agrees with the Dirac operator on* $N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_{\Sigma}$*, for which the Clifford multiplication is the* G_2 *multiplication* $\times$ *and the connection is the induced connection from* M*.*

Proof. Given any $p \in \Sigma$, we may further assume the "good" frame $\{W_\alpha\}_{\alpha=1,2,\dots 7}$ on $B_\varepsilon(p)$ in Proposition 17 satisfies

$$\nabla_{W_i}^{T\Sigma_0} W_j(p) = \nabla_{W_i}^{N_{\Sigma_0/C}} W_k(p) = \nabla_{W_i}^{N_{\mathcal{C}/M}|_{\Sigma_0}} W_{k+2}(p) = 0 \tag{57}$$

for $2 \le i, j \le 3$ and $4 \le k \le 5$, where the $\nabla^{T\Sigma_0}$,$\nabla^{N_{\Sigma_0/C_0}}$, and $\nabla^{N_{\mathcal{C}/M}|_{\Sigma_0}}$ are orthogonal projections of the Levi-Civita connection ∇ on M to $T\Sigma_0, N_{\Sigma_0/C_0}$, and $N_{\mathcal{C}/M}|_{\Sigma_0}$, respectively (with respect to the metric g). To achieve (57), we can first require $\nabla_{W_i}^{\Sigma_0} W_j(p) = \nabla_{W_i}^{N_{\Sigma_0/C}} W_4(p) = 0$ for $2 \le i, j \le 3$, and then use $W_{i+4} := W_i \times W_4$ for $1 \le i \le 3$ to show all other covariant derivatives in (57) vanish at p. Let $Z = \frac{1}{\sqrt{2}}(W_2 - iW_3) \in T\Sigma_{\mathbb{C}}^{1,0}$, $\overline{Z} = \frac{1}{\sqrt{2}}(W_2 + iW_3) \in T\Sigma_{\mathbb{C}}^{0,1}$, and $\overline{Z}^* = \frac{W_2^* - iW_3^*}{\sqrt{2}} \in T^*\Sigma_{\mathbb{C}}^{0,1}$, then $\left\langle \overline{Z}^*, \overline{Z} \right\rangle = 1$. We recall that the real bundle isomorphism $f : \wedge_{\mathbb{C}}^{0,1}(N_{\Sigma/C}) \simeq N_{\mathcal{C}/M}|_{\Sigma}$ from [**17**] (up to factor $-\frac{1}{\sqrt{2}}$) in local frame was given by

$$\begin{array}{rlcl} f: & \wedge_{\mathbb{C}}^{0,1}(N_{\Sigma/C}) = W_4 \otimes_{\mathbb{C}} T^*\Sigma^{0,1} & \longrightarrow & W_4 \times T\Sigma^{1,0} = N_{\mathcal{C}/M}|_{\Sigma}, \\ & W_4 \otimes_{\mathbb{C}} \overline{Z}^* & \mapsto & -\frac{1}{\sqrt{2}} W_4 \times Z, \end{array} \tag{58}$$

which is complex conjugate linear and independent on the choice of $W_4 \in N_{\Sigma/C}$. So we have the map

$$\Phi := id \oplus f : N_{\Sigma/C} \oplus \wedge_{\mathbb{C}}^{0,1}(N_{\Sigma/C}) \to N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_{\Sigma}$$

whose action on the basis of (rank two *real vector bundles*) $N_{\Sigma/C}$ and $\wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right)$ is the following:

$$\begin{aligned}
&\Phi : N_{\Sigma/C} \overset{id}{\to} N_{\Sigma/C}, \quad \{W_4, W_5\} \to \{W_4, W_5\}, \\
(59)& \\
&\Phi : \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right) \overset{f}{\to} N_{\mathcal{C}/M}|_{\Sigma}, \quad \left\{W_4 \otimes \overline{Z}^*, W_5 \otimes \overline{Z}^*\right\} \to \{W_6, W_7\}.
\end{aligned}$$

This is because under f,

$$\begin{aligned}
W_4 \otimes \left(\frac{W_2^* - iW_3^*}{\sqrt{2}}\right) &\to -\frac{1}{\sqrt{2}}\frac{W_4 \times W_2 - W_4 \times iW_3}{\sqrt{2}} \\
&= -\frac{-W_6 - (W_1 \times W_4) \times W_3}{2} = W_6, \\
W_5 \otimes \left(\frac{W_2^* - iW_3^*}{\sqrt{2}}\right) &= J_n W_4 \otimes \left(\frac{W_2^* - iW_3^*}{\sqrt{2}}\right) \\
&\to -J_n\left(W_6\right) = -W_1 \times W_6 = W_7,
\end{aligned}$$

where the second row is from the tensor property of f and the last row is because $f : \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right) \to N_{\mathcal{C}/M}|_{\Sigma}$ is complex conjugate linear. So for the sections

$$\begin{aligned}
U &= V^4 W_4 + V^5 W_5 + V^6 W_4 \otimes \overline{Z}^* + V^7 W_5 \\
&\quad \otimes \overline{Z}^* \in \Gamma\left(N_{\Sigma/M} \oplus \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right)\right), \\
V &= V^4 W_4 + V^5 W_5 + V^6 W_6 + V^7 W_7 \in \Gamma\left(N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_{\Sigma}\right),
\end{aligned}$$

we have

$$\Phi\left(U\right) = V.$$

We compute the Dolbeault Dirac operator using the above frame. The Clifford multiplication c on the Dolbeault complex $N_{\Sigma/C} \oplus \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right)$ at p is (see Section 1.4.3 in [**31**])

$$\begin{aligned}
&c\left(\frac{W_2 + iW_3}{\sqrt{2}}\right) : V^4 W_4 + V^5 W_5 \overset{\sqrt{2}e\left(\overline{Z}^*\right)}{\to} \sqrt{2}\left(V^4 W_4 \otimes \overline{Z}^* + V^5 W_5 \otimes \overline{Z}^*\right), \\
&c\left(\frac{W_2 - iW_3}{\sqrt{2}}\right) : V^6 W_4 \otimes \overline{Z}^* + V^7 W_5 \otimes \overline{Z}^* \overset{-\sqrt{2}i\left(\overline{Z}\right)}{\to} \sqrt{2}\left(-V^6 W_4 - V^7 W_5\right),
\end{aligned}$$

where $e\left(\overline{Z}^*\right)$ is the wedge by $\overline{Z}^*$ and $i\left(\overline{Z}\right)$ is the contraction by $\overline{Z}$. Using (57) at p and complex linearity, we have

$$\begin{aligned}
&\nabla_{\left(\frac{W_2 + iW_3}{\sqrt{2}}\right)}\left(V^4 W_4 + V^5 W_5\right)(p) \\
&= \frac{1}{\sqrt{2}}\left(V^4_2 W_4 + V^4_3 J_n W_4 + V^5_2 W_5 + V^5_3 J_n W_5\right) \\
&= \frac{1}{\sqrt{2}}\left(\left(V^4_2 - V^5_3\right) W_4 + \left(V^5_2 + V^4_3\right) W_5\right)
\end{aligned}$$

and

$$c\left(\frac{W_2+iW_3}{\sqrt{2}}\right)\circ\nabla_{\left(\frac{W_2+iW_3}{\sqrt{2}}\right)}\left(V^4W_4+V^5W_5\right)(p)$$
$$=\left(V_2^4-V_3^5\right)W_4\otimes\overline{Z}^*+\left(V_2^5+V_3^4\right)W_5\otimes\overline{Z}^*.$$

Similarly, we have

$$\nabla_{\left(\frac{W_2-iW_3}{\sqrt{2}}\right)}\left(V^6W_4\otimes\overline{Z}^*+V^7W_5\otimes\overline{Z}^*\right)(p)$$
$$=\frac{1}{\sqrt{2}}\left[\left(V_2^6+V_3^7\right)W_4\otimes\overline{Z}^*+\left(-V_3^6+V_2^7\right)W_5\otimes\overline{Z}^*\right]$$

and

$$c\left(\frac{W_2-iW_3}{\sqrt{2}}\right)\circ\nabla_{\left(\frac{W_2-iW_3}{\sqrt{2}}\right)}\left(V^6W_4\otimes\overline{Z}^*+V^7W_5\otimes\overline{Z}^*\right)(p)$$
$$=-\left(V_2^6+V_3^7\right)W_4-\left(-V_3^6+V_2^7\right)W_5.$$

The Dolbeault Dirac operator $\overline{\partial}$ on $N_{\Sigma/M}\oplus\wedge_{\mathbb{C}}^{0,1}\left(N_{\Sigma/M}\right)$ is defined by

$$\overline{\partial}=c\left(\frac{W_2+iW_3}{\sqrt{2}}\right)\circ\nabla_{\left(\frac{W_2+iW_3}{\sqrt{2}}\right)}+c\left(\frac{W_2-iW_3}{\sqrt{2}}\right)\circ\nabla_{\left(\frac{W_2-iW_3}{\sqrt{2}}\right)},$$

so at p we have

$$\overline{\partial}U(p)=-\left(V_2^6+V_3^7\right)W_4+\left(V_3^6-V_2^7\right)W_5$$
$$+\left(V_2^4-V_3^5\right)W_4\otimes Z^*+\left(V_2^5+V_3^4\right)W_5\otimes Z^*,$$

and

$$\Phi\left(\overline{\partial}U(p)\right)=-\left(V_2^6+V_3^7\right)W_4+\left(V_3^6-V_2^7\right)W_5$$
$$+\left(V_2^4-V_3^5\right)W_6+\left(V_2^5+V_3^4\right)W_7.$$

On the other hand, the twisted Dirac operator on $N:=N_{\Sigma/C}\oplus N_{\mathcal{C}/M}|_\Sigma$ is

$$D:=W_2\times\nabla_{W_2}^N+W_3\times\nabla_{W_3}^N. \tag{60}$$

Using (57) at p, we have

$$\begin{aligned}DV(p)&=\left(W_2\times\nabla_{W_2}^N+W_3\times\nabla_{W_3}^N\right)\left(V^4W_4+V^5W_5+V^6W_6+V^7W_7\right)\\&=\left(V_2^4W_2\times W_4+V_3^4W_3\times W_4\right)+\left(V_2^5W_2\times W_5+V_3^5W_3\times W_5\right)\\&+\left(V_2^6W_2\times W_6+V_3^6W_3\times W_6\right)+\left(V_2^7W_2\times W_7+V_3^7W_3\times W_7\right)\\&=-\left(V_2^6+V_3^7\right)W_4+\left(V_3^6-V_2^7\right)W_5\\&+\left(V_2^4-V_3^5\right)W_6+\left(V_3^4+V_2^5\right)W_7.\end{aligned}\tag{61}$$

Therefore

$$\Phi\left(\overline{\partial}U(p)\right)=DV(p).$$

Note that Dirac operators are independent on the choice of orthonormal basis at p, and our $p \in \Sigma$ is arbitrary, so $\Phi\left(\overline{\partial}U\right) = DV$ on Σ. The proof of the proposition is completed. q.e.d.

We now make the connection to Subsection 3.1. Taking $L = N_{\Sigma/C}$ in that subsection, then, $\mathbb{S}^{+} = N_{\Sigma/C}$ and $\mathbb{S}^{-} = \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right)$. Our assumption that Σ is regular implies $\dim H^{0,1}_{\overline{\partial}}\left(N_{\Sigma/C}\right) = 0$, for $H^{0,1}_{\overline{\partial}}\left(N_{\Sigma/C}\right)$ corresponds to the cokernel of $\overline{\partial}$. By the Dolbeault isomorphism, we have

$$\dim H^{1}\left(\Sigma, N_{\Sigma/C}\right) = \dim H^{0,1}_{\overline{\partial}}\left(N_{\Sigma/C}\right) = 0.$$

Since the dimension for the Seiberg-Witten moduli is 0, by the equivalence to Gromov-Witten moduli we have

$$\dim H^{0}\left(\Sigma, N_{\Sigma/C}\right) = \dim H^{1}\left(\Sigma, N_{\Sigma/C}\right) = 0,$$

for we only count J_n-holomorphic curves Σ of index 0. Hence for

$$\overline{\partial} : \Omega^{0}\left(N_{\Sigma/C}\right) \to \Omega^{0,1}\left(N_{\Sigma/C}\right)$$
$$\overline{\partial}^{*} : \Omega^{0,1}\left(N_{\Sigma/C}\right) \to \Omega^{0}\left(N_{\Sigma/C}\right)$$

in Subsection 3.1, $\ker \overline{\partial}$ and $\ker \overline{\partial}^{*}$ are trivial. Moreover, by Theorem 13, the linear operator

$$\mathcal{D} : L^{1,2}_{-}\left(\mathtt{A}_{\varepsilon}, \mathbb{S}\right) \to L^{2}\left(\mathtt{A}_{\varepsilon}, \mathbb{S}\right)$$

is one-to-one and onto.

4.2. Linearization of instanton equation and comparison with the operator $\mathcal{D}$. To prove the main theorem (Theorem 27), we will construct a map

$$F_{\varepsilon} : C^{m,\alpha}_{-}\left(N_{A'_{\varepsilon}/M}\right) \to C^{m-1,\alpha}\left(N_{A'_{\varepsilon}/M}\right)$$

such that the solution to the equation $F_{\varepsilon}(V) = 0$ will give rise to an associative submanifold (instanton) with boundary lying on $C_0 \cup C_{\varepsilon}$. The spaces $C^{m,\alpha}\left(N_{A'_{\varepsilon}/M}\right)$ and $C^{m,\alpha}_{-}\left(N_{A'_{\varepsilon}/M}\right)$ are defined by

$$C^{m}\left(N_{A'_{\varepsilon}/M}\right) := \left\{ V \in \Gamma\left(N_{A'_{\varepsilon}/M}\right) \middle| \, \|V\|_{C^{m}\left(N_{A'_{\varepsilon}/M}\right)} < +\infty \right\},$$

$$C^{m}_{-}\left(N_{A'_{\varepsilon}/M}\right) := \left\{ V \in \Gamma\left(N_{A'_{\varepsilon}/M}\right) \middle| \begin{array}{c} \|V\|_{C^{m}\left(N_{A'_{\varepsilon}/M}\right)} < +\infty, \text{ and} \\ V|_{\varphi(\{0\}\times\Sigma)} \subset TC_0, V|_{\varphi(\{\varepsilon\}\times\Sigma)} \subset TC_{\varepsilon}. \end{array} \right\}.$$

We construct a three dimensional submanifold $\mathtt{A}'_{\varepsilon} = \varphi\left(\mathtt{A}_{\varepsilon}\right) \subset M$ by flowing Σ along with C_t. For the spinor bundle $\mathbb{S} \to \mathtt{A}_{\varepsilon}$, we will construct an exponential-like map $\widetilde{\exp} : \mathbb{S} \to M$, with the following properties of the differential $d\widetilde{\exp}|_{\mathtt{A}_{\varepsilon}}$ on $\{0\} \times \Sigma$:

1) on fiber directions of $\mathbb{S}$, $d\widetilde{\exp}|_{\{0\}\times\Sigma} = (id, f) : N_{\Sigma/C} \oplus \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right) \to N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_{\Sigma}$, where $f : \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right) \to N_{\mathcal{C}/M}|_{\Sigma}$ is the real vector bundle isomorphism in (58);

2) on base directions of $\mathbb{S}$, $d\widetilde{\exp}|_{\{0\}\times\Sigma} = id : T\Sigma \to T\Sigma$ and $d\widetilde{\exp}|_{\{0\}\times\Sigma} : \frac{\partial}{\partial x_1} \to n(z)$;
3) on the boundary $\{0,\varepsilon\}\times\Sigma$ of $\mathtt{A}_\varepsilon$, $\widetilde{\exp}|_{\{0\}\times\Sigma}(\mathbb{S}^+\oplus 0) \subset C_0$, and $\widetilde{\exp}|_{\{\varepsilon\}\times\Sigma}(\mathbb{S}^+\oplus 0) \subset C_\varepsilon$.

The construction of the map $\widetilde{\exp}$ is somewhat technical. To keep the main flow of our paper, we postpone it to the appendix.

To make the linear theory developed in the previous section applicable, we compare the linearization $DF_\varepsilon(0)$ with the operator $\mathcal{D}$ in Section 3.1 by the following diagram to get the ε-dependent bound of its right inverse:

$$\begin{array}{ccc} C_-^{m,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S}) & \xrightarrow{\mathcal{D}} & C_-^{m-1,\alpha}(\mathtt{A}_\varepsilon,\mathbb{S}) \\ d\widetilde{\exp}\downarrow & & \downarrow d\widetilde{\exp} \\ C_-^{m,\alpha}\left(N_{\mathtt{A}'_\varepsilon/M}\right) & \xrightarrow{DF_\varepsilon(0)} & C^{m-1,\alpha}\left(N_{\mathtt{A}'_\varepsilon/M}\right) \end{array}$$

Now we define the nonlinear map F_ε with the important property that elements in $F_\varepsilon^{-1}(0)$ with small norm correspond to associative submanifolds in M near A'_ε for small ε. Given any $C_0\cup C_\varepsilon$, we modify the metric g near C_0 and C_ε to make them *totally geodesic*. We denote this new smooth metric by g_ε, and we make g_ε C^1-continuously depend on ε in our construction. Let $\exp^{g_\varepsilon}$ be the exponential map of g_ε. Then $\exp^{g_\varepsilon}$ has the following properties:

1) For sections V of $N_{\mathtt{A}'_\varepsilon/M}$ with C^0 norm smaller than a fixed constant δ_0 (depending on the uniform injectivity radius of the family of metrics $\{g_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$),
$$\exp^{g_\varepsilon} V : \mathtt{A}'_\varepsilon \to M$$
is a smooth embedding.
2) For $V\in C_-^{m,\alpha}\left(N_{A'_\varepsilon/M}\right)$, let
$$\mathtt{A}_\varepsilon(V) := \left(\exp^{g_\varepsilon} V\right)\left(\mathtt{A}'_\varepsilon\right). \tag{62}$$
Then $\mathtt{A}_\varepsilon(V)$ is a submanifold of M nearby $\mathtt{A}'_\varepsilon$ satisfying the boundary condition
$$\partial\mathtt{A}_\varepsilon(V)\subset C_0\cup C_\varepsilon. \tag{63}$$
3) For small $t\ge 0$, the family of embeddings $\exp^{g_\varepsilon}(tV) : \mathtt{A}'_\varepsilon \to M$ satisfies
$$\left.\frac{d}{dt}\right|_{t=0}\exp^{g_\varepsilon}(tV) = V. \tag{64}$$

Next we define $F_\varepsilon : C_-^{m,\alpha}\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right) \to C^{m-1,\alpha}\left(\mathtt{A}'_\varepsilon, N_{\mathtt{A}'_\varepsilon/M}\right)$,
$$F_\varepsilon(V) = *_{\mathtt{A}'_\varepsilon}\circ\perp_{\mathtt{A}'_\varepsilon}\circ\left(T_V\circ\left(\exp^{g_\varepsilon} V\right)^*\tau\right), \tag{65}$$
where

1) $\left(\exp^{g_\varepsilon} V\right)^*$ pulls back the *differential form part* of τ,

2) $T_V : T_{\exp_p^{g_\varepsilon}(tV)}M \to T_pM$ pulls back the *vector part* of τ by the parallel transport with respect to g along the path $\exp_p^{g_\varepsilon}(tV)$,
3) $\perp_{\mathtt{A}'_\varepsilon} : TM|_{\mathtt{A}'_\varepsilon} \to N_{\mathtt{A}'_\varepsilon/M}$ is the orthogonal projection with respect to g,
4) $*_{\mathtt{A}'_\varepsilon} : \Omega^3(\mathtt{A}'_\varepsilon) \to \Omega^0(\mathtt{A}'_\varepsilon)$ is the quotient by the volume form $dvol_{\mathtt{A}'_\varepsilon}$ induced from g.

We stress that g_ε is only used to construct a map $\exp^{g_\varepsilon} : N_{\mathtt{A}'_\varepsilon/M} \to M$ satisfying the coassociative boundary condition (63) and derivative condition (64). For our covariant derivatives, parallel transport, orthogonal projection, and volume form, we still use the original metric g.

To better understand F_ε, we let

$$P_\varepsilon := *_{\mathtt{A}'_\varepsilon} \circ \perp_{\mathtt{A}'_\varepsilon} : \Gamma\left(TM|_{\mathtt{A}'_\varepsilon}\right) \otimes \Omega^3\left(\mathtt{A}'_\varepsilon\right) \to \Gamma\left(N_{\mathtt{A}'_\varepsilon/M}\right),$$

and

$$F^{g_\varepsilon}(V) := T_V \circ \left(\exp^{g_\varepsilon} V\right)^* \tau. \tag{66}$$

Then

$$F_\varepsilon(V) = P_\varepsilon \circ F^{g_\varepsilon}(V).$$

At any $p \in \mathtt{A}'_\varepsilon$, both $\perp_{\mathtt{A}'_\varepsilon}$ and $*_{\mathtt{A}'_\varepsilon}$ are linear operators between finite dimensional spaces, so $\|P_\varepsilon\| \leq C$, where the constant C only depends on φ and is uniform for all $0 < \varepsilon \leq \varepsilon_0$. We notice that P_ε does not involve V, so the essential part of $F_\varepsilon(V)$ is $F^{g_\varepsilon}(V)$.

By Proposition 8, if $\mathtt{A}'_\varepsilon$ is sufficiently close to being associative, then there exists $\delta > 0$ such that for $\|V\|_{C^{1,\alpha}_-\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} < \delta$,

$$F_\varepsilon(V) = 0 \Leftrightarrow F^{g_\varepsilon}(V) = 0 \Leftrightarrow \mathtt{A}_\varepsilon(V) \text{ associative.}$$

We also note that $\partial \mathtt{A}_\varepsilon(V) \subset C_0 \cup C_\varepsilon$ for $V \in C^{1,\alpha}_-\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)$.

Before we compute $F^{g_\varepsilon\prime}(0)V$, we observe the following useful fact. The original exponential map $\exp^g : N_{\mathtt{A}'_\varepsilon/M} \to M$ does not satisfy the coassociative boundary condition (63), but satisfies the same derivative condition (64):

$$\left.\frac{d}{dt}\right|_{t=0} \exp^g(tV) = V = \left.\frac{d}{dt}\right|_{t=0} \exp^{g_\varepsilon}(tV).$$

For smooth maps $f : A'_\varepsilon \to M$ and a smooth form τ, the nonlinear map $\Gamma : f \to f^*\tau$ is differentiable with respect to f. Therefore

$$\left.\frac{d}{dt}\right|_{t=0} \left(\exp^g(tV)\right)^* \tau = \Gamma'(0)V = \left.\frac{d}{dt}\right|_{t=0} \left(\exp^{g_\varepsilon}(tV)\right)^* \tau. \tag{67}$$

Recall that the $F(V)$ (6) defined in our proof of McLean's theorem is

$$F(V) := T_V \circ \left(\exp^g V\right)^* \tau,$$

where the parallel transport T_V is with respect to g along the geodesic $\exp^g(tV)$. $F^{g_\varepsilon}(V)$ and $F(V)$ are different nonlinear maps, but (67) says that

$$F^{g_\varepsilon\prime}(0)\,V = F'(0)\,V. \tag{68}$$

We give an alternative proof of (68) in the following lemma. The proof gives more information of $F^{g_\varepsilon\prime}(0)\,V$ when there is a good frame field for the vector-valued form τ.

Lemma 19. *For any $V \in C_-^{1,\alpha}\left(N_{A'_\varepsilon/M}\right)$, $F^{g_\varepsilon\prime}(0)\,V = F'(0)\,V$.*

Proof. For any $p \in A'_\varepsilon$, we choose a frame field $\{W_a\}_{1\le\alpha\le7}$ in its neighborhood $B \subset A'_\varepsilon$, and then extend V and $\{W_a\}_{1\le\alpha\le7}$ to M by the parallel transport with respect to g along the curve $\exp_p^{g_\varepsilon}(tV)$. We write $\tau = \omega^\alpha \otimes W_\alpha$ in the neighborhood of p in M, following Einstein's summation convention. Similar to (12), we compute

$$F^{g_\varepsilon\prime}(0)\,V = d\left(i_V\omega^\alpha\right)\otimes W_\alpha + i_V d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_V W_\alpha,$$

where the covariant derivative ∇ is with respect to g, and we have used that $\frac{d}{dt}|_{t=0}\exp^{g_\varepsilon}(tV) = V$. By the parallel property of τ and W_α with respect to g, we have

$$i_V d\omega^\alpha \otimes W_\alpha = 0 = \omega^\alpha \otimes \nabla_V W_\alpha$$

(also see Remark 10, item 3). The first term $d\left(i_V\omega^\alpha\right)|_{A'_\varepsilon}$ only depends on the restriction of V and ω^α on A'_ε (Remark 10, item 1). Therefore,

$$F^{g_\varepsilon\prime}(0)\,V = d\left(i_V\omega^\alpha\right)\otimes W_\alpha|_{A'_\varepsilon} = F'(0)\,V.$$

q.e.d.

Remark 20. To apply the implicit function theorem to $F_\varepsilon(V)$, in the remaining part of our paper we will only need the estimate of $F'_\varepsilon(0)$, and the quadratic estimate (80) of $F'_\varepsilon(V)$. Because of the above lemma, to compute $F'_\varepsilon(0) = P_\varepsilon \circ F^{g_\varepsilon\prime}(0)$, we can replace the metric g_ε by g in (66). This will simplify the exposition in many places. For our quadratic estimate (80), the proof uses no feature of the G_2 metric g and is valid for any Riemannian metrics, including g_ε. The constant C in (80) is uniform for $\{g_\varepsilon\}_{0\le\varepsilon\le\varepsilon_0}$, since this is a compact family of metrics C^1-continuously depending on ε. So although we used g_ε in the definition of $F_\varepsilon(V)$ (65), from now on we will pretend g_ε is g in $F_\varepsilon(V)$, and we will simply write $\exp^g$ as exp.

Proposition 21. *For any section V_1 of $\mathbb{S}$ and section $V_2 := d\widetilde{\exp}\cdot V_1$ of $N_{\mathtt{A}'_\varepsilon/M}$, we have*

$$\left\|F'(0)\,V_2 - \left(d\widetilde{\exp}\circ \mathcal{D}V_1\right)\otimes dvol_{A'_\varepsilon}\right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} \le C\varepsilon^{1-\alpha}\left\|V_1\right\|_{C^{1,\alpha}(A_\varepsilon,\mathbb{S})},$$

and

$$\left\| F'_\varepsilon (0) V_2 - (\widetilde{d\exp}) \circ \mathcal{D} V_1 \right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} \leq C \varepsilon^{1-\alpha} \|V_1\|_{C^{1,\alpha}(A_\varepsilon, \mathbb{S})},$$

where $\mathcal{D}$ is the operator on $\mathbb{S}$ in the linear model, and the constant C is uniform for all ε.

Proof. For each $p = \varphi(0,z)$ in $\Sigma_0 := \varphi(\{0\} \times \Sigma) \subset A'_\varepsilon$, we choose "good" frame $\{W_\alpha\}_{\alpha=1,2,\dots 7}$ on $B_\varepsilon(z) \subset \Sigma$ as before. Then we extend the frame to $U_\varepsilon(p) := \varphi([0,\varepsilon] \times B_\varepsilon(z))$ such that

$$W_\alpha(\varphi(t,z)) = T_\gamma \cdot W_\alpha(\varphi(0,z)) \quad \text{for } \alpha = 1,2,\dots 7,$$

where T_γ is the parallel transport along the path $\gamma := \varphi([0,t] \times \{z\})$ by the Levi-Civita connection on M. We further extend $\{W_\alpha\}_{\alpha=1,2,\dots 7}$ to a tubular neighborhood of $U_\varepsilon(p) \subset M$ by parallel transport in fiber directions of $N_{A'_\varepsilon/M}$. Both $F'(0) V_2$ and $\mathcal{D} V_1$ are globally defined on A'_ε and A_ε respectively. We are going to compare them in $U_\varepsilon(p)$ using the "good" frame. We write $\tau = \omega^\alpha \otimes W_\alpha$ in the neighborhood of p. By the parallel property of $\{W_\alpha\}_{\alpha=1,2,\dots 7}$, the 3-forms ω^α are similar to the standard ones in $\operatorname{Im}\mathbb{O}$, in the sense that dx_i is replaced by $(W_i)^*$ for $i = 1,2,\dots 7$. We still have the formula (12) at q:

$$\begin{aligned} F'(0) V_2(q) &= d(i_{V_2}\omega^\alpha) \otimes W_\alpha(q) \\ &\quad + i_{V_2} d\omega^\alpha \otimes W_\alpha(q) + \omega^\alpha \otimes \nabla_{V_2} W_\alpha(q). \end{aligned} \tag{69}$$

The first term $d(i_{V_2}\omega^\alpha) \otimes W_\alpha(q)$ is the principal symbol part of the differential operator $F'(0)$. We claim that

$$d(i_{V_2}\omega^\alpha) \otimes W_\alpha = (\widetilde{d\exp} \cdot \mathcal{D} V_1) \otimes dvol_{A'_\varepsilon} + E(q) V_1 \tag{70}$$

where $E(q)$ is a smooth tensor on $U_\varepsilon(p)$ with $\|E\|_{C^1(U_\varepsilon(p))} \leq C_3 \varepsilon$. To see this, we first compute $d(i_{V_2}\omega^\alpha) \otimes W_\alpha$. For section V_2 of $N_{A'_\varepsilon/M}$, we write

$$V_2(q) = \Sigma_{\alpha=4}^7 \phi^\alpha(q) W_\alpha(q).$$

Similar to our proof of McLean's theorem, we have

$$d(i_{V_2}\omega^\alpha) \otimes W_\alpha(q)|_{T_q A'_\varepsilon} = \mathcal{D} V_2(q)\, dvol_{A'_\varepsilon} + E_1(q) V_2 \tag{71}$$

where

$$\begin{aligned} \mathcal{D} V_2(q) &= -\left(\phi_1^5 + \phi_2^6 + \phi_3^7\right) W_4 + \left(\phi_1^4 + \phi_3^6 - \phi_2^7\right) W_5 \\ &\quad + \left(\phi_2^4 - \phi_3^5 + \phi_1^7\right) W_6 + \left(\phi_3^4 + \phi_2^5 - \phi_1^6\right) W_7, \end{aligned} \tag{72}$$

and

$$\phi_i^k(q) := d\phi^k(q)(W_i).$$

The reason is the following: From our construction of W_α, $\nabla_{W_i} W_j(p) = 0$ for $1 \leq i,j \leq 7$, and ω^α are similar to the standard ones in $\operatorname{Im}\mathbb{O}$. Thus when $q = p$,

$$d(i_{V_2}\omega^\alpha) \otimes W_\alpha(p)|_{T_p A'_\varepsilon} = \mathcal{D} V_2(p)\, dvol_{A'_\varepsilon}.$$

If q is ε-close to p, then in (71) the error term $E_1(q)$ is of order ε in C^1 norm, because $\nabla_{W_i} W_j(q) = o(\varepsilon)$ in C^1 and $\operatorname{span}\{W_\alpha(q)\}_{\alpha=1,2,3}$ has ε-order deviation from $T_q A'_\varepsilon$ in C^1. We want to show

$$\mathcal{D}V_2(q) = d\widetilde{\exp} \circ \left[\left(h^{-1/2}(z)\, e_1 \cdot \frac{\partial}{\partial x_1} + \bar{\partial}\right) V_1\right] + E_2(q)\, V_2, \tag{73}$$

where the error term $E_2(q)$ is of order ε in C^1 norm, and the $\bar{\partial}$ is the Dolbeault Dirac operator on $N_{\Sigma/C} \oplus \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right)$. Similar to our argument for $E_1(q)$, it is enough to show

$$\mathcal{D}V_2(p) = d\widetilde{\exp}|_{\{0\}\times\Sigma} \circ \left[\left(h^{-1/2}(z)\, e_1 \cdot \frac{\partial}{\partial x_1} + \bar{\partial}\right) V_1(0,z)\right].$$

We observe that

$$\mathcal{D}V_2(p) = DV_2(p) - \phi_1^5 W_4 + \phi_1^4 W_5 + \phi_1^7 W_6 - \phi_1^6 W_7$$

from (72) and (61), where D is the twisted Dirac operator (60) of $N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma$ over Σ in the previous subsection. By Proposition 18, we have

$$d\widetilde{\exp}|_{\{0\}\times\Sigma} \cdot \bar{\partial} V_1 = \Phi\left(\bar{\partial} V_1\right) = DV_2.$$

So it is enough to prove, at $p = \varphi(0,z)$, that

$$d\widetilde{\exp} \circ \left[h^{-1/2}(z)\, e_1 \cdot \frac{\partial}{\partial x_1} V_1(0,z)\right] = \left(-\phi_1^5 W_4 + \phi_1^4 W_5 + \phi_1^7 W_6 - \phi_1^6 W_7\right)(p). \tag{74}$$

From

$$V_2(q) = \Sigma_{\alpha=4}^7 \phi^\alpha(q)\, W_\alpha(q) = \Sigma_{\alpha=4}^7 \phi^\alpha(q) \cdot T_\gamma W_\alpha(\varphi(0,z))$$

we get

$$\begin{aligned} V_1(t,z) &= \Sigma_{\alpha=4}^7 \phi^\alpha(\varphi(t,z)) \cdot (d\widetilde{\exp})^{-1}\, T_\gamma W_\alpha(\varphi(0,z)) \\ &= \Sigma_{\alpha=4}^7 \phi^\alpha(\varphi(t,z)) \cdot \Phi^{-1} W_\alpha(\varphi(0,z)) + E_3(q)\, V_2, \end{aligned}$$

where $\left\{\Phi^{-1} W_\alpha(\varphi(0,z))\right\}_{\alpha=4}^7$ is regarded as a frame of $\mathbb{S} \to A_\varepsilon$ that is invariant along the t direction, and $E_3(q)$ is of order ε in C^1 norm. The second identity of $V_1(t,z)$ is because the maps $d\widetilde{\exp}$ and T_γ, at $t=0$, are maps Φ and the identity map respectively on $\operatorname{span}\{W_\alpha\}_{4\le\alpha\le7}$, and for $0 \le t \le \varepsilon$ we have the error term $E_3(q)$ of desired order by smoothness of φ, ϕ^α, and W_α. Let

$$\psi^\alpha(x_1,z) := \phi^\alpha(\varphi(x_1,z)), \text{ and } \psi_1^\alpha(x_1,z) = \frac{\partial}{\partial x_1}\psi^\alpha(x_1,z).$$

We have

$$\begin{aligned} & e_1 \cdot \frac{\partial}{\partial x_1}\left[\Sigma_{\alpha=4}^7 \psi^\alpha(x_1,z) \cdot \Phi^{-1} W_\alpha(\varphi(0,z))\right] \\ &= \Phi^{-1}\left[W_1 \times \Sigma_{\alpha=4}^7 \psi_1^\alpha(x_1,z) \cdot W_\alpha(\varphi(0,z))\right] \\ &= \Phi^{-1}\left[-\psi_1^5 W_4 + \psi_1^4 W_5 + \psi_1^7 W_6 - \psi_1^6 W_7\right]. \end{aligned} \tag{75}$$

By the chain rule, for $4 \leq \alpha \leq 7$,

$$\psi_1^\alpha(0,z) = \nabla_{W_1}\phi^\alpha(\varphi(0,z)) \cdot h(z)^{\frac{1}{2}} = h(z)^{\frac{1}{2}}\,\phi_1^\alpha(\varphi(0,z)) \tag{76}$$

where the factor $h(z)^{\frac{1}{2}}$ is from $\left\langle W_1, \frac{d}{dt}\varphi(t,z)\,|_{t=0}\right\rangle = h(z)^{\frac{1}{2}}$. Comparing (74) and (75), and noticing (76), we have

$$d\widetilde{\exp}|_{\{0\}\times\Sigma} \circ \mathcal{D}V_1 = \mathcal{D}V_2.$$

Therefore, for q that is ε-close to p, claim (70) is proved. So we get

$$\begin{aligned} &\left\| d\left(i_{V_2}\omega^\alpha\right) \otimes W_\alpha - \left(d\widetilde{\exp} \circ \mathcal{D}V_1\right) \otimes dvol_{A'_\varepsilon} \right\|_{C^\alpha(U_\varepsilon(p))} \\ &= \left\| E(q)\,V_2 \right\|_{C^\alpha(U_\varepsilon(p))} \\ &\leq C_1\varepsilon^{1-\alpha}\left\|V_2\right\|_{C^{1,\alpha}(U_\varepsilon(p))}, \end{aligned} \tag{77}$$

where the constant C_1 is uniform for all ε. The remaining term

$$BV_2 := i_{V_2}d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_{V_2}W_\alpha$$

is a 0-th order linear operator on V_2, where $B = B(q)$ is a smooth tensor on A'_ε. Since $\nabla\tau = 0$, the same as in our proof in Theorem 9, we have

$$i_W d\omega^\alpha(p) = \nabla_W W_\alpha(p) = 0 \text{ for all } W \in N_{A'_\varepsilon/M}(p),$$

so we have $B(p)\,|_{N_{A'_\varepsilon/M}(p)} = 0$. Since $\mathrm{dist}(p,q) \leq C_0\varepsilon$ for some uniform constant C_0, and $N_{A'_\varepsilon/M}$ is a smooth fiber bundle over A'_ε, we conclude that the operator norm

$$\left\| B(q)\,|_{N_{A'_\varepsilon/M}(q)} \right\| \leq C_2\varepsilon$$

for some uniform constant C_2. From this it is easy to prove

$$\left\|BV_2\right\|_{C^\alpha(U_\varepsilon(p))} \leq C_3\varepsilon^{1-\alpha}\left\|V_2\right\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}. \tag{78}$$

The constant C_3 is uniform for all ε, by the compactness of Σ and finite covering of A'_ε by $U_\varepsilon(p)$. Putting (78) and (77) in (69), we have

$$\begin{aligned} &\left\| F'(0)\,V_2 - \left(d\widetilde{\exp} \cdot \mathcal{D}V_1\right) \otimes dvol_{A'_\varepsilon} \right\|_{C^\alpha(U_\varepsilon(p))} \\ &\leq C_4\varepsilon^{1-\alpha}\left\|V_2\right\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} \\ &\leq C_5\varepsilon^{1-\alpha}\left\|V_1\right\|_{C^{1,\alpha}(A_\varepsilon,\mathbb{S})} \end{aligned}$$

where the constants C_4 and C_5 are uniform for all ε. Using finitely many $U_\varepsilon(p)$ covering A'_ε and then taking supremum, we have

$$\left\| F'(0)\,V_2 - \left(d\widetilde{\exp} \cdot \mathcal{D}V_1\right) \otimes dvol_{A'_\varepsilon} \right\|_{C^\alpha(A'_\varepsilon)} \leq C\varepsilon^{1-\alpha}\left\|V_1\right\|_{C^{1,\alpha}(A_\varepsilon,\mathbb{S})}. \tag{79}$$

Applying P_ε on the left-hand side of the above inequality, and noticing that $(d\widetilde{\exp})\,|_{\{0\}\times\Sigma} \circ \mathcal{D}V_1 = \mathcal{D}V_2$ is a section of $N_{A'_\varepsilon/M}$, we have

$$P_\varepsilon \circ \left(\left(d\widetilde{\exp}\right) \circ \mathcal{D}V_1(t,z) \otimes dvol_{A'_\varepsilon}\right) = \left(d\widetilde{\exp}\right) \circ \mathcal{D}V_1(t,z) + E_3(q)\,V_1,$$

where $E_3(q)$ is of order ε in C^1 norm, because at $q = \varphi(t,z)$ the vector $(d\widetilde{\exp}) \circ \mathcal{D}V_1(t,z)$ may have a component of order ε in C^1 norm orthogonal to $N_{A'_\varepsilon/M}(q)$. So we get

$$\begin{aligned}
&\left\| F'_\varepsilon(0) V_2 - (d\widetilde{\exp}) \circ \mathcal{D}V_1 \right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} \\
&\leq \left\| P_\varepsilon \cdot \left(F'(0) V_2 - (d\widetilde{\exp}) \circ \mathcal{D}V_1 \otimes dvol_{A'_\varepsilon}\right) \right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} \\
&+ \left\| E_3(q) V_1 \right\|_{C^\alpha(A'_\varepsilon)} \\
&\leq C\varepsilon^{1-\alpha} \left\| V_1 \right\|_{C^{1,\alpha}(A_\varepsilon, \mathbb{S})}
\end{aligned}$$

where C is a uniform constant independent on ε. q.e.d.

Proposition 22. *There exists a right inverse* $\widetilde{Q_\varepsilon}^{true}$ *of* $F'_\varepsilon(0)$*, such that* $\left\| \widetilde{Q_\varepsilon}^{true} \right\| \leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}$*, where the constant* C *is uniform for all* $0 < \varepsilon \leq \varepsilon_0$*.*

Proof. From Theorem 14, for operator $\mathcal{D}$ on spinor bundle $\mathbb{S}$ over $\mathtt{A}_\varepsilon$, there is a right inverse Q_ε of $\mathcal{D}$ such that $\|Q_\varepsilon\| \leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}$. Let

$$\begin{aligned}
\widetilde{\mathcal{D}} &= d\widetilde{\exp} \circ \mathcal{D} \circ (d\widetilde{\exp})^{-1}, \\
\widetilde{Q_\varepsilon} &= d\widetilde{\exp} \circ Q_\varepsilon \circ (d\widetilde{\exp})^{-1},
\end{aligned}$$

then

$$\left\| \widetilde{Q_\varepsilon} \right\| \leq \|d\widetilde{\exp}\| \cdot C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} \cdot \left\| (d\widetilde{\exp})^{-1} \right\| \leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)},$$

and

$$\begin{aligned}
F'_\varepsilon(0) \widetilde{Q_\varepsilon} - id &= \left(F'_\varepsilon(0) - \widetilde{\mathcal{D}} + \widetilde{\mathcal{D}}\right) \widetilde{Q_\varepsilon} - id \\
&= \left(F'_\varepsilon(0) - \widetilde{\mathcal{D}}\right) \widetilde{Q_\varepsilon} + \widetilde{\mathcal{D}}\widetilde{Q_\varepsilon} - id \\
&= \left(F'_\varepsilon(0) - \widetilde{\mathcal{D}}\right) \widetilde{Q_\varepsilon},
\end{aligned}$$

where the last identity is because $\widetilde{\mathcal{D}}\widetilde{Q_\varepsilon} = d\widetilde{\exp} \circ \mathcal{D}Q_\varepsilon \circ (d\widetilde{\exp})^{-1} = id$. From the previous proposition, $\left\| \left(F'_\varepsilon(0) - \widetilde{\mathcal{D}}\right) \right\| \leq C\varepsilon^{1-\alpha}$. Therefore

$$\begin{aligned}
\left\| F'_\varepsilon(0) \widetilde{Q_\varepsilon} - id \right\| &\leq \left\| \left(F'_\varepsilon(0) - \widetilde{\mathcal{D}}\right) \right\| \left\| \widetilde{Q_\varepsilon} \right\| \\
&\leq C\varepsilon^{1-\alpha} \cdot C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} < \frac{1}{2}
\end{aligned}$$

when ε is sufficiently small, by our assumption that $1 - \left(\frac{3}{p} + 3\alpha\right) > 0$. So $\widetilde{Q_\varepsilon}$ is an approximate right inverse of $F'_\varepsilon(0)$. The true right inverse

$\widetilde{Q}^{true}$ of $F'_\varepsilon(0)$ is $\widetilde{Q}^{true} = \widetilde{Q_\varepsilon}\left(F'_\varepsilon(0)\widetilde{Q_\varepsilon}\right)^{-1}$ and

$$\left\|\widetilde{Q}^{true}\right\| \leq \left\|\widetilde{Q_\varepsilon}\right\| \left\|\left(F'_\varepsilon(0)\widetilde{Q_\varepsilon}\right)^{-1}\right\| \leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} \cdot 2 \leq C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}.$$

q.e.d.

4.3. Quadratic estimates.

Proposition 23. *There exists $\delta_0 > 0$ such that for all sections V_0, V of $N_{A'_\varepsilon/M}$ that $\|V_0\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)} < \delta_0$, we have*

$$\begin{aligned}&\left\|F'_\varepsilon(V_0)V - F'_\varepsilon(0)V\right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\\ &\leq C\|V_0\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\|V\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)},\end{aligned} \tag{80}$$

where the constant C is independent on ε.

Proof. Because $F_\varepsilon(V) = P_\varepsilon \circ F(V)$ and P_ε is a bounded linear operator independent on V, it is enough to prove the quadratic estimate of $F(V)$, namely

$$\begin{aligned}&\left\|F'(V_0)V - F'(0)V\right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\\ &\leq C\|V_0\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\|V\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}.\end{aligned}$$

For any $p = \varphi(t,z)$ on A'_ε, we choose a normal frame field $\{W_\alpha\}_{\alpha=1,2,\ldots,7}$ in its neighborhood $B_{\delta_0}(p)$ where $\tau = \omega^\alpha \otimes W_a$, with $\delta_0 =$ the injectivity radius of M. For the point $q := \exp_p V_0$ and if $|V_0| < \delta_0$, then we can assume this neighborhood covers q. Let the submanifold

$$A_\varepsilon(V) =: (\exp V)\left(A'_\varepsilon\right),$$

which is similar to (62), except that the section V here is in $N_{A'_\varepsilon/M}$ instead of $\mathbb{S}$, and the $\exp : N_{A'_\varepsilon/M} \to M$ is the exponential map on M. For the family of embeddings of submanifolds

$$\psi_t := \exp(V_0 + tV) : \mathtt{A}_\varepsilon \to A_\varepsilon(V_0 + tV) \subset M,$$

we have

$$\begin{aligned}\psi_0(\mathtt{A}_\varepsilon) &= A_\varepsilon(V_0),\\ \left.\frac{d}{dt}\right|_{t=0}\psi_t(q) &= (d\exp_p V_0)V(p) := V_1(q)\end{aligned}$$

for any $p \in A'_\varepsilon = A_\varepsilon(0)$. Arguing as in Theorem 9, from $\nabla\tau = 0$ we have $\nabla W_\alpha(q) = d\omega^\alpha(q) = 0$. We have

$$\begin{aligned}
&F'(V_0)\,V(p)\\
&= \left.\frac{d}{dt}\right|_{t=0} F(V_0+tV)(p)\\
&= \left.\frac{d}{dt}\right|_{t=0} \left[\exp(V_0+tV)^*\,\omega^\alpha(p)\otimes T_{V_0+tV}W_\alpha\left(\exp_p(V_0+tV)\right)\right]\\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\exp(V_0+tV)^*\,\omega^\alpha\right)\otimes W_\alpha(p)\\
&+ (\exp V_0)^*\,\omega^\alpha(p)\otimes \left.\frac{d}{dt}\right|_{t=0} T_{V_0+tV}W_\alpha\left(\exp_p(V_0+tV)\right),
\end{aligned} \tag{81}$$

where in the last equality we have used $T_{V_0+tV}W_\alpha\left(\exp_p(V_0+tV)\right) = W_\alpha(p)$ by the parallel property of W_α. For the first term of (81), by the property of the exponential map, namely $d\exp_q(0) = id_{T_qM}$, we have

$$\exp_p(V_0+tV) = \exp_q\left(tV_1 + t\,|V|\,\beta(V_0,V,t)\right)\circ\exp_p V_0$$

where $V_1 = \left(d\exp_{A'_\varepsilon} V_0\right)V$ is a vector field on $A_\varepsilon(V_0)$, $\beta(V_0,V,t)$ and $\beta'(V_0,V,t)$ are uniformly bounded, and $\lim_{t\to 0}\beta(V_0,V,t) = 0$. Therefore

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} \exp(V_0+tV)^*\,\omega^\alpha(p)\\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\exp_{A'_\varepsilon} V_0\right)^* \exp_{A_\varepsilon(V_0)}\left(tV_1 + t\,|V|\,\beta(V_0,V,t)\right)^*\omega^\alpha\\
&= \left(\exp_p V_0\right)^* L_{V_1}\omega^\alpha\\
&= \left(\exp_p V_0\right)^* \left(d\left(i_{V_1}\omega^\alpha\right) + i_{V_1}d\omega^\alpha\right),
\end{aligned}$$

where in the third equality we have used that $\lim_{t\to 0}\beta(V_0,V,t) = 0$. For the second term of (81), by the Gauss-Bonnet formula, which compares the parallel transports along different paths by curvature integration, we have

$$\begin{aligned}
&T_{V_0+tV}W_\alpha\left(\exp_p(V_0+tV)\right)\\
&= T_{V_0}\circ T_{tV_1+t|V|\beta(V_0,V,t)}W_\alpha\left(\exp_q\left(tV_1+t\,|V|\,\beta(V_0,V,t)\right)\right)\\
&+ \int_{\Delta(V_0,tV)} R(\sigma)\,d\sigma W_\alpha(p),
\end{aligned}$$

where $R(\sigma)$ is the Ricci curvature tensor, $\Delta(V_0,tV)$ is the 2-dimensional geodesic triangle with the vertices $p, \exp_p V_0$, and $\exp_p(V_0+tV)$, and σ

is the area element. Therefore

$$\begin{aligned}
&\left.\frac{d}{dt}\right|_{t=0} T_{V_0+tV} W_\alpha \left(\exp_p \left(V_0+tV\right)\right) \\
&= T_{V_0} \circ \left.\frac{d}{dt}\right|_{t=0} T_{tV_1} W_\alpha \left(\exp_q tV_1\right) + \left.\frac{d}{dt}\right|_{t=0} \int_{\Delta(V_0,tV)} R(\sigma)\, d\sigma W_\alpha(p),
\end{aligned}$$

where in the first term we have dropped the higher order term $t\,|V|\,\beta\,(V_0,V,t)$ since it is irrelevant for the derivative at 0. If we denote

$$B(V_0,V) := \left.\frac{d}{dt}\right|_{t=0} \int_{\Delta(V_0,tV)} R(\sigma)\, d\sigma,$$

by the area formula of $\Delta\,(V_0,tV)$ it can be shown that

$$|B(V_0,V)\, W_\alpha(p)| \le C_5\, |V_0|\, |V|\, |W_\alpha|\,(p),$$

where C_5 is a constant only depending on (M,g). Plugging these into (81), we have

$$\begin{aligned}
&F'(V_0)\,|_{A_\varepsilon(0)} V(p) \\
&= (\exp V_0)^* L_{V_1}\omega^\alpha \otimes W_\alpha(p) + (\exp V_0)^* \omega^\alpha \otimes T_{V_0} \nabla_{V_1} W_\alpha(q) \\
&\quad + B(V_0,V)\, W_\alpha(p) \\
&= ((\exp V_0)^* \otimes T_{V_0}) \circ [L_{V_1}\omega^\alpha \otimes W_\alpha(q) + \omega^\alpha(q) \otimes \nabla_{V_1} W_\alpha(q)] \\
&\quad + B(V_0,V)\, W_\alpha(p) \\
&= ((\exp V_0)^* \otimes T_{V_0}) \circ [(d(i_{V_1}\omega^\alpha) + i_{V_1} d\omega^\alpha) \otimes W_\alpha(q) \\
&\quad + \omega^\alpha(q) \otimes \nabla_{V_1} W_\alpha(q)] \\
&\quad + B(V_0,V)\, W_\alpha(p).
\end{aligned} \tag{82}$$

Here we have used that $T_{V_0} W_\alpha(q) = W_\alpha(p)$. Equation (82) can be rewritten as

$$\begin{aligned}
F'(V_0)\,|_{A_\varepsilon(0)} V(p) = ((\exp V_0)^* \otimes T_{V_0}) \circ \left[F'(0)\,|_{A_\varepsilon(V_0)} V_1(q)\right] \\
+ B(V_0,V)\, W_\alpha(p),
\end{aligned}$$

which means the derivative $F'(V_0)$ on $A_\varepsilon(0)$ can be expressed by the derivative $F'(0)$ on $A_\varepsilon(V_0)$ via the transform $(\exp V_0)^* \otimes T_{V_0}$, up to the curvature term $B(V_0,V)\, W_\alpha(p)$. If $V_0 = 0$, then $q = p$ and $V_1 = V$, together with

$$i_V d\omega^\alpha(p) = \nabla_V W_\alpha(p) = B(0,V) = 0,$$

the formula (82) is simplified as

$$F'(0)\,|_{A'_\varepsilon} V(p) = d(i_V\omega^\alpha) \otimes W_\alpha(p).$$

Therefore

$$F'(V_0)V(p) - F'(0)\,V(p)$$
$$= [(\exp V_0)^* \, d(i_{V_1}\omega^\alpha) - d(i_V\omega^\alpha)]\,|_{A_{\varepsilon(0)}} \otimes W_\alpha(p) + \tag{83}$$
$$((\exp V_0)^* \otimes T_{V_0}) \circ [i_{V_1} d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_{V_1} W_\alpha]\,(p) + B(V_0, V)W_\alpha(p). \tag{84}$$

For the second line (84), it is a 0-th order linear differential operator on $V \in C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right)$, where

$$V_1 = (d\exp V_0)\,|_{A_\varepsilon(0)} V = E\,(V_0)\,V.$$

We notice that for the linear operator

$$H : C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right) \to C^\alpha\left(\Gamma\left(Hom\left(TM|_{A'_\varepsilon}, TM|_{A_\varepsilon(V_0)}\right)\right)\right)$$

that sends

$$V_0 \to ((\exp V_0)^* \otimes T_{V_0}) \circ \left[i_{E(V_0)(\cdot)} d\omega^\alpha \otimes W_\alpha + \omega^\alpha \otimes \nabla_{E(V_0)(\cdot)} W_\alpha\right] + B\,(V_0, \cdot)\,W_\alpha,$$

H is a bounded operator since the terms $\exp V_0, (\exp V_0)^*, T_{V_0}, E(V_0)$, and $B(V_0, \cdot)$ as elements in C^α are differentiable for $V_0 \in C^{1,\alpha}(\Gamma(N_{A'_\varepsilon/M}))$ with bounded Frechet derivatives. The bound of the derivatives is uniform on ε, so $\|H\|$ is uniformly bounded. Since $H(V_0) = 0$ when $V_0 = 0$, we have

$$\|H\,(V_0,V)\|_{C^\alpha} \leq C_8\,\|V_0\|_{C^{1,\alpha}(A'_\varepsilon)}\,\|V\|_{C^{1,\alpha}(A'_\varepsilon)}\,.$$

For the first line (83), $[(\exp V_0)^*\, d\,(i_{V_1}\omega^\alpha) - d\,(i_V\omega^\alpha)]$ is a first order linear differential operator on V $\left(V_1 = (d\exp V_0)\,|_{A_\varepsilon(0)} V\right)$, and only involves the covariant derivatives of V. In the next lemma we will show

$$\|(\exp V_0)^*\, d\,(i_{V_1}\omega^\alpha) - d\,(i_V\omega^\alpha)\|_{C^\alpha(A'_\varepsilon)} \leq C_7\,\|V_0\|_{C^{1,\alpha}(A'_\varepsilon)}\,\|V\|_{C^{1,\alpha}(A'_\varepsilon)}\,.$$

Combining the two lines we get

$$\begin{aligned}&\left\|F'(V_0)\,V - F'(0)\,V\right\|_{C^\alpha\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\\ &\leq C\,\|V_0\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\,\|V\|_{C^{1,\alpha}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}.\end{aligned}$$

q.e.d.

Lemma 24. *For any $V_0, V \in \Gamma\left(N_{A'_\varepsilon/M}\right)$ with $\|V_0\|_{C^{1,\alpha}(A'_\varepsilon)} \leq \delta_0$, we have*

$$\|(\exp V_0)^*\, d\,(i_{V_1}\omega^\alpha) - d\,(i_V\omega^\alpha)\|_{C^\alpha(A'_\varepsilon)} \leq C_7\,\|V_0\|_{C^{1,\alpha}(A'_\varepsilon)}\,\|V\|_{C^{1,\alpha}(A'_\varepsilon)}\,,$$

where the constants δ_0 and C_7 are independent on ε.

Proof. Consider the 3 linear maps from T_pM to T_qM, where $q = \exp_p V_0$:

$$\begin{aligned} Pal_{V_0}: T_pM \to T_qM,&\ \text{parallel transport along the geodesic} \\ &\ \exp_p tV_0 \text{ for } 0 \le t \le 1, \\ d(\exp V_0): T_pM \to T_qM,&\ \text{tangent map of the diffeomorphism} \\ &\ \exp V_0 : A'_\varepsilon \to A_\varepsilon(V_0), \\ d\left(\exp_p\right)(V_0): T_{V_0}(T_pM) \to T_qM,&\ \text{tangent map of the exponential map} \\ &\ \exp_p : T_pM \to M, \end{aligned}$$

where in the last one we identify $T_{V_0}(T_pM) \simeq T_pM$. Note that for diffeomorphism $d(\exp V_0)$ we need a vector field V_0 on A'_ε, but for $d\left(\exp_p\right)(V_0)$ we only need $V_0 \in T_pM$, so they are essentially different maps. But the 3 maps are very close to each other when V_0 is small. More precisely, for any $V_0, V \in \Gamma\left(N_{A'_\varepsilon/M}\right)$ with $\|V_0\|_{C^{1,\alpha}(A'_\varepsilon)} \le \delta_0$, we have comparison of the maps as in the following:

$$\begin{aligned} d(\exp V_0)V &= Pal_{V_0}V + h_1(V_0)V, \\ d\left(\exp_p\right)(V_0)V &= d(\exp V_0)V + h_2(V_0)V, \end{aligned}$$

where for each $i = 1, 2$, the error term

$$h_i(\cdot) : C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right) \to C^\alpha\left(\Gamma\left(Hom\left(TM|_{A'_\varepsilon}, TM|_{A_\varepsilon(V_0)}\right)\right)\right)$$

is differentiable with respect to the variable V_0, and

$$h_i(V_0) = 0 \text{ for } V_0 = 0.$$

This is because for any fixed p, $Pal_{V_0(p)}$ and $d\left(\exp_p\right)(V_0)$ smoothly depend on V_0, and for the compact family $p \in A'_\varepsilon$, Pal_{V_0} and $d\left(\exp_p\right)(V_0)$ inherit the smooth dependence on $V_0 \in C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right)$. For $d(\exp V_0)$, since

$$d(\exp V_0)(x) = F_1(x, V_0)\,dx + F_2(x, V_0)\,dV_0(x)$$

where $F_1(x,y) = \frac{\partial}{\partial x}\exp_x y$ and $F_2(x,y) = \frac{\partial}{\partial y}\exp_x y$ are bounded on A_ε, we see $d(\exp V_0) \in C^\alpha$ smoothly depends on $V_0 \in C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right)$ with bounded Frechet derivative. This especially implies that

$$\|h_i(V_0)V\|_{C^a} \le C_8\|V_0\|_{C^{1,\alpha}(A'_\varepsilon)}\|V\|_{C^{1,\alpha}(A'_\varepsilon)} \tag{85}$$

for $i = 1, 2$. We have

$$\begin{aligned} V_1 &= d(\exp V_0)V + h_2(V_0)V, \\ d(\exp V_0)W_\alpha(p) &= W_a(q) + h_1(V_0)W_\alpha(p), \\ (\exp V_0)^*\omega^\alpha(q) &= \omega^\alpha(p) + h_3(V_0)\omega^\alpha(q) \end{aligned}$$

using the parallel property of W_a and τ (hence ω^α). (Here $h_3(V_0)$ depends on $V_0 \in C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right)$ smoothly by similar reason of h_1 and

h_2. Since ω^α is a 3-form, $h_3(V_0)$ can have terms of cubic power order of V_0.) Therefore

$$\begin{aligned}
&(\exp V_0)^* d(i_{V_1}\omega^\alpha) - d(i_V\omega^\alpha)\\
&= d\left[(\exp V_0)^*\left(i_{d(\exp V_0)V+h_2(V_0)V}\omega^\alpha\right) - i_V\omega^\alpha\right]\\
&= d\left[(\exp V_0)^* i_{d(\exp V_0)V}\omega^\alpha + (\exp V_0)^* i_{h_2(V_0)V}\omega^\alpha - i_V\omega^\alpha\right]\\
&= d\left[i_V(\exp V_0)^*\omega^\alpha + (\exp V_0)^* i_{h_2(V_0)V}\omega^\alpha - i_V\omega^\alpha\right]\\
&= d\left[i_V(\omega^\alpha + h_3(V_0)\omega^\alpha) + (\exp V_0)^* i_{h_2(V_0)V}\omega^\alpha - i_V\omega^\alpha\right]\\
&= d\left[i_V(h_3(V_0)\omega^\alpha(q)) + (\exp V_0)^* i_{h_2(V_0)V}\omega^\alpha(q)\right].
\end{aligned}$$

Using the property (85) of $h_i\,(i=1,2,3)$, we observe that each term in the above last identity in C^α norm smoothly depends on $V_0 \in C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right)$ and linearly on $V \in C^{1,\alpha}\left(\Gamma\left(N_{A'_\varepsilon/M}\right)\right)$. Also when $V_0=0$, we have $(\exp V_0)^* d(i_{V_1}\omega^\alpha) - d(i_V\omega^\alpha) = 0$. Therefore

$$\left\|(\exp V_0)^* d(i_{V_1}\omega^\alpha) - d(i_V\omega^\alpha)\right\|_{C^\alpha(A'_\varepsilon)} \le C\left\|V_0\right\|_{C^{1,\alpha}(A'_\varepsilon)}\left\|V\right\|_{C^{1,\alpha}(A'_\varepsilon)}.$$

q.e.d.

Remark 25. We also have some point estimates tied to the feature that τ is a 3-form. It is well known that for any smooth embeddings $\varphi : A_\varepsilon \to M$ and smooth sections V_0 of $TM|_{A'_\varepsilon}$ with $|V_0|$ smaller than the injectivity radius of M,

$$\left|d\exp_\varphi V_0\right| \le C_6\left(|d\varphi| + |\nabla V_0|\right),$$

where C_6 is a constant only depending on (M,g). Hence for the map $\exp V_0 : TM|_{A'_\varepsilon} \to M$ and any section V of $TM|_{A'_\varepsilon}$, at $p \in A'_\varepsilon$ we have

$$\left|(d\exp V_0)(p)V\right| \le C_6\left(|d\varphi| + |\nabla V_0|\right)|V(p)|.$$

So for the first term (83), we have

$$\begin{aligned}
&\left|\left[(\exp V_0)^* d(i_{V_1}\omega^\alpha) - d(i_V\omega^\alpha)\right]\otimes W_\alpha(p)\right|\\
&\le C_7\left(|d\varphi| + |\nabla V_0|\right)^3\left(|\nabla V||V_0| + |V||\nabla V_0| + |V||V_0|\right)(p),
\end{aligned}$$

where the power 3 is because ω^α is a 3-form. For the next row (84), it is a 0-th order linear differential operator on V, where $V_1 = (d\exp V_0)|_{A_\varepsilon(0)}V$. By the C^2 smoothness of $\tau = \omega^\alpha \otimes W_\alpha$ and

$$i_W d\omega^\alpha(p) = \nabla_W W_\alpha(p) = 0$$

for any $W \in \Gamma\left(N_{A'_\varepsilon/M}\right)$, at $q = \exp_p V_0$ we have

$$\left|i_{V_1}d\omega^\alpha(q)\right| \le C_4|V_0||V|, \quad \left|\nabla_{V_1}W_\alpha(q)\right| \le C_4|V_0||V|.$$

Hence

$$\begin{aligned}
&\left|\left((\exp V_0)^*\otimes T_{V_0}\right)\circ\left[i_{V_1}d\omega^\alpha\otimes W_\alpha + \omega^\alpha\otimes\nabla_{V_1}W_\alpha\right](p) + B(V_0,V)W_\alpha(p)\right|\\
&\le C_6^3\left(|d\varphi| + |\nabla V_0|\right)^3\cdot 2C_4|V_0||V|(p) + C_5|V_0||V|(p)
\end{aligned}$$

where the power 3 is because ω^α is a 3-form. Combining all, we have

$$\begin{aligned}&\left|F'(V_0)V - F'(0)V\right|(p)\\ &\le \left[C_7\left(|d\varphi| + |\nabla V_0|\right)^3\left(|V_0||V| + |\nabla V_0||V| + |V_0||\nabla V|\right) + C_5|V_0||V|\right](p).\end{aligned}$$

Because of the cubic power terms, it is difficult to get a similar quadratic estimate in the L^p setting, namely

$$\begin{aligned}&\left\|F'(V_0)V - F'(0)V\right\|_{L^p\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\\ &\le C\|V_0\|_{W^{1,p}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}\|V\|_{W^{1,p}\left(A'_\varepsilon, N_{A'_\varepsilon/M}\right)}.\end{aligned}$$

This is one of the key reasons that we choose the Schauder setting for implicit function theorem, where the right inverse bound of $F'(0)$ is much harder to obtain than in the L^p setting. In contrast, the Cauchy-Riemann operator of J-holomorphic curves is more linear in the L^p setting: it has the quadratic estimate (see Proposition 3.5.3 in [**30**])

$$\left\|F'(V_0)V - F'(0)V\right\|_{L^p(\Sigma)} \le C\|V_0\|_{W^{1,p}(\Sigma)}\|V\|_{W^{1,p}(\Sigma)}.$$

4.4. Perturbation argument. To find the zeros of F_ε, we are going to apply the following quantitative version of the implicit function theorem (cf. Theorem 15.6 [**9**] or Proposition A3.4 in [**30**]).

Theorem 26. *Let $(X, |\cdot|_X)$ and $(Y, |\cdot|_Y)$ be Banach spaces and $F : B_r(0) \subset X \to Y$ a C^1-map, such that*

1) *$(DF(0))^{-1}$ is a bounded linear operator with $\left\|(DF(0))^{-1}F(0)\right\| \le A$ and $\left\|(DF(0))^{-1}\right\| \le B$;*
2) *$\|DF(x) - DF(0)\| \le \kappa|x|_X$ for all $x \in B_r(0)$;*
3) *$2\kappa AB < 1$ and $2A < r$.*

Then F has a unique zero in $B_{2A}(0)$.

To apply the above theorem, we define the map

$$\tilde{F}_\varepsilon := \varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}F_\varepsilon : C^{1,\alpha}_-\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right) \longrightarrow C^\alpha\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right).$$

Then Proposition 22 implies that

$$\left\|\left(D\tilde{F}_\varepsilon(0)\right)^{-1}\right\| \le \varepsilon^{\frac{3}{p}+2\alpha}\cdot C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} = C.$$

By Proposition 23, for $\|V_0\|_{C^{1,\alpha}_-\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right)} \le \delta_0$ and any $V \in C^{1,\alpha}_-\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right)$,

$$\begin{aligned}&\left\|\left(D\tilde{F}_\varepsilon(V_0) - D\tilde{F}_\varepsilon(0)\right)V\right\|_{C^\alpha\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right)}\\ &\le C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}\|V_0\|_{C^{1,\alpha}_-\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right)}\|V\|_{C^{1,\alpha}_-\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right)}.\end{aligned}$$

For $F(V) = T_V \circ (\exp V)^* \tau$, fixing a volume form on $\mathtt{A}'_\varepsilon$, we can regard $F(V)$ as a section of $TM|_{\mathtt{A}'_\varepsilon}$. We have

$$\begin{aligned}\|F(0)\|_{C^\alpha(\mathtt{A}'_\varepsilon,TM)} &= \|\varphi^*\tau\|_{C^\alpha(\mathtt{A}'_\varepsilon,TM)} \\ &= C\left\|(\varphi^*\tau)(0,z) + \int_0^{x_1} \frac{\partial}{\partial x_1}(\varphi^*\tau)(s,z)\,ds\right\|_{C^\alpha(\mathtt{A}'_\varepsilon,TM)} \\ &= C\left\|\int_0^{x_1} \frac{\partial}{\partial x_1}\varphi^*\tau(s,z)\,ds\right\|_{C^\alpha(\mathtt{A}'_\varepsilon,TM)} \le C\varepsilon^{1-\alpha}, \end{aligned} \tag{86}$$

where $(\varphi^*\tau)(0,z) = 0$ is because $\varphi(0\times\Sigma)$ is J_n-holomorphic and then $\tau|_{T\mathtt{A}'_\varepsilon|_{\varphi(0\times\Sigma)}} = 0$, and the last inequality is because for

$$H(x_1,z) := \int_0^{x_1} \frac{\partial}{\partial x_1}\varphi^*\tau(s,z)\,ds$$

where $f(s,z) := \frac{\partial}{\partial x_1}\varphi^*\tau(s,z)$ is smooth on $[0,\varepsilon]\times\Sigma$, we have

$$\begin{aligned}\|H\|_{C^0(\mathtt{A}_\varepsilon)} &\le \int_0^\varepsilon \|f\|_{C^0(\mathtt{A}_\varepsilon)}\,ds = \|f\|_{C^0(\mathtt{A}_\varepsilon)}\,\varepsilon, \\ [H]^z_{\alpha;\mathtt{A}_\varepsilon} &\le \int_0^\varepsilon [f]^z_{\alpha;\mathtt{A}_\varepsilon}\,ds = [f]^z_{\alpha;\mathtt{A}_\varepsilon}\,\varepsilon, \\ [H]^{x_1}_{\alpha;\mathtt{A}_\varepsilon} &\le C\,[H]^{x_1}_{1;\mathtt{A}_\varepsilon}\,\varepsilon^{1-\alpha} \le \|f\|_{C^0(\mathtt{A}_\varepsilon)}\,\varepsilon^{1-\alpha}.\end{aligned}$$

(Here $[f]^z_{\alpha;\mathtt{A}_\varepsilon}$, $[f]^{x_1}_{\alpha;\mathtt{A}_\varepsilon}$ are the Schauder components of f in the z and x_1 directions of $\mathtt{A}_\varepsilon$.) Note that (86) implies that the tangent space of $\mathtt{A}'_\varepsilon$ is $\varepsilon^{1-\alpha}$-close to being associative.

By our construction

$$F_\varepsilon(V) = *_{\mathtt{A}'_\varepsilon} \circ \perp_{\mathtt{A}'_\varepsilon} \circ F(V),$$

where $*_{\mathtt{A}'_\varepsilon} \circ \perp_{\mathtt{A}'_\varepsilon}$ is a bounded linear operator and T_V is a parallel transport, we have

$$\begin{aligned}\left\|\tilde{F}_\varepsilon(0)\right\|_{C^\alpha\left(\mathtt{A}_\varepsilon, N_{A'_\varepsilon/M}\right)} &\le C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}\|F(0)\|_{C^\alpha\left(\mathtt{A}_\varepsilon, N_{A'_\varepsilon/M}\right)} \\ &\le C\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)}\varepsilon^{1-\alpha} = C\varepsilon^{1-\left(\frac{3}{p}+3\alpha\right)}.\end{aligned}$$

Then we have

$$\left\|D\tilde{F}_\varepsilon(0)^{-1}\right\|\left\|\tilde{F}_\varepsilon(0)\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})}\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} \le C\varepsilon^{1-\frac{6}{p}-5\alpha}.$$

Note we have chosen that $0 < \frac{3}{p} + 3\alpha < \frac{1}{2}$ (say $\alpha = 1/12$ and $p > 12$) in the beginning, so $1 - \frac{6}{p} - 5\alpha > 0$, then

$$\begin{aligned}&\left\|D\tilde{F}_\varepsilon(0)^{-1}\right\|\left\|\tilde{F}_\varepsilon(0)\right\|_{C^\alpha(\mathtt{A}_\varepsilon,\mathbb{S})}\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} \\ &\le C\varepsilon^{1-\left(\frac{3}{p}+3\alpha\right)}\varepsilon^{-\left(\frac{3}{p}+2\alpha\right)} = C\varepsilon^{1-\frac{6}{p}-5\alpha} \to 0\end{aligned}$$

as $\varepsilon \to 0$. Theorem 26 then implies that there is a unique $V_\varepsilon \in \Gamma\left(N_{A'_\varepsilon/M}\right)$ with

$$\|V_\varepsilon\|_{C^{1,\alpha}\left(\mathtt{A}'_\varepsilon, N_{A'_\varepsilon/M}\right)} \leq 2C\varepsilon^{1-\left(\frac{3}{p}+3\alpha\right)} \tag{87}$$

that solves $\tilde{F}_\varepsilon(V_\varepsilon) = 0$, i.e. $F_\varepsilon(V_\varepsilon) = 0$. Note that (86) and (87) together imply that the tangent space of $\mathtt{A}_\varepsilon(V_\varepsilon)$ is $\varepsilon^{1-\left(\frac{3}{p}+3\alpha\right)}$-close to being associative. By Proposition 8, for almost associative submanifolds $\mathtt{A}_\varepsilon(V_\varepsilon)$,

$$F_\varepsilon(V_\varepsilon) = 0 \Leftrightarrow F(V_\varepsilon) = 0,$$

while the latter means that $\mathtt{A}_\varepsilon(V_\varepsilon)$ is associative.

Finally, we obtain our main result:

Theorem 27. *Suppose that M is a G_2-manifold and C_t is a one parameter family of coassociative submanifolds in M. Suppose that the self-dual two form $\eta = dC_t/dt|_{t=0} \in \Omega^2_+(C)$ is nonvanishing, then it defines an almost complex structure J on C_0.*

For any regular J-holomorphic curve Σ in C_0, there is an instanton A_ε in M which is diffeomorphic to $[0,\varepsilon] \times \Sigma$ and $\partial A_\varepsilon \subset C_0 \cup C_\varepsilon$, for all sufficiently small positive ε.

Finally, we expect that any instanton A in M bounding $C_0 \cup C_t$ and with small volume must arise in the above manner. Namely, we need to prove a ε-regularity result for instantons.

A few remarks are in order: First, counting such small instantons is basically a problem in four manifold theory because of Bryant's result [**7**], which says that the zero section C in $\Lambda^2_+(C)$ is always a coassociative submanifold for an incomplete G_2-metric on its neighborhood, provided that the bundle $\Lambda^2_+(C)$ is topologically trivial. Second, when η has zeros, the above Theorem 27 should still hold true. However, using the present approach to prove it would require a good understanding of the Seiberg-Witten theory on any four manifold with a degenerated symplectic form as in the Taubes program. But at least for regular J_η-holomorphic curves Σ in C_0 that are *away from the zero locus* of η, the instantons $\mathtt{A}'_\varepsilon$ in Theorem 27 still exist nearby to $\Sigma \subset M$, for our gluing analysis only involves the local geometry of Σ in M. Third, if we do not restrict to instantons of small volume, then we have to take into account the compactification of the moduli of instantons which has not been established yet, e.g. *bubbling* phenomenon as for pseudo-holomorphic curves, and gluing of instantons of big and small volumes similar to [**32**] in the Floer trajectory case. Nevertheless, one would expect that if the volume of $\mathtt{A}_t$'s is small, then bubbling cannot occur; thus they would converge to a J_η-holomorphic curve in C_0.

5. Appendix: The exponential-like map $\widetilde{\exp}$

We construct the exponential-like map $\widetilde{\exp} : \mathbb{S} \to M$ satisfying the properties 1∼3 in Subsection 4.2.

For any section $V = (u, v)$ of the spinor bundle $\mathbb{S} \to A_\varepsilon$, since the bundle $\mathbb{S} \to \mathtt{A}_\varepsilon$ is a Cartesian product of the spinor bundle $\mathbb{S}_\Sigma \to \Sigma$ with the interval $[0, \varepsilon]$, we may alternatively regard $V(t, z)$ as a one parameter family of sections of the bundle $\mathbb{S}_\Sigma \to \Sigma$ for the parameter $t \in [0, \varepsilon]$. Our goal is to obtain the deformation of $A'_\varepsilon \subset M$ from V. The idea is to deform $\mathtt{A}_\varepsilon \subset \mathtt{C} := [0, \varepsilon] \times C$ using only the u component, then map the deformed $\mathtt{A}_\varepsilon$ to M by φ, and at last deform it in M by the v component. The coassociative boundary condition is preserved under the map $\widetilde{\exp}$, since we separate the u and v deformations before and after the map φ, respectively. The precise description is in order.

For each fixed $t \in [0, \varepsilon]$, $u(t, z)$ and $v(t, z)$ are sections of the spinor bundle $\mathbb{S}_\Sigma \to \Sigma$. Using the real vector bundle isomorphism

$$(id, f) : N_{\Sigma/C} \oplus \wedge^{0,1}_{\mathbb{C}} \left(N_{\Sigma/C}\right) \simeq N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_\Sigma,$$

$u(t, z)$ and $f(v(t, z))$ are sections of $N_{\Sigma/C}$ and $N_{\mathcal{C}/M}|_\Sigma$, respectively. For any fixed $z \in \Sigma$, using the u component, we have a one parameter deformation $\exp^C_z u(t, z)$ in C for $0 \le t \le \varepsilon$, where $\exp^C$ is the exponential map associated to the induced metric of C in M. Alternatively, we can view the deformation in the product space $\mathtt{C} := [0, \varepsilon] \times C$, such that the line $[0, \varepsilon] \times \{z\} \subset \mathtt{C}$ is deformed to the curve $\left\{ \left(t, \exp^C_z u(t, z)\right) | 0 \le t \le \varepsilon \right\} \subset \mathtt{C}$. Let the curve γ_u in M be

$$\gamma_u(t, z) := \varphi\left(t, \exp^C_z u(t, z)\right), \text{ for } 0 \le t \le \varepsilon \text{ and } z \in \Sigma.$$

It is clear that

$$\gamma_u(t, z) \subset C_t := \varphi(t, C)$$

since $\exp^C_z u(t, z) \subset C$.

For the v component, for each fixed $t \in [0, \varepsilon]$, we have $f(v(t, z)) \in N_{\mathcal{C}/M}|_\Sigma(z)$. Let

$$T_{\gamma_u(t,z)} : N_{\mathcal{C}/M}|_{\gamma_u(0,z)} \longrightarrow N_{\mathcal{C}/M}|_{\gamma_u(t,z)}$$

be the parallel transport along the curve $\gamma_u(s, z)$ for $0 \le s \le t$ with respect to the connection of $N_{\mathcal{C}/M}$ induced from the metric g on M. We define the exponential-like map $\widetilde{\exp}$ as follows:

$$\begin{array}{ccc} \widetilde{\exp} : \mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^- & \longrightarrow & M \\ V(t, z) = (u(t, z), v(t, z)) & & \exp^M_{\gamma_u(t,z)}\left(T_{\gamma_u(t,z)} f(v(t, z))\right) \end{array}$$

where $\exp^M$ is the exponential map in M. It follows from our construction that for any $V \in C^m_-(\mathtt{A}_\varepsilon, \mathbb{S})$ and $x \in \partial \mathtt{A}_\varepsilon = \{0, \varepsilon\} \times \Sigma$, $\widetilde{\exp} V$ *satisfies the boundary condition* $\widetilde{\exp} V|_{\partial \mathtt{A}_\varepsilon} \subset C_0 \cup C_\varepsilon$, because

$$(\widetilde{\exp} V)(x) = \exp^M_{\gamma_u(x)}(0) = \gamma_u(x) \subset \varphi(\{0, \varepsilon\} \times C) = C_0 \cup C_\varepsilon.$$

By the definition of $\widetilde{\exp}$, it is easy to see $\widetilde{\exp}|_{\mathtt{A}_\varepsilon} = \varphi$, because $\mathtt{A}_\varepsilon$ is the zero section of $\mathbb{S}$. Hence on base directions of $\mathbb{S} \to \mathtt{A}_\varepsilon$, $d\widetilde{\exp}|_{\mathtt{A}_\varepsilon} = d\varphi|_{\mathtt{A}_\varepsilon}$, especially

$$d\widetilde{\exp}|_{\{0\}\times\Sigma} = id : T\Sigma \to T\Sigma \text{ and } d\widetilde{\exp}|_{\{0\}\times\Sigma} : \frac{\partial}{\partial x_1} \to n\,(z)\,.$$

On fiber directions, that $d\widetilde{\exp}|_{\{0\}\times\Sigma} = (id, f) : N_{\Sigma/C} \oplus \wedge^{0,1}_{\mathbb{C}}\left(N_{\Sigma/C}\right) \to N_{\Sigma/C} \oplus N_{\mathcal{C}/M}|_{\Sigma}$ follows from $d\exp^M(0) = id$.

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