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Quantum Pieri rules for tautological subbundles

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Abstract

We give quantum Pieri rules for quantum cohomology of Grassmannians of classical types, expressing the quantum product of Chern classes of the tautological subbundles with general cohomology classes. We derive them by showing the relevant genus zero, three-pointed Gromov–Witten invariants coincide with certain classical intersection numbers.

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1. Introduction

The complex Grassmannian $Gr(k, n+1)$ parameterizes k-dimensional complex vector subspaces of $\mathbb{C}^{n+1}$. It can be written as $X = G/P$ with G being a complex Lie group of type A, i.e. $G = SL(n+1, \mathbb{C})$, and P being a maximal parabolic subgroup of G. We will continue to call such X's as *Grassmannians* even when G is not of type A. Indeed when G is a classical Lie group of type B, C or D, such a Grassmannian parameterizes subspaces in a vector space which are isotropic with respect to a non-degenerate skew-symmetric or symmetric bilinear form.

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Therefore it is usually called an *isotropic Grassmannian*. Recall that the tautological subbundle $\mathcal{S}$ over any point $[V] \in Gr(k, n+1)$ is just the k-dimensional vector subspace V itself. And it restricts to the tautological subbundle $\mathcal{S}$ of any isotropic Grassmannian.

The cohomology ring $H^*(X, \mathbb{Z})$ of an isotropic Grassmannian $X = G/P$, or more generally a generalized flag variety, has a natural basis consisting of Schubert cohomology classes σ^u, labeled by a subset of the Weyl group W of G. The (small) quantum cohomology ring $QH^*(X)$ of X, as a vector space, is isomorphic to $H^*(X) \otimes \mathbb{Q}[t]$. The quantum ring structure is a deformation of the ring structure on $H^*(X)$ by incorporating three-pointed, genus zero Gromov–Witten invariants of X. Since $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, the homology class of a holomorphic curve in X is labeled by its degree d. In the case of $X = IG(k, 2n)$ being a Grassmannian of type C_n, the Schubert cohomology classes $\sigma^u = \sigma^{\mathbf{a}}$ can also be labeled by *shapes* $\mathbf{a}$, which are certain pairs of partitions. Every nonzero Chern class $c_p(\mathcal{S}^*) = (-1)^p c_p(\mathcal{S}) = \sigma^p$ (up to a scale factor of 2) is then a special Schubert class given by a special shape p, and they generate the quantum cohomology ring $QH^*(IG(k, 2n))$. One of the main results of the present paper is the following formula.

Quantum Pieri rule for tautological subbundles of $IG(k, 2n)$. *(See Theorem 4.4.) For any shape* $\mathbf{a}$ *and every special shape p, in $QH^*(IG(k, 2n))$, we have*

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum 2^{e(\mathbf{a},\mathbf{b})} \sigma^{\mathbf{b}} + t \sum 2^{e(\tilde{\mathbf{a}},\tilde{\mathbf{c}})} \sigma^{\mathbf{c}}.$$

Here $\tilde{\mathbf{a}}$ and $\tilde{\mathbf{c}}$ are shapes associated to $\mathbf{a}$ and $\mathbf{c}$ respectively; $e(\mathbf{a}, \mathbf{b})$ and $e(\tilde{\mathbf{a}}, \tilde{\mathbf{c}})$ are cardinalities of certain combinatorial sets, determined by the classical Pieri rules of Pragacz and Ratajski [27]. We have also obtained similar formulas for Grassmannians of type B and D, details of which are given in Section 4.

The aforementioned *quantum Pieri rule* is a quantum version of the classical Pieri rule for isotropic Grassmannians. The famous classical Pieri rules are known firstly for complex Grassmannians (see e.g. [15]). For $X = Gr(k, n+1)$, they describe the cup product of a general Schubert class in $H^*(X)$ with $c_p(\mathcal{S}^*)$ or $c_p(\mathcal{Q})$, where $\mathcal{Q}$ is the tautological quotient bundle over X given by the exact sequence $0 \to \mathcal{S} \to \mathbb{C}^{n+1} \to \mathcal{Q} \to 0$. It was generalized for other partial flag varieties of type A, firstly given by Lascoux and Schützenberger [22], and was also generalized for Grassmannians X of type B, C or D. Note that there is also a tautological quotient bundle $\mathcal{Q}$ over X. When X parameterizes maximal isotropic subspaces (roughly speaking) there is no difference between the Chern classes of $\mathcal{S}^*$ and $\mathcal{Q}$, and the classical Pieri rules has been given by Hiller and Boe [16]. When X parameterizes non-maximal isotropic subspaces, the classical Pieri rules with respect to $c_p(\mathcal{S}^*)$ have been given by Pragacz and Ratajski [27,28], while the classical Pieri rules with respect to $c_p(\mathcal{Q})$ are just covered in the recent work of Buch, Kresch and Tamvakis [5] on quantum Pieri rules. In contrast to complex Grassmannians, knowing either of them cannot deduce the other one. There is also a previous work of Sertöz [29] as well as a generalized classical Pieri rule given by Bergeron and Sottile [1], which gives the formula for multiplying a Schubert class on a complete flag variety of type B or C by a special Schubert class pulled back from the Grassmannian of maximal isotropic subspaces.

The story of quantum Pieri rules are almost parallel to the story of the classical Pieri rules. The quantum Pieri rules are also known firstly for complex Grassmannians, which were firstly given by Bertram [2]. They were generalized by Ciocan-Fontanine [11] for other partial flag varieties of type A, and by Kresch and Tamvakis [19,20] for those X that parameterize maximal isotropic subspaces. Recently in [5], Buch, Kresch and Tamvakis have given us the quantum Pieri rules with respect to $c_p(\mathcal{Q})$ for those X that parameterize non-maximal isotropic subspaces. In contrast to complex Grassmannians, (in general) the quantum Pieri rules with respect to $c_p(\mathcal{Q})$ do not imply the quantum Pieri rules with respect to $c_p(\mathcal{S}^*)$ and vice versa.

Our quantum Pieri rules are consequences of the following main technical result.

Main Theorem. *Let $X = G/P$ be a Grassmannian of type B, C or D, and $\mathcal{S}$ denote the tautological subbundle over X. Let σ^u, σ^v be Schubert classes in $QH^*(X)$ with $\sigma^u = (-1)^p c_p(\mathcal{S})$ (possibly up to a scale factor of $\frac{1}{2}$, see Section 3.1 for more details) for any p. In the quantum product*

$$\sigma^u \star \sigma^v = \sigma^u \cup \sigma^v + \sum_{d\geqslant 1} N_{u,v}^{w,d} t^d \sigma^w,$$

all the degree d Gromov–Witten invariants $N_{u,v}^{w,d}$ coincide with certain classical intersection numbers. More precisely, we have

(1) *If $d = 1$, then there exist $u_1, v_1, w_1 \in W$ such that $N_{u,v}^{w,1} = N_{u_1,v_1}^{w_1,0}$.*
(2) *If $d = 2$, then there exist $u_2, v_2, w_2 \in W$ such that $N_{u,v}^{w,2} = N_{u_2,v_2}^{w_2,0}$.*
(3) *If $d \geqslant 3$, then $N_{u,v}^{w,d} = 0$.*

Here $N_{u_i,v_i}^{w_i,0}$'s are classical intersection numbers of the corresponding Schubert varieties in the complete flag variety G/B, where $B \subset P$ is a Borel subgroup of G. The elements u_i, v_i, w_i can be explicitly written down in terms of u, v, w as given in Theorem 3.13 and Theorem 3.21. In fact $N_{u,v}^{w,2}$ also vanishes for some cases.

The above theorem is an application of the main results of [24], where the authors studied the "quantum to classical" principle for flag varieties of general type. Roughly speaking, the "quantum to classical" principle says that certain three-pointed genus zero Gromov–Witten invariants are classical intersection numbers. Such phenomenon, probably for the first time, occurred in the proof of quantum Pieri rule for partial flag varieties of type A by Ciocan-Fontanine [11], and later occurred in the elementary proof of quantum Pieri rule for complex Grassmannians by Buch [3] and the work [19,20] of Kresch and Tamvakis on Lagrangian and orthogonal Grassmannians. This principle has been studied mainly for Grassmannians in the works (especially) by Buch, Kresch, and Tamvakis [4,5], by Chaput, Manivel, and Perrin [9,10] and by Buch and Mihalcea [7,8]. There are relevant studies for some other cases by Coskun [12] and by Li and Mihalcea [25].

The proofs of our quantum Pieri rules are combinatorial in nature. The ideas of all these proofs are the same, namely we obtain all the theorems by showing that the relevant Gromov–Witten invariants of degree d vanish unless d is small enough (for instance $d \leqslant 2$), and for such a small d they coincide with certain classical intersection numbers. Moreover, all these relevant classical intersection numbers are exactly or can be calculated from certain structure constants in classical Pieri rules of same type. We should note that in [23], the authors established natural filtered algebra structures on $QH^*(G/B)$. Using structures of these filtrations, we obtained relationships among three-pointed genus zero Gromov–Witten invariants for G/B in [24], which enable us to carry out the above ideas in real proofs. Finally, we should also note that our quantum Pieri rules for type B, D are not quite satisfying, as signs are involved in some cases.

This paper is organized as follows. In Section 2, we fix the notations and review the main results of [24]. In Section 3, we reduce all the relevant Gromov–Witten invariants to certain classical intersection numbers, for the quantum Pieri rules for Grassmannians of type B, C or

D with respect to $c_p(\mathcal{S}^*)$. In Section 4, we obtain the quantum Pieri rules by computing those classical intersection numbers in Section 3 combinatorially. Finally in Appendix A, we reprove the well-known quantum Pieri rules for Grassmannians of type A (i.e. complex Grassmannians).

2. Preliminary results

2.1. Notations

We recall notations in [24] which we will use here as well. Readers can refer to Section 2.1 of [24] and references therein for more details.

Let G be a simply-connected complex simple Lie group of rank n, $B \subset G$ be a Borel subgroup and $\mathfrak{h}$ be the corresponding Cartan subalgebra. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ be the simple roots with the associated Dynkin diagram $Dyn(\Delta)$ being the same as in Section 11.4 of [17]. Let $\{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ be the simple coroots, $\{\chi_1, \ldots, \chi_n\}$ be the fundamental weights and R^+ be the set of positive roots in the root system R. Denote $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ and $\rho = \sum_{i=1}^n \chi_i$. The Weyl group W is generated by the simple reflections s_i's on $\mathfrak{h}^*$ defined by $s_i(\beta) = s_{\alpha_i}(\beta) := \beta - \langle \beta, \alpha_i^\vee \rangle \alpha_i$ for each i, where $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h} \to \mathbb{C}$ is the natural pairing. Each parabolic subgroup $P \supset B$ is in one-to-one correspondence with a subset $\Delta_P \subset \Delta$. Let $\ell : W \to \mathbb{Z}_{\geqslant 0}$ be the length function, W_P denote the Weyl subgroup generated by $\{s_\alpha \mid \alpha \in \Delta_P\}$ and W^P denote the minimal length representatives of the cosets W/W_P. Let ω_P denote the (unique) longest element in W_P.

The (co)homology of a (generalized) flag variety $X = G/P$ is torsion free and it has an additive basis of Schubert (co)homology classes σ_u's (resp. σ^u's) indexed by W^P. Note that $\sigma^u \in H^{2\ell(u)}(X, \mathbb{Z})$ and that $H_2(X, \mathbb{Z}) = \bigoplus_{\alpha_i \in \Delta \setminus \Delta_P} \mathbb{Z}\sigma_{s_i}$ can be canonically identified with $Q^\vee/Q_P^\vee$, where $Q_P^\vee := \bigoplus_{\alpha \in \Delta_P} \mathbb{Z}\alpha^\vee$. For each $\alpha_j \in \Delta \setminus \Delta_P$, we introduce a formal variable $q_{\alpha_j^\vee + Q_P^\vee}$. For $\lambda_P = \sum_{\alpha_j \in \Delta \setminus \Delta_P} a_j \alpha_j^\vee + Q_P^\vee \in H_2(X, \mathbb{Z})$, we denote $q_{\lambda_P} = \prod_{\alpha_j \in \Delta \setminus \Delta_P} q_{\alpha_j^\vee + Q_P^\vee}^{a_j}$. The **(small) quantum cohomology** $QH^*(X) = (H^*(X) \otimes \mathbb{Q}[\mathbf{q}], \star)$ of X (see e.g. [13] for more details) is a commutative ring and has a $\mathbb{Q}[\mathbf{q}]$-basis of Schubert classes $\sigma^u = \sigma^u \otimes 1$. The structure coefficients $N_{u,v}^{w,\lambda_P}$ for the quantum product

$$\sigma^u \star \sigma^v = \sum_{w \in W^P, \lambda_P \in Q^\vee/Q_P^\vee} N_{u,v}^{w,\lambda_P} q_{\lambda_P} \sigma^w$$

are three-pointed genus zero Gromov–Witten invariants and they are all **nonnegative**.

When $P = B$, we have $\Delta_P = \emptyset$, $Q_P^\vee = 0$, $W_P = \{1\}$ and $W^P = W$. In this case, we simply denote $\lambda = \lambda_P$ and $q_j = q_{\alpha_j^\vee}$. It is well known that $N_{u,v}^{w,\lambda} = 0$ unless both of the following hold:

(1) $\ell(w) + \langle 2\rho, \lambda \rangle = \ell(u) + \ell(v)$ (which comes from the dimension constraint);
(2) λ is *effective*, i.e. $\lambda = \sum_{j=1}^n a_j \alpha_j^\vee$ with $a_j \in \mathbb{Z}_{\geqslant 0}$ for all j.

2.2. Preliminaries

In this subsection, we collect some known propositions. As we will see in the next section, we give the quantum Pieri rules for Grassmannians of classical types, based on the main result of [24] (see Proposition 2.1), the Peterson–Woodward comparison formula (see Proposition 2.4) and the quantum Chevalley formula (see Proposition 2.5).

As in [24], given any simple root $\alpha \in \Delta$, we define a map sgn_α as follows:

$$\mathrm{sgn}_\alpha : W \to \{0, 1\}; \qquad \mathrm{sgn}_\alpha(w) = \begin{cases} 1, & \text{if } \ell(w) - \ell(ws_\alpha) > 0, \\ 0, & \text{if } \ell(w) - \ell(ws_\alpha) \leqslant 0. \end{cases}$$

It is well known that (see e.g. [18]) $\mathrm{sgn}_\alpha(w) = 0$ if and only if $w(\alpha) \in R^+$.

The following proposition relates numbers of rational curves representing "different" homology classes of G/B.

Proposition 2.1. *(See Theorem 2.2 of [24].) For any $u, v, w \in W$ and $\lambda \in Q^\vee$, we have*

(1) $N_{u,v}^{w,\lambda} = 0$ *unless* $\mathrm{sgn}_\alpha(w) + \langle \alpha, \lambda \rangle \leqslant \mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v)$ *for all* $\alpha \in \Delta$.
(2) *Suppose* $\mathrm{sgn}_\alpha(w) + \langle \alpha, \lambda \rangle = \mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v) = 2$ *for some* $\alpha \in \Delta$, *then*

$$N_{u,v}^{w,\lambda} = N_{us_\alpha, vs_\alpha}^{w,\lambda-\alpha^\vee}, \qquad \textit{whenever } \mathrm{sgn}_\alpha(w) = 0 \textit{ or } 1; \tag{2.1}$$

$$N_{u,v}^{w,\lambda} = N_{u,vs_\alpha}^{ws_\alpha,\lambda-\alpha^\vee}, \qquad \textit{if } \mathrm{sgn}_\alpha(w) = 0; \tag{2.2}$$

$$N_{u,v}^{w,\lambda} = N_{u,vs_\alpha}^{ws_\alpha,\lambda}, \qquad \textit{if } \mathrm{sgn}_\alpha(w) = 1. \tag{2.3}$$

As a consequence, we obtain the next vanishing criterion for the Gromov–Witten invariants[1] $N_{u,v}^{w,\lambda}$.

Corollary 2.2. *For any $u, v, w \in W$ and $\lambda \in Q^\vee$, we have $N_{u,v}^{w,\lambda} = 0$ whenever there exists $\alpha \in \Delta$ such that one of the following holds.*

(1) $\langle \alpha, \lambda \rangle = 2$ *and* $N_{u,vs_\alpha}^{ws_\alpha,\lambda-\alpha^\vee} = 0$;
(2) $\langle \alpha, \lambda \rangle = 1$, $\mathrm{sgn}_\alpha(u) = 0$ *and* $N_{u,vs_\alpha}^{ws_\alpha,\lambda-\alpha^\vee} = 0$;
(3) $\langle \alpha, \lambda \rangle = 0$, $\mathrm{sgn}_\alpha(u) = \mathrm{sgn}_\alpha(v) = 0$ *and* $N_{u,vs_\alpha}^{ws_\alpha,\lambda} = 0$.

Proof. Note that sgn_α is a map frow W to $\{0, 1\}$. Thus if any one of the above three assumptions holds, we have $\mathrm{sgn}_\alpha(w) + \langle \alpha, \lambda \rangle \geqslant \mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v)$.

When the inequality ">" holds, we are already done by using Proposition 2.1 (1). Now we assume the equality "=" holds. As a consequence, if assumption (2) holds, then we have $\mathrm{sgn}_\alpha(w) = 0$ and $\mathrm{sgn}_\alpha(v) = 1$. Applying Proposition 2.1 (2) for $u' = v$, $v' = us_\alpha$ and $w' = ws_\alpha$, we deduce that $N_{u,v}^{w,\lambda} = N_{v,u}^{w,\lambda} = N_{u',v's_\alpha}^{w's_\alpha,\lambda} = N_{u's_\alpha,v's_\alpha}^{w',\lambda-\alpha^\vee} = N_{vs_\alpha,u}^{ws_\alpha,\lambda-\alpha^\vee} = 0$. Similarly, we can show $N_{u,v}^{w,\lambda} = 0$ whenever either assumption (1) or assumption (3) holds. □

We will use Proposition 2.1 and Corollary 2.2 very frequently in Section 3. Whenever necessary, we will point out what we are applying explicitly, by using the words "Applying (*reference*) to $(u', v', w', \lambda', \alpha')$".

The next identity for certain classical intersection numbers is also a direct consequence of Proposition 2.1.

[1] By "Gromov–Witten invariants", we always mean "three-pointed genus zero Gromov–Witten invariants" in this paper.

Corollary 2.3. *Let $u, v, w \in W$ and $\alpha \in \Delta$. Suppose $\mathrm{sgn}_\alpha(w) = \mathrm{sgn}_\alpha(u) + 1 = 1$, then $N_{u,v}^{w,0}$ is equal to $N_{u,vs_\alpha}^{ws_\alpha,0}$ if $\mathrm{sgn}_\alpha(v) = 1$, or 0 otherwise.*

Proof. If $\mathrm{sgn}_\alpha(v) = 0$, then $N_{u,v}^{w,0} = 0$ follows directly from Proposition 2.1 (1). If $\mathrm{sgn}_\alpha(v) = 1$, then we have $N_{v,us_\alpha}^{ws_\alpha,\alpha^\vee} = N_{vs_\alpha,u}^{ws_\alpha,0} = N_{v,u}^{w,0}$, by Proposition 2.1 (2). □

The next comparison formula tells us that every Gromov–Witten invariant ($N_{u,v}^{w,\lambda_P}$) for G/P equals a certain Gromov–Witten invariant ($N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}$) for G/B.

Proposition 2.4 *(Peterson–Woodward comparison formula). (See [31]; see also [26,21].)*

(1) *Let $\lambda_P \in Q^\vee/Q_P^\vee$. Then there is a unique $\lambda_B \in Q^\vee$ such that $\lambda_P = \lambda_B + Q_P^\vee$ and $\langle \gamma, \lambda_B\rangle \in \{0, -1\}$ for all $\gamma \in R_P^+$ $(= R^+ \cap \bigoplus_{\beta\in\Delta_P} \mathbb{Z}\beta)$.*
(2) *Denote $\Delta_{P'} := \{\beta \in \Delta_P \mid \langle\beta, \lambda_B\rangle = 0\}$. For every $u, v, w \in W^P$, we have*

$$N_{u,v}^{w,\lambda_P} = N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}.$$

Here ω_P (resp. $\omega_{P'}$) is the longest element in the Weyl subgroup W_P (resp. $W_{P'}$).

Thanks to the above comparison formula, we obtain an injection of vector spaces

$$\psi_{\Delta,\Delta_P} : QH^*(G/P) \to QH^*(G/B) \quad \text{defined by } q_{\lambda_P}\sigma^w \mapsto q_{\lambda_B}\sigma^{w\omega_P\omega_{P'}}.$$

We denote by P_α the parabolic subgroup (containing B) that corresponds to the special case of a singleton subset $\{\alpha\} \subset \Delta$, and simply denote $\psi_\alpha = \psi_{\Delta,\{\alpha\}}$. Note that $R_{P_\alpha}^+ = \{\alpha\}$ and $Q_{P_\alpha}^\vee = \mathbb{Z}\alpha^\vee$. In addition, we have the natural fibration $P_\alpha/B \to G/B \to G/P_\alpha$ with $P_\alpha/B \cong \mathbb{P}^1$.

The (Peterson's) quantum Chevalley formula, proved in [14], describes the quantum product of two Schubert classes when one of them is given by a simple reflection.

Proposition 2.5 *(Quantum Chevalley formula for G/B). For $u \in W$, $1 \leqslant i \leqslant n$,*

$$\sigma^u \star \sigma^{s_i} = \sum \langle \chi_i, \gamma^\vee\rangle \sigma^{us_\gamma} + \sum \langle \chi_i, \gamma^\vee\rangle q_{\gamma^\vee}\sigma^{us_\gamma},$$

where the first sum is over positive roots γ for which $\ell(us_\gamma) = \ell(u) + 1$, and the second sum is over positive roots γ for which $\ell(us_\gamma) = \ell(u) + 1 - \langle 2\rho, \gamma^\vee\rangle$.

When Δ is of A-type, the above formula is also called the quantum Monk formula.

In addition, we will need the next two lemmas.

Lemma 2.6. *(See Lemma 3.9 of [23].) Let $v \in W$ and $\gamma \in R^+$ satisfy $\ell(vs_\gamma) = \ell(v) + 1 - \langle 2\rho, \gamma^\vee\rangle$. Then for any $\alpha_j \in \Delta$ with $\langle \alpha_j, \gamma^\vee\rangle > 0$, we have $\ell(vs_\gamma s_j) = \ell(vs_\gamma) + 1$.*

Lemma 2.7. *For $v \in W^P$ and $\alpha \in \Delta_P$, we have $\mathrm{sgn}_\alpha(v) = 0$.*

Proof. It is a well-known fact that $v \in W^P$ and $\alpha \in \Delta_P$ imply $v(\alpha) \in R^+$. Thus, $\ell(vs_\alpha) > \ell(v)$, and then the statement follows. □

3. Quantum Pieri rules for Grassmannians of classical types: classical aspects

In this section, we study the quantum Pieri rules for Grassmannians of classical types, which describe the quantum product of general Schubert classes with the Chern classes of the dual of the tautological subbundles. We will only deal with Grassmannians of type B, C, D here, and will reprove the well-known quantum Pieri rules for Grassmannians of type A (i.e. complex Grassmannians) [2] in the appendix. Furthermore, we will only reduce all the relevant Gromov–Witten invariants in the quantum Pieri rules to certain classical intersection numbers. As we will see in next section, further reductions can be taken so that these rules can be reformulated in a traditional way.

3.1. Grassmannians of type B, C, D

In this subsection, we review some facts on Grassmannians of type C, B, D, i.e. the quotients of Lie groups G of the aforementioned types by their maximal parabolic subgroups P. More details on these facts can be found for example in [30] and [5]. We also illustrate the idea of our proof of quantum Pieri rules.

Every such Grassmannian X parameterizes isotropic subspaces in a vector space $E = \mathbb{C}^N$ equipped with a standard non-degenerated bilinear form $(\cdot,\cdot)$ which is skew-symmetric in the C case and symmetric in the B or D cases. Thus it is usually called an *isotropic Grassmannian* and it can be described explicitly as follows. The maximal parabolic subgroup P corresponds to a subset $\Delta_P = \Delta \setminus \{\alpha_k\}$ (we use the convention of labeling the base as in Humphreys' book [17]) and the space X is given by

(i) $IG(k, 2n) = \{V \leqslant \mathbb{C}^{2n} \mid \dim_{\mathbb{C}} V = k,\ (V, V) = 0\}$ for type C_n;
(ii) $OG(k, 2n+1) = \{V \leqslant \mathbb{C}^{2n+1} \mid \dim_{\mathbb{C}} V = k,\ (V, V) = 0\}$ for type B_n;
(iii) $OG(k, 2n+2) = \{V \leqslant \mathbb{C}^{2n+2} \mid \dim_{\mathbb{C}} V = k,\ (V, V) = 0\}$, if G is of type D_{n+1} and $1 \leqslant k \leqslant n-1$;
(iv) a connected component $OG^o(n+1, 2n+2)$ of $OG(n+1, 2n+2)$, if G is of type D_{n+1} and $k \in \{n, n+1\}$.

In the first three cases, we have $N = 2n$, $2n+1$ and $2n+2$ respectively. For convenience, we have assumed G to be of type D_{n+1}, rather than D_n, in cases (iii) and (iv). Furthermore when this holds, we can always assume $k \leqslant n-1$, since case (iv) can be reduced to case (ii) (see Remark 3.1). Customarily, $IG(n, 2n)$ (resp. $OG(n, 2n+1)$, $OG^o(n+1, 2n+2)$) is called a *Lagrangian* (resp. *odd orthogonal*, *even orthogonal*) *Grassmannian*.

There are tautological bundles over the isotropic Grassmannian G/P:

$$0 \to \mathcal{S} \to E \to \mathcal{Q} \to 0.$$

The Chern classes of the dual of the tautological subbundle $\mathcal{S}$ are given by the Schubert classes $c_p(\mathcal{S}^*) = \sigma^u$ (where $1 \leqslant p \leqslant k$) with

$$u := s_{k-p+1} \cdots s_{k-1} s_k,$$

possibly up to a scale factor of 2. Precisely, for cases (i), (ii) and (iii), we always have $c_p(\mathcal{S}^*) = \sigma^u$, except for the special case of $k = n$ for case (ii). Furthermore for this exceptional case, we have $c_p(\mathcal{S}^*) = 2\sigma^u$ (see e.g. [27,28]). Note that case (iv) has been reduced to the exceptional case. In the next two subsections, we will show the classical aspects of the quantum Pieri rules with respect to $c_p(\mathcal{S}^*)$ (or equivalently $c_p(\mathcal{S}) = (-1)^p c_p(\mathcal{S}^*)$). That is, we give a formula for the quantum product $\sigma^u \star \sigma^v$ of a general Schubert class σ^v in $QH^*(G/P)$ with such a special Schubert class σ^u, in which we reduce all the Gromov–Witten invariants to classical intersection numbers.

Note that the quantum Pieri rules with respect to $c_p(\mathcal{Q})$ have been given by Buch, Kresch and Tamvakis [5]. In contrast to complex Grassmannians, (quantum) Pieri rules with respect to $c_p(\mathcal{Q})$ do not imply (quantum) Pieri rules with respect to $c_p(\mathcal{S}^*)$, whenever $1 < k < n$ (see the next remark and note that $c_1(\mathcal{S}^*) = c_1(\mathcal{Q})$).

Remark 3.1. (See e.g. [30] and [5].)

(1) $OG(n+1, 2n+2)$ has two isomorphic connected components, either of which is projectively isomorphic to $OG(n, 2n+1)$.
(2) $OG(n, 2n+2)$ is a flag variety $G/\bar{P}$ of D_{n+1}-type with $\Delta_{\bar{P}} = \Delta \setminus \{\alpha_n, \alpha_{n+1}\}$.
(3) For any Lagrangian or orthogonal Grassmannian (i.e. for $k = n$), we have $c_p(\mathcal{S}^*) = c_p(\mathcal{Q})$ whenever $p \leqslant \operatorname{rank}(\mathcal{S})$.

The idea of our proof of such a formula is as follows. For the isotropic Grassmannian G/P, we have $H_2(G/P, \mathbb{Z}) = Q^\vee/Q_P^\vee \cong \mathbb{Z}$ and consequently $QH^*(G/P)$ contains only one quantum variable $t := q_{\alpha_k^\vee + Q_P^\vee}$. Thus we can write

$$\sigma^u \star \sigma^v = \sigma^u \cup \sigma^v + \sum_{w \in W^P, d \geqslant 1} N_{u,v}^{w,d} \sigma^w t^d.$$

Here we have $N_{u,v}^{w,d} = N_{u,v}^{w,\lambda_P}$ where $\lambda_P := d\alpha_k^\vee + Q_P^\vee$, compared with the previous notations. In addition, we have $N_{u,v}^{w,\lambda_P} = N_{u,v}^{\tilde{w},\lambda_B}$, where $\tilde{w} = w\omega_P\omega_{P'}$ and λ_B are given by the Peterson–Woodward comparison formula. We can show

(1) $N_{u,v}^{w,d} = 0$ unless d is small enough (for instance $d \leqslant 2$).
(2) For a small d, $N_{u,v}^{\tilde{w},\lambda_B}$ is equal to a certain classical intersection number $N_{u',v'}^{w',0}$ for which the classical Pieri rules (or other known formulas) can be applied.

The dimension constraint may also be helpful. That is, we have $N_{u,v}^{w,d} = 0$ unless $\ell(u) + \ell(v) = \ell(w) + d \cdot \deg t$. Here we have $\deg t = 2n + 1 - k$ (resp. $2n - k$, $2n + 1 - k$) for case (i) (resp. (ii), (iii)) if $k < n$, and $\deg t = n + 1$ (resp. $2n$) for case (i) (resp. (ii) or (iv)) if $k = n$.

In fact, the above method can also been used to recover the well-known quantum Pieri rules for complex Grassmannians. Details will be given in Appendix A.

Note that whenever referring to $N_{u,v}^{\tilde{w},\lambda_B}$ (resp. $N_{u,v}^{w,\lambda_P}$ or $N_{u,v}^{w,d}$), we are discussing the quantum product $\sigma^u \star_B \sigma^v$ in $QH^*(G/B)$ (resp. $\sigma^u \star_P \sigma^v$ in $QH^*(G/P)$).

Due to the above assumptions on Δ, the Dynkin diagram of $\{\alpha_1, \dots, \alpha_{n-1}\}$ is of type A_{n-1} in the standard way. As a consequence, we have the following fact on certain products in the Weyl subgroup generated by $\{s_1, \dots, s_{n-1}\}$.

Lemma 3.2. *(See Lemma 3.3 of [23].) For $1 \leqslant i \leqslant j \leqslant m < n$ and $1 \leqslant r \leqslant m$, we have*

$$(s_i s_{i+1} \cdots s_j)(s_r s_{r+1} \cdots s_m) = \begin{cases} (s_r s_{r+1} \cdots s_m)(s_i s_{i+1} \cdots s_j), & \text{if } r \geqslant j+2 \\ s_i s_{i+1} \cdots s_m, & \text{if } r = j+1 \\ (s_{r+1} s_{r+2} \cdots s_m)(s_i s_{i+1} \cdots s_{j-1}), & \text{if } i \leqslant r \leqslant j \\ (s_r s_{r+1} \cdots s_m)(s_{i-1} s_i \cdots s_{j-1}), & \text{if } r < i. \end{cases}$$

3.2. Classical aspects of quantum Pieri rules for Grassmannians of type C

Throughout this subsection, we consider a Grassmannian of type C_n. Precisely, we consider the isotropic Grassmannian $G/P = IG(k, 2n)$. Thus the base Δ is of type C_n. Unless otherwise stated, we will always use the following definition of u in the rest of this section.

Definition 3.3. Fix $1 \leqslant p \leqslant k$, we define

$$u := s_{k-p+1} \cdots s_{k-1} s_k.$$

To show Theorem 3.13, the main result of this subsection, we need to compute all the Gromov–Witten invariants $N_{u,v}^{w,d}$ for the quantum product $\sigma^u \star \sigma^v$. Recall that for a given d, we have $N_{u,v}^{w,d} = N_{u,v}^{w,\lambda_P} = N_{u,v}^{\tilde{w},\lambda_B}$, where $\lambda_P = d\alpha_k^\vee + Q_P^\vee$, and $\tilde{w} = w\omega_P\omega_{P'}$, λ_B are both defined in Proposition 2.4. For each j, we simply denote $\mathrm{sgn}_j := \mathrm{sgn}_{\alpha_j}$.

Lemma 3.4. *Write $d = mk + r$ where $m, r \in \mathbb{Z}$ with $1 \leqslant r \leqslant k$. Then we have $\lambda_B = \lambda'$ with $\lambda' := m\sum_{j=1}^{k-1} j\alpha_j^\vee + \sum_{j=1}^{r-1} j\alpha_{k-r+j}^\vee + d\sum_{j=k}^{n} \alpha_j^\vee$.*

Proof. It is easy to check that $\langle \alpha_i, \lambda' \rangle = -1$ if $i = k - r$, or 0 otherwise. Hence, $\langle \gamma, \lambda' \rangle \in \{0, -1\}$ for all $\gamma \in R_P^+$. Thus the statement follows from the uniqueness of λ_B (see Proposition 2.4). □

Lemma 3.5. *With the same notations as in Lemma 3.4, we have $\langle \alpha_k, \lambda_B \rangle = m + 1$ if $k < n$, or $2(m + 1)$ if $k = n$.*

Proof. If $k = n$, we have $\langle \alpha_k, \lambda_B \rangle = \langle \alpha_n, m(n - 1)\alpha_{n-1}^\vee + (r - 1)\alpha_{n-1}^\vee + (mn + r)\alpha_n^\vee \rangle = 2(m + 1)$, by noting $\langle \alpha_n, \alpha_j^\vee \rangle = 0$ for all $j < n - 1$. If $k < n$, we have $\langle \alpha_k, \lambda_B \rangle = \langle \alpha_k, m(k - 1)\alpha_{k-1}^\vee + (r - 1)\alpha_{k-1}^\vee + (mk + r)\alpha_k^\vee + (mk + r)\alpha_{k+1}^\vee \rangle = m + 1$. □

Lemma 3.6. *Suppose $d \geqslant k + 1$, then we have $N_{u,v}^{w,d} = 0$ for any $w \in W^P$.*

Proof. Since $d \geqslant k + 1$, we have $d = mk + r$ with $1 \leqslant r \leqslant k$ and $m \geqslant 1$.

When $k = n$, we have $\langle \alpha_k, \lambda_B \rangle = 2(m + 1) > 2$ by Lemma 3.5. Thus we have $N_{u,v}^{w,d} = N_{u,v}^{\tilde{w},\lambda_B} = 0$ by Proposition 2.1 (1).

When $k < n$, we have $\langle \alpha_k, \lambda_B \rangle = m + 1 \geqslant 2$, by Lemma 3.5 again. If ">" holds, then we are already done by using Proposition 2.1 (1) again. Note $v \in W^P$ and $\alpha_{k+1} \in \Delta_P$. By Lemma 2.7, we have $\mathrm{sgn}_{k+1}(v) = 0$. If $\langle \alpha_k, \lambda_B \rangle = 2$, then we have $\mathrm{sgn}_{k+1}(us_k) + \mathrm{sgn}_{k+1}(v) = 0 < 1 \leqslant \mathrm{sgn}_{k+1}(\tilde{w}s_k) + \langle \alpha_{k+1}, \lambda_B - \alpha_k^\vee \rangle$. Thus we have $N_{us_k,v}^{\tilde{w}s_k,\lambda_B-\alpha_k^\vee} = 0$ by Proposition 2.1 (1). Consequently, we still have $N_{u,v}^{w,d} = N_{u,v}^{\tilde{w},\lambda_B} = 0$ by Corollary 2.2 (1). □

Remark 3.7. The above lemma also follows directly from the dimension count.

Lemma 3.8. *Let $u', v', w' \in W$ and $\lambda \in Q^\vee$. For the quantum product $\sigma^{u'} \star \sigma^{v'}$ in $QH^*(G/B)$, the structure constant $N_{u',v'}^{w',\lambda}$ vanishes, if both* (a) *and* (b) *hold*:

$$\text{(a) } \langle \alpha_n, \lambda \rangle = 2, \quad \langle \alpha_{n-1}, \lambda \rangle = 0; \qquad \text{(b) } \mathrm{sgn}_n\big(u's_ns_{n-1}\big) = 0, \quad \mathrm{sgn}_{n-1}\big(v'\big) = 0.$$

Proof. Note that $\mathrm{sgn}_n(u's_ns_{n-1})+\mathrm{sgn}_n(v')=0+\mathrm{sgn}_n(v')\leqslant 1<2=\langle\alpha_n,\lambda\rangle=\langle\alpha_n,\lambda-\alpha_n^\vee-\alpha_{n-1}^\vee\rangle$. By Proposition 2.1 (1), we have $N_{u's_ns_{n-1},v'}^{w's_ns_{n-1},\lambda-\alpha_n^\vee-\alpha_{n-1}^\vee}=0$. Since $\mathrm{sgn}_{n-1}(v')=0$ and $\langle\alpha_{n-1},\lambda-\alpha_n^\vee\rangle=-\langle\alpha_{n-1},\alpha_n^\vee\rangle=1$, we have $N_{u's_n,v'}^{w's_n,\lambda-\alpha_n^\vee}=0$ by Corollary 2.2 (2). Consequently, we have $N_{u',v'}^{w',\lambda}=0$ by Corollary 2.2 (1). $\square$

The next proposition shows us that t is the largest power t^d appearing in the quantum product $\sigma^u\star\sigma^v$ in $QH^*(G/P)$.

Proposition 3.9. *Suppose $d\geqslant 2$, then we have $N_{u,v}^{w,d}=0$ for any $w\in W^P$.*

Proof. We can assume $2\leqslant d\leqslant k$ due to Lemma 3.6. It suffices to show $N_{u,v}^{\tilde{w},\lambda_B}=0$, where we note $\lambda_B=\sum_{j=1}^{d-1}j\alpha_{k-d+j}^\vee+d\sum_{j=k}^n\alpha_j^\vee$. Consequently, $\langle\alpha_{k-1},\lambda_B\rangle=0$.

Suppose $k=n$, then $\langle\alpha_n,\lambda_B\rangle=2$. Clearly, we have $\mathrm{sgn}_n(us_ns_{n-1})=0$ and $\mathrm{sgn}_{n-1}(v)=0$. Thus we are done by Lemma 3.8.

Now we assume $k<n$. Since $\mathrm{sgn}_{k+1}(u)=\mathrm{sgn}_{k+1}(v)=0$ and $\langle\alpha_{k+1},\lambda_B\rangle=0$, it suffices to show $N_{us_{k+1},v}^{\tilde{w}s_{k+1},\lambda_B}=0$, due to Corollary 2.2 (3). Note $\mathrm{sgn}_k(us_{k+1})=0$ and $\langle\alpha_k,\lambda_B\rangle=1$. Then it suffices to show $N_{us_{k+1},vs_k}^{\tilde{w}s_{k+1}s_k,\lambda_B-\alpha_k^\vee}=0$, due to Corollary 2.2 (2). For any $1\leqslant i\leqslant m\leqslant n$, we denote $v_i^{(m)}:=s_ms_{m-1}\cdots s_{m-i+1}$. By induction on i, we reduce the above statement to the following one. To show $N_{us_{k+1},vs_k}^{\tilde{w}s_{k+1}s_k,\lambda_B-\alpha_k^\vee}=N_{us_{k+1},vv_1^{(k)}}^{\tilde{w}s_{k+1}v_1^{(k)},\lambda_B-\sum_{j=1}^{1}\alpha_{k-j+1}^\vee}=0$, it suffices to show $N_{us_{k+1},vv_d^{(k)}}^{\tilde{w}s_{k+1}v_d^{(k)},\lambda_B-\sum_{j=1}^{d}\alpha_{k-j+1}^\vee}=0$. Furthermore by induction on m, it suffices to show $N_{u',v'}^{w',\lambda'_B}=0$, in which $u'=us_{k+1}\cdots s_{n-1}s_n$, $v'=vv_d^{(k)}\cdots v_d^{(n-1)}$, $w'=\tilde{w}s_{k+1}\cdots s_{n-1}s_nv_d^{(k)}\cdots v_d^{(n-1)}$ and $\lambda'_B=\lambda_B-\sum_{j=1}^d\alpha_{k-j+1}^\vee-\cdots-\sum_{j=1}^d\alpha_{n-1-j+1}^\vee=\sum_{j=1}^d j\alpha_{n-d+j}^\vee$. (Here we always use Corollary 2.2 (2), (3) for the inductions.)

Note that $\langle\alpha_n,\lambda'_B\rangle=2$, $\langle\alpha_{n-1},\lambda'_B\rangle=0$ and $\mathrm{sgn}_n(u's_ns_{n-1})=0$. In addition, we note that $d\geqslant 2$, so that $v_d^{(j)}(\alpha_j)=\alpha_{j-1}$ for each $k\leqslant j\leqslant n-1$. Thus we have $v'(\alpha_{n-1})=v(\alpha_{k-1})\in R^+$ and consequently $\mathrm{sgn}_{n-1}(v')=0$. Hence, we do have $N_{u',v'}^{w',\lambda'_B}=0$, by using Lemma 3.8. $\square$

Remark 3.10. Proposition 3.9 (resp. Proposition 3.20) can also be proved by using Theorem 1.3 (d) (resp. Theorem 2.3 (d) and Theorem 3.3 (d)) of [5].

The next lemma also works with exactly the same arguments, for either of the cases: (1) Δ is of type B_n; (2) Δ is of type D_{n+1} and $k<n$.

Lemma 3.11. *Let $u'=s_{j-i+1}\cdots s_{j-1}s_j$ where $1\leqslant i\leqslant j<k$, and λ be effective with $\langle\chi_j,\lambda\rangle=0$. If $\lambda\neq 0$, then we have $N_{u',v'}^{w',\lambda}=0$ for any $v',w'\in W$.*

Proof. Note that the product $(\sigma^{s_j})^i:=\sigma^{s_j}\star\cdots\star\sigma^{s_j}$ of i copies of σ^{s_j} is the summation of $\sigma^{u'}$ and other nonnegative terms. Hence, $(\sigma^{s_j})^i\star\sigma^{v'}=\sigma^{u'}\star\sigma^{v'}+(\text{other } \textit{nonnegative} \text{ terms})=N_{u',v'}^{w',\lambda}q_\lambda\sigma^{w'}+(\text{other } \textit{nonnegative} \text{ terms})$. On the other hand, we have

$$\begin{aligned}(\sigma^{s_j})^i \star \sigma^{v'} &= \sum_{\gamma_i}\cdots\sum_{\gamma_1} N_{s_j,v'}^{v's_{\gamma_1},\mu_1} N_{s_j,v's_{\gamma_1}}^{v's_{\gamma_1}s_{\gamma_2},\mu_2}\cdots N_{s_j,v's_{\gamma_1}\cdots s_{\gamma_{i-1}}}^{v's_{\gamma_1}\cdots s_{\gamma_i},\mu_i} q_{\mu_1+\cdots+\mu_i}\sigma^{v's_{\gamma_1}\cdots s_{\gamma_i}}\\ &= \sum_{\gamma_i}\cdots\sum_{\gamma_1}\prod_{h=1}^{i}\langle\chi_j,\gamma_h^\vee\rangle q_{\mu_1+\cdots+\mu_i}\sigma^{v's_{\gamma_1}\cdots s_{\gamma_i}},\end{aligned}$$

by the quantum Chevalley formula. Here for $1\leqslant h\leqslant i$, $\gamma_h\in R^+$, $\mu_h\in\{0,\gamma_h^\vee\}$, and they satisfy $\ell(v's_{\gamma_1}\cdots s_{\gamma_h})=\ell(v's_{\gamma_1}\cdots s_{\gamma_{h-1}})+1-\langle 2\rho,\mu_h\rangle$. If $N_{u',v'}^{w',\lambda}\neq 0$ for some v', w', then there exists a sequence $(\gamma_1,\ldots,\gamma_i)$ such that $\lambda=\sum_{h=1}^i\mu_h$ and $\prod_{h=1}^i\langle\chi_j,\gamma_h^\vee\rangle\neq 0$. Since $\lambda\neq 0$, there exists $1\leqslant h'\leqslant i$ such that $\mu_{h'}\neq 0$ and $0=\langle\chi_j,\lambda\rangle=\langle\chi_j,\sum_{h=1}^i\mu_h\rangle\geqslant\langle\chi_j,\mu_{h'}\rangle=\langle\chi_j,\gamma_{h'}^\vee\rangle>0$. Contradiction. □

The next well-known fact, characterizing $\omega_P\omega_{P'}$, works for Δ of *any type*.

Lemma 3.12. *(See e.g. Lemma 3.5 of [23].) An element $\bar{w}\in W_P$ is equal to $\omega_P\omega_{P'}$, if both of the following hold*: (i) $\ell(\bar{w})=\ell(\omega_P\omega_{P'})$; (ii) $\bar{w}(\alpha)\in R^+$ *for all* $\alpha\in\Delta_{P'}$.

In the rest of this subsection, we fix the positive root $\gamma:=\alpha_n+2\sum_{j=k}^{n-1}\alpha_j$. Note that $\gamma^\vee=\alpha_k^\vee+\alpha_{k+1}^\vee+\cdots+\alpha_n^\vee$ and $\langle 2\rho,\gamma^\vee\rangle=2n-2k+2$.

Theorem 3.13 *(Classical aspects of quantum Pieri rules for Grassmannians of type C_n). Let σ^u,σ^v be Schubert classes in the quantum cohomology of the isotropic Grassmannian $G/P=IG(k,2n)$. Recall $u=s_{k-p+1}\cdots s_{k-1}s_k$, where $1\leqslant p\leqslant k$. We have*

$$\sigma^u\star\sigma^v=\sigma^u\cup\sigma^v+\begin{cases} t\sum_{w\in W^P}N_{us_k,vs_\gamma}^{ws_1\cdots s_{k-1},0}\sigma^w, & \text{if } \ell(vs_\gamma)=\ell(v)-2n+2k-1\\ 0, & \text{otherwise.}\end{cases}$$

Proof. Due to Proposition 3.9, we have $\sigma^u\star\sigma^v=\sigma^u\cup\sigma^v+t\sum_{w\in W^P}N_{u,v}^{w,1}\sigma^w$. Thus by the Peterson–Woodward comparison formula, it suffices to compute the Gromov–Witten invariants $N_{u,v}^{w,1}=N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}$ with respect to $\lambda_P=\alpha_k^\vee+Q_P^\vee$. By Lemma 3.4, we have $\lambda_B=\sum_{j=k}^n\alpha_j^\vee=\gamma^\vee$, so that $\Delta_{P'}=\Delta_P\setminus\{\alpha_{k-1}\}$. Consequently, we have $\ell(\omega_P\omega_{P'})=|R_P^+|-|R_{P'}^+|=k-1$. Hence, we conclude $\omega_P\omega_{P'}=s_1s_2\cdots s_{k-1}$ by (easily) checking the assumptions in Lemma 3.12. Therefore, it is sufficient to compute $N_{u,v}^{ws_1\cdots s_{k-1},\gamma^\vee}q_{\gamma^\vee}\sigma^{ws_1\cdots s_{k-1}}$ in the product $\sigma^u\star_B\sigma^v$ in $QH^*(G/B)$. By abuse of notations, we simply denote "$\star_B$" as "$\star$" here. We claim

(1) the contribution $N_{u,v}^{ws_1\cdots s_{k-1},\gamma^\vee}$ for $q_{\gamma^\vee}\sigma^{ws_1\cdots s_{k-1}}$ from $\sigma^u\star\sigma^v$ is the same as the contribution $N_{us_k,s_k,v}^{ws_1\cdots s_{k-1},\gamma^\vee}$ for $q_{\gamma^\vee}\sigma^{ws_1\cdots s_{k-1}}$ from $\sigma^{us_k}\star\sigma^{s_k}\star\sigma^v$;
(2) $N_{us_k,s_k,v}^{w',\gamma^\vee}=N_{s_k,v}^{vs_\gamma,\gamma^\vee}\cdot N_{us_k,vs_\gamma}^{w',0}$, for any $w'\in W$.

Assuming these claims, if $\ell(vs_\gamma)\neq\ell(v)+1-\langle 2\rho,\gamma^\vee\rangle$, then $N_{s_k,v}^{vs_\gamma,\gamma^\vee}=0$. As a consequence, $N_{u,v}^{w,1}=N_{u,v}^{ws_1\cdots s_{k-1},\gamma^\vee}=0\cdot N_{us_k,vs_\gamma^\vee}^{ws_1\cdots s_{k-1},0}=0$ for any $w\in W^P$. Hence, $\sigma^u\star_P\sigma^v=\sigma^u\cup\sigma^v$. If $\ell(vs_\gamma)=\ell(v)+1-\langle 2\rho,\gamma^\vee\rangle$, then we have $N_{s_k,v}^{vs_\gamma,\gamma^\vee}=\langle\chi_k,\gamma^\vee\rangle=1$, by the quantum Chevalley formula. Thus $N_{u,v}^{ws_1\cdots s_{k-1},\gamma^\vee}=N_{us_k,vs_\gamma}^{ws_1\cdots s_{k-1},0}$. In addition, we note $\ell(v)+1-\langle 2\rho,\gamma^\vee\rangle=\ell(v)-2n+2k-1$. Hence, we are done.

It remains to show claims (1) and (2). If $\ell(u)=1$, then $us_k=1$ and we are done. Thus we assume $\ell(u)>1$ in the rest, and show claim (1) first. Note that $u=s_{k-p+1}\cdots s_{k-1}s_k$ is of length p. By the quantum Chevalley formula, we have $\sigma^{us_k}\star\sigma^{s_k}=\sigma^u+\sigma^{s_kus_k}$. It suffices to show $\sigma^{s_kus_k}\star\sigma^v$ makes no contribution for $q_{\gamma^\vee}\sigma^{ws_1\cdots s_{k-1}}$. Indeed, we have $\mathrm{sgn}_j(s_kus_k)=0$ whenever $j\geqslant k$, by noting $s_kus_k(\alpha_j)=s_ks_{k-p+1}\cdots s_{k-2}s_{k-1}(\alpha_j)\in R^+$. Since $\langle\alpha_k,\gamma^\vee\rangle=1$, $N_{s_kus_k,v}^{ws_1\cdots s_{k-1},\gamma^\vee}=0$ follows if $N_{s_kus_k,vs_k}^{ws_1\cdots s_{k-1}s_k,\gamma^\vee-\alpha_k^\vee}=0$, by Corollary 2.2 (2). Repeating this reduction, it suffices to show $N_{s_kus_k,vs_k\cdots s_{n-1}}^{ws_1\cdots s_{n-1},\alpha_n^\vee}=0$, which does follow by using Proposition 2.1 (1) with respect to sgn_n. Thus claim (1) follows.

The contribution $N_{us_k,s_k,v}^{w',\gamma^\vee}$ for $q_{\gamma^\vee}\sigma^{w'}$ from $\sigma^{us_k}\star\sigma^{s_k}\star\sigma^v=(\sigma^{s_k}\star\sigma^v)\star\sigma^{us_k}$ is given by $N_{us_k,s_k,v}^{w',\gamma^\vee}=\sum_{w''\in W,\lambda\in Q^\vee}N_{s_k,v}^{w'',\lambda}N_{w'',us_k}^{w',\gamma^\vee-\lambda}$ (which contains only finitely many nonzero terms). Hence, claim (2) becomes a direct consequence of the quantum Chevalley formula and Lemma 3.11. □

Remark 3.14. Using Proposition 2.1, we can also show $N_{u,v}^{ws_1\cdots s_{k-1},\gamma^\vee}=N_{u',v'}^{w',0}$ where $u'=s_{k-p+1}\cdots s_n$, $v'=vs_k\cdots s_n$ and $w'=ws_1\cdots s_{k-1}s_{k+1}\cdots s_ns_k\cdots s_n$. As a consequence, we can apply a generalized classical Pieri rule given by Bergeron and Sottile [1] to express $N_{u',v'}^{w',0}$ more explicitly.

Example 3.15. For $X=IG(2,8)$, we take $u=s_1s_2$, $v=s_3s_4s_3s_1s_2$ and $w=\mathrm{id}$. Then $vs_\gamma=vs_2s_3s_4s_3s_2=s_1s_2$, so that $\ell(v)=2\neq 0=\ell(v)-2\cdot 4+2\cdot 2-1$. Thus $N_{u,v}^{w,1}=0$. (In terms of notations in Example 1.3 of [5], we have $\sigma^u=\sigma_{1,1},\sigma^v=\sigma_{4,1}$ and $N_{u,v}^{w,1}=\langle\sigma_{1,1},\sigma_{4,1},\sigma_{6,5}\rangle_1$.)

Denote by $\tilde P\supset B$ the parabolic subgroup corresponding to the subset $\Delta\setminus\{\alpha_{k-1}\}$. That is, $G/\tilde P=IG(k-1,2n)$. (When $k=1$, we mean $\tilde P=G$, i.e. $W^{\tilde P}=\{\mathrm{id}\}$.) Recall that $\gamma^\vee=\sum_{j=k}^n\alpha_j^\vee$ so that $\ell(v)+1-\langle 2\rho,\gamma^\vee\rangle=\ell(v)-2n+2k-1$.

Lemma 3.16. *For any $v\in W^P$, the following are equivalent*:

$$\text{(a) } \ell(vs_\gamma)=\ell(v)+1-\langle 2\rho,\gamma^\vee\rangle;\qquad \text{(b) } vs_\gamma(\alpha_k)\in R^+;\qquad \text{(c) } vs_\gamma\in W^{\tilde P}.$$

Proof. Note that $\gamma^\vee=\sum_{j=k}^n\alpha_j^\vee=s_ks_{k+1}\cdots s_{n-1}(\alpha_n^\vee)$. We conclude that $s_\gamma=s_ks_{k+1}\cdots s_n\cdots s_{k+1}s_k$ and $\ell(s_\gamma)=\langle 2\rho,\gamma^\vee\rangle-1=2n-2k+1$.

Suppose assumption (a) holds first. Note that $\langle\alpha_k,\gamma^\vee\rangle>0$. By Lemma 2.6, we have $\ell(vs_\gamma s_k)=\ell(vs_\gamma)+1$. Hence, $vs_\gamma(\alpha_k)\in R^+$. That is, assumption (b) holds.

Assume (b) holds now. Note that $\langle\alpha_i,\gamma^\vee\rangle=0$ for any $\alpha_i\in\Delta\setminus\{\alpha_{k-1},\alpha_k\}$, so that $vs_\gamma(\alpha_i)=v(\alpha_i)\in R^+$. Hence, (c) follows.

Assume (c) holds, equivalently, $vs_\gamma(\alpha_i)\in R^+$ for all $\alpha_i\in\Delta\setminus\{\alpha_{k-1}\}$. Then we have $v\neq 1$, because otherwise $vs_\gamma(\alpha_k)=\alpha_k-\langle\alpha_k,\gamma^\vee\rangle\gamma\notin R^+$. As a consequence, we have $v(\alpha_k)\in -R^+$ (as $v\in W^P$). Rewrite $(\alpha_k,\dots,\alpha_n,\dots,\alpha_k)$ as $(\beta_1,\dots,\beta_{2n-2k+1})$. To show (a), it suffices to show $\ell(vs_\gamma s_{\beta_1}\cdots s_{\beta_{j-1}}s_{\beta_j})=\ell(vs_\gamma s_{\beta_1}\cdots s_{\beta_{j-1}})+1$ (or equivalently to show $vs_\gamma s_{\beta_1}\cdots s_{\beta_{j-1}}(\beta_j)\in R^+$) for all $1\leqslant j\leqslant 2n-2k+1$. Since $\alpha_k=\beta_1$, this holds when $j=1$. When $2\leqslant j\leqslant 2n-2k$, we note that $s_{\beta_1}\cdots s_{\beta_{j-1}}(\beta_j)=\alpha_k+\beta$ for a positive root β in the root subsystem with respect to the subbase $\{\alpha_{k+1},\dots,\alpha_n\}$. Thus $vs_\gamma s_{\beta_1}\cdots s_{\beta_{j-1}}(\beta_j)=vs_\gamma(\alpha_k)+vs_\gamma(\beta)=vs_\gamma(\alpha_k)+$

$v(\beta) \in R^+$. When $j = 2n - 2k + 1$, we have $vs_\gamma s_{\beta_1} \cdots s_{\beta_{j-1}}(\beta_j) = vs_\gamma s_\gamma s_{\beta_{2n-2k+1}}(\beta_{2n-2k+1}) = -v(\beta_{2n-2k+1}) = -v(\alpha_k) \in R^+$. Thus we are done. □

Thanks to the above lemma, the assumption "$\ell(vs_\gamma) = \ell(v) - 2n + 2k - 1$" in Theorem 3.13 is equivalent to the assumption "$vs_\gamma \in W^{\tilde{P}}$". This indicates us that $N_{u,v}^{w,1} = N_{us_k, vs_\gamma}^{ws_1 \cdots s_{k-1}, 0}$ is a classical intersection number involved in the cup product $\sigma^{s_{k-p+1} \cdots s_{k-1}} \cup \sigma^{vs_\gamma}$ in $H^*(IG(k-1, 2n))$. As a consequence, the classical Pieri rule in [27] can still be applied. In particular, we can reformulate Theorem 3.13 in a more traditional way, which will be described in Theorem 4.4.

3.3. *Classical aspects of quantum Pieri rules for Grassmannians of type B, D*

Throughout this subsection, we consider a Grassmannian of type B_n or D_{n+1}. Precisely, we consider the isotropic Grassmannian $G/P = OG(k, 2n+1)$ (resp. $OG(k, 2n+2)$) for Δ of type B_n (resp. D_{n+1}). Note that the even orthogonal Grassmannian $OG^o(n+1, 2n+2)$ is isomorphic to the odd orthogonal Grassmannian $OG(n, 2n+1)$. It suffices to deal with either of them only. Hence, when Δ is of D_{n+1}-type, we can always assume $k \leqslant n-1$. In other words, whenever referring to "$k = n$", we are dealing with Δ of B_n-type, unless otherwise stated. As before, we need to compute the Gromov–Witten invariants $N_{u,v}^{w,d} = N_{u,v}^{w,\lambda_P} = N_{u,v}^{\tilde{w},\lambda_B}$ (where $\lambda_P = d\alpha_k^\vee + Q_P^\vee$) for the quantum product $\sigma^u \star \sigma^v$ in $QH^*(G/P)$.

Let $[x]$ denote the integer satisfying $0 \leqslant x - [x] < 1$. In order to state the results uniformly, we denote

$$\bar{\alpha}_n^\vee := \alpha_n^\vee \quad \big(\text{resp. } \alpha_n^\vee + \alpha_{n+1}^\vee\big) \quad \text{and} \quad s_{\bar{\alpha}_n} := s_n \quad (\text{resp. } s_n s_{n+1})$$

when Δ is of type B_n (resp. D_{n+1}). Furthermore, we denote $D = d$ (resp. $2d$) if $k < n$ (resp. $k = n$).

With the same arguments as for Lemma 3.4, we have

Lemma 3.17. *Write $D = mk + r$ where $m, r \in \mathbb{Z}$ with $1 \leqslant r \leqslant k$. Then we have*

$$\lambda_B = m \sum_{j=1}^{k-1} j\alpha_j^\vee + \sum_{j=1}^{r-1} j\alpha_{k-r+j}^\vee + d\alpha_k^\vee + \begin{cases} 2[\frac{d}{2}] \sum_{j=k+1}^{n-1} \alpha_j^\vee + [\frac{d}{2}]\bar{\alpha}_n^\vee, & \text{if } k < n \\ 0, & \text{if } k = n. \end{cases}$$

Consequently, we have $\langle \alpha_k, \lambda_B \rangle = m + 1 + D - 2[\frac{D}{2}]$; for $k+1 \leqslant n$, $\langle \alpha_{k+1}, \lambda_B \rangle = -d + 2[\frac{d}{2}]$; for $1 \leqslant i \leqslant k-1$, $\langle \alpha_i, \lambda_B \rangle = -1$ if $i = k - r$, or 0 otherwise.

Recall $u = s_{k-p+1} \cdots s_{k-1} s_k$ where $1 \leqslant p \leqslant k$.

Lemma 3.18. *If $D \geqslant k+1$, then we have $N_{u,v}^{w,d} = 0$ for any $w \in W^P$.*

Proof. Use the same notations as in Lemma 3.17. If $D > 2k$, then we have $\langle \alpha_k, \lambda_B \rangle \geqslant m + 1 > 2$ and consequently $N_{u,v}^{w,d} = 0$ by Proposition 2.1. Now we assume $k + 1 \leqslant D \leqslant 2k$, so that $m = 1$.

Suppose $k < n$, that is, $D = d$. Note that $\langle \alpha_k, \lambda_B \rangle = 1 + 1 + d - 2[\frac{d}{2}] = 3$ (resp. 2) if d is odd (resp. even). Thus when d is odd, we are already done. When d is even, we note that $\mathrm{sgn}_{k+1}(us_k) + \mathrm{sgn}_{k+1}(v) = 0 < 1 \leqslant \mathrm{sgn}_{k+1}(\tilde{w}s_k) + \langle \alpha_{k+1}, \lambda_B - \alpha_k^\vee \rangle$. Thus we have $N_{us_k, v}^{\tilde{w}s_k, \lambda_B - \alpha_k^\vee} = 0$ by using Proposition 2.1 (1). As a consequence, we have $N_{u,v}^{\tilde{w},\lambda_B} = 0$, by using Corollary 2.2 (1). That is, $N_{u,v}^{w,d} = 0$.

Suppose $k=n$, that is, $D=2d$. Note that $\langle\alpha_n,\lambda_B\rangle=2$, $\langle\alpha_{n-1},\lambda_B\rangle=0$ and consequently $\langle\alpha_{n-1},\lambda_B-\alpha_n^\vee\rangle=2$. Since $\mathrm{sgn}_{n-1}(us_n)+\mathrm{sgn}_{n-1}(v)\leqslant 1+0<2=\langle\alpha_{n-1},\lambda_B-\alpha_n^\vee\rangle$, we have $N_{us_n,v}^{\tilde{w}s_n,\lambda_B-\alpha_n^\vee}=0$ by Proposition 2.1 (1). Thus we have $N_{u,v}^{\tilde{w},\lambda_B}=0$ by Corollary 2.2 (1). □

Lemma 3.19. *Let $u',v',w'\in W$ and $\lambda\in Q^\vee$. For the quantum product $\sigma^{u'}\star\sigma^{v'}$ in $QH^*(G/B)$, the structure constant $N_{u',v'}^{w',\lambda}$ vanishes, if both* (a) *and* (b) *hold*:

(a) $\langle\alpha_{n-1},\lambda\rangle=\langle\alpha_{n-2},\lambda\rangle=0$; $\langle\alpha_j,\lambda\rangle=1$, *whenever* $j\geqslant n$.
(b) $\mathrm{sgn}_i(u')=\mathrm{sgn}_i(v')=0$ *for* $i\in\{n-2,n-1\}$; $\mathrm{sgn}_j(u's_{n-1})=0$, *if* $j\geqslant n$.

Proof. Since $\mathrm{sgn}_{n-2}(v')=0$, we have $v's_{\bar{\alpha}_n}(\alpha_{n-2})=v'(\alpha_{n-2})\in R^+$. Thus $\mathrm{sgn}_{n-2}(v's_{\bar{\alpha}_n})=0$. Consequently, we have $\mathrm{sgn}_{n-2}(u')+\mathrm{sgn}_{n-2}(v's_{\bar{\alpha}_n})=0<1=\langle\alpha_{n-2},-\alpha_{n-1}^\vee\rangle=\langle\alpha_{n-2},\lambda-\bar{\alpha}_n^\vee-\alpha_{n-1}^\vee\rangle$. By Proposition 2.1 (1), we have $N_{u',v's_{\bar{\alpha}_n}}^{w's_{n-1}s_{\bar{\alpha}_n}s_{n-1},\lambda-\bar{\alpha}_n^\vee-\alpha_{n-1}^\vee}=0$. Since $\langle\alpha_{n-1},\lambda-\bar{\alpha}_n^\vee\rangle=-\langle\alpha_{n-1},\bar{\alpha}_n^\vee\rangle=2$, we have $N_{u's_{n-1},v's_{\bar{\alpha}_n}}^{w's_{n-1}s_{\bar{\alpha}_n},\lambda-\bar{\alpha}_n^\vee}=0$ by Corollary 2.2 (1). Note that $\mathrm{sgn}_j(u's_{n-1})=0$ and $\langle\alpha_j,\lambda-\bar{\alpha}_n^\vee+\alpha_j^\vee\rangle=1+\langle\alpha_j,-\bar{\alpha}_n^\vee+\alpha_j^\vee\rangle=1$, whenever $j\geqslant n$. Using Corollary 2.2 (2), we deduce $N_{u's_{n-1},v'}^{w's_{n-1},\lambda}=0$. Then we have $N_{u',v'}^{w',\lambda}=0$ by Corollary 2.2 (3). □

Proposition 3.20. *If $D\geqslant 3$, then we have $N_{u,v}^{w,d}=0$ for any $w\in W^P$.*

Proof. We can assume $3\leqslant D\leqslant k$, due to Lemma 3.18.

Suppose $k=n$. Then $D=2d$ and $\lambda_B=\sum_{j=1}^{2d-1}j\alpha_{n-2d+j}^\vee+d\alpha_n^\vee$. It is easy to check that all the assumptions in Lemma 3.19 hold for u,v,λ_B. Thus we are done.

Suppose $k<n$ now. Then $D=d$. Recall that $u=s_{k-p+1}\cdots s_k$ with $\ell(u)=p$.

Assume d is odd. Then $\lambda_B=\sum_{j=1}^{d-1}j\alpha_{k-d+j}^\vee+d\alpha_k^\vee+(d-1)\sum_{j=k+1}^{n-1}\alpha_j^\vee+\frac{d-1}{2}\bar{\alpha}_n^\vee$. Consequently, we have $\langle\alpha_k,\lambda_B\rangle=2$ and $\langle\alpha_j,\lambda_B\rangle=0$ for each $k-d+1\leqslant j\leqslant k-1$. Denote $\bar{d}:=\min\{p,d\}$, $u':=us_ks_{k-1}\cdots s_{k-\bar{d}+1}$ and $\lambda:=\lambda_B-\sum_{j=1}^{\bar{d}}\alpha_{k-\bar{d}+j}^\vee$. We claim $N_{u',v}^{\tilde{w}s_ks_{k-1}\cdots s_{k-\bar{d}+1},\lambda}=0$. (Indeed, if $p\leqslant d$, then $u'=1$ and therefore the claim follows, by noting $\lambda\neq 0$. Note that $p\leqslant k$. If $p>d$, then $\bar{d}=d$, $u'=s_{k-p+1}s_{k-p+2}\cdots s_{k-d}$ and we note that $\langle\chi_{k-d},\lambda\rangle=0$. Thus the claim still follows by Lemma 3.11.) Note that $\mathrm{sgn}_{k-\bar{d}+1}(v)=0$ and $\langle\alpha_{k-\bar{d}+1},\lambda+\alpha_{k-\bar{d}+1}^\vee\rangle=1$. Applying Corollary 2.2 (2) to $(v,u's_{k-\bar{d}+1},\tilde{w}s_ks_{k-1}\cdots s_{k-\bar{d}+2},\lambda+\alpha_{k-\bar{d}+1}^\vee,\alpha_{k-\bar{d}+1})$, we obtain $N_{u's_{k-\bar{d}+1},v}^{\tilde{w}s_ks_{k-1}\cdots s_{k-\bar{d}+2},\lambda+\sum_{j=1}^h\alpha_{k-\bar{d}+j}^\vee}=0$ for $h=1$. By induction, we conclude $N_{u's_{k-\bar{d}+1}\cdots s_{k-\bar{d}+h},v}^{\tilde{w}s_ks_{k-1}\cdots s_{k-\bar{d}+h+1},\lambda+\sum_{j=1}^h\alpha_{k-\bar{d}+j}^\vee}=0$ for each $1\leqslant h\leqslant\bar{d}-1$. In particular, we have $N_{us_k,v}^{\tilde{w}s_k,\lambda_B-\alpha_k^\vee}=0$ when $h=\bar{d}-1$. Since $\langle\alpha_k,\lambda_B\rangle=2$, we have $N_{u,v}^{\tilde{w},\lambda_B}=0$ by Corollary 2.2 (1).

Assume d is even. Then $\lambda_B=\sum_{j=1}^{d-1}j\alpha_{k-d+j}^\vee+d\sum_{j=k}^{n-1}\alpha_j^\vee+\frac{d}{2}\bar{\alpha}_n^\vee$. Consequently, we have $\langle\alpha_k,\lambda_B\rangle=1$ and $\langle\alpha_j,\lambda_B\rangle=0$ for any $j\notin\{k,k-d\}$. Using exactly the same arguments as in the third paragraph of the proof of Proposition 3.9, we conclude that it suffices to show $N_{u',v'}^{w',\lambda_B'}=0$, in order to show $N_{u,v}^{\tilde{w},\lambda_B}=0$. Here $u'=us_{k+1}\cdots s_{n-1}s_{\bar{\alpha}_n}$, $v'=vv_d^{(k)}\cdots v_d^{(n-1)}$, $w'=\tilde{w}s_{k+1}\cdots s_{n-1}s_{\bar{\alpha}_n}v_d^{(k)}\cdots v_d^{(n-1)}$ and $\lambda_B'=\sum_{j=1}^{d-1}j\alpha_{n-d+j}^\vee+\frac{d}{2}\bar{\alpha}_n^\vee$, where $v_d^{(i)}:=s_is_{i-1}\cdots s_{i-d+1}$ for any $k\leqslant i\leqslant n-1$. Hence, we are done, by using Lemma 3.19 with respect to u',v',λ_B'.

(Indeed, we have $v_d^{(i)}(\alpha_i) = \alpha_{i-1}$ and $v_d^{(i)}(\alpha_{i-1}) = \alpha_{i-2}$ for each $k \leqslant i \leqslant n-1$, by noting $d \geqslant 3$. Thus we have $v'(\alpha_{n-1}) = v(\alpha_{k-1}) \in R^+$, $v'(\alpha_{n-2}) = v(\alpha_{k-2}) \in R^+$ and consequently $\mathrm{sgn}_{n-1}(v') = \mathrm{sgn}_{n-2}(v') = 0$. It is easy to check that all the remaining assumptions in Lemma 3.19 hold for u', v', λ'_B.) □

Theorem 3.21 *(Classical aspects of quantum Pieri rules for Grassmannians of type B_n, D_{n+1}). Let σ^u, σ^v be Schubert classes in the quantum cohomology of the isotropic Grassmannian $G/P = OG(k, N)$, where $N = 2n+1$ (resp. $2n+2$) for Δ of type B_n (resp. D_{n+1}). Recall $u = s_{k-p+1} \cdots s_{k-1}s_k$, where $1 \leqslant p \leqslant k$. (Note that $c_p(\mathcal{S}^*) = \sigma^u$, possibly up to a scale factor of 2, where $\mathcal{S}$ denotes the tautological subbundle over $OG(k, N)$.) Then in the quantum product*

$$\sigma^u \star \sigma^v = \sigma^u \cup \sigma^v + \sum_{w \in W^P, d \geqslant 1} N_{u,v}^{w,d} t^d \sigma^w,$$

all the Gromov–Witten invariants $N_{u,v}^{w,d}$ coincide with certain classical intersection numbers. More precisely, we have

(1) *If $d = 1$, then we have*

$$N_{u,v}^{w,1} = N_{u_1,v_1}^{w_1,0}$$

with the elements $u_1, v_1, w_1 \in W$ given by

$$(u_1, v_1, w_1) = \begin{cases} (us_k, vs_ks_{k+1} \cdots s_{n-1}s_{\bar{\alpha}_n}s_{n-1} \cdots s_{k+1}, ws_1 \cdots s_{k-1}), & \text{if } k < n \\ (u, vs_ns_{n-1}, ws_2 \cdots s_{n-1}s_1 \cdots s_{n-2}s_{n-1}s_n), & \text{if } k = n, \end{cases}$$

provided that $\ell(u) + \ell(v) = \ell(w) + \deg t$, and zero otherwise.

(2) *If $d = 2$, then we have*

$$N_{u,v}^{w,2} = N_{u,v_2}^{w_2,0}$$

with $v_2 = vs_k \cdots s_{n-1}s_{\bar{\alpha}_n}s_{n-1} \cdots s_1$ and $w_2 = ws_1 \cdots s_{n-1}s_{\bar{\alpha}_n}s_{n-1} \cdots s_k$, provided that $k < n$ and $\ell(u) + \ell(v) = \ell(w) + 2\deg t$, and zero otherwise.

(3) *If $d \geqslant 3$, then we have $N_{u,v}^{w,d} = 0$.*

Proof. Recall that we have $N_{u,v}^{w,d} = N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}$, where $\omega_P\omega_{P'}, \lambda_B$ are elements associated to $\lambda_P = d\alpha_k^\vee + Q_P^\vee$, defined by the Peterson–Woodward comparison formula. Furthermore, we have $N_{u,v}^{w,d} = 0$ unless $\ell(u) + \ell(v) = \ell(w) + d \cdot \deg t$, because of the dimension constraint. By Proposition 3.20, we have $N_{u,v}^{w,d} = 0$, for either of the cases: (1) $d \geqslant 3$ and $k < n$; (2) $d \geqslant 2$ and $k = n$. For the remaining cases, we assume $d = 1$ first.

When $k < n$, we have $\lambda_B = \alpha_k^\vee$ (by Lemma 3.17) and consequently $\Delta_{P'} = \Delta_P \setminus \{\alpha_{k-1}, \alpha_{k+1}\}$. Denote $v' := s_{k+1} \cdots s_{n-1}s_{\bar{\alpha}_n}s_{n-1} \cdots s_{k+1}$. By direct calculations, we conclude that $\ell(\omega_P\omega_{P'}) = |R_P^+| - |R_{P'}^+| = \ell(s_1 \cdots s_{k-1}v')$ and $s_1 \cdots s_{k-1}v'(\alpha) \in R^+$ for all $\alpha \in \Delta_{P'}$. Thus we have $\omega_P\omega_{P'} = s_1 \cdots s_{k-1}v'$ by Lemma 3.12. Furthermore by Proposition 2.1 (2), we have $N_{u,v}^{w,1} = N_{u,v}^{w\omega_P\omega_{P'},\alpha_k^\vee} = N_{us_k,vs_k}^{w\omega_P\omega_{P'},0}$. Note that $v' = (v')^{-1}$ and $\mathrm{sgn}_j(us_k) = 0$ for all $j > k$. It follows directly from Corollary 2.3 that $N_{us_k,vs_k}^{w\omega_P\omega_{P'},0} = N_{us_k,vs_kv'}^{ws_1\cdots s_{k-1},0}$ if $\ell(vs_kv') = \ell(vs_k) - \ell(v')$, or 0 otherwise. In the latter case, we have $N_{us_k,vs_kv'}^{ws_1\cdots s_{k-1},0} = 0$ from the dimension constraint (by noting $\ell(us_k) + \ell(vs_k) = \ell(w) + \deg t - 2 = \ell(w\omega_P\omega_{P'}) = \ell(ws_1 \cdots s_{k-1}) + \ell(v')$). Thus (1) follows in this case.

When $k=n$ (in this case Δ is of B_n-type by our assumption), we have $\lambda_B=\alpha_{n-1}^\vee+\alpha_n^\vee$, so that $\Delta_{P'}=\Delta_P\setminus\{\alpha_{n-2}\}$. Using Lemma 3.2, we obtain $\omega_P\omega_{P'}=s_2\cdots s_{n-1}s_1\cdots s_{n-2}$. Consequently, we have $N_{u,v}^{w,1}=N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}=N_{u,v}^{w_1s_ns_{n-1},\lambda_B}=: a_1$. Furthermore, we note that $\ell(\omega_P\omega_{P'})+\langle 2\rho,\lambda_B\rangle=2n=\deg t=\ell(u)+\ell(v)-\ell(w)\leqslant n+\ell(v)$. Thus $\ell(v)\geqslant 2$ and consequently $\operatorname{sgn}_n(v)=\operatorname{sgn}_{n-1}(vs_n)=1$. Denote $a_2:=N_{us_{n-1},v}^{w_1s_n,\lambda_B}$, $a_3:=N_{us_{n-1},vs_n}^{w_1,\lambda_B-\alpha_n^\vee}$ and $a_4:=N_{u,vs_ns_{n-1}}^{w_1,\lambda_B-\alpha_n^\vee-\alpha_{n-1}^\vee}$. We can show the following identities.

i) $a_1=a_2$. Indeed, if $a_2=0$, then we have $a_1=0$, by noting $\operatorname{sgn}_{n-1}(u)=\operatorname{sgn}_{n-1}(v)=\langle\alpha_{n-1},\lambda_B\rangle=0$ and using Corollary 2.2 (3). If $a_2\neq 0$, then we have $\operatorname{sgn}_{n-1}(w_1s_n)=\operatorname{sgn}_{n-1}(us_{n-1})+\operatorname{sgn}_{n-1}(v)-\langle\alpha_{n-1},\lambda_B\rangle=\operatorname{sgn}_{n-1}(us_{n-1})=1$. Thus $\operatorname{sgn}_{n-1}(w_1s_ns_{n-1})=0$. Note $\langle\alpha_{n-1},\alpha_{n-1}^\vee+\lambda_B\rangle=2$, $\operatorname{sgn}_{n-1}(us_{n-1})=1$ and $\operatorname{sgn}_{n-1}(vs_{n-1})=1$. Applying "$(u,v,w,\lambda,\alpha)$" in Eqs. (2.1) and (2.2) of Proposition 2.1 (2) to $(us_{n-1},vs_{n-1},w_1s_ns_{n-1},\alpha_{n-1}^\vee+\lambda_B,\alpha_{n-1})$, we have $N_{us_{n-1},vs_{n-1}}^{w_1s_ns_{n-1},\alpha_{n-1}^\vee+\lambda_B}=N_{us_{n-1}s_{n-1},vs_{n-1}s_{n-1}}^{w_1s_ns_{n-1},\lambda_B}=N_{u,v}^{w_1s_ns_{n-1},\lambda_B}$ and $N_{us_{n-1},vs_{n-1}}^{w_1s_ns_{n-1},\alpha_{n-1}^\vee+\lambda_B}=N_{us_{n-1},vs_{n-1}s_{n-1}}^{w_1s_ns_{n-1}s_{n-1},(\alpha_{n-1}^\vee+\lambda_B)-\alpha_{n-1}^\vee}=N_{us_{n-1},v}^{w_1s_n,\lambda_B}$. Hence, $N_{u,v}^{w_1s_ns_{n-1},\lambda_B}=N_{us_{n-1},v}^{w_1s_n,\lambda_B}$. That is, $a_1=a_2$.

ii) $a_2=a_3$. Indeed, if $a_3=0$, then we have $a_2=0$ by Corollary 2.2 (2). If $a_3\neq 0$, then we have $\ell(w_1)+2=\ell(us_{n-1})+\ell(vs_n)=\ell(u)+\ell(v)=\ell(w)+2n=\ell(w\omega_P\omega_{P'})+4=\ell(w_1s_ns_{n-1})+4$. Hence, $\ell(w_1s_ns_{n-1})=\ell(w_1)-2$ and consequently we have $\operatorname{sgn}_n(w_1)=1$. Note that $\operatorname{sgn}_n(us_{n-1})=\operatorname{sgn}_n(vs_n)=0$ and $\langle\alpha_n,\alpha_{n-1}^\vee\rangle=-1$. Then we have $a_2=a_3$ by Proposition 2.1 (2).

iii) $a_3=a_4$. If $\operatorname{sgn}_{n-1}(w_1)=0$, then we are done by Proposition 2.1 (2). If $\operatorname{sgn}_{n-1}(w_1)=1$, then we have $a_3=0$ and $a_4=0$ by Proposition 2.1 (1), so that we are also done.

Hence, we have $a_1=a_4$. That is, $N_{u,v}^{w,1}=N_{u_1,v_1}^{w_1,0}$.

It remains to deal with the case when $d=2$ and $k<n$. In this case, we have $\lambda_B=\alpha_{k-1}^\vee+2\sum_{j=k}^{n-1}\alpha_j^\vee+\bar\alpha_n^\vee$, which satisfies $\langle\alpha,\lambda_B\rangle=0$ for all $\alpha\in\Delta_P\setminus\{\alpha_{k-2}\}$. By Lemma 3.12 again, we conclude $\omega_P\omega_{P'}=s_2\cdots s_{k-1}s_1\cdots s_{k-2}$ (by which we mean the unit 1 if $k-2\leqslant 0$). Note $\langle\alpha_{k+1},\alpha_{k+1}^\vee+\lambda_B\rangle=2$, $\operatorname{sgn}_{k+1}(w\omega_P\omega_{P'})=0$, $\operatorname{sgn}_{k+1}(us_{k+1})=1$ and $\operatorname{sgn}_{k+1}(vs_{k+1})=1$. Applying "$(u,v,w,\lambda,\alpha)$" in Eqs. (2.1) and (2.2) of Proposition 2.1 (2) to $(us_{k+1},vs_{k+1},w\omega_P\omega_{P'},\alpha_{k+1}^\vee+\lambda_B,\alpha_{k+1})$, we have $N_{us_{k+1},vs_{k+1}}^{w\omega_P\omega_{P'},\alpha_{k+1}^\vee+\lambda_B}=N_{us_{k+1}s_{k+1},vs_{k+1}s_{k+1}}^{w\omega_P\omega_{P'},\alpha_{k+1}^\vee+\lambda_B-\alpha_{k+1}^\vee}=N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}$ and $N_{us_{k+1},vs_{k+1}}^{w\omega_P\omega_{P'},\alpha_{k+1}^\vee+\lambda_B}=N_{us_{k+1},vs_{k+1}s_{k+1}}^{w\omega_P\omega_{P'}s_{k+1},\alpha_{k+1}^\vee+\lambda_B-\alpha_{k+1}^\vee}=N_{us_{k+1},v}^{w\omega_P\omega_{P'}s_{k+1},\lambda_B}$. Hence, we have $N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}=N_{us_{k+1},v}^{w\omega_P\omega_{P'}s_{k+1},\lambda_B}$. Then by using induction and the same arguments above, we conclude

$$N_{u,v}^{w,2}=N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}=N_{us_{k+1}\cdots s_{n-1}s_{\bar\alpha_n},v}^{w\omega_P\omega_{P'}s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n},\lambda_B}.$$

Denote $(\beta_1,\ldots,\beta_{2n-2k}):=(\alpha_k,\ldots,\alpha_{n-1},\alpha_{k-1},\ldots,\alpha_{n-2})$ and set

$$u'':=us_{k+1}\cdots s_{n-1}s_{\bar\alpha_n},\qquad v'':=vs_{\beta_1}\cdots s_{\beta_{2n-2k}},$$
$$w'':=w\omega_P\omega_{P'}s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n}s_{\beta_1}\cdots s_{\beta_{2n-2k}},\qquad \lambda_B'':=\lambda_B-\sum_{i=1}^{2n-2k}\beta_i^\vee.$$

Since $v''(\alpha_{n-1})=v(\alpha_{k-1})\in R^+$, we have $\operatorname{sgn}_{n-1}(v'')=0$. Note that $\lambda_B''=\alpha_{n-1}^\vee+\bar\alpha_n^\vee$. Applying "$(u,v,w,\lambda,\alpha)$" in Eqs. (2.1) and (2.2) of Proposition 2.1 (2) to $(u''s_{n-1},v''s_{n-1},w'',\alpha_{n-1}^\vee+\lambda_B'',\alpha_{n+1})$, we conclude $N_{u'',v''}^{w'',\lambda_B''}=N_{u''s_{n-1},v''}^{w''s_{n-1},\lambda_B''}$ Applying "(u,v,w,λ,α)" in Eqs. (2.3) and (2.1) of Proposition 2.1 (2) to $(v'', u''s_{n-1}s_{\bar\alpha_n}, w''s_{n-1}s_{\bar\alpha_n}, \lambda''_B, \bar\alpha_n)$ (where we use these inductions for twice, i.e., for α_n and α_{n+1} respectively, in the case of type D_{n+1}), we conclude $N_{u''s_{n-1},v''}^{w''s_{n-1},\lambda''_B} = N_{u''s_{n-1},v''s_{\bar\alpha_n}}^{w''s_{n-1}s_{\bar\alpha_n},\lambda''_B-\bar\alpha_n^\vee}$. Applying Eq. (2.1) of Proposition 2.1 (2), we conclude $N_{u''s_{n-1},v''s_{\bar\alpha_n}}^{w''s_{n-1}s_{\bar\alpha_n},\lambda''_B-\bar\alpha_n^\vee} = N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0}$. (In order to apply equations in Proposition 2.1 (2) above, we have assumed the grading equality "$\mathrm{sgn}_\alpha(u) + \mathrm{sgn}_\alpha(v) = \mathrm{sgn}_\alpha(w) + \langle\alpha, \lambda\rangle$" holds for all the following four structure constants. If it does not hold for any one of the structure constants, it is easy to show all of them are equal to 0, so that we always have the following equalities as expected.) That is, we have

$$N_{u'',v''}^{w'',\lambda''_B} = N_{u''s_{n-1},v''}^{w''s_{n-1},\lambda''_B} = N_{u''s_{n-1},v''s_{\bar\alpha_n}}^{w''s_{n-1}s_{\bar\alpha_n},\lambda''_B-\bar\alpha_n^\vee} = N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0}.$$

Hence, (2) follows directly from Lemma 3.22 and the next claim:

$$N_{us_{k+1}\cdots s_{n-1}s_{\bar\alpha_n},v}^{w\omega_P\omega_{P'}s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n},\lambda_B} = N_{u'',v''}^{w'',\lambda''_B}. \qquad (\circledast)$$

(Indeed if $N_{u'',v''}^{w'',\lambda''_B} = 0$, then we can show $N_{u'',v''s_{\beta_{2n-2k}}\cdots s_{\beta_{2n-2k-h+1}}}^{w''s_{\beta_{2n-2k}}\cdots s_{\beta_{2n-2k-h+1}},\lambda''_B+\sum_{i=1}^h\beta_{2n-2k-i+1}^\vee} = 0$ for $1 \leqslant h \leqslant 2n-2k$, by using Corollary 2.2 (2) and induction on h. In particular for $h = 2n-2k$, we have $N_{u'',v''s_{\beta_{2n-2k}}\cdots s_{\beta_1}}^{w''s_{\beta_{2n-2k}}\cdots s_{\beta_1},\lambda_B} = 0$, hence $(\circledast)$. Since $\ell(u) + \ell(v) = \ell(w) + 2\deg t$, we have $\ell(u'') + (v) = \ell(w\omega_P\omega_{P'}s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n}) + \langle 2\rho, \lambda_B\rangle$. Thus if $N_{u'',v''}^{w'',\lambda''_B} \neq 0$, then we have $\ell(u'') + \ell(v'') = \ell(w'') + \langle 2\rho, \lambda''_B\rangle$ and consequently $\ell(u'') + \ell(v''s_{\beta_{2n-2k}}\cdots s_{\beta_h}) = \ell(w''s_{\beta_{2n-2k}}\cdots s_{\beta_h}) + \langle 2\rho, \sum_{i=h}^{2n-2k}\beta_i^\vee\rangle$ for all $1 \leqslant h \leqslant 2n-2k$. Hence, we have $\langle\beta_h, \lambda_B - \sum_{i=1}^{h-1}\beta_i^\vee\rangle = 1$, $\mathrm{sgn}_{\beta_h}(w\omega_P\omega_{P'}s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n}s_{\beta_1}\cdots s_{\beta_{h-1}}) = 0$, $\mathrm{sgn}_{\beta_h}(u'') = 0$ and $\mathrm{sgn}_{\beta_h}(vs_{\beta_1}\cdots s_{\beta_{h-1}}) = 1$. By using Proposition 2.1 (2) and induction on h, we still conclude that $(\circledast)$ holds.) □

Lemma 3.22. *Using the same assumptions and notations as in Theorem 3.21 (and in the proof of it), we have*

$$N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0} = N_{u,v_2}^{w_2,0}.$$

Furthermore if $N_{u,v_2}^{w_2,0} \neq 0$, then we have $v_2 \in W^P$ and $\ell(v_2) = \ell(v) - \ell(v^{-1}v_2)$.

Proof. By Lemma 3.2, we have $v''s_{\bar\alpha_n}s_{n-1}s_{n-2}\cdots s_1 = v_2\cdot(s_ks_{k+1}\cdots s_{n-1})$ and $w''s_{n-1}s_{\bar\alpha_n} = ws_1\cdots s_{k-1}s_{k+1}\cdots s_{n-1}s_k\cdots s_{n-2}s_{\bar\alpha_n}s_{n-1}s_{\bar\alpha_n}s_1\cdots s_{n-2}$. Denote $w_3 := w''s_{n-1}s_{\bar\alpha_n}s_{n-2}\cdots s_1 = ws_1\cdots s_{k-1}s_{k+1}\cdots s_{n-1}s_k\cdots s_{n-2}s_{\bar\alpha_n}s_{n-1}s_{\bar\alpha_n}$.

Assume $N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0} \neq 0$ first, then $\ell(u'') + \ell(v''s_{\bar\alpha_n}s_{n-1}) = \ell(w''s_{n-1}s_{\bar\alpha_n})$. Observe that $\ell(u'') = \ell(u) + \ell(u^{-1}u'')$ and $\ell(w^{-1}w''s_{n-1}s_{\bar\alpha_n}) + \ell(v^{-1}v''s_{\bar\alpha_n}s_{n-1}) = 2\deg t + \ell(u^{-1}u'')$. Combining the assumption $\ell(u) + \ell(v) = \ell(w) + 2\deg t$, we conclude that both $\ell(w''s_{n-1}s_{\bar\alpha_n}) = \ell(w) + \ell(w^{-1}w''s_{n-1}s_{\bar\alpha_n})$ and $\ell(v''s_{\bar\alpha_n}s_{n-1}) = \ell(v) - \ell(v^{-1}v''s_{\bar\alpha_n}s_{n-1})$ hold. Furthermore by Corollary 2.3, we have

$$N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0} = N_{u'',v''s_{\bar\alpha_n}s_{n-1}s_{n-2}\cdots s_1}^{w''s_{n-1}s_{\bar\alpha_n}\cdot s_{n-2}\cdots s_1,0} = N_{u'',v_2s_ks_{k+1}\cdots s_{n-1}}^{w_3,0} \qquad (*)$$

and

$$\ell\big(v''s_{\bar\alpha_n}s_{n-1}s_{n-2}\cdots s_1\big) = \ell\big(v''s_{\bar\alpha_n}s_{n-1}\big) - \ell(s_{n-2}\cdots s_1). \qquad (*')$$

Note $\ell(v)-\ell(v^{-1}v_2s_ks_{k+1}\cdots s_{n-1})\leqslant \ell(v_2s_ks_{k+1}\cdots s_{n-1})=\ell(v''s_{\bar\alpha_n}s_{n-1}s_{n-2}\cdots s_1)=\ell(v)-\ell(v^{-1}v''s_{\bar\alpha_n}s_{n-1})-\ell(s_{n-2}\cdots s_1)=\ell(v)-\ell(v^{-1}v_2s_ks_{k+1}\cdots s_{n-1})$. Hence, the equality holds and consequently $\ell(v_2s_ks_{k+1}\cdots s_{n-1})=\ell(v_2)-\ell(s_ks_{k+1}\cdots s_{n-1})$. Then by Corollary 2.3 again, we have

$$N_{u'',v_2s_ks_{k+1}\cdots s_{n-1}}^{w_3,0}=N_{u'',v_2}^{w_3s_{n-1}\cdots s_{k+1}s_k,0}. \tag{$**$}$$

Note that $s_{\bar\alpha_n}s_{n-1}s_{\bar\alpha_n}s_{n-1}=s_{n-1}s_{\bar\alpha_n}s_{n-1}s_{\bar\alpha_n}$. We have $w_3s_{n-1}\cdots s_{k+1}s_k=w_2s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n}$ with $\ell(w_2s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n})=\ell(w_2)+\ell(s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n})$. Since $v\in W^P$, we have $v_2(\alpha_j)\in R^+$ for all $j>k$. Thus by Corollary 2.3, we have

$$N_{u'',v_2}^{w_3s_{n-1}\cdots s_{k+1}s_k,0}=N_{us_{k+1}\cdots s_{n-1}s_{\bar\alpha_n},v_2}^{w_2s_{k+1}\cdots s_{n-1}s_{\bar\alpha_n},0}=N_{u,v_2}^{w_2,0}. \tag{$***$}$$

In particular, we have $N_{u,v_2}^{w_2,0}=N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0}\neq 0$.

Now we assume $N_{u,v_2}^{w_2,0}\neq 0$, which implies $\ell(u)+\ell(v_2)=\ell(w_2)$. Note that $(w^{-1}w_2)^{-1}=v^{-1}v_2$ and $\ell(v^{-1}v_2)=\deg t$. Since $\ell(u)+\ell(v)=\ell(w)+2\deg t$, we have $\ell(w_2)=\ell(w)+\ell(w^{-1}w_2)$ and $\ell(v_2)=\ell(v)-\ell(v^{-1}v_2)$. Thus we have $\ell(v_2s_1)=\ell(v_2)+1$ and consequently $v_2(\alpha_1)\in R^+$. Note $v\in W^P$. It is easy to check that $v_2(\alpha)\in R^+$ for all $\alpha\in\Delta_P\setminus\{\alpha_1\}$. Hence, $v_2\in W^P$ and consequently $w_2\in W^P$. Thus $(***)$ follows directly from Corollary 2.3. Then $(**)$ also follows from Corollary 2.3, by noting $\ell(w_3s_{n-1}\cdots s_{k+1}s_k)=\ell(w_3)+\ell(s_{n-1}\cdots s_{k+1}s_k)$ and $N_{u'',v_2}^{w_3s_{n-1}\cdots s_{k+1}s_k,0}\neq 0$. Furthermore, we conclude that $(*)$ holds, by noting $(*')$ and using Corollary 2.3. In particular, we have $N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0}=N_{u,v_2}^{w_2,0}\neq 0$.

If none of the above two assumptions holds, we still have $N_{u'',v''s_{\bar\alpha_n}s_{n-1}}^{w''s_{n-1}s_{\bar\alpha_n},0}=N_{u,v_2}^{w_2,0}$, both of which vanish. □

Remark 3.23. Recall that the odd orthogonal Grassmannian $OG(n,2n+1)$ is isomorphic to the even orthogonal Grassmannian $OG^o(n+1,2n+2)$. It suffices to deal with either of them. The former case has been covered in the above theorem. The later case has been dealt with earlier by Kresch and Tamvakis in [20].

As indicated by Lemma 3.22, we can use the classical Pieri rules given by Pragacz and Ratajski to interpret the classical intersection numbers $N_{u,v_2}^{w_2,0}$ explicitly. For $N_{u_1,v_1}^{w_1,0}$, when $k=n$ we can make use of the generalized classical Pieri rules given by Bergeron and Sottile (see Theorem D of [1]); when $k<n$, we can use the classical Chevalley formula for $k\in\{1,2\}$. However, a classical Pieri formula analogous with the one given by Bergeron and Sottile [1] is still lacking in general. It will be desirable to derive such a formula.

Remark 3.24. In our proof of Theorem 3.21, we make use of Proposition 2.1 to reduce $N_{u,v}^{w,1}$ to a classical intersection number for the two step flag variety $OF(k-1,k+1;N)$ first. For this step, there is another approach using the well-known fact that the parameter space of lines on the Grassmannian $OG(k,N)$ is $OF(k-1,k+1;N)$, as pointed out explicitly by Buch, Kresch and Tamvakis in [5].

4. Quantum Pieri rules for Grassmannians of classical types: reformulations in traditional ways

Historically, the Schubert classes for complex Grassmannians are labeled by partitions, and the (quantum) Pieri rule therein expresses the (quantum) product of a special partition and a general partition. There are similar notions for isotropic Grassmannians. In this section, we derive our quantum Pieri rules for Grassmannians of type C_n and B_n, by reformulating Theorem 3.13 and Theorem 3.21 in a traditional way (i.e. in terms of "partitions"). In addition, we use the same notations as in Section 3.1 and always assume $k<n$. The remaining cases will be discussed in Section 4.3.

4.1. *Quantum Pieri rules for Grassmannians of type C_n*

We first review some parameterizations of the minimal length representatives W^P for the isotropic Grassmannian $G/P=IG(k,2n)$, following [27] (see also [30] and [5]). Each element x in the permutation group S_n is represented by its image $(x(1),\ldots,x(n))$, or simply $(x_1,\ldots,x_n)$. The Weyl group W of type C_n is isomorphic to the hyperoctahedral group $S_n\ltimes\mathbb{Z}_2^n$ of barred permutations, which is an extension of the permutation group $S_n=\langle\dot{s}_1,\ldots,\dot{s}_{n-1}\rangle$ by an element $\dot{s}_n$ acting on the right by $(x_1,\ldots,x_n)\dot{s}_n=(x_1,\ldots,x_{n-1},\bar{x}_n)$. Here $\dot{s}_i$ denote the transposition $(i,i+1)$ for each $1\leqslant i\leqslant n-1$. Each element w in W^P can be identified with a sequence of the form $(y_1,\ldots,y_{k-m},\bar{z}_m,\ldots,\bar{z}_1,v_1,\ldots,v_{n-k})$, where $y_1<\cdots<y_{k-m}$, $z_m>\cdots>z_1$ and $v_1<\cdots<v_{n-k}$, as follows. Let $\epsilon_i:=-\mathbf{e}_i\in\{-1,1\}\mathbf{e}_1\oplus\cdots\oplus\{-1,1\}\mathbf{e}_n=\mathbb{Z}_2^n$ for each $1\leqslant i\leqslant n$. Write $w=u_1s_nu_2s_n\cdots u_js_nu_{j+1}$ where u_j's are all in S_n. Denote $a_i:=(u_{i+1}u_{i+2}\cdots u_{j+1})^{-1}(n)$ for each $1\leqslant i\leqslant j$. Then $w\in W$ is identified with the barred permutation $(u_1u_2\cdots u_{j+1},\epsilon_{a_1}\epsilon_{a_2}\cdots\epsilon_{a_j})\in S_n\ltimes\mathbb{Z}_2^n$. The inequalities among entries in the identified sequence automatically hold as a consequence of the property $w\in W^P$. The element w in W^P can also be identified with an element $\mu=(\mu^t//\mu^b)$ in the set $\mathcal{P}_k$ of *shapes*. That is, μ^t and μ^b are strict partitions inside $(n-k)$ by n rectangle and k by n rectangle respectively, and they satisfy the inequality $\mu^t_{n-k}\geqslant\ell(\mu^b)+1$. Here $\mu^t=(\mu^t_1,\ldots,\mu^t_{n-k})$ and $m:=\ell(\mu^b)$ denotes the length of the partition μ^b. Precisely, we have $\mu^b_j=n+1-z_j$ for each $1\leqslant j\leqslant m$, and $\mu^t_r=n+1-v_r+\sharp\{i\mid z_i<v_r,\ i=1,\ldots,m\}$ for each $1\leqslant r\leqslant n-k$. For such a shape μ in $\mathcal{P}_k$, we have $|\mu|:=|\mu^t|+|\mu^b|-\binom{n-k+1}{2}$. There is a particular reduced expression of the corresponding $w=w_\mu\in W^P$ with $\ell(w_\mu)=|\mu|$, given by

$$\begin{aligned}w_\mu=&(s_{n-\mu^b_m+1}\cdot s_{n-\mu^b_m+2}\cdot\cdots\cdot s_{n-1}\cdot s_n)\cdot\cdots\\&\cdot(s_{n-\mu^b_1+1}\cdot s_{n-\mu^b_1+2}\cdot\cdots\cdot s_{n-1}\cdot s_n)\cdot(s_{n-\mu^t_{n-k}+1}\cdot\cdots\cdot s_{n-2}\cdot s_{n-1})\\&\cdot(s_{n-\mu^t_{n-k-1}+1}\cdot\cdots\cdot s_{n-3}\cdot s_{n-2})\cdot\cdots\cdot(s_{n-\mu^t_1+1}\cdot\cdots\cdot s_{k-1}\cdot s_k).\end{aligned}\qquad(\star)$$

In particular, for the special class $c_p(\mathcal{S}^*)=\sigma^u$, $u=s_{k-p+1}\cdots s_k=u_\mu$ corresponds to the special shape $\mu=((n-k+p,n-k-1,\ldots,1)//\emptyset)$. Usually, such a special μ is simply denoted as $p\in\mathcal{P}_k$. We note that the quantum Pieri rules with respect to the Chern classes $c_i(\mathcal{Q})$ (where $1\leqslant i\leqslant 2n-k$) have been given by Buch, Kresch and Tamvakis [5].

Remark 4.1. In terms of notations in [30], $c_i(\mathcal{Q})=\sigma^{u'}$ (up to a scale factor of 2) with u' corresponding to the special shape $(1^{\min(i,n-k)}|\max(i-n+k,0))$. In general, the notation of shapes $(\mathbf{a}|\mathbf{b})$ in [30] is slightly different from the notation $(\mu^t//\mu^b)$ in [27]. For the same $w\in W^P$, we have $\mathbf{b} = \mu^b$ and $\mathbf{a} = (a_1, \dots, a_{n-k})$ with $a_r = \mu_r^t + r - n + k - 1$ for each $1 \leqslant r \leqslant n-k$. In addition, w can also be identified with the $(n-k)$-strict partition $\mathbf{a}' + \mathbf{b}$ introduced in [5], where $\mathbf{a}'$ is the transpose of $\mathbf{a}$.

Let $\tilde{P}$ denote the standard parabolic subgroup that corresponds to the subset $\Delta \setminus \{\alpha_{k-1}\}$. The minimal length representatives $W^{\tilde{P}}$ are identified with shapes in $\mathcal{P}_{k-1}$. (Note $G/\tilde{P} = IG(k-1, 2n)$ in this subsection.) To reformulate Theorem 3.13 and Theorem 3.21 in terms of shapes, we need the following lemma, which tells us about the explicit identifications.

Lemma 4.2.

(1) $s_{k-p+1} \cdots s_{k-1}$ *corresponds to the special shape* $p-1 \in \mathcal{P}_{k-1}$.

(2) *Let* $v \in W^P$ *correspond to* $\mathbf{a} = (\mathbf{a}^t // \mathbf{a}^b) \in \mathcal{P}_k$. *Denote* $m := \ell(\mathbf{a}^b)$, $\gamma := \alpha_n + 2\sum_{j=k}^{n-1} \alpha_j$, $v_1 := v s_\gamma s_k$ *and* $v_2 := v s_k \cdots s_{n-1} s_n s_{n-1} \cdots s_1$.

(a) $\ell(v s_\gamma) = \ell(v) - 2n + 2k - 1$ *if and only if* $a_1^b \geqslant a_1^t$. *Furthermore when this holds,* $v s_\gamma \in W^{\tilde{P}}$ *corresponds to the shape*

$$\tilde{\mathbf{a}} = \big((a_1^b, a_1^t - 1, a_2^t - 1, \dots, a_{n-k}^t - 1) // (a_2^b, a_3^b, \dots, a_m^b)\big) \in \mathcal{P}_{k-1}.$$

(b) $\ell(v_1) = \ell(v) - 2n + 2k$ *if and only if* $a_1^b \geqslant a_2^t$. *Furthermore if this holds and* $\ell(v s_\gamma) = \ell(v_1) + 1$, *then* $a_1^t > a_1^b$ *and* $v_1 \in W^{\tilde{P}}$ *corresponds to*

$$\bar{\mathbf{a}} = \big((a_1^t, a_1^b, a_2^t - 1, \dots, a_{n-k}^t - 1) // (a_2^b, a_3^b, \dots, a_m^b)\big) \in \mathcal{P}_{k-1}.$$

(c) *If* $v_2 \in W^P$ *and* $\ell(v_2) = \ell(v) - \ell(v^{-1} v_2)$, *then* $a_1^b = n$ *and* v_2 *corresponds to the shape*

$$\hat{\mathbf{a}} = \big((a_1^t - 1, a_2^t - 1, \dots, a_{n-k}^t - 1) // (a_2^b, a_3^b, \dots, a_m^b)\big) \in \mathcal{P}_k.$$

(3) *Let* $w \in W^P$ *correspond to* $\mathbf{c} = (\mathbf{c}^t // \mathbf{c}^b) \in \mathcal{P}_k$. *Denote* $w_1 := w s_1 \cdots s_{k-1}$.

(a) $w_1 \in W^{\tilde{P}}$ *if and only if* $c_1^t < n$. *Furthermore when this holds,* w_1 *corresponds to the shape*

$$\tilde{\mathbf{c}} = \big((n, c_1^t, c_2^t, \dots, c_{n-k}^t) // \mathbf{c}^b\big) \in \mathcal{P}_{k-1}.$$

(b) *If* $w_1 s_k \in W^{\tilde{P}}$ *if and only if* $c_1^t = n$, $c_2^t \leqslant n-2$. *Furthermore when this holds,* $w_1 s_k$ *corresponds to*

$$\bar{\mathbf{c}} = \big((n, n-1, c_2^t, \dots, c_{n-k}^t) // \mathbf{c}^b\big) \in \mathcal{P}_{k-1}.$$

(4) *Let* $w' \in W^P$ *correspond to* $\mathbf{d} = (\mathbf{d}^t // \mathbf{d}^b) \in \mathcal{P}_k$. *Denote* $m' := \ell(\mathbf{d}^b)$ *and* $w_2 := w' s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k$. *If* $w_2 \in W^P$ *and* $\ell(w_2) = \ell(w') + \ell(s_1 \cdots s_{n-1} s_n s_{n-1} \cdots s_k)$, *then* $d_1^b < n$ *and* w_2 *corresponds to the shape*

$$\hat{\mathbf{d}} = \big((d_1^t + 1, d_2^t + 1, \dots, d_{n-k}^t + 1) // (n, d_1^b, d_2^b, \dots, d_{m'}^b)\big) \in \mathcal{P}_k.$$

Proof. Clearly, statement (1) follows.

Note that for every case in Lemma 3.2, the expression on the right-hand side of the equality is reduced. Furthermore, we note that $\ell(s_\gamma) = 2n - 2k + 1$ and $v s_\gamma = v s_k s_{k+1} \cdots s_{n-1} s_n s_{n-1} \cdots s_{k+1} s_k = v_1 s_k$. Hence, $\ell(v s_\gamma) = \ell(v) - \ell(s_\gamma)$ holds if and only if both $\ell(v s_k s_{k+1} \cdots s_n) =$

$\ell(v) - \ell(s_k s_{k+1} \cdots s_n)$ and $\ell(vs_\gamma) = \ell(vs_k s_{k+1} \cdots s_n) - \ell(s_{n-1} \cdots s_{k+1} s_k)$ hold. Using $(\star)$ and Lemma 3.2, we have

$$\begin{aligned} vs_\gamma &= (vs_k s_{k+1} \cdots s_n) \cdot (s_{n-1} \cdots s_{k+1} s_k) \\ &= \big((s_{n-a_m^b+1} \cdots s_{n-1} s_n) \cdots (s_{n-a_2^b+1} \cdots s_{n-1} s_n)(s_{n-a_1^b+1} \cdots s_{n-2} s_{n-1}) \\ &\quad \cdot (s_{n-a_{n-k}^t+1} \cdots s_{n-3} s_{n-2}) \cdots (s_{n-a_1^t+1} \cdots s_{k-2} s_{k-1})\big) \cdot (s_{n-1} \cdots s_{k+1} s_k) \\ &= (s_{n-a_m^b+1} \cdots s_{n-1} s_n) \cdots (s_{n-a_2^b+1} \cdots s_{n-1} s_n) \cdot (s_{n-a_{n-k}^t+2} \cdots s_{n-2} s_{n-1}) \\ &\quad \cdot (s_{n-a_{n-k-1}^t+2} \cdots s_{n-3} s_{n-2}) \cdots (s_{n-a_1^t+2} \cdots s_{k-1} s_k) \cdot (s_{n-a_1^b+1} \cdots s_{k-2} s_{k-1}), \end{aligned}$$

the right-hand side of the last equality in which gives a reduced expression of vs_γ. In other words, we have $n - a_1^b + 1 \leqslant \min\{n - a_{n-k}^t + 1, \ldots, n - a_2^t + 1, n - a_1^t + 1, n\} = n - a_1^t + 1$. That is, $a_1^b \geqslant a_1^t$. Since $\mathbf{a} \in \mathcal{P}_k$, we have $\tilde{\mathbf{a}} = ((a_1^b, a_1^t - 1, a_2^t - 1, \ldots, a_{n-k}^t - 1)//(a_2^b, a_3^b, \ldots, a_m^b)) \in \mathcal{P}_{k-1}$, corresponding to vs_γ. Thus statement (2a) holds.

The remaining parts are also consequences of the formula $(\star)$ and Lemma 3.2. The arguments for them are also similar. □

For any shapes $\mu, \nu \in \mathcal{P}_k$, we denote by $e_k(\mu, \nu)$ the cardinality of the set of *components that are not extremal, not related and have no* $(\nu - \mu)$*-boxes over them*. The relevant notions, together with the notion of "ν *compatible with* μ", can be found on page 152 and page 153 of [27]. We always skip the subscription part of $e_k(\mu, \nu)$, whenever $e(\mu, \nu)$ is well understood. In addition, we denote the Schubert cohomology class σ^{w_μ} as σ^μ. The following classical Pieri rule was given by Pragacz and Ratajski.

Proposition 4.3. *(See Theorem 2.2 of [27].) For every* $\mathbf{a} \in \mathcal{P}_k$ *and* $p \leqslant k$*, we have*

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum 2^{e(\mathbf{a},\mathbf{b})} \sigma^{\mathbf{b}},$$

where the sum is over all shapes $\mathbf{b} \in \mathcal{P}_k$ *compatible with* $\mathbf{a}$ *such that* $|\mathbf{b}| = |\mathbf{a}| + p$.

Combining the above proposition and (statements (1), (2a), (3a) of) Lemma 4.2, we can reformulate Theorem 3.13 directly as follows.

Theorem 4.4 *(Quantum Pieri rule for* $IG(k, 2n)$*). Let* $\mathbf{a} \in \mathcal{P}_k$ *and* $p \leqslant k$*. Using the same notations as in Lemma 4.2, we have*

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum 2^{e(\mathbf{a},\mathbf{b})} \sigma^{\mathbf{b}} + t \sum 2^{e(\tilde{\mathbf{a}},\tilde{\mathbf{c}})} \sigma^{\mathbf{c}},$$

where the first summation is over all shapes $\mathbf{b} \in \mathcal{P}_k$ *with* $|\mathbf{b}| = |\mathbf{a}| + p$ *such that* $\mathbf{b}$ *is compatible with* $\mathbf{a}$*, and the second summation is over all shapes* $\mathbf{c} \in \mathcal{P}_k$ *with* $|\mathbf{c}| = |\mathbf{a}| + p - 2n + k - 1$ *such that* $\tilde{\mathbf{c}} \in \mathcal{P}_{k-1}$ *is compatible with* $\tilde{\mathbf{a}} \in \mathcal{P}_{k-1}$*. (Here we denote the second sum as* 0 *if* $\tilde{\mathbf{a}} \notin \mathcal{P}_{k-1}$*.)*

4.2. Quantum Pieri rules for Grassmannians of type B_n

The Weyl group W of type B_n is also isomorphic to the hyperoctahedral group $S_n \ltimes \mathbb{Z}_2^n$ of barred permutations. The minimal length representatives W^P for the isotropic Grassmannian $G/P = OG(k, 2n+1)$ can also be identified with shapes in $\mathcal{P}_k$ as well as other parameterizations described for $IG(k, 2n)$ in the same way as in Section 4.1. Therefore we can use all the same notations as in Section 4.1 but replace $IG(k, 2n)$ by $OG(k, 2n+1)$.

In this subsection, we obtain our quantum Pieri rule for $OG(k, 2n+1)$ (which may involve signs in some cases), by reformulating part of Theorem 3.21. For any shapes $\mu, \nu \in \mathcal{P}_k$, we denote by $e'(\mu, \nu)$ the cardinality of the set of *components that are not related and have no* $(\nu - \mu)$*-boxes over them*. The following classical Pieri rule for $OG(k, 2n+1)$ was also given by Pragacz and Ratajski.

Proposition 4.5. *(See Theorem 10.1 of [27].) For every* $\mathbf{a} \in \mathcal{P}_k$ *and* $p \leqslant k$*, we have*

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum 2^{e'(\mathbf{a},\mathbf{b})} \sigma^{\mathbf{b}},$$

where the sum is over all shapes $\mathbf{b} \in \mathcal{P}_k$ *compatible with* $\mathbf{a}$ *such that* $|\mathbf{b}| = |\mathbf{a}| + p$.

Remark 4.6. The rational cohomology of the complete flag varieties of type B_n and C_n are isomorphic to each other. As a consequence, there is a relationship between the classical intersection numbers for these two flag varieties, which was explicitly described in Section 3 of [1] (see also Section 2.2 of [5]). This provides one way to obtain the information on classical intersection numbers for $OG(k, 2n+1)$ from that for $IG(k, 2n)$.

Recall that the minimal length representatives $W^{\tilde{P}}$ are identified with shapes in $\mathcal{P}_{k-1}$. (Note $G/\tilde{P} = OG(k-1, 2n+1)$ in this subsection.)

Definition 4.7. Let $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denote the canonical basis of $\mathbb{Z}^n$ and $\vec{\mu} \in \mathbb{Z}^n$ denote the canonical vector associated to a given $\mu = (\mu^t // \mu^b) \in \mathcal{P}_{k-1}$; that is, $\vec{\mu} = \sum_{i=1}^{n-k+1} \mu_i^t \mathbf{e}_i + \sum_{j=1}^{m} \mu_j^b \mathbf{e}_{n-k+1+j}$ where $m := \ell(\mu^b)$. We define the sets $S(\mu)$, $\Gamma_1(\mu)$ and $\Gamma_2(\mu)$ associated to μ as follows:

$$\begin{aligned} S(\mu) &:= \{\nu \in \mathcal{P}_{k-1} \mid \vec{\nu} = \vec{\mu} - \mathbf{e}_j \text{ for some } 2 \leqslant j \leqslant n-k+1+m\} \\ &\quad \cup \left\{ \nu \in \mathcal{P}_{k-1} \;\middle|\; \begin{cases} \vec{\nu} = \vec{\mu} - (f+1)\mathbf{e}_{n-k+1+j} + f\mathbf{e}_i \\ \mu_i^t = \mu_j^b + j - f - 1 \end{cases} \right. \\ &\quad \left. \text{for some } \begin{Bmatrix} f > 0, \quad 1 \leqslant j \leqslant m \\ 1 \leqslant i \leqslant n-k+1 \end{Bmatrix} \right\}; \\ \Gamma_2(\mu) &:= \left\{ \nu \in \mathcal{P}_{k-1} \;\middle|\; \exists 1 \leqslant j \leqslant m \text{ such that } \begin{cases} \vec{\nu} = \vec{\mu} - \mathbf{e}_{n-k+1+j} \\ \mu_t^i \neq \mu_j^t + j - 1, \quad \forall 2 \leqslant i \leqslant n-k+1 \end{cases} \right\} \\ &\quad \cup \left\{ \nu \in \mathcal{P}_{k-1} \;\middle|\; \begin{cases} \vec{\nu} = \vec{\mu} - (f+1)\mathbf{e}_{n-k+1+j} + f\mathbf{e}_1 \\ \mu_1^t = \mu_j^b + j - f - 1 \end{cases} \right. \\ &\quad \left. \text{for some } f > 0,\ 1 \leqslant j \leqslant m \right\}. \end{aligned}$$

In addition, we note $\Gamma_2(\mu) \subset S(\mu)$ and denote $\Gamma_1(\mu) := S(\mu) \setminus \Gamma_2(\mu)$.

Lemma 4.8. *Suppose* $v', w' \in W^{\tilde{P}}$ *correspond to the shapes* $\nu, \mu \in \mathcal{P}_{k-1}$ *respectively; then* $N_{s_k, v'}^{w', 0} \neq 0$ *if and only if* $\nu \in S(\mu)$*. Furthermore,* $N_{s_k, v'}^{w', 0} = 1$ *if* $\nu \in \Gamma_1(\mu)$*, or* 2 *if* $\nu \in \Gamma_2(\mu)$*.*

Proof. By the classical Chevalley formula (i.e. the classical part of Proposition 2.5), we have $N_{s_k,v'}^{w',0} \neq 0$ only if $w' = v's_{\gamma'}$ for some positive root γ' satisfying $\ell(v's_{\gamma'}) = \ell(v') + 1$, and when this holds, we have $N_{s_k,v'}^{w',0} = \langle \chi_k, (\gamma')^\vee \rangle$. Thus v' is obtained by deleting a unique simple reflection from the reduced expression of w' that is given by ($\star$) (with respect to $k-1$). Furthermore, such an induced expression of v' is reduced. Using Lemma 3.2 (together with the definition of μ as a shape in $\mathcal{P}_{k-1}$), we can derive another reduced expression of v' in the form of ($\star$), from the induced reduced expression of v'. In particular, we conclude that $\nu \in S(\mu) \cup \{((\mu_1^t - 1, \mu_2^t, \ldots, \mu_{n-k+1}^t)//\mu^b)\}$ (details for which are similar to the proof of statement a) of Proposition 3.4 of [23]).

Note that if v' is obtained by deleting a simple reflection s_j from the ℓth position, then we have $\gamma' = x^{-1}(\alpha_j^\vee)$ where x is the product of simple reflections from the $(\ell+1)$th position to the end of the reduced expression of w'. Assume $\nu \in \Gamma_2(\mu)$. If $\vec{\nu} = \vec{\mu} - \mathbf{e}_{n-k+1+j}$ for some $1 \leqslant j \leqslant m$, then v' is obtained by deleting the simple reflection $s_{n-\mu_j^b+1}$.

(1) If $n - \mu_j^b + 1 = n$, then $\mu_j^b = 1$. Consequently, we have $j = m$ and $(\gamma')^\vee = (w')^{-1} s_n(\alpha_n^\vee) = \alpha_n^\vee + 2\sum_{i=k-1}^{n-1} \alpha_i^\vee$. Thus $\langle \chi_k, (\gamma')^\vee \rangle = 2$.
(2) If $n - \mu_j^b + 1 \neq n$, then it is less than n and $(\gamma')^\vee = x^{-1}(\alpha_n^\vee) = \gamma_1^\vee$, where for each $1 \leqslant i \leqslant n-k+1$ we denote by $\gamma_i^\vee$ the following coroot

$$(s_{i+k-2} \cdots s_{n-\mu_i^t+1}) \cdots (s_{n-1} \cdots s_{n-\mu_{n-k+1}^t+1})\left(\alpha_n^\vee + 2\sum_{h=n-j+1}^{n-1} \alpha_h^\vee + \sum_{h=n-\mu_j^b+2-j}^{n-j} \alpha_h^\vee\right).$$

Denote $\mu_{n-k+2}^t := 1$ and $\gamma_{n-k+2}^\vee := \alpha_n^\vee + 2\sum_{h=n-j+1}^{n-1} \alpha_h^\vee + \sum_{h=n-\mu_j^b+2-j}^{n-j} \alpha_h^\vee$. Note that μ^t is a strict partition. Since $\nu \in \Gamma_2(\mu)$, $n - \mu_i^t + 1 \neq n - \mu_j^b + 2 - j$ for all $2 \leqslant i \leqslant n-k+1$; consequently, there exists a unique $2 \leqslant r \leqslant n-k+2$ such that $(n - \mu_i^t + 1) - (n - \mu_j^b + 2 - j)$ is positive if $i \geqslant r$, or negative if $2 \leqslant i < r$. Note that $k + i - j - 1 \geqslant k + i - m - 1 \geqslant k + i - \mu_{n-k+1}^t = n - (\mu_{n-k+1}^t + n - k + 1 - i) + 1 \geqslant n - \mu_i^t + 1 > n - \mu_j^b + 2 - j$, for each $n - k + 1 \geqslant i \geqslant r$. By induction on i (descendingly), we conclude $\gamma_i^\vee = \alpha_n^\vee + 2\sum_{h=k+i-j-1}^{n-1} \alpha_h^\vee + \sum_{h=n-\mu_j^b+2-j}^{k+i-j-2} \alpha_h^\vee$ for all $n-k+2 \geqslant i \geqslant r$. In particular, we obtain $\gamma_r^\vee$. Furthermore, for each $r \geqslant i \geqslant 2$, we have $n - \mu_j^b + 1 + (i - r) > n - \mu_{r-1}^t + 1 + (i - r) = n - (\mu_{r-1}^t + r - i) + 1 \geqslant n - \mu_{i-1}^t + 1$. Thus, by induction on i (descendingly), we conclude $\gamma_i^\vee = \alpha_n^\vee + 2\sum_{h=k+i-j-1}^{n-1} \alpha_h^\vee + \sum_{h=n-\mu_j^b+2-j+i-r}^{k+i-j-2} \alpha_h^\vee$ for all $r \geqslant i \geqslant 2$. In particular, we obtain $\gamma_2^\vee$, for which we note $k + 2 - j - 1 \leqslant k$. Thus

$$\langle \chi_k, (\gamma')^\vee \rangle = \langle \chi_k, s_{k-1}s_{k-2} \cdots s_{n-\mu_1^t+1}(\gamma_2^\vee) \rangle = \langle \chi_k, \gamma_2^\vee \rangle = 2.$$

Otherwise, we have $\vec{\nu} = \vec{\mu} - (f+1)\mathbf{e}_{n-k+1+j} + f\mathbf{e}_1$ for some $f > 0$ In this case, v' is obtained by deleting $s_{n-\mu_j^b+f+1}$, and we have $(\gamma')^\vee = (\gamma_1')^\vee$ with $(\gamma_{n-k+2}')^\vee = \alpha_n^\vee + 2\sum_{h=n-j+1}^{n-1} \alpha_h^\vee + \sum_{h=n-\mu_j^b+2-j+f}^{n-j} \alpha_h^\vee$ and $(\gamma_i')^\vee = (s_{i+k-2}s_{i+k-3} \cdots s_{n-\mu_i^t+1})((\gamma_{i+1}')^\vee)$ for each $n-k+1 \geqslant$

$i \geqslant 1$. Note that $k+i-j-1 \geqslant n-\mu_i^t+1 > n-\mu_1^t+1 = n-\mu_j^b+2-j+f$ for each $n-k+1 \geqslant i \geqslant 2$. Using the same arguments as for case (2), we can show $\langle \chi_k, (\gamma')^\vee\rangle = \langle \chi_k, s_{k-1}s_{k-2}\cdots s_{n-\mu_1^t+1}(\gamma_2'^\vee)\rangle = \langle \chi_k, \gamma_2'^\vee\rangle = 2$.

Hence, if $\nu \in \Gamma_2(\mu)$, then we have $N_{s_k,v'}^{w',0} = \langle \chi_k, (\gamma')^\vee\rangle = 2$. Similarly, we can show $\langle \chi_k, (\gamma')^\vee\rangle = 1$ if $\nu \in \Gamma_1(\mu)$, or 0 if $\nu = ((\mu_1^t-1, \mu_2^t, \ldots, \mu_{n-k+1}^t)//\mu^b)$.

Hence, the statement follows. □

Theorem 4.9 *(Quantum Pieri rule for $OG(k, 2n+1)$). Let $\mathbf{a} \in \mathcal{P}_k$ and $p \leqslant k$. Using the same notations as in Lemma 4.2, we have*

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum 2^{e'(\mathbf{a},\mathbf{b})}\sigma^{\mathbf{b}} + t\sum N_{p,\mathbf{a}}^{\mathbf{c},1}\sigma^{\mathbf{c}} + t^2\sum 2^{e'(\hat{\mathbf{a}},\hat{\mathbf{d}})}\sigma^{\mathbf{d}}.$$

Here the first summation is over all shapes $\mathbf{b} \in \mathcal{P}_k$ with $|\mathbf{b}| = |\mathbf{a}| + p$ such that $\mathbf{b}$ is compatible with $\mathbf{a}$; the third summation occurs only if $\hat{\mathbf{a}} \in \mathcal{P}_k$, and when this holds, the summation is over all shapes $\mathbf{d} \in \mathcal{P}_k$ with $|\mathbf{d}| = |\mathbf{a}| + p - 4n + 2k$ such that $\hat{\mathbf{d}} \in \mathcal{P}_k$ is compatible with $\hat{\mathbf{a}}$; the second summation is over all shapes $\mathbf{c} \in \mathcal{P}_k$ with $|\mathbf{c}| = |\mathbf{a}| + p - 2n + k$. Furthermore, we have

$$N_{p,\mathbf{a}}^{\mathbf{c},1} = \begin{cases} 2^{e'(\tilde{\mathbf{a}},\tilde{\mathbf{c}})}, & \text{if } a_1^t > a_1^b \geqslant a_2^t \\ 2^{e'(\tilde{\mathbf{a}},\tilde{\mathbf{c}})}, & \text{if } a_1^b \geqslant a_1^t,\ c_1^t = n \text{ and } c_2^t \leqslant n-2 \\ M, & \text{if } a_1^b \geqslant a_1^t \text{ and } c_1^t < n \\ 0, & \text{otherwise.} \end{cases}$$

with

$$M = \sum_{\mu} 2^{e'(\tilde{\mathbf{a}},\mu)} + \sum_{\mu} 2^{1+e'(\tilde{\mathbf{a}},\mu)} - \sum_{\nu} 2^{e'(\nu,\tilde{\mathbf{c}})} - \sum_{\nu} 2^{1+e'(\nu,\tilde{\mathbf{c}})}$$

where the first sum is over $\{\mu \mid \mu \in \Gamma_1(\tilde{\mathbf{c}})\}$, the second sum is over $\{\mu \mid \mu \in \Gamma_2(\tilde{\mathbf{c}})\}$, the third sum is over $\{\nu \mid \tilde{\mathbf{a}} \in \Gamma_1(\nu)\}$, and the last sum is over $\{\nu \mid \tilde{\mathbf{a}} \in \Gamma_2(\nu)\}$.

Proof. The first summation is provided from the classical Pieri rule. Note $\deg t = 2n - k$ for $OG(k, 2n+1)$. The third summation follows directly from Theorem 3.21, Lemma 3.22, (statements (2c), (4) of) Lemma 4.2 and Proposition 4.5.

Because of the dimension constraint, $N_{p,\mathbf{a}}^{\mathbf{c},1} = N_{u,v}^{w,1} = N_{us_k,v_1}^{w_1,0}$ is nonzero only if $\ell(v_1) = \ell(v) - 2n + 2k$ (see Lemma 4.2 for the notations). When this holds, we have $a_1^b \geqslant a_2^t$. Note $a_1^t > a_2^t$.

Assume $a_1^t > a_1^b$; then $v_1 \in W^{\tilde{P}}$, so that $N_{us_k,v_1}^{w_1,0} \neq 0$ only if $w_1 \in W^{\tilde{P}}$. Thus in this case, we have $N_{p,\mathbf{a}}^{\mathbf{c},1} = 2^{e'(\tilde{\mathbf{a}},\tilde{\mathbf{c}})}$ by using statements (2b) and (3a) of Lemma 4.2 together with the classical Pieri rule for $OG(k-1, 2n+1)$.

Assume $a_1^b \geqslant a_1^t$; then $v_1 = vs_\gamma s_k$ with $vs_\gamma \in W^{\tilde{P}}$, so that $\mathrm{sgn}_k(v_1) = 1$.

(1) Suppose $w_1 \notin W^{\tilde{P}}$; then by Corollary 2.3, we conclude that $N_{us_k,v_1}^{w_1,0} \neq 0$ only if $\mathrm{sgn}_k(w_1) = 1$, and $N_{us_k,v_1}^{w_1,0} = N_{us_k,vs_\gamma}^{w_1s_k,0}$ when this holds. In particular, we have $w_1s_k \in W^{\tilde{P}}$. Thus if $c_1^t = n$ and $c_2^t \leqslant n-2$, then we have $N_{p,\mathbf{a}}^{\mathbf{c},1} = 2^{e'(\tilde{\mathbf{a}},\tilde{\mathbf{c}})}$, by using Lemma 4.2 and Proposition 4.5.

(2) Suppose $w_1 \in W^{\tilde{P}}$; that is, $c_1^t < n$. Using Corollary 2.3 again, we conclude $\sigma^{s_k} \cup \sigma^{vs_\gamma} = \sigma^{vs_\gamma s_k} + \sum_{v' \in W^{\tilde{P}}} N_{s_k,vs_\gamma}^{v',0}\sigma^{v'}$. Hence, $N_{us_k,v_1}^{w_1,0} = N_{us_k,vs_\gamma s_k}^{w_1,0} = \sum_{w' \in W^{\tilde{P}}} N_{s_k,w'}^{w_1,0} N_{us_k,vs_\gamma}^{w',0} -$

$\sum_{v'\in W^{\tilde{P}}} N_{s_k,vs_\gamma}^{v',0} N_{us_k,v'}^{w_1,0}$. Thus in this case, the statement follows from Lemma 4.8, Lemma 4.2 and Proposition 4.5. □

Remark 4.10. One might want to use the method of Buch, Kresch, and Tamvakis on page 372 of [5] to compute $N_{p,\mathbf{a}}^{\mathbf{c},1}$, but replace $OF(m, m+1; 2n+1)$ (resp. $OG(m+1, 2n+1)$) by $OF(m-1, m; 2n+1)$ (resp. $OG(m-1, 2n+1)$). However, such a method does not work here, since the identity $\varphi_{2*}\pi_2^*\pi_{2*}\varphi_2^*\varphi_1^*\tau_\mu = \pi^*\pi_*\varphi_1^*\tau_\mu$ used in their proof does not hold any more in our case.

4.3. Remarks

As in the previous two subsections, we have derived the quantum Pieri rules for Grassmannians of type C_n and B_n for $k < n$. The remaining cases for these two types are about the Lagrangian Grassmannian $IG(n, 2n)$ and the odd orthogonal Grassmannian $OG(n, 2n+1)$. Theorem 3.13 also works for $IG(n, 2n)$, therefore we can also reformulate it in the way of Theorem 4.4, by writing down the explicit identifications as needed and using the classical Pieri rule of Hiller and Boe [16]. For $OG(n, 2n+1)$, we can also reformulate Theorem 3.21 in a nice way like Theorem 4.4, by using generalized classical Pieri rules (Theorem D) given by Bergeron and Sottile [1]. The former case has been done by Kresch and Tamvakis in [19], and the latter case has also been done by them in an equivalent way in [20]. Hence, we skip the details here.

When Δ is of type D_{n+1}, the minimal length representatives W^P for the isotropic Grassmannian $OG(k, 2n+2)$ can also be identified with "shapes", which consist in two types. There are parallel propositions to Lemma 4.2, Lemma 4.8 and Theorem 4.9 with similar arguments. Since the notations are more involved and the theorem as expected may also involve signs in some cases, we skip the details.

Assuming the classical Pieri rules with respect to the Chern classes $c_p(\mathcal{Q})$ of the tautological quotient bundle $\mathcal{Q}$ over the isotropic Grassmannians, we can also reprove the quantum Pieri rule of Buch, Kresch, and Tamvakis for type B_n and D_{n+1}. For instance for $G/P = OG(k, 2n)$ where $k < n$, the Chern classes of the tautological quotient bundle $\mathcal{Q}$ are given by σ^u (up to a scale factor of 2) for u of the following form (see Section 6 of [6])

$$s_k,\ s_{k+1}s_k,\ \ldots,\ s_ns_{n-1}\cdots s_k,\ s_{n-1}s_ns_{n-1}\cdots s_k,\ \ldots,\ s_1\cdots s_{n-1}s_ns_{n-1}\cdots s_k.$$

Taking any one of the above u and any $v \in W^P$, we can show the following quantum Pieri rule given by Buch, Kresch and Tamvakis.

Proposition 4.11. *(See Theorem 2.4 of [5].)*

$$\sigma^u \star \sigma^v = \sigma^u \cup \sigma^v + \sum_{\substack{w\in W^P\\ \ell(u)+\ell(v)=\ell(w)+2n-k}} N_{us_k,v_1}^{w_1,0}\sigma^w t + \sum_{\substack{w\in W^P\\ \ell(u)+\ell(v)=\ell(w)+4n-2k}} N_{u,v_2}^{w_2,0}\sigma^w t^2,$$

in which we have $v_1 = vs_ks_{k-1}\cdots s_1$, $w_1 = ws_{k+1}\cdots s_{n-1}s_ns_{n-1}\cdots s_{k+1}$, $v_2 = vs_k\cdots s_{n-1}s_{\bar{\alpha}_n}s_{n-1}\cdots s_1$ *and* $w_2 = ws_1\cdots s_{n-1}s_{\bar{\alpha}_n}s_{n-1}\cdots s_k$.

We sketch an alternative proof of the above proposition as follows:

(1) Using similar arguments for Proposition 3.20, we can show that $N_{u,v}^{w,d} = 0$ if $d \geqslant 3$.
(2) Using Proposition 2.1, we can show $N_{u,v}^{w,1} = N_{us_k,v_1}^{w_1,0}$ and $N_{u,v}^{w,2} = N_{u,v_2}^{w_2,0}$. (The arguments here become much simpler than the proof of Theorem 3.21.)

Furthermore for each $i \in \{1, 2\}$, it follows directly from the dimension constraint that $N_{u,v_i}^{w_i,0} \neq 0$ only if $\ell(v_i) = \ell(v) - \ell(v^{-1}v_i)$. When this holds, we have:

(3) $v_1 \in W^{\tilde{P}}$ where $G/\tilde{P} = OG(k+1, 2n+1)$. Furthermore, let λ denote the $(n-k)$-strict partition corresponding to v; then v_1 exactly corresponds the $(n-k-1)$-strict partition $\bar{\lambda}$ defined on page 372 of [5].

(4) Statements (2c) and (4) of Lemma 4.2, which give identifications between W^P and shapes (and therefore other parameterizations), can be applied directly for v_2 and w_2 respectively.

For $IG(k, 2n)$ (i.e. Grassmannians of type C_n), we can also show that there are most degree 0 and 1 Gromov–Witten invariants occurring in the quantum Pieri rule; furthermore, we can reduce the degree 1 Gromov–Witten invariants to certain classical intersection numbers, for $1 \leqslant p \leqslant n-k+1$.

We can also use Proposition 2.1 to compute $N_{u,v}^{w,d}$ for certain u, v, w, d, in which none of $u, v \in W^P$ is special.

Example 4.12. For $X = IG(3, 10)$, we take $w = s_2$, $u = s_2s_1s_4s_3s_2s_5s_4s_3$ and $v = s_1s_5s_4s_3s_2s_4s_5s_4s_3$. By using Proposition 2.1 directly, we can show $N_{u,v}^{w,2} = N_{u,v'}^{w',0}$ where $v' = s_1s_3$ and $w' = s_3s_2s_1s_5s_4s_3s_2s_5s_4s_3$. We can easily compute the classical intersection number $N_{u,v'}^{w',0}$, for instance by using the Chevalley formula with the observation that $\sigma^{s_1s_3} = \sigma^{s_1} \cup \sigma^{s_3}$. As a consequence, we have $N_{u,v}^{w,2} = N_{u,v'}^{w',0} = 1$. (Note that in terms of the notations in Example 1.5 of [5], we have $\sigma^u = \sigma_{4,2,2}$, $\sigma^v = \sigma_{5,3,1}$ and $\sigma^w = \sigma_1$.)

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Appendix A

In this appendix, we reprove the well-known quantum Pieri rules for Grassmannians of type A (i.e. complex Grassmannians) with the same method used in the present paper.

Recall that $Gr(k, n+1) = SL(n+1, \mathbb{C})/P$ with P being the (standard) maximal parabolic subgroup corresponding $\Delta_P = \Delta \setminus \{\alpha_k\}$. The Weyl group W is canonically isomorphic to the permutation group S_{n+1}, by mapping s_j to the transposition $(j, j+1)$ for each $1 \leqslant j \leqslant n$. Customarily, Schubert classes in $H^*(Gr(k, n+1))$ are labeled by the set

$$\mathcal{P}_k = \left\{ \mathbf{a} = (a_1, \ldots, a_k) \,\middle|\, n+1-k \geqslant a_1 \geqslant a_2 \geqslant \cdots \geqslant a_k \geqslant 0 \right\}$$

of partitions inside the k by $(n+1-k)$ rectangle. Precisely, for each $u \in W^P$, we can rewrite the Schubert cohomology class σ^u as $\sigma^{\mathbf{a}(u)}$ with

$$\mathbf{a}(u) = \big(u(k) - k, u(k-1) - (k-1), \dots, u(2) - 2, u(1) - 1\big)$$

via the canonical identification between W^P and $\mathcal{P}_k$ (see e.g. [14]). For the tautological bundles, $0 \to \mathcal{S} \to \mathbb{C}^{n+1} \to \mathcal{Q} \to 0$, the ith Chern class $c_i(\mathcal{Q})$ (resp. $c_i(\mathcal{S}^*)$) coincides with the Schubert cohomology class $\sigma^{\mathbf{a}(u)} = \sigma^u$ where $u = s_{k+i-1}s_{k+i-2}\cdots s_{k+1}s_k$ and consequently $\mathbf{a}(u) = (i, 0, \dots, 0) =: i$ for each $1 \leqslant i \leqslant n+1-k$ (resp. $u = s_{k-i+1}s_{k-i+2}\cdots s_{k-1}s_k$ and consequently $\mathbf{a}(u) = (1^i)$ for each $1 \leqslant i \leqslant k$) (see e.g. [3]). The following quantum Pieri rule with respect to $c_p(\mathcal{Q})$ was first proved by Bertram [2].

Proposition A.1 *(Quantum Pieri rule). For any p, $\mathbf{a} \in \mathcal{P}_k$, we have*

$$\sigma^p \star \sigma^{\mathbf{a}} = \sum \sigma^{\mathbf{b}} + t \sum \sigma^{\mathbf{c}},$$

where the first is summation over all $\mathbf{b} \in \mathcal{P}_k$ satisfying $|\mathbf{b}| = |\mathbf{a}| + p$ and $n+1-k \geqslant b_1 \geqslant a_1 \geqslant b_2 \geqslant a_2 \geqslant \cdots \geqslant b_k \geqslant a_k$, and the second summation is over all $\mathbf{c} \in \mathcal{P}$ satisfying $|\mathbf{c}| = |\mathbf{a}| + p - (n+1)$ and $a_1 - 1 \geqslant c_1 \geqslant a_2 - 1 \geqslant c_2 \geqslant \cdots \geqslant a_k - 1 \geqslant c_k$.

We will reprove the above proposition, rather than a quantum Pieri rule with respect to $c_p(\mathcal{S}^*)$, for the following two reasons.

(1) The tautological quotient bundle over $Gr(k, n+1) = Gr(k, \mathbb{C}^{n+1})$ is isomorphic to the dual of the tautological subbundle over the dual Grassmannian $Gr(n+1-k, (\mathbb{C}^{n+1})^*) \cong Gr(n+1-k, n+1)$. Hence, it is sufficient to know quantum Pieri rules with respect to either $c_p(\mathcal{Q})$ or $c_p(\mathcal{S}^*)$ for all complex Grassmannians.
(2) From our alternative proof, we can see that one point of our method relies on the combinatorial property of u, rather than the geometric property (of being a Chern class of $\mathcal{Q}$ or $\mathcal{S}^*$) for σ^u. In particular, our method could have more general applications (see e.g. Theorem 1.2 of [24]).

We will use the classical Pieri rule (as given by the first summation in the above formula) and the next two lemmas, which are easily deduced from Proposition 2.1. In addition, we use all the same notations as in Section 3.1 but set $u = s_{k+p-1}s_{k+p-2}\cdots s_k$ (where $1 \leqslant p \leqslant n+1-k$) and $v \in W^P$ with $\mathbf{a} = \mathbf{a}(v)$. We need to compute the Gromov–Witten invariants $N_{u,v}^{w,d}$ for the quantum product $\sigma^u \star \sigma^v$.

Lemma A.2. *If $d \geqslant 2$, then we have $N_{u,v}^{w,d} = 0$ for any $w \in W^P$.*

Proof. Recall that $N_{u,v}^{w,d} = N_{u,v}^{w,\lambda_P} = N_{u,v}^{w\omega_P\omega_{P'},\lambda_B}$ where $\lambda_P = d\alpha_k^\vee + Q_P^\vee$. Furthermore, we denote $d = m_1k + r_1 = m_2(n-k+1) + r_2$ where $1 \leqslant r_1 \leqslant k$ and $1 \leqslant r_2 \leqslant n-k+1$. Then we conclude $\lambda_B = m_1\sum_{j=1}^{k-1} j\alpha_j^\vee + \sum_{j=1}^{r_1-1} j\alpha_{k-r_1+j}^\vee + d\alpha_k^\vee + m_2\sum_{j=1}^{n-k} j\alpha_{n+1-j}^\vee + \sum_{j=1}^{r_2-1} j\alpha_{k+r_2-j}^\vee$ (by noting $\langle\alpha_i, \lambda_B\rangle = -1$ if $i \in \{k-r_1, k+r_2\}$, or 0 otherwise). When $k = 1$ (resp. n), then $\langle\alpha_k, \lambda_B\rangle = d + m_2 + 1$ (resp. $d + m_1 + 1$) is larger than 2 and thus we are done by Proposition 2.1 (1). When $2 \leqslant k \leqslant n-1$, we have $\langle\alpha_k, \lambda_B\rangle = m_1 + m_2 + 2$. We still have $N_{u,v}^{w,d} = 0$ unless $m_1 = m_2 = 0$ and $\mathrm{sgn}_k(w\omega_P\omega_{P'}) = 0$. When both conditions hold, we have $N_{u,v}^{w\omega_P\omega_{P'},\lambda_B} = N_{us_k,v}^{w\omega_P\omega_{P'}s_k,\lambda_B-\alpha_k^\vee}$ by Proposition 2.1 (2). Furthermore, we have $\mathrm{sgn}_{k-1}(us_k) = \mathrm{sgn}_{k-1}(v) = 0 < 1 = \langle\alpha_{k-1}, \lambda_B - \alpha_k^\vee\rangle$. Hence, we have $N_{us_k,v}^{w\omega_P\omega_{P'}s_k,\lambda_B-\alpha_k^\vee} = 0$ by using Proposition 2.1 (1) again. □

Note that $\ell(u)=p$ and the degree of the quantum variable is $\deg t=n+1$.

Lemma A.3. *For any $w\in W^P$ with $\ell(w)=\ell(v)+p-(n+1)$, we have*

$$N_{u,v}^{w,1}=N_{us_k,vs_ks_{k-1}\cdots s_2s_1}^{ws_ns_{n-1}\cdots s_{k+1},0}.$$

Proof. Note that for $\lambda_P=\alpha_k^\vee+Q_P^\vee$, we have $\lambda_B=\alpha_k^\vee$ and consequently $\Delta_{P'}=\Delta_P\setminus\{\alpha_{k-1},\alpha_{k+1}\}$. Then conclude $\omega_P\omega_{P'}=s_ns_{n-1}\cdots s_{k+1}s_1s_2\cdots s_{k-1}$ by checking the assumptions in Lemma 3.12 directly. Thus we have $N_{u,v}^{w,1}=N_{u,v}^{w\omega_P\omega_{P'},\alpha_k^\vee}=N_{us_k,vs_k}^{w\omega_P\omega_{P'},0}$, by using Proposition 2.1 (2).

Note that $\mathrm{sgn}_{k-1}(us_k)=0$ and $\mathrm{sgn}_{k-1}(w\omega_P\omega_{P'})=1$. By Corollary 2.3, we have $N_{us_k,vs_k}^{w\omega_P\omega_{P'},0}=N_{us_k,vs_ks_{k-1}}^{w\omega_P\omega_{P'}s_{k-1},0}$ if $\ell(vs_ks_{k-1})=\ell(vs_k)-1$, or 0 otherwise. By induction and using Corollary 2.3 repeatedly, we conclude $N_{us_k,vs_k}^{w\omega_P\omega_{P'},0}=N_{us_k,vs_ks_{k-1}\cdots s_1}^{w\omega_P\omega_{P'}s_{k-1}\cdots s_1,0}$ if $\ell(vs_ks_{k-1}\cdots s_1)=\ell(vs_k)-(k-1)$, or 0 otherwise. Furthermore, we note $\ell(us_k)+\ell(vs_k)=\ell(w\omega_P\omega_{P'})=\ell(w\omega_P\omega_{P'}s_{k-1}\cdots s_1)+(k-1)$. Thus if $\ell(vs_ks_{k-1}\cdots s_1)\neq\ell(vs_k)-(k-1)$, then we have $N_{us_k,vs_ks_{k-1}\cdots s_1}^{w\omega_P\omega_{P'}s_{k-1}\cdots s_1,0}=0$, due to the dimension constraint. Hence, the statement follows. □

It remains to apply the classical Pieri rule and reformulate the product $\sigma^u\star\sigma^v$ in terms of $\sigma^p\star\sigma^{\mathbf{a}}$ (labeled by partitions).

Proof of Proposition A.1. Denote $\mathbf{c}=\mathbf{c}(w)\in\mathcal{P}_k$. It follows directly from Proposition 2.1 (1) and the dimension constraint that $N_{us_k,vs_ks_{k-1}\cdots s_2s_1}^{ws_ns_{n-1}\cdots s_{k+1},0}=0$ unless $\tilde{\mathbf{a}}=\tilde{\mathbf{a}}(vs_ks_{k-1}\cdots s_2s_1)$ and $\tilde{\mathbf{c}}=\tilde{\mathbf{c}}(ws_ns_{n-1}\cdots s_{k+1})$ are both partitions in $\mathcal{P}_{k+1}$ for $Gr(k+1,n+1)$. As a consequence, we have that $\tilde{a}_i=vs_ks_{k-1}\cdots s_2s_1(k+2-i)-(k+2-i)=v(k+1-i)-(k+1-i)-1=a_i-1$ for each $1\leqslant i\leqslant k$, and that $\tilde{c}_j=ws_ns_{n-1}\cdots s_{k+1}(k+2-j)-(k+2-j)=w(k+2-j)-(k+2-j)=c_{j-1}$ for each $2\leqslant j\leqslant k+1$. Note that $|\tilde{\mathbf{c}}|=\ell(ws_ns_{n-1}\cdots s_{k+1})=\ell(w)+n-k=|\mathbf{c}|+n-k$. We have $\tilde{c}_1=n-k$. For $|\tilde{\mathbf{a}}|=\ell(vs_ks_{k-1}\cdots s_1)=\ell(v)-k=|\mathbf{a}|-k$, we conclude $\tilde{a}_{k+1}=0$. In addition, we note that us_k corresponds to the special partition $p-1$ in $\mathcal{P}_{k+1}$ for $Gr(k+1,n+1)$. Hence, the non-classical part of the quantum product $\sigma^p\star\sigma^{\mathbf{a}}$ is exactly the second half as in the statement, by using the classical Pieri rule for $\sigma^{p-1}\cup\sigma^{\tilde{\mathbf{a}}}$ in $H^*(Gr(k+1,n+1))$. □

Remark A.4. This is the same approach as taken in the elementary proof of the quantum Pieri rule by Buch [3], where he uses a different method to obtain Lemma A.2 and Lemma A.3.

References

[1] N. Bergeron, F. Sottile, A Pieri-type formula for isotropic flag manifolds, Trans. Amer. Math. Soc. 354 (2002) 2659–2705.

[2] A. Bertram, Quantum Schubert calculus, Adv. Math. 128 (2) (1997) 289–305.

[3] A.S. Buch, Quantum cohomology of Grassmannians, Compos. Math. 137 (2) (2003) 227–235.

[4] A.S. Buch, A. Kresch, H. Tamvakis, Gromov–Witten invariants on Grassmannians, J. Amer. Math. Soc. 16 (4) (2003) 901–915 (electronic).

[5] A.S. Buch, A. Kresch, H. Tamvakis, Quantum Pieri rules for isotropic Grassmannians, Invent. Math. 178 (2) (2009) 345–405.

[6] A.S. Buch, A. Kresch, H. Tamvakis, A Giambelli formula for isotropic Grassmannians, arXiv:0811.2781 [math.AG].

[7] A.S. Buch, L.C. Mihalcea, Quantum K-theory of Grassmannians, Duke Math. J. 156 (3) (2011) 501–538.

[8] A.S. Buch, L.C. Mihalcea, Curve neighborhoods and the quantum Chevalley formula, preprint available at arXiv:1303.6013 [math.AG].
[9] P.E. Chaput, L. Manivel, N. Perrin, Quantum cohomology of minuscule homogeneous spaces, Transform. Groups 13 (1) (2008) 47–89.
[10] P.-E. Chaput, N. Perrin, On the quantum cohomology of adjoint varieties, Proc. Lond. Math. Soc. (3) 103 (2) (2011) 294–330.
[11] I. Ciocan-Fontanine, On quantum cohomology rings of partial flag varieties, Duke Math. J. 98 (3) (1999) 485–524.
[12] I. Coskun, The quantum cohomology of flag varieties and the periodicity of the Schubert structure constants, Math. Ann. 346 (2) (2010) 419–447.
[13] W. Fulton, R. Pandharipande, Notes on stable maps and quantum cohomology, in: Proc. Sympos. Pure Math., vol. 62, part 2, Amer. Math. Soc., Providence, RI, 1997.
[14] W. Fulton, C. Woodward, On the quantum product of Schubert classes, J. Algebraic Geom. 13 (4) (2004) 641–661.
[15] P. Griffiths, J. Harris, Principles of Algebraic Geometry, John Wiley & Sons, Inc., New York, 1994.
[16] H. Hiller, B. Boe, Pieri formula for SO_{2n+1}/U_n and Sp_n/U_n, Adv. Math. 62 (1) (1986) 49–67.
[17] J.E. Humphreys, Introduction to Lie Algebras and Representation Theory, Grad. Texts in Math., vol. 9, Springer-Verlag, New York, Berlin, 1980.
[18] J.E. Humphreys, Reflection Groups and Coxeter Groups, Cambridge University Press, Cambridge, UK, 1990.
[19] A. Kresch, H. Tamvakis, Quantum cohomology of the Lagrangian Grassmannian, J. Algebraic Geom. 12 (4) (2003) 777–810.
[20] A. Kresch, H. Tamvakis, Quantum cohomology of orthogonal Grassmannians, Compos. Math. 140 (2) (2004) 482–500.
[21] T. Lam, M. Shimozono, Quantum cohomology of G/P and homology of affine Grassmannian, Acta Math. 204 (1) (2010) 49–90.
[22] A. Lascoux, M.-P. Schützenberger, Polynômes de Schubert, C. R. Acad. Sci. Paris Sér. I Math. 294 (13) (1982) 447–450.
[23] N.C. Leung, C. Li, Functorial relationships between $QH^*(G/B)$ and $QH^*(G/P)$, J. Differential Geom. 86 (2) (2010) 303–354.
[24] N.C. Leung, C. Li, Classical aspects of quantum cohomology of generalized flag varieties, Int. Math. Res. Not. IMRN 2012 (16) (2012) 3706–3722.
[25] C. Li, L.C. Mihalcea, K-theoretic Gromov–Witten invariants of lines in homogeneous spaces, Int. Math. Res. Not. IMRN (2013), Art. ID rnt090; preprint at arXiv:1206.3593 [math.AG].
[26] D. Peterson, Quantum cohomology of G/P, Lecture notes at MIT, 1997 (notes by J. Lu and K. Rietsch).
[27] P. Pragacz, J. Ratajski, A Pieri-type theorem for Lagrangian and odd orthogonal Grassmannians, J. Reine Angew. Math. 476 (1996) 143–189.
[28] P. Pragacz, J. Ratajski, A Pieri-type formula for even orthogonal Grassmannians, Fund. Math. 178 (1) (2003) 49–96.
[29] S. Sertöz, A triple intersection theorem for the varieties $SO(n)/P_d$, Fund. Math. 142 (3) (1993) 201–220.
[30] H. Tamvakis, Quantum cohomology of isotropic Grassmannians, in: Geometric Methods in Algebra and Number Theory, in: Progr. Math., vol. 235, Birkhäuser Boston, Boston, MA, 2005, pp. 311–338.
[31] C.T. Woodward, On D. Peterson's comparison formula for Gromov–Witten invariants of G/P, Proc. Amer. Math. Soc. 133 (6) (2005) 1601–1609.