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Kac–Moody $\widetilde{E}_k$-bundles over elliptic curves and del Pezzo surfaces with singularities of type A

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Abstract Given an elliptic curve Σ, flat E_k-bundles over Σ are in one-to-one correspondence with smooth del Pezzo surfaces of degree $9 - k$ containing Σ as an anti-canonical curve. This correspondence was generalized to Lie groups of any type. In this article, we show that there is a similar correspondence between del Pezzo surfaces of degree 0 with an A_d-singularity containing Σ as an anti-canonical curve and Kac–Moody $\widetilde{E}_k$-bundles over Σ with $k = 8 - d$. In the degenerate case where surfaces are rational elliptic surfaces, the corresponding $\widetilde{E}_k$-bundles over Σ can be reduced to E_k-bundles.

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0 Introduction

Let X_n be a blow-up of $\mathbb{P}^2$ at *n possibly infinitely near points* [9]. The Picard lattice of X_n is a rank $n+1$ lattice of index $(1, n)$ with respect to the intersection form $\langle , \rangle$ on X_n. Let K be the canonical class. When $n \leq 8$, the restriction of $\langle , \rangle$ to $K^\perp$ is negative definite, and the sub-lattice $K^\perp$ is a root lattice of the exceptional Lie algebra E_n.

When these blown-up points are in general position, $-K$ is ample, and X_n is a *del Pezzo surface*. These are important surfaces in algebraic geometry, as well as in physics. For instance, del Pezzo surfaces arise in M-theory [10,11] and the mysterious duality [13]. For such an X_n, there is a natural Lie algebra bundle of E_n-type over it [17]. Furthermore, there are two fundamental representation bundles over X_n constructed using lines and rulings on X_n.

Motivated from the duality between F-theory and string theory, given any fixed elliptic curve Σ, there is a one-to-one correspondence between flat E_n-bundles on Σ and del Pezzo surfaces of degree $9-n$ containing Σ as an anti-canonical curve [5,6,8] (this was most likely known by Looijenga in some form in the 1970's). Friedman–Morgan [7] realized this identification by constructing a principal bundle over del Pezzo surfaces X, possibly with rational double points, for a conformal form of E_n, whose restriction to the smooth anti-canonical curve Σ is the corresponding principal bundle. This correspondence has been generalized from E_n to compact Lie groups which are simply-laced in [17] and non-simply-laced in [18].

The motivation of this paper is in two aspects. One is the natural question: what happens when we blow up $\mathbb{P}^2$ at more than 8 points? Is there still some 'duality'? Another one is the so-called *magic triangle* in [10,11] (for the explanation of the magic triangle, see Remark 34). We want to see how the triangle behaves in the affine case.

In this article, we show that the correspondence between moduli spaces can be generalized to Kac–Moody Lie groups of untwisted affine E_n-type. For this, we first consider the surface X_9. It turns out that on X_9, there are very abundant and very interesting structures.

In this case, we obtain a Kac–Moody root lattice of untwisted affine E_8-type (which is denoted by $\widetilde{E}_8$), and all (-2)-classes form a real affine root system. Moreover, the anti-canonical class $-K$ is the null root, which generates the imaginary root system. Over X_9, we will define a natural affine Lie algebra bundle of $\widetilde{E}_8$-type and a natural representation bundle constructed using the set of exceptional classes.

When there is an A_d-chain of (-2)-curves (say, $C_1, \ldots, C_d$) in X_9 (denote such a surface by $\widetilde{X}_{k,d}$ with $k+d=8$, see Sect. 2.2), contracting this A_d-chain, we obtain a singular surface $X_{k,d}$ with an A_d-singularity. On a general X_k, $|-K| \cong \mathbb{P}^d$, and therefore there is a unique anti-canonical curve Σ passing through a general d-jet point $p^{[d]}$ in X_k. Indeed such a pair $(X_k, p^{[d]})$ corresponds to a general $\widetilde{X}_{k,d}$ (Proposition 25).

On such a surface $X_{k,d}$ (equivalently on $\widetilde{X}_{k,d}$), the sub-lattice $K^\perp$ (respectively, $\{K, C_1, \ldots, C_d\}^\perp$) of the Picard lattice is an untwisted affine root lattice of type $\widetilde{E}_k$. Here we make the following convention

$$\widetilde{E}_5 = \widetilde{D}_5,\ \widetilde{E}_4 = \widetilde{A}_4,\ \text{and}\ \widetilde{E}_3 = A_2 \times \widetilde{A}_1.$$

Table 1 The split magic triangle of E_n-type

	d = 0	d = 1	d = 2	d = 3	d = 4	d = 5	d = 6	d = 7	d = 8
k = 0	+								
k = 1	$\mathbb{R}$ or A_1	+							
k = 2	$\mathbb{R} \times A_1$	$\mathbb{R}$							
k = 3	$A_1 \times A_2$	$\mathbb{R} \times A_1$ or A_2	A_1						
k = 4	A_4	$\mathbb{R} \times A_2$	$\mathbb{R} \times A_1$	$\mathbb{R}$	+				
k = 5	D_5	$A_1 \times A_3$	$\mathbb{R} \times A_1^2$	$\mathbb{R}^2$ or A_1^2	$\mathbb{R}$				
k = 6	E_6	A_5	A_2^2	$\mathbb{R} \times A_1^2$	$\mathbb{R} \times A_1$	A_1			
k = 7	E_7	D_6	A_5	$A_1 \times A_3$	$\mathbb{R} \times A_2$	$\mathbb{R} \times A_1$ or A_2	$\mathbb{R}$	+	
k = 8	E_8	E_7	E_6	D_5	A_4	$A_1 \times A_2$	$\mathbb{R} \times A_1$	$\mathbb{R}$ or A_1	+

We exclude $\widetilde{E}_2, \widetilde{E}_1$, as $E_1 = \mathbb{R}$ and $E_2 = A_1 \times \mathbb{R}$ have non-simple factors (see Table 1). The set $\{\alpha \in Pic(X_{k,d}) \mid \alpha^2 = -2, \alpha \cdot K = 0\}$ forms a real affine root system of $\widetilde{E}_k$-type. This is related to *Julia's magic triangle* [10,11]. Using this, a natural Lie algebra bundle of $\widetilde{E}_k$-type over $X_{k,d}$ (or $\widetilde{X}_{k,d}$) can be constructed. (Thus, we see that this triangle could be realized at the level of bundles, see Remark 34. Also we have interesting applications.) Moreover, in all cases, the anti-canonical divisor (which is always effective) is the null root.

In the following, we state our main results. Let X_9 be the surface obtained by blowing up $\mathbb{P}^2$ at the 9 points $x_1, \ldots, x_9$. Let $\{\alpha_1, \ldots, \alpha_d\}$ be an A_d-chain of (-2)-classes. Denote $\Delta^{re}(\mathfrak{g})$ (resp. $\Delta^{im}(\mathfrak{g})$) the real root system (resp. the imaginary root system) of a Kac–Moody Lie algebra $\mathfrak{g}$.

Theorem 1 (Theorem 20 and Proposition 23)

(i) *The sub-lattice $K^\perp$ is a root lattice of affine E_8-type with the real root system $\Delta^{re} = \{\alpha \in K^\perp | \alpha^2 = -2\}$, the imaginary root system $\Delta^{im} = \{nK | n \neq 0, n \in \mathbb{Z}\}$ and the null root $-K$.*

(ii) *The sub-lattice $\{\alpha_1, \ldots, \alpha_d, K\}^\perp$ of $K^\perp$ is a root lattice of affine E_k-type ($k = 8 - d$) with $\Delta^{re} = \{\alpha \in \{K, \alpha_1, \ldots, \alpha_d\}^\perp | \alpha^2 = -2\}$, $\Delta^{im} = \{nK | n \neq 0, n \in \mathbb{Z}\}$ and the null root $-K$.*

(iii) *Over $X_{k,d}$, there is a canonically defined affine Lie algebra bundle $\widetilde{\mathscr{E}}_k$, and the set of exceptional classes defines a fundamental representation bundle of $\widetilde{\mathscr{E}}_k$ when $k = 8$.*

Suppose $\widetilde{X}_{k,d}$ is *general* (Sect. 2.2). In this article, a general or strictly semi-general $\widetilde{X}_{k,d}$ is also called *a del Pezzo surface of degree 0 with an A_d-singularity*. For instance, a general $\widetilde{X}_{8,0}$ is obtained by blowing up $\mathbb{P}^2$ at 9 points *in general position*. The Weyl group $W(\widetilde{E}_k)$ acts on the set of exceptional configurations simply transitively. Thus, the restriction of such affine Lie algebra bundles to the fixed anti-canonical curve (which is a smooth elliptic curve) determines a correspondence between two different types of moduli spaces $\mathcal{M}_\Sigma^{\widetilde{E}_k}$ and $\mathcal{S}_{\Sigma,k}$, where $\mathcal{M}_\Sigma^{\widetilde{E}_k}$ is the moduli space (Sect. 3) of holomorphic $Lie(\widetilde{E}_k)$-bundles over Σ (see Definition 26) and $\mathcal{S}_{\Sigma,k}$ is the moduli space (Sect. 4) of general $\widetilde{X}_{k,d}$ with $\widetilde{E}_k$-configuration containing an anti-canonical curve isomorphic to Σ. By Proposition 28,

$$\mathcal{M}_{\Sigma}^{\widetilde{E}_k} \cong Hom(\Lambda(\widetilde{E}_k), \Sigma)/(\{\pm 1\} \times W(\widetilde{E}_k)),$$

where Λ denotes the root lattice. By restriction of the $Lie(\widetilde{E}_k)$-bundle over $X_{k,d}$ to its anti-canonical curve Σ, we have a map $\Phi : \mathcal{S}_{\Sigma,k} \to \mathcal{M}_{\Sigma}^{\widetilde{E}_k}$.

Theorem 2 (Theorem 31)

(i) *The restriction map Φ induces an embedding of $\mathcal{S}_{\Sigma,k}$ into*

$$Hom(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k).$$

(ii) *Moreover, this embedding can be extended to an identification from the natural compactification $\overline{\mathcal{S}}_{\Sigma,k}$ of $\mathcal{S}_{\Sigma,k}$ onto $Hom(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k)$.*

(iii) *There is a covering map of degree two: $\overline{\mathcal{S}}_{\Sigma,k} \twoheadrightarrow \mathcal{M}_{\Sigma}^{\widetilde{E}_k}$.*

These spaces are in fact analytic spaces, and fine moduli spaces, following a general argument (for example, see Looijenga's paper [21]). We will deal with such problems in future. In this article, we just concentrate on the correspondence and the lattice analysis, and use freely the phrase 'moduli space'.

It is natural to study how the correspondence Φ extends to $\overline{\mathcal{S}}_{\Sigma,k} \backslash \mathcal{S}_{\Sigma,k}$. For this, let E_k be the corresponding finite type Lie group of $\widetilde{E}_k$, and

$$\mathcal{M}_{\Sigma}^{E_k} \cong Hom(\Lambda(E_k), \Sigma)/W(E_k)$$

be the moduli space of flat E_k-bundles over Σ. Let δ be the null root $-K$ (see Theorem 1). By Proposition 36,

$$\mathcal{M}_{\Sigma}^{E_k} \cong Hom_0(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k),$$

where $Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$ is the subgroup of $Hom(\Lambda(\widetilde{E}_k), \Sigma)$ consisting of those f's such that $f(\delta) = 0$.

Denote $\mathcal{S}'_{\Sigma,k}$ the moduli space of strictly semi-general surfaces $\widetilde{X}_{k,d}$ with an $\widetilde{E}_k$-configuration and containing an anti-canonical curve isomorphic to Σ (Sect. 5). The natural compactification of $\mathcal{S}'_{\Sigma,k}$ in the following theorem is one of the largest irreducible components in $\overline{\mathcal{S}}_{\Sigma,k} \backslash \mathcal{S}_{\Sigma,k}$. By restriction of the $Lie(\widetilde{E}_k)$-bundle over $X_{k,d}$ to its anti-canonical curve Σ, we have a map $\Phi : \mathcal{S}'_{\Sigma,k} \to \mathcal{M}_{\Sigma}^{E_k}$. Then we have

Theorem 3 (Theorem 38) *The restriction map Φ induces an open dense embedding from $\mathcal{S}'_{\Sigma,k}$ into $\mathcal{M}_{\Sigma}^{E_k}$. Moreover the embedding can be extended to an isomorphism from the natural compactification $\overline{\mathcal{S}'}_{\Sigma,k}$ onto $\mathcal{M}_{\Sigma}^{E_k}$.*

This article is arranged as follows. In Sect. 1, we study the lattice structure on a blow-up S of $\mathbb{P}^2$ at 9 points. In Sect. 2, we construct the affine Lie algebra bundle and its representation bundle (associated to the set of exceptional classes) over S.

In Sect. 3, we study some elementary properties of $Lie(\widetilde{E}_k)$-bundles over elliptic curves. In Sect. 4, we study the relation between $\overline{\mathcal{S}}_{\Sigma,k}$ and $\mathcal{M}_{\Sigma}^{\widetilde{E}_k}$ by restriction of such bundles to a fixed elliptic curve $\Sigma \in |-K_S|$. In Sect. 5, we study the correspondence in the special degenerate case where the surfaces are strictly semi-general. A short review of untwisted affine Lie algebras is presented as an appendix.

1 Rational surfaces

1.1 Root lattices on smooth rational surfaces

Let X_n be the blow-up of $\mathbb{CP}^2$ (usually abbreviated to $\mathbb{P}^2$) at n points $x_1, \ldots, x_n$. The Picard group $Pic(X_n) \cong H^2(X_n, \mathbb{Z})$ of X_n is a lattice of rank $n+1$ with generators $h, l_1, \ldots, l_n$, where h is the class of lines in $\mathbb{P}^2$, and l_i is the exceptional class determined by the blow-up at x_i. So $h^2 = 1 = -l_i^2$, and $h \cdot l_i = 0 = l_i \cdot l_j, i \neq j$. The canonical class is $K_n = -3h + l_1 + \cdots + l_n$, which is also denoted as K when there is no confusion. Let $K_n^{\perp}$ be the sublattice of $Pic(X_n)$ perpendicular to K_n.

Denote

$$
\begin{aligned}
P_n &= \{\alpha \in H^2(X_n, \mathbb{Z}) \mid \alpha \cdot K_n = 0\}, \\
R_n &= \{\alpha \in H^2(X_n, \mathbb{Z}) \mid \alpha \cdot K_n = 0,\ \alpha^2 = -2\} \subset P_n, \\
I_n &= \{e \in H^2(X_n, \mathbb{Z}) \mid e^2 = -1 = e \cdot K_n\}, \\
C_n &= \{\zeta_n = (e_1, \ldots, e_n) \mid e_i \in I_n,\ e_i \cdot e_j = 0,\ i \neq j\}, \text{ and} \\
W(R_n) &= \text{ the group generated by the reflections } s_\alpha, \text{ where } \alpha \in R_n.
\end{aligned}
$$

The reflection s_α is defined by $s_\alpha(x) := x + (x \cdot \alpha)\alpha$, for any $x \in H^2(X_n, \mathbb{Z})$. An element $\zeta_n = (e_1, \ldots, e_n) \in C_n$ is called an *exceptional system* (*of length* n); when X_n is considered as a successive blow-up of $\mathbb{P}^2$ at n points such that $e_1, \ldots, e_n$ are the associated exceptional classes, $\zeta_n = (e_1, \ldots, e_n) \in C_n$ is called an *exceptional configuration*; when each e_i is a smooth irreducible curve, ζ_n is called a *configuration of exceptional curves*.

It is well-known that R_n is a root system of type E_n for $n \leq 8$. In the following we study the case where $n = 9$.

Lemma 4 [4] *Let* $\alpha = a_0 h - \sum_{i=1}^n a_i l_i \in Pic(X_n)$ *(for any* n*).*

(i) *If* α *is effective, then* $a_0 = \alpha \cdot h \geq 0$*. Conversely, if* $a_0 \geq -2$ *and* $\alpha^2 \geq \alpha \cdot K$*, then* α *is effective.*
(ii) *For any divisor* $\alpha \in K^{\perp}$*,* α^2 *is even.*

Proof (i). By definition of the blow-ups, if α is effective, then a_0 must be non-negative.

By Riemann–Roch,

$$
\chi(\mathcal{O}(\alpha)) := h^0(\mathcal{O}(\alpha)) - h^1(\mathcal{O}(\alpha)) + h^2(\mathcal{O}(\alpha)) = 1 + 1/2(\alpha^2 - \alpha K).
$$

By Serre's duality, $h^2(\mathcal{O}(\alpha)) = h^0(\mathcal{O}(K-\alpha))$. But $K-\alpha = (-3-a_0)h+\sum(1+a_i)l_i$. $a_0 \geq -2$ implies $-3-a_0 \leq -1$. So if $a_0 \geq -2$, then $h^0(\mathcal{O}(K-\alpha)) = 0$, and if additionally $\alpha^2 \geq \alpha \cdot K$, then $\chi(\mathcal{O}(\alpha)) \geq 1$. Therefore $h^0(\mathcal{O}(\alpha)) \geq 1$. Hence α is effective.

(ii) For any divisor $\alpha \in K^{\perp}$, let $\alpha = a_0 h - \sum a_i l_i$. Then $\alpha K = 0$ implies $3a_0 = \sum a_i$. So $\alpha^2 = a_0^2 - \sum a_i^2 = 2\sum_{i<j} a_i a_j - 8a_0^2$ is even.

□

Lemma 5 *Let $x = ah - (a_1 l_1 + \cdots + a_9 l_9) \in I_9$. Then*

(i) *$a \geq 0$, and $a = 0$ if and only if $x = l_i$, for some i;*
(ii) *$a_i \geq -1$, and $a_i = -1$ if and only if $x = l_i$;*
(iii) *x is effective.*

Proof First, we prove that $a \geq 0$. Since $x \in I_9$, we have $x^2 = x \cdot K_9 = -1$. Thus we obtain two equations:

$$3a - \left(\sum a_i\right) = 1, \tag{1}$$

$$a^2 - \left(\sum a_i^2\right) = -1. \tag{2}$$

From (1) and (2), we have

$$(3a-1)^2 = \left(\sum a_i\right)^2 \leq 9\left(\sum a_i^2\right) = 9(a^2+1).$$

Thus, $a \geq -\frac{8}{6}$. Then $a \geq -1$, since a is an integer. Next we show that $a \neq -1$. Otherwise, if $a = -1$, then by (2), there exist two integers i, j with $1 \leq i < j \leq 9$, such that $a_i = \pm 1$, $a_j = \pm 1$ and $a_k = 0$ for all $k \neq i, j$. Considering (1), we obtain a contradiction. Hence we have $a \geq 0$. The proof of the other part of (i) is easy.

Assume $a_9 \leq -1$. Then $a_1^2 + \cdots + a_8^2 = a^2 + 1 - a_9^2 \leq a^2$, $a_1 + \cdots + a_8 = 3a - 1 - a_9 \geq 3a$. So $9a^2 \leq (a_1 + \cdots + a_8)^2 \leq 8(a_1^2 + \cdots + a_8^2) \leq 8a^2$, since $a \geq 0$ by (i). Then $a = 0 = a_i (i = 1, \ldots, 8)$ and $a_9 = -1$. Therefore by Lemma 4, x is effective. Thus (ii) and (iii) are proved. □

Lemma 6 *$\alpha \in R_9$ implies $\alpha + mK_9 \in R_9$ for $m \in \mathbb{Z}$.*

Proof Let $\alpha \in R_9$. Then $\alpha^2 = -2$, $\alpha \cdot K_9 = 0$. So $(\alpha + mK_9)^2 = \alpha^2 + 2\alpha \cdot (mK_9) + (mK_9)^2 = -2$ and $(\alpha + mK_9) \cdot K_9 = 0$, since $\alpha \cdot K_9 = K_9^2 = 0$. □

Proposition 7 *Consider P_8 as the sublattice of P_9 perpendicular to l_9. Then*

$$R_9 = \{\alpha + mK_9 | \alpha \in R_8, m \in \mathbb{Z}\}.$$

Proof Let $\beta = ah - (a_1 l_1 + \cdots + a_9 l_9) \in R_9$. Then $\beta - a_9 K_9 \in R_9$ by Lemma 6. But $\beta - a_9 K_9$ contains no l_9 terms. Thus $\alpha = \beta - a_9 K_9 \in R_8$, since $\alpha^2 = -2$ and $\alpha \cdot K_8 = \alpha \cdot K_9 = 0$. Then $\beta = \alpha + a_9 K_9$, where $\alpha \in R_8$. The inclusion $\{\alpha + mK_9 | \alpha \in R_8, m \in \mathbb{Z}\} \subseteq R_9$ is obvious. □

Proposition 8 (i) *The restriction $\langle , \rangle|_{P_9}$ of the intersection form to $P_9 = K_9^{\perp}$ is non-positive definite and degenerate. The radical of $\langle , \rangle|_{P_9}$ is $\{nK_9 | n \in \mathbb{Z}\}$.*

(ii) *All the sets I_9, R_9 and $W(R_9)$ are infinite sets.*

(iii) *The isotropic subgroup of l_9 in $W(R_9)$ is $W(R_8)$.*

(iv) *$W(R_9)$ acts transitively on I_9; $W(R_9)$ acts transitively on R_9.*

(v) *$W(R_9)$ acts simply transitively on C_9.*

Proof (i) Let $\alpha = a_0 h - \sum a_i l_i \in K_9^{\perp}$. Then $\alpha \cdot K_9 = -3a_0 + \sum a_i = 0$, and thus $3a_0 = \sum a_i$. Thus $\alpha^2 = a_0^2 - \sum a_i^2 = 1/9\left(\sum a_i\right)^2 - \sum a_i^2 \le (n/9 - 1)\sum a_i^2$. So if $n \le 9$, $\alpha^2 \le 0$. If $n < 9$ and $\alpha \ne 0$, then $\alpha^2 < 0$. When $n = 9$, $K_9^2 = 0$ implies $K_9 \in K_9^{\perp}$. So K_9 is an element in the radical. Since $P_8 = l_9^{\perp} \in P_9$ and the form $\langle , \rangle|_{P_8}$ is negative definite, the radical of $\langle , \rangle|_{P_9}$ is of rank 1. For (ii)–(v), see [4]. □

Given three distinct points x_1, x_2, x_3 on $\mathbb{P}^2$, we have the well-known *quadratic Cremona transformation* centered at x_1, x_2, x_3. Under this (birational) transformation, we obtain three new points y_1, y_2, y_3 on another $\mathbb{P}^2$. For accurate construction, see the classical text book [9]. A *Cremona transformation* is defined as the product of a finite number of quadratic Cremona transformations.

The following definition is taken from [23] (see also [9], page 409).

Definition 9 Let $x_1, \ldots, x_n$ be n distinct points on $\mathbb{P}^2$. These n points are said to be *non-special with respect to Cremona transformations* if for any Cremona transformation T with centers within x_i's, the points $y_1, \ldots, y_n$ corresponding to x_i's under T are distinct points such that no three points among $y_1, \ldots, y_n$ are collinear.

Definition 10 We say that the nine points $x_1, \ldots, x_9$ are *in general position*, if they satisfy the following three conditions:

(i) they are distinct points on $\mathbb{P}^2$;

(ii) they are non-special with respect to Cremona transformations;

(iii) there is a unique cubic curve passing through all of them.

If they satisfy only (i) and (ii), but not (iii), we say that they are *in strictly semi-general position*.

Note that the above definition is slightly different from that in [9]. And the conditions (i) and (ii) imply that any 8 of these 9 points are *in general position* (see Proposition 9 of [23]). That is, no lines pass through three of them, no conics pass through six of them, and no cubic curves pass through eight of them with one of the eight points being a double point [4]. In fact, eight points are in general position if and only if they satisfy the conditions (i) and (ii).

Proposition 11 *Let $X = X_9$ be the blow-up of $\mathbb{P}^2$ at 9 points $x_1, \ldots, x_9$ in general position. Then there is a unique irreducible anti-canonical curve (denoted by Σ). Assume Σ is smooth. Then on X we have*

(i) *For any effective divisor D, one has $D\Sigma \ge 0$.*

(ii) *There are no irreducible curves C with $C^2 \le -2$. If C is an irreducible curve with $C^2 = -1$, then $CK = -1$ and C is exceptional.*

(iii) *For any irreducible curve C with $C\Sigma = 0$, one has $C^2 = 0$, and therefore $C = \Sigma$.*

(iv) *For any $\alpha \in R_9$, α is not effective.*

Proof Since the 9 points $x_1, \dots, x_9 \in \mathbb{P}^2$ are in general position, by the condition (iii) in Definition 10, there is an irreducible cubic curve $\Sigma' \subset \mathbb{P}^2$ passing through all of these points. By the condition (ii) in Definition 10, Σ' is of multiplicity 1 at each of these points. Thus, the strict transform of Σ' under the blow-up map $X \to \mathbb{P}^2$ is linearly equivalent to the anti-canonical divisor $-K_9$. Hence $-K_9$ is represented by an irreducible curve Σ, and the uniqueness of Σ' determines the uniqueness of Σ.

(i) Let $D = \sum a_i D_i$ with D_i irreducible and $a_i \geq 0$, for any i. Then $D_i\Sigma \geq 0$ unless $D_i = \Sigma$. If $D_i = \Sigma$, then $D_i\Sigma = K^2 = 0$. In all cases, we have $D_i\Sigma \geq 0$.

(ii) This is just Proposition 12 in [23].

(iii) Since $0 \leq g(C) = 1 + 1/2(C^2 + CK)$, we have $0 \geq C^2 \geq -2$. So by Lemma 4, $C^2 = -2$ or $C^2 = 0$. By (ii), $C^2 = 0$. Then C is an element in the radical. So $C = -mK$. We show that $C = -K = \Sigma$ if C is irreducible. From the short exact sequence

$$0 \to \mathcal{O}(-mK) \to \mathcal{O}(-(m+1)K) \to \mathcal{O}_\Sigma(-(m+1)K) \to 0$$

we have an exact sequence

$$0 \to H^0(X, \mathcal{O}(-mK)) \to H^0(X, \mathcal{O}(-(m+1)K)) \\ \to H^0(\Sigma, \mathcal{O}_\Sigma(-(m+1)K)).$$

When $m \geq 0$, $H^0(\Sigma, \mathcal{O}_\Sigma(-(m+1)K)) = 0$, since $x_1, \dots, x_9$ are in general position. So

$$h^0(X, \mathcal{O}(-mK)) = h^0(X, \mathcal{O}(-(m+1)K)) = h^0(X, \mathcal{O}) = 1.$$

Thus, the linear system $|-mK|$ contains a unique element, for any $m \geq 1$. Hence if $C \in |-mK|$ is irreducible, then m=1.

(iv) Suppose $\alpha \in R_9$ is effective. Let $\alpha = \sum D_i$ with D_i irreducible. Then $0 = \alpha K = \sum D_i K$. By (i), for each i, $D_i K \leq 0$. So for each i, $D_i\Sigma = 0$. Then by (iii), $D_i = \Sigma$, contradicting to the hypothesis that $\alpha \in R_9$.

□

Corollary 12 *Let $X = X_9$ be the blow-up of $\mathbb{P}^2$ at 9 points in general position with a smooth anti-canonical curve. Then any exceptional divisor D on X is represented by an irreducible exceptional curve.*

Proof According to Lemma 5, $D \in I_9$ is effective. Assume $D = \sum D_i \in I_9$ with D_i irreducible. Then $\sum D_i K = -1$ and $D_i K \leq 0$ by Proposition 11, (i). We can assume $D_1 K = -1$ and $D_i K = 0$ for $i > 1$. Then $D_i = -K$, by Proposition 11, (iii). Thus $D_1 = D + mK$, $m \geq 0$ and $D_1^2 = (D + mK)^2 = -1 - 2m \leq -1$. Since by Proposition 11, (ii), there is no irreducible curve C with $C^2 \leq -2$, $D_1^2 = -1$ and $m = 0$. Hence $D = D_1$ is an irreducible exceptional curve. □

Remark 13 Corollary 12 is also a corollary of general results in [23]. Here we give a simple proof in our case. In the rational elliptic fibration case, the same result holds when the blown up points are in strictly semi-general position [12].

Remark 14 [4,22] For $n \leq 8$, the following three conditions are equivalent to each other:

(i) $x_1, \dots, x_n$ are in general position;
(ii) on X_n each exceptional divisor is effective and irreducible;
(iii) the anti-canonical class is ample.

Theorem 15 *P_9 is a lattice with generators $\Pi_9 = \{\alpha_1 = h - l_1 - l_2 - l_3, \alpha_k = l_{k-1} - l_k, k = 1, \cdots, 9\}$.$R_8$ is a root system of type E_8, which has R_9 (respectively, the Dynkin diagram of Π_9) as its affine real root system (resp., affine Dynkin diagram). The highest root of R_8 is $-\alpha_0 = -K_9 - (l_8 - l_9)$. So the imaginary roots are the set $\{m\delta | m \neq 0, m \in \mathbb{Z}\}$, where $\delta = -\alpha_0 + (l_8 - l_9) = -K_9$ is the null root.*

Proof For the proof that R_8 is a root system of type E_8 with given simple roots, see [22]. $H^2(X_9, \mathbb{Z})$ is a lattice with $\mathbb{Z}$-basis $h, l_1, \dots, l_9$. Obviously, $\{e_0 = l_1, e_1 = \alpha_1, \dots, e_9 = \alpha_9\}$ forms another $\mathbb{Z}$-basis. Take any $x \in P_9 \subset H^2(X_n, \mathbb{Z})$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K_9 = 0$ implies $a_0 = 0$. So, as a lattice, P_9 has generators Π_9.

R_9 is the affine root system of R_8 according to the analysis of root systems of Kac–Moody algebras in [14], or see the appendix. □

Corollary 16 *The set $R_9 \cup \{m(-K_9) \mid m \neq 0, m \in \mathbb{Z}\}$ forms a root system of (untwisted) affine E_8-type (that is, $\widetilde{E}_8$-type) with $\Delta^{re} = R_9 = \{\alpha \in K_9^{\perp} | \alpha^2 = -2\}$ and $\Delta^{im} = \{m(-K_9) \mid m \neq 0, m \in \mathbb{Z}\}$.*

1.2 Root lattices on rational surfaces with singularities of type A

Assume $\gamma_1, \dots, \gamma_d \in R_n$ form an A_d-chain, $0 \leq d \leq 8$, which means that the intersection matrix of the $\gamma_i, i = 1, \dots, d$, is a Cartan matrix of A_d-type.

Lemma 17 *Suppose $\gamma_1, \dots, \gamma_d \in R_8$ form an A_d-chain. Let $k = 8 - d$. Then the sub-lattice $\{\gamma_1, \dots, \gamma_d\}^{\perp}$ forms a root lattice of E_k-type.*

Proof This is just the last row of the so-called *split magic triangle* [10,11]. Let $e_1 = -K - \gamma_1, e_2 = -K - \gamma_1 - \gamma_2, \dots, e_d = -K - \gamma_1 - \cdots - \gamma_d$. Then it is easy to check that $e_1, \dots, e_d$ are all exceptional classes, and they are disjoint with each other. Therefore we can assume that $\{e_1, \dots, e_d\} = \{l_8, \dots, l_{k+1}\}$. Moreover

$$\{\gamma_1, \dots, \gamma_d\}^{\perp} = \{e_1, \dots, e_d\}^{\perp} \cap R_8.$$

But the right-hand side is just a root system of E_k. □

Proposition 18 [10] *Consider the surface obtained by blowing up $\mathbb{P}^2$ at 8 points. Let $(e_1, \dots, e_d)$ be an exceptional system of length d (that is, $e_i e_j = -\delta_{ij}, e_i K = -1$, for any i, j), and $(\beta_1, \dots, \beta_l)$ be an A_l-chain of (-2)-classes, such that $e_i \cdot \beta_j = 0$*

for any i, j. Then there exist an A_d-chain $(e'_1, \ldots, e'_d)$ of (-2)-classes and an exceptional system $(\beta'_1, \ldots, \beta'_l)$ with $e'_i \cdot \beta'_j = 0$, such that

$$\{e_1, \ldots, e_d, \beta_1, \ldots, \beta_l\}^\perp = \{e'_1, \ldots, e'_d, \beta'_1, \ldots, \beta'_l\}^\perp.$$

Lemma 19 *Let $\alpha \in R_9$, then there is a unique $n \in \mathbb{Z}$, such that $\alpha - nK_9 \in \{l_9\}^\perp$. In this case $\alpha_0 := \alpha - nK_9 \in R_8$.*

Proof Let $\alpha = a_0 h - \sum_{i=1}^{9} a_i l_i$. Then $\alpha \cdot l_9 = a_9$ and $K_9 \cdot l_9 = -1$, so $n = a_9$ is unique such that $\alpha_0 \in \{l_9\}^\perp$. Moreover $\alpha_0^2 = (\alpha - nK_9)^2 = -2$, and $\alpha_0 \cdot K_8 = (\alpha - nK_9)(K_9 - l_9) = 0$. So $\alpha_0 \in R_8$. □

Theorem 20 *Let $0 \le d \le 5$. Assume that $\gamma_1, \ldots, \gamma_d$ form an A_d-chain in R_9. Then the sub-lattice $\Lambda(\widetilde{E}_k) := \{\gamma_1, \ldots, \gamma_d, K\}^\perp \subset P_9$ is a root lattice of $\widetilde{E}_k$-type $(k = 8 - d)$ with the real root system $\Delta^{re}(\widetilde{E}_k) = \{\alpha \in \Lambda(\widetilde{E}_k) | \alpha^2 = -2\}$ and the null root $\delta = -K$.*

Proof Let $\alpha, \beta \in R_9$ and $\alpha = \alpha_0 + uK_9$, $\beta = \beta_0 + vK_9$, where $\alpha_0, \beta_0 \in R_8$, by the above lemma. Since $\beta \cdot \alpha = \beta_0 \cdot \alpha_0$, we can assume $\gamma_i \in R_8$ for $i = 1, \ldots, d$. By the same reason, for any $\gamma = \gamma_0 + wK_9 \in R_9$ with $\gamma_0 \in R_8$, $\gamma \perp \gamma_i$ if and only if $\gamma_0 \perp \gamma_i$. Therefore $\{\gamma_1, \ldots, \gamma_d\}^\perp = \{\zeta + mK_9 \mid \zeta \in R_8, \zeta \perp \gamma_i, i = 1, \ldots, d\}$. By Lemma 17, the set $\{\zeta \in R_8 \mid \zeta \perp \gamma_i, i = 1, \ldots, d\}$ is a root system of E_k-type. Therefore $\{\gamma_1, \ldots, \gamma_d\}^\perp = \{\zeta + mK_9 \mid \zeta \in R(E_k)\}$, where $R(E_k)$ means the root system of E_k-type. Hence $\{\gamma_1, \ldots, \gamma_d\}^\perp$ is a real root system of $\widetilde{E}_k$-type.

It remains to check that there exists a root basis $\Pi(\widetilde{E}_k)$ which is a basis of the lattice $\Lambda(\widetilde{E}_k)$. We fix a root basis of $\widetilde{E}_8$ as $\Pi(\widetilde{E}_8) = \{\alpha_1 = h - l_1 - l_2 - l_3, \alpha_i = l_{i-1} - l_i (i = 2, \ldots, 9)\}$ (Notice that $\alpha_9 + \alpha_h = -K$ where α_h is the highest root of E_8). For $d \le 5$, since $W(E_8)$ acts on the set of A_d-chains transitively by [1], we can assume our A_d-chain is $\{\alpha_{9-d}, \ldots, \alpha_8\}$. Furthermore, we can assume our A_d-chain is $\{\alpha_{10-d}, \ldots, \alpha_9\}$, by an action of $W(\widetilde{E}_8)$. Then $\Pi(\widetilde{E}_{8-d}) = \{\alpha_1, \ldots, \alpha_{8-d}, -K - \alpha_h(E_{8-d})\}$ is a basis of the lattice $\Lambda(\widetilde{E}_{8-d})$ where $\alpha_h(E_{8-d})$ is the highest root of E_{8-d}.

$-K$ is the null root since $-K$ is the generator for the radical of the intersection form and $-K - \alpha_h(E_{8-d})$ is the root associated to the extended node where α_h is the highest root of $\widetilde{E}_{8-d}$. □

Remark 21 There is always an effective anti-canonical divisor $\Sigma \in |-K_9|$. When the blown up points $x_1, \ldots, x_9$ are in general or strictly semi-general position, Σ can be taken as a smooth elliptic curve. Therefore the imaginary root system is generated by the class of an elliptic curve Σ.

2 The affine Lie algebra bundles over rational surfaces

2.1 Lie($\widetilde{\boldsymbol{E}}_{\mathbf{8}}$)-bundles and a natural representation bundle

Since we have a root system of $\widetilde{E}_8$-type in the Picard lattice on X_9, we can construct a $Lie(\widetilde{E}_8)$-bundle over X_9. Let $\mathscr{E}_8$ be the pull-back of the E_8-bundle [17] over X_8

(the blow-up at $x_1, \ldots, x_8$) via the blow-up map $\pi_9 : X_9 \to X_8$.

$$\widetilde{\mathscr{E}}_8 = (\mathcal{O}^{\oplus 8} \oplus \mathcal{O}) \oplus \bigoplus_{\alpha \in \Delta^{re}} \mathcal{O}(\alpha) \bigoplus_{\beta \in \Delta^{im}} \mathcal{O}(\beta)^{\oplus 8}$$
$$\cong \mathscr{E}_8 \otimes \left(\bigoplus_{n \in \mathbb{Z}} \mathcal{O}(nK_9) \right) \oplus \mathcal{O}. \quad (3)$$

We define a fiberwise Lie algebra structure on $\widetilde{\mathscr{E}}_8$ as follows. Let U be a trivialization open set (such a trivialization must exist), such that $\widetilde{\mathscr{E}}_8|_U = \mathscr{E}_8|_U \otimes (U \times (\oplus_n \mathbb{C}_{nK})) \oplus (U \times \mathbb{C})$. Take a local basis $e_i^U, i = 1, \ldots, \mathrm{rank}\mathscr{E}_8$ of $\mathscr{E}_8|_U$, e_{nK}^U of $\mathbb{C}_{nK}$, e_c of $\mathbb{C}$, compatible with the tensor product, for example, $e_{nK}^U \otimes e_{mK}^U = e_{(m+n)K}^U$. Then define

$$\left[e_i^U e_{nK}^U + \lambda e_c, e_j^U e_{mK}^U + \mu e_c \right] = \left[e_i^U, e_j^U \right]_0 e_{(n+m)K}^U + n\delta_{n+m,0}(e_i^U, e_j^U) e_c. \quad (4)$$

Here $[,]_0$ is the Lie bracket on $\mathscr{E}_8$, and $(,)$ is the Killing form on E_8.

Remark 22 Exceptional E_k-bundles $\mathscr{E}_k$ over del Pezzo surfaces of degree $9-k$ were defined in [17,18].

Proposition 23 (i) *The formula* (2) *defines a fiberwise affine Lie algebra structure which is compatible with any trivialization.*
(ii) *The isomorphism in the formula* (1) *is independent of the pull-back.*

Proof (i) Let V be another trivialization of $\widetilde{\mathscr{E}}_8$, and f_i^{UV}, f_{nK}^{UV} be respectively the transition functions, that is $e_i^V = f_i^{UV} e_i^U$, $e_{nK}^V = f_{nK}^{UV} e_{nK}^U$. Then on $U \bigcap V$ we should have

$$[e_i^V e_{nK}^V + \lambda e_c, e_j^V e_{mK}^V + \mu e_c] = [e_i^V, e_j^V]_0 e_{nK}^V e_{mK}^V + n\delta_{n+m,0}(e_i^V, e_j^V) e_c.$$

The left hand side is (note that $f_{mK}^{UV} = (f_K^{UV})^m$)

$$\begin{aligned} &\left[e_i^V e_{nK}^V + \lambda e_c, e_j^V e_{mK}^V + \mu e_c \right] \\ &= \left[f_i^{UV} e_i^U f_{nK}^{UV} e_{nK}^U + \lambda e_c, f_j^{UV} e_j^U f_{mK}^{UV} e_{nK}^U + \mu e_c \right] \\ &= f_i^{UV} f_j^{UV} \left(f_K^{UV} \right)^{m+n} \left[e_i^U, e_j^U \right]_0 e_{(m+n)K}^U + n\delta_{n+m,0} \left(f_i^{UV} f_{nK}^{UV} e_i^U, f_j^{UV} f_{mK}^{UV} e_j^U \right) e_c \\ &= f_i^{UV} f_j^{UV} \left(f_K^{UV} \right)^{m+n} \left[e_i^U, e_j^U \right]_0 e_{(n+m)}^U + n\delta_{n+m,0} f_i^{UV} f_j^{UV} f_{(n+m)K}^{UV} \left(e_i^U, e_j^U \right) e_c. \end{aligned}$$

One can see that this is exactly equal to the right hand side, since $f_{(n+m)K}^{UV} = 1$ when $n + m = 0$.

(ii) Let $W(\widetilde{E}_8) = W(R_9)$ be the affine Weyl group. Then the group $\{\pm 1\} \times W(\widetilde{E}_8)$ acts on root bases in R_9 simply transitively (Lemma 27). Fiberwise our definition is exactly the same as the definition of untwisted affine Lie algebras. By Proposition 8, (v), two different pull-backs differ by an action of

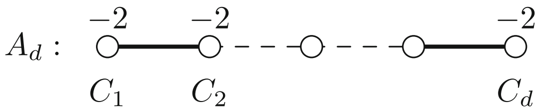

Fig. 1 The dual graph of the canonical resolution of an A_d-singularity

a unique element of the affine Weyl group. The action of the Weyl group induces automorphisms of the Lie algebra. That is, let $g_\alpha := \exp(\mathrm{ad}\ \mathrm{e}_\alpha)\exp(\mathrm{ad}\ \mathrm{e}_{-\alpha})\exp(\mathrm{ad}\ \mathrm{e}_\alpha)$. Then $g_\alpha(e_\beta) = e_{s_\alpha(\beta)}$ and $g_\alpha([X, Y]) = [g_\alpha(X), g_\alpha(Y)]$. So the definition is independent of the pull-back. □

Proposition 24 *The exceptional classes define a natural irreducible integrable representation bundle over* X_9*:*

$$\mathcal{W} = \bigoplus_{\xi \in I_9} \mathcal{O}(\xi).$$

Proof The action is defined as: $e^U_\alpha(e^U_\xi) = e^U_{\alpha+\xi}$ if $\alpha + \xi \in I_9$; $e^U_\alpha(e^U_\xi) = 0$ otherwise. And define $h_\alpha(e^U_\xi) = (\alpha \cdot \xi)e^U_\xi$. Here $h_\alpha, e^U_\alpha, e^U_\xi$ are respectively local bases of $(\mathcal{O}^{\oplus 8} \oplus \mathcal{O})$, $\mathcal{O}(\alpha)$ and $\mathcal{O}(\xi)$ on a trivialization open subset U. One can easily check that this action is independent of choices of trivialization open subsets. And l_9 is the highest weight vector with the highest weight $\lambda := -\langle \cdot, l_9 \rangle$ associated to the root basis $\{-\alpha_1, \ldots, -\alpha_8\}$, where $\alpha_1 = h - l_1 - l_2 - l_3$, $\alpha_i = l_{i-1} - l_i$, $i = 2, \ldots, 8$. □

2.2 Lie($\widetilde{\boldsymbol{E}}_{\boldsymbol{k}}$)-bundles over rational surfaces with an A_d-singularity

Rational double points in surfaces are classified according to their dual graphs defined as follows. Given a rational double point singularity, its canonical resolution consists of a tree of (-2)-curves (i.e. rational curves of self-intersection number -2). The dual graph is defined as the dual graph of this tree with respect to the intersection form. Such graphs are always the Dynkin diagrams of ADE-type. For example, the canonical resolution of an A_d-singularity is a chain of (-2)-curves of length d, we simply call it an A_d-chain of (-2)-curves. Its dual graph is illustrated in Fig. 1.

On the other hand, contracting an A_d-chain of (-2)-curves always gives an A_d-singularity. Denote $\widetilde{X}_{k,d}$ ($k = 8 - d$) the blow-up of $\mathbb{P}^2$ at 9 (possibly infinitely near) points with an A_d-chain of (-2)-curves. Then we can contract this A_d-chain to an A_d-singularity. Let $X_{k,d}$ ($k = 8-d$) be the resulting surface. Then $\widetilde{X}_{k,d}$ is the canonical resolution of $X_{k,d}$. On $X_{k,d}$, the set $\Delta^{re}(\widetilde{E}_k) := \{\alpha \in Pic(X_{k,d}) \,|\, \alpha \cdot K = 0, \alpha^2 = -2\}$ is a real root system of $\widetilde{E}_k$-type, and $\Delta^{im}(\widetilde{E}_k) := \{mK | m \in \mathbb{Z}\backslash\{0\}\}$ is the imaginary root system with the null root $-K$, according to Theorem 20.

Thus similar to the $d = 0$ case, we can construct an affine Lie algebra bundle of $\widetilde{E}_k$-type over $X_{k,d}$ (or $\widetilde{X}_{k,d}$, equivalently) as follows:

$$\widetilde{\mathscr{E}}_k = (\mathcal{O}^{\oplus k} \oplus \mathcal{O}) \oplus \bigoplus_{\alpha \in \Delta^{re}(\widetilde{E}_k)} \mathcal{O}(\alpha) \bigoplus_{\beta \in \Delta^{im}(\widetilde{E}_k)} \mathcal{O}(\beta)^{\oplus k}.$$

For a generic $\widetilde{X}_{k,d}$, if it has an exceptional configuration $(l_1, \dots, l_9)$ such that $(l_{k+1} - l_{k+2}), \dots, (l_8 - l_9)$ form the unique A_d-chain of (-2)-curves, and the linear system $|-K_{\widetilde{X}_{k,d}}|$ consists of a unique element, we call such an $\widetilde{X}_{k,d}$ *general*. In this article, a general $\widetilde{X}_{k,d}$ is also called *a del Pezzo surface of degree 0 with an* A_d*-singularity*. Such a configuration is called an $\widetilde{E}_k$-configuration. When $d = 0$, a general $\widetilde{X}_{8,0}$ with smooth anti-canonical curve is just a blow-up of $\mathbb{P}^2$ at 9 points in general position, and each exceptional configuration is an $\widetilde{E}_8$-configuration.

In the following, we show that such surfaces are related to *d-jets*.

Let X be a surface. Let $J_d \to X_k$ be the fiber bundle of d-jets of germs of parameterized curves in X, that is, the set of equivalence classes of holomorphic maps $f : (\mathbb{C}, 0) \to (X, p)$, with the equivalence relation $f \sim g$ given by all derivatives $f^{(j)}(0) = g^{(j)}(0)$ coinciding for $0 \le j \le d$, when computed in some local coordinate systems of $\mathbb{C}$ near 0. The projection map $J_d \to X$ is simply $f \mapsto f(0)$.

Let $p^{[d]}$ be a d-jet at $p \in X$, namely an element in the fiber of J_d over p. Then $p^{[d]}$ is defined by the equivalence class of a holomorphic map f. Thus we have $d+1$ derivatives $f^{(j)}(0)$, $j = 0, \dots, d$. We define the *blow-up of X at a d-jet* $p^{[d]}$ as the surface $\widetilde{X} := X^{(d+1)}$ obtained by the following procedure. We first blow up X at the point p to obtain $X^{(1)}$ with the exceptional curve $E^{(1)}$. Note that $f^{(1)}(0)$ corresponds to a unique point $p^{(1)} \in E^{(1)}$. Then we blow up $X^{(1)}$ at $p^{(1)}$ to obtain $X^{(2)}$ with the exceptional curve $E^{(2)}$. And $f^{(2)}(0)$ corresponds to a unique point $p^{(2)} \in E^{(2)}$. Thus inductively we obtain a unique surface $X^{(d+1)}$. Note that the construction of the blow-up at a jet is independent of the choice of the function f.

Therefore, blowing up X at $p^{[d]}$, we obtain an $\widetilde{X}$ with an exceptional configuration $(e_1, \dots, e_{d+1})$ determined by $p^{[d]}$, such that $e_1 - e_2, \dots, e_d - e_{d+1}$ form an A_d-chain of (-2)-classes.

For a d-jet $p^{[d]}$ on a del Pezzo surface X_k, if there is a unique anti-canonical curve passing through $p^{[d]}$ and in the blow-up of X_k at $p^{[d]}$ every irreducible curve which is not contained in the above A_d-chain has a self-intersection number at least -1, then we call such a $p^{[d]}$ *general*.

We say that two pairs $(X_k, p^{[d]})$ and $(X'_k, p'^{[d]})$ are *equivalent to each other* if there is an isomorphism $\phi : \widetilde{X}_k \to \widetilde{X}'_k$, where $\widetilde{X}_k$ (respectively $\widetilde{X}'_k$) is the blow-up of X_k (resp. X'_k) at $p^{[d]}$ (resp. $p'^{[d]}$), such that the exceptional configuration determined by the d-jet on $\widetilde{X}_k$ corresponds to that on $\widetilde{X}'_k$ under ϕ.

Proposition 25 *The isomorphism classes of general* $\widetilde{X}_{k,d}$ *with* $\widetilde{E}_k$*-configuration are in one-to-one correspondence with the equivalence classes of the pair* $(X_k, p^{[d]})$ *with* $p^{[d]}$ *general.*

Proof First we consider the case $d = 0$. In this case, a general $\widetilde{X}_{8,0}$ is obtained by blowing up $\mathbb{P}^2$ at 9 points $x_1, \dots, x_9$ in general position. Blowing up any eight points $x_{i_1}, \dots, x_{i_8}$ we obtain a del Pezzo surface X_8. Then take $p^{[0]} = p = x_{i_9}$. Then we obtain a pair $(X_8, p^{[0]})$ with $p^{[0]}$ general. On the other hand, for a pair $(X_8, p^{[0]})$, blowing up X_8 at p, we obtain a general $\widetilde{X}_{8,0}$.

For any d, the surface $\widetilde{X}_{k,d}$ is obtained by blowing up $\mathbb{P}^2$ at 9 points $x_1, \dots, x_9$ where $x_1, \dots, x_{k+1}$ are in general position and $x_{k+2}, \dots, x_9$ are infinitely near to x_{k+1}. Then $(x_{k+1}, \dots, x_9)$ determines a d-jet $p^{[d]}$ on X_k where $p = x_{k+1}$, and X_k is the blow-up of $\mathbb{P}^2$ at $x_1, \ldots, x_k$. $\widetilde{X}_{k,d}$ is general implies that $p^{[d]}$ is general. Conversely, given a pair $(X_k, p^{[d]})$, where X_k is a del Pezzo surface of degree $d+1$ obtained by blowing up $\mathbb{P}^2$ at $x_1, \ldots, x_k$. The condition that $p^{[d]}$ is general implies that the blow-up of X_k at $p^{[d]}$ is obtained by blowing up X_k at the same point $p \in C(d+1)$ times, where C is the unique anti-canonical curve passing through $p^{[d]}$. Thus we obtain a unique A_d-chain of (-2)-curves which are $l_{k+1} - l_{k+2}, \ldots, l_8 - l_9$ for an exceptional configuration $(l_1, \ldots, l_9)$. Then we obtain a general $\widetilde{X}_{k,d}$. □

3 Lie($\widetilde{E}_k$)-bundles over elliptic curves

Let Σ be a smooth anti-canonical curve in $\widetilde{X}_{k,d}$. The restriction $\widetilde{\mathscr{E}}_k|_\Sigma$ is an infinite direct sum of holomorphic line bundles of degree 0 on Σ, which we call a *$Lie(\widetilde{E}_k)$-bundle* over Σ. More precisely,

Definition 26 For $k \le 8$, a vector bundle $\mathscr{V}$ of infinite rank over Σ is called a *holomorphic $Lie(\widetilde{E}_k)$-bundle*, if it satisfies the following two conditions:

(1) It is completely split into a direct sum of line bundles of degree zero.
(2) It has a fiberwise affine E_k Lie algebra structure, and this structure is compatible with any trivialization.

Two holomorphic $Lie(\widetilde{E}_k)$-bundles $\mathscr{V}$ and $\mathscr{V}'$ are said to be *isomorphic* to each other, if they are isomorphic as vector bundles, and this isomorphism preserves the fiberwise Lie algebra structure.

By definition, once we choose a root basis, we can write a holomorphic $Lie(\widetilde{E}_k)$-bundle $\mathscr{V}$ uniquely up to isomorphisms as

$$\mathscr{V} := \mathcal{O}^{\oplus k} \oplus \mathcal{O} \oplus \bigoplus_{\alpha \in \Delta^{re}} \mathcal{L}_\alpha \oplus \bigoplus_{\beta \in \Delta^{im}} \mathcal{L}_\beta^{\oplus k},$$

where the fiberwise Lie bracket is defined as usual.

Let $\mathcal{M}_\Sigma^{\widetilde{E}_k}$ be the moduli space of holomorphic $Lie(\widetilde{E}_k)$-bundles over Σ up to isomorphism defined as above. Let $\Lambda(\widetilde{E}_k)$ be the root lattice of $\widetilde{E}_k$.

Lemma 27 [14] *The group $\{\pm 1\} \times W(\widetilde{E}_k)$ acts on the set of all root bases of the root lattice $\Lambda(\widetilde{E}_k)$ simply transitively.*

Proposition 28 *Fix $0 \in \Sigma$. Then $\mathcal{M}_\Sigma^{\widetilde{E}_k} \cong Hom(\Lambda(\widetilde{E}_k), \Sigma)/(\{\pm 1\} \times W(\widetilde{E}_k))$.*

Proof A holomorphic $Lie(\widetilde{E}_k)$-bundle over Σ is uniquely determined by its direct summands $\mathcal{L}_{\alpha_i}$ and $\mathcal{L}_\delta$, where $\alpha_i, i = 1, \ldots, k$ are the simple roots and δ is the null root. We have a map $\Lambda(\widetilde{E}_k) \to Jac(\Sigma) \cong \Sigma$ defined by $\alpha \mapsto \mathcal{L}_\alpha$. This is obviously an abelian group homomorphism. And two different root bases differ by an action of a unique element of $\{\pm 1\} \times W(\widetilde{E}_k)$, according to Lemma 27. Thus the result follows. □

Let $T(\widetilde{E}_k)$ be a maximal torus of $\widetilde{E}_k$.

Lemma 29

$$Hom(\pi_1(\Sigma), T(\widetilde{E}_k)) \cong Hom(\Lambda(\widetilde{E}_k), \Sigma).$$

Proof Given a root lattice Λ, we denote the coroot lattice of Λ by Λ_c. Then $Hom(\pi_1(\Sigma), T(\widetilde{E}_k)) = Hom(\pi_1(\Sigma), U(1) \otimes \Lambda_c(\widetilde{E}_k)) \cong Hom(\pi_1(\Sigma), U(1)) \otimes \Lambda_c(\widetilde{E}_k)$ and $Hom(\pi_1(\Sigma), U(1)) \cong Jac(\Sigma) \cong \Sigma$. □

Remark 30 The moduli space of flat G-bundles over Σ is isomorphic to

$$Hom(\pi_1(\Sigma), G)/ad(G).$$

In the finite dimensional case, $Hom(\pi_1(\Sigma), G)/ad(G) \cong Hom(\pi_1(\Sigma), T)/W$, since a commuting pair of elements of G can be diagonalized simultaneously [3]. But it is not true when G is of infinite dimension. By Lemma 29

$$Hom(\pi_1(\Sigma), T(\widetilde{E}_k))/(\{\pm 1\} \times W(\widetilde{E}_k)) \cong Hom(\Lambda(\widetilde{E}_k), \Sigma)/(\{\pm 1\} \times W(\widetilde{E}_k)).$$

Thus the $Lie(\widetilde{E}_k)$-bundles considered by us are in fact those flat bundles such that the image of $\pi_1(\Sigma)$ can be diagonalized.

4 The correspondence between moduli spaces

In this section, we establish the correspondence between del Pezzo surfaces of degree 0 with A_d-singularity containing an anti-canonical curve isomorphic to Σ and Kac–Moody $Lie(\widetilde{E}_k)$-bundles over Σ.

Let S be a smooth rational surface. An $\widetilde{E}_k$-*configuration* on S is an exceptional system $(e_1, \ldots, e_9)$ such that we can consider S as a blow-up of $\mathbb{P}^2$ at 9 points in turn with corresponding exceptional classes $e_1, \ldots, e_9$, and $e_{k+1} - e_{k+2}, \ldots, e_8 - e_9$ form an A_d-chain ($d + k = 8$ and $0 \le d \le 5$) of irreducible (-2)-curves.

Let Σ be a fixed smooth anti-canonical curve in S, and $\mathcal{S}_{\Sigma,k}$ be the moduli space of equivalence classes of general rational surfaces $\widetilde{X}_{k,d}$ with $\widetilde{E}_k$-configuration containing Σ as an anti-canonical curve. Here two surfaces $\widetilde{X}_{k,d}$ and $\widetilde{X}'_{k,d}$ are said to be equivalent to each other, if there is an isomorphism $F : \widetilde{X}_{k,d} \xrightarrow{\sim} \widetilde{X}'_{k,d}$, such that $\iota' = F \circ \iota$, where ι (resp. ι') is the embedding of Σ into $\widetilde{X}_{k,d}$ (resp. $\widetilde{X}'_{k,d}$). Fix the identity $0 \in \Sigma$ to be an inflection point with respect to the embedding $\Sigma \hookrightarrow \mathbb{P}^2$ induced by a blow-up.

By restriction of the affine Lie algebra bundle $\widetilde{\mathscr{E}}_k$ over Σ, we obtain a holomorphic $Lie(\widetilde{E}_k)$-bundle over $\Sigma \subset S$ which is an infinite direct sum of line bundles of degree 0 over Σ. The restriction gives us a map $\Phi : \mathcal{S}_{\Sigma,k} \to Hom(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k)$. Fixing a cyclic level subgroup of order $3(d+1)$ of Σ, then we have

Theorem 31 (i) *There is a natural embedding*

$$\Phi : \mathcal{S}_{\Sigma,k} \hookrightarrow Hom(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k),$$

given by the restriction of $\widetilde{\mathscr{E}}_k$ *from* $\widetilde{X}_{k,d}$ *to* Σ.

(ii) *Φ extends to an identification $\overline{\Phi} : \overline{\mathcal{S}}_{\Sigma,k} \cong Hom(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k)$, where $\overline{\mathcal{S}}_{\Sigma,k}$ is a natural compactification, by adjoining equivalence classes of all rational surfaces S with $\widetilde{E}_k$-configuration such that $\Sigma \in |-K_S|$.*

(iii) *Each curve in a surface corresponding to a point on $\overline{\mathcal{S}}_{\Sigma,k} \backslash \mathcal{S}_{\Sigma,k}$ has a self-intersection number at least -2.*

Proof (i) Let $X_{k,d} \in \mathcal{S}_{\Sigma,k}$. Then $X_{k,d}$ is a blow-up of $\mathbb{P}^2$ at 9 points $x_1, \ldots, x_9$ with corresponding exceptional classes $l_1, \ldots, l_9$ such that $\beta_1 = l_{k+1} - l_{k+2}, \ldots, \beta_d = l_8 - l_9$ are irreducible (-2)-curves which form an A_d-chain. Then $\{K, \beta_1, \ldots, \beta_d\}^{\perp} \subset P_9$ is a root lattice of affine E_k-type with simple roots $\alpha_1 = l_1 - l_2, \ldots, \alpha_k = l_{k-1} - l_k$ and the null root $\delta = -K$.

Thus we can define a map $f_S : \Lambda(\widetilde{E}_k) \to \Sigma$ as follows. First we have a map $f'_S : \Lambda(\widetilde{E}_k) \to Jac(\Sigma)$. Let $f'_S(x) = \mathcal{O}(x)|_{\Sigma}$. Since $\Sigma \in |-K_S|$, $deg(\mathcal{O}(x)|_{\Sigma}) = x \cdot \Sigma = 0$. Then $\mathcal{O}(x)|_{\Sigma} \in Jac(\Sigma)$. As we have fixed $0 \in \Sigma$, we have an isomorphism $\psi : Jac(\Sigma) \cong \Sigma$. Let $f_S = \psi \circ f'_S : \Lambda(\widetilde{E}_k) \to \Sigma$. This map is a homomorphism of abelian groups: $f_S(x+y) = \psi(\mathcal{O}(x+y)|_{\Sigma}) = \psi(\mathcal{O}(x)|_{\Sigma}) + \psi(\mathcal{O}(y)|_{\Sigma}) = f_S(x) + f_S(y)$. By Lemma 33, given any two $\widetilde{E}_k$-configurations ζ, ζ', there is a unique element ρ of $W(\widetilde{E}_k)$, such that $\zeta = \rho(\zeta')$. Hence we have a well-defined map $\Phi : \mathcal{S}_{\Sigma,k} \to Hom(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k)$ defined by $S \mapsto f_S$. Next we prove the map Φ is injective. Φ is defined by a system of linear equations on Σ:

$$\begin{cases} -x_1 - x_2 - x_3 = f_S(\alpha_1), \\ x_1 - x_2 = f_S(\alpha_2), \\ \ldots \\ x_{k-1} - x_k = f_S(\alpha_k), \\ x_{k+1} = \cdots = x_9, \\ -x_1 - \cdots - x_9 = f_S(\delta). \end{cases}$$

This system of linear equations has a unique solution, since its determinant is of absolute value $3(d+1)$, and the identity of Σ is taken as a inflection point on $\Sigma \in |-K_S|$. Thus Φ is injective, by Lemma 33.

(ii) Given $f \in Hom(\Lambda(\widetilde{E}_k), \Sigma)$, we embed Σ into $\mathbb{P}^2$ as an anti-canonical curve such that 0 is an inflection point. The above system of linear equations has a solution $x_1, \ldots, x_k, x_{k+1} = \cdots = x_9 \in \Sigma \in |-K_{\mathbb{P}^2}|$. Note that $x_{k+1} = \cdots = x_9 \in \Sigma$ determines a d-jet $p^{[d]}$ with $p = x_{k+1}$ and $d = 8 - k$. Blowing up $x_1, \ldots, x_k, p^{[d]}$, we obtain a rational surface S with $\widetilde{E}_k$-configuration such that $f = f_S$.

(iii) Let $S \in \overline{\mathcal{S}}_{\Sigma,k}$. Then $\Sigma \in |-K_S|$. For any irreducible curve C in S, by the genus formula, $g(C) = 1 + 1/2(C^2 + CK)$. Then $C^2 \geq -2 + C\Sigma \geq -2$ since $C\Sigma \geq 0$. □

Since $\mathcal{M}_{\Sigma}^{\widetilde{E}_k} \cong Hom(\Lambda(\widetilde{E}_k), \Sigma)/(\{\pm 1\} \times W(\widetilde{E}_k))$ by Proposition 28, we have the following immediate corollary.

Corollary 32 *There is a covering map of degree 2 from $\overline{\mathcal{S}}_{\Sigma,k}$ onto $\mathcal{M}_{\Sigma}^{\widetilde{E}_k}$.*

It remains to prove the following lemma which is needed in the above proof.

Lemma 33 *Fix a general $\widetilde{X}_{k,d}$. The Weyl group $W(\widetilde{E}_k)$ acts on the set of $\widetilde{E}_k$-configurations simply transitively.*

Proof Denote $\Pi(\widetilde{E}_k) = \{\alpha_0, \ldots, \alpha_k\}$ the extended root basis of $\widetilde{E}_k$ with α_0 the extended root.

For $k = 8$, see Corollary 12 and Proposition 8.

For $k = 7$, let $\zeta = (l_1, \ldots, l_9)$ be an $\widetilde{E}_7$-configuration. Then $\Pi(\widetilde{E}_7) = \{h - l_1 - l_8 - l_9, l_i - l_{i+1} (i = 1, \ldots, 6), h - l_1 - l_2 - l_3\}$. We first verify that for any generator $s_\alpha \in W(\widetilde{E}_7)$, $s_\alpha(\zeta)$ is still an $\widetilde{E}_7$-configuration. $W(\widetilde{E}_7)$ is generated by s_α where α can be taken as $l_i - l_{i+1}$, $i = 1, \ldots, 6$ and $h - l_1 - l_2 - l_3$, $h - l_1 - l_8 - l_9$. It suffices to verify it for $\alpha_0 = h - l_1 - l_8 - l_9$. Then $s_\alpha(\zeta) = (h - l_8 - l_9, l_2, \ldots, l_7, h - l_1 - l_9, h - l_1 - l_8)$. It is easy to see that this is an $\widetilde{E}_7$-configuration. Thus the simplicity follows since $W(\widetilde{E}_7)$ is a subgroup of $W(\widetilde{E}_8)$. For any two $\widetilde{E}_7$-configurations $\zeta = (l_1, \ldots, l_9)$ and $\zeta' = (l'_1, \ldots, l'_9)$, we have a unique $\rho \in W(\widetilde{E}_8)$ such that $\zeta' = \rho(\zeta)$. We must have $\rho \in W(\widetilde{E}_7)$ since $W(\widetilde{E}_7)$ is the subgroup of $W(\widetilde{E}_8)$ fixing the (-2)-curve $l_8 - l_9$ (ρ maps a root basis of $\widetilde{E}_7$ to another root basis of $\widetilde{E}_7$ and fixes δ).

For other k, we only need to verify that for the root α_0 associated to the extended node, $s_{\alpha_0}(\zeta)$ is still an $\widetilde{E}_k$-configuration if so is ζ, since the remaining arguments are the same.

For $k = 6$, let $\zeta = (l_1, \ldots, l_9)$ be an $\widetilde{E}_6$-configuration. Then $\Pi(\widetilde{E}_6) = \{h - l_7 - l_8 - l_9, l_i - l_{i+1} (i = 1, \ldots, 5), h - l_1 - l_2 - l_3\}$. Let $\alpha_0 = h - l_7 - l_8 - l_9$. Then $s_{\alpha_0}(\zeta) = (l_1, \ldots, l_6, h - l_8 - l_9, h - l_7 - l_9, h - l_7 - l_8)$. This is again an $\widetilde{E}_6$-configuration since $h - l_7 - l_8$ is an irreducible (-1)-curve.

For $k = 5$, let $\zeta = (l_1, \ldots, l_9)$ be an $\widetilde{E}_5$-configuration. Then $\Pi(\widetilde{E}_5) = \{2h - l_1 - l_2 - l_6 - l_7 - l_8 - l_9, l_i - l_{i+1} (i = 1, \ldots, 4), h - l_1 - l_2 - l_3\}$. Let $\alpha_0 = 2h - l_1 - l_2 - l_6 - l_7 - l_8 - l_9$. Then $s_{\alpha_0}(\zeta) = (e_1, \ldots, e_9)$ with $e_1 = 2h - l_2 - l_6 - l_7 - l_8 - l_9$, $e_2 = 2h - l_1 - l_6 - l_7 - l_8 - l_9$, $e_3 = l_3$, $e_4 = l_4$, $e_5 = l_5$, $e_6 = 2h - l_1 - l_2 - l_7 - l_8 - l_9$, $e_7 = 2h - l_1 - l_2 - l_6 - l_8 - l_9$, $e_8 = 2h - l_1 - l_2 - l_6 - l_7 - l_9$, $e_9 = 2h - l_1 - l_2 - l_6 - l_7 - l_8$. This is again an $\widetilde{E}_5 = \widetilde{D}_5$-configuration.

For $k = 4$, let $\zeta = (l_1, \ldots, l_9)$ be an $\widetilde{E}_4$-configuration. Then $\Pi(\widetilde{E}_4) = \{2h - l_1 - l_5 - l_6 - l_7 - l_8 - l_9, l_i - l_{i+1} (i = 1, \ldots, 3), h - l_1 - l_2 - l_3\}$. Let $\alpha_0 = 2h - l_1 - l_5 - l_6 - l_7 - l_8 - l_9$. Then $s_{\alpha_0}(\zeta) = (e_1, \ldots, e_9)$ with $e_1 = 2h - l_5 - l_6 - l_7 - l_8 - l_9$, $e_2 = l_2$, $e_3 = l_3$, $e_4 = l_4$, $e_5 = 2h - l_1 - l_6 - l_7 - l_8 - l_9$, $e_6 = 2h - l_1 - l_5 - l_7 - l_8 - l_9$, $e_7 = 2h - l_1 - l_5 - l_6 - l_8 - l_9$, $e_8 = 2h - l_1 - l_5 - l_6 - l_7 - l_9$, $e_9 = 2h - l_1 - l_5 - l_6 - l_7 - l_8$. This is again an $\widetilde{E}_4 = \widetilde{A}_4$-configuration.

For $k = 3$, let $\zeta = (l_1, \ldots, l_9)$ be an $\widetilde{E}_3$-configuration. Then $\Pi(\widetilde{E}_3) = \{2h - l_4 - l_5 - l_6 - l_7 - l_8 - l_9, l_1 - l_2, l_2 - l_3, h - l_1 - l_2 - l_3\}$. Let $\alpha_0 = 2h - l_4 - l_5 - l_6 - l_7 - l_8 - l_9$. Then $s_{\alpha_0}(\zeta) = (e_1, \ldots, e_9)$ with $e_1 = l_1$, $e_2 = l_2$, $e_3 = l_3$, $e_j = 2h + l_j - \sum_{i=4}^{9} l_i$, $j = 4, \ldots, 9$. This is still an $\widetilde{E}_3 = \widetilde{A}_1 \times A_2$-configuration. □

Remark 34 Let k, d be arbitrary integers between 0 and 8 without requiring that $k + d = 8$. Let $S = S_{k,d}$ be a rational surface with degree $K_S^2 = 9 - k$ and an A_d-chain of (-2)-curves. Then we have a $\mathfrak{g}_{k,d}$-Lie algebra bundle over S, where $\mathfrak{g}_{k,d}$ is defined as follows: for $k = 9$, $\mathfrak{g}_{k,d} = Lie(\widetilde{E}_{8-d})$, which is of affine type; for $k \leq 8$, $\mathfrak{g}_{k,d}$ is the Lie algebra of the group appearing at the corresponding position in Julia's magic triangle (see [10,11]), which is of finite type. If we fix a smooth elliptic curve $\Sigma \in |-K_S|$, by restriction, we obtain a holomorphic $\mathfrak{g}_{k,d}$ bundle over Σ. The following table (Table 1) is the magic triangle, which has a symmetry across the diagonal. Starting from E_8, the rows are obtained by contracting A_d-chains of (-2)-curves, and the columns are obtained by blowing down (-1)-curves successively.

5 The correspondence in the strictly semi-general case and comparing with the moduli space of flat E_k-bundles

In this section, we study how the correspondence Φ extends to $\overline{\mathcal{S}}_{\Sigma,k} \backslash \mathcal{S}_{\Sigma,k}$. We find in the strictly semi-general case (denote by $\mathcal{S}'_{\Sigma,k}$ the corresponding moduli space), the correspondence Φ extends to a one-to-one correspondence from $\mathcal{S}'_{\Sigma,k}$ to flat E_k-bundles over Σ. In fact, this case is related to rational elliptic surfaces.

The surface corresponding to a point on $\mathcal{S}'_{\Sigma,k}$ admits an elliptic fibration structure. This happens precisely when X_9 is the blow-up of $\mathbb{P}^2$ at 9 points which are the intersection points of two cubic curves in $\mathbb{P}^2$. In this case, below we will show that $\widetilde{\mathscr{E}}_k|_\Sigma \cong \mathscr{E}_k|_\Sigma \otimes \mathbb{C}[t, t^{-1}] \oplus \mathcal{O}_\Sigma$. This suggests that the restriction should produce an identification between the moduli space of such surfaces with an anti-canonical curve Σ, and the moduli space of flat E_k-bundles over Σ.

Let $0 \le d \le 5$ and $k = 8 - d$. Recall we denote $\widetilde{X}_{k,d}$ the blow-up of $\mathbb{P}^2$ at 9 (possibly infinitely near) points $x_1, \dots, x_9$ with an A_d-chain of (-2)-curves. We say that $\widetilde{X}_{k,d}$ is *strictly semi-general* if it has an exceptional configuration $(l_1, \dots, l_9)$ such that $(l_{k+1} - l_{k+2}), \dots, (l_8 - l_9)$ form the unique A_d-chain of (-2)-curves, and the linear system $|-K_{\widetilde{X}_{k,d}}|$ defines a rational elliptic fibration. Such a surface is also called a *del Pezzo surface*. Such an exceptional configuration is still called an $\widetilde{E}_k$*-configuration* as before. Note that when $d = 0$, a strictly semi-general $\widetilde{X}_{8,0}$ is just a rational elliptic fibration which is a blow-up of $\mathbb{P}^2$ at 9 points *in strictly semi-general position* (Definition 10). Fix any smooth fiber and denote it by Σ. Fix an inflection point on Σ as the origin.

Lemma 35 *For a strictly semi-general $\widetilde{X}_{k,d}$, the Weyl group $W(\widetilde{E}_k)$ acts on the set of $\widetilde{E}_k$-configurations simply transitively.*

Proof The proof is the same as that of Lemma 33. □

Let $Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$ be the subgroup of $Hom(\Lambda(\widetilde{E}_k), \Sigma)$ consisting of the homomorphisms which map the null root δ to $0 \in \Sigma$. Recall that $\widetilde{\mathscr{E}}_k$ is the $Lie(\widetilde{E}_k)$-algebra bundle over $\widetilde{X}_{k,d}$.

Proposition 36 (i) *The restriction of $\mathcal{O}(K_S)$ on $\Sigma \in |-K_S|$ is trivial and $\sum x_i = 0$ on Σ.*

(ii) *Let $\oplus_{\mathbb{Z}}\mathcal{O}_\Sigma$ be the infinite direct sum of the same bundle $\mathcal{O}_\Sigma$ where the summands are indexed by $\mathbb{Z}$. Then*

$$\widetilde{\mathscr{E}}_k|_\Sigma \cong \mathscr{E}_k|_\Sigma \otimes (\oplus_{\mathbb{Z}}\mathcal{O}_\Sigma) \oplus \mathcal{O}_\Sigma \cong \mathscr{E}_k|_\Sigma \otimes \mathbb{C}[t, t^{-1}] \oplus \mathcal{O}_\Sigma,$$

where $\mathbb{C}[t, t^{-1}]$ is the Laurent polynomial ring.

(iii) $Hom_0(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k) \cong Hom(\Lambda(E_k), \Sigma)/W(E_k)$.

Proof (i) This is trivial since we choose an inflection point as the origin of Σ.

(ii) It follows from (i).

(iii) Firstly, we show that the subgroup $Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$ is stable under the action of $W(\widetilde{E}_k)$. Take any generator $s_{\alpha+n\delta} \in W(\widetilde{E}_k)$. For any $f \in Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$, $s_{\alpha+n\delta}(f)(\delta) = f(s_{\alpha+n\delta}(\delta)) = f(\delta) = 0$. Secondly, note that $\Lambda(\widetilde{E}_k) = \mathbb{Z}\langle\alpha_1, \ldots, \alpha_k, \delta\rangle$ and $\Lambda(E_k) = \mathbb{Z}\langle\alpha_1, \ldots, \alpha_k\rangle$. For $f \in Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$, f is uniquely determined by the $f(\alpha_i)$ and $f(\delta) = 0$. Thus we obtain a $\phi(f) \in Hom(\Lambda(E_k), \Sigma)$ defined by $\phi(f)(\alpha_i) := f(\alpha_i)$. Obviously, ϕ defines an isomorphism

$$\phi : Hom_0(\Lambda(\widetilde{E}_k), \Sigma) \cong Hom(\Lambda(E_k), \Sigma).$$

Lastly, we show that for any two $f, g \in Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$, there is $\rho \in W(\widetilde{E}_k)$ such that $f = \rho(g)$ if and only if there exists a $\rho' \in W(E_k)$ such that $\phi(f) = \rho'(\phi(g))$. Suppose $f = \rho(g)$. Without loss of generality, we assume $\rho = s_{\alpha+n\delta}$ where α is a root of E_k. Then we take $\rho' = s_\alpha$. Thus $\rho'(\phi(g))(x) = \phi(g)(\rho'(x)) = \phi(g)(x + (\alpha, x)\alpha) = \phi(g)(x) + (x, \alpha)\phi(g)(\alpha) = g(x) + (x, \alpha)g(\alpha)$. On the other hand, $f = \rho(g)$ implies that $\phi(f)(x) = \phi(\rho(g))(x) = \rho(g)(x) = g(\rho(x)) = g(x+(x, \alpha+n\delta)(\alpha+n\delta)) = g(x) + (x, \alpha)g(\alpha)$ since $g(\delta) = 0$ and $(x, \delta) = 0$ (because $\delta = -K_9$). Hence there exists a $\rho' \in W(E_k)$ such that $\phi(f) = \rho'(\phi(g))$. The other direction is obvious, since $W(E_k)$ is a subgroup of $W(\widetilde{E}_k)$. So ϕ induces a bijection

$$\overline{\phi} : Hom_0(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k) \cong Hom(\Lambda(E_k), \Sigma)/W(E_k).$$

□

Lemma 37 $Hom_0(\Lambda(\widetilde{E}_k), \Sigma)/(\{\pm 1\} \times W(\widetilde{E}_k)) = Hom_0(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k)$.

Proof $\Lambda(\widetilde{E}_k) = \mathbb{Z}\langle\alpha_1, \ldots, \alpha_k, \delta\rangle$. Since $\{\alpha_1, \ldots, \alpha_k\}$ is a root basis of finite type E_k, there is a $\rho \in W(E_k)$ such that $\rho(\alpha_i) = -\alpha_i, i = 1, \ldots, k$. Then ρ acts on $Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$ as -1. We verify this statement. For any $f \in Hom_0(\Lambda(\widetilde{E}_k), \Sigma), \rho(f)(\alpha_i) = f(\rho^{-1}(\alpha_i)) = f(-\alpha_i) = -f(\alpha_i)$. And $\rho(f)(\delta) = f(\rho^{-1}(\delta)) = f(\delta) = 0 = -f(\delta)$. Thus $\rho(f) = -f$. □

Let $\mathcal{S}'_{\Sigma,k}$ be the moduli space of strictly semi-general surfaces $\widetilde{X}_{k,d}$ with $\widetilde{E}_k$-configuration containing an anti-canonical curve isomorphic to Σ. Note that $\mathcal{S}'_{\Sigma,k}$ is contained in $\overline{\mathcal{S}}_{\Sigma,k} \backslash \mathcal{S}_{\Sigma,k}$. As usual, let $\mathcal{M}_\Sigma^{E_k}$ be the moduli space of flat E_k bundles over Σ. Fix a cyclic level $(9-k)$ subgroup of Σ. It is known that [17]

$$\mathcal{M}_\Sigma^{E_k} \cong Hom(\Lambda(E_k), \Sigma)/W(E_k).$$

The restriction of the $Lie(\widetilde{E}_k)$-bundle over $\widetilde{X}_{k,d}$ to $\Sigma \subset \widetilde{X}_{k,d}$ induces a map $\Psi : \mathcal{S}'_{\Sigma,k} \to \mathcal{M}_\Sigma^{E_k}$.

Theorem 38 (i) *The restriction map* Ψ *induces an open dense embedding from* $\mathcal{S}'_{\Sigma,k}$ *into* $\mathcal{M}_{\Sigma}^{E_k}$.

(ii) *Moreover the embedding can be extended to a bijection* $\overline{\Psi}$ *from the natural compactification* $\overline{\mathcal{S}'}_{\Sigma,k}$ *onto* $\mathcal{M}_{\Sigma}^{E_k}$.

(iii) *Each irreducible curve in a surface corresponding to a point on* $\overline{\mathcal{S}'}_{\Sigma,k}$ *has a self-intersection number at least* -2.

Proof (i) Given an element in $\mathcal{S}'_{\Sigma,k}$. We have a rational elliptic fibration S which is a blow-up of $\mathbb{P}^2$ at 9 points $x_1, \ldots, x_9 \in \Sigma$ where $\Sigma \in |-K_{\mathbb{P}^2}|$ and $x_1 + \cdots + x_9 = 0$, such that $l_1, \ldots, l_9$ are the corresponding exceptional classes. The restriction determines a map $f'_S : \Lambda(\widetilde{E}_k) \to Jac(\Sigma)$, which is defined as $L \mapsto \mathcal{O}(L)|_\Sigma$. Since $deg(\mathcal{O}(L)|_\Sigma) = L \cdot \Sigma = L \cdot (-K) = 0$, we have $\mathcal{O}(L)|_\Sigma \in Jac(\Sigma)$ when $L \in K^\perp$. It is a homomorphism of abelian groups. And $f'_S(\delta) = \mathcal{O}_\Sigma(-K) = \mathcal{O}_\Sigma$. As we have fixed $0 \in \Sigma$, we have an isomorphism $J : Jac(\Sigma) \cong \Sigma$. Define $f_S = J \circ f'_S$. Then $f_S \in Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$ since $f_S(\delta) = J(\mathcal{O}_\Sigma) = 0$. Two exceptional configurations on S differ by an element of the Weyl group $W(\widetilde{E}_k)$, by Lemma 35. So we obtain a well-defined map $\Psi : \mathcal{S}'_{\Sigma,k} \to Hom_0(\Lambda(\widetilde{E}_k), \Sigma)/W(\widetilde{E}_k)$ defined by $S \mapsto f_S$.
The map Ψ is injective. It is defined by a system of linear equations:

$$\begin{cases} -x_1 - x_2 - x_3 = f_S(\alpha_1), \\ x_1 - x_2 = f_S(\alpha_2), \\ \cdots \\ x_{k-1} - x_k = f_S(\alpha_k), \\ x_{k+1} = \cdots = x_9, \\ x_1 + \cdots + x_9 = f_S(\delta) = 0. \end{cases}$$

The determinant is $\pm 3(1+d)$. Since the determinant is non-zero, the system of linear equations has a unique solution. This implies (i).

(ii) Given $f \in Hom_0(\Lambda(\widetilde{E}_k), \Sigma)$, we embed Σ into $\mathbb{P}^2$ as an anti-canonical curve such that 0 is an inflection point. The above system of linear equations has a solution $x_1, \ldots, x_k, x_{k+1} = \cdots = x_9 \in \Sigma \in |-K_{\mathbb{P}^2}|$. Note that $x_{k+1} = \cdots = x_9 \in \Sigma$ determines a d-jet $p^{[d]}$ with $p = x_{k+1}$ and $d = 8 - k$. Blowing up $x_1, \ldots, x_k, p^{[d]}$, we obtain a rational surface S with elliptic fibration structure and with $\widetilde{E}_k$-configuration such that $f = f_S$.

(iii) The last statement follows from the genus formula. For any $S \in \overline{\mathcal{S}'}_{\Sigma,k}$, we know that $|-K_S|$ contains a smooth curve Σ. Then for any irreducible curve C in S, $0 \le g(C) = 1 + 1/2(C^2 + CK)$ implies that $C^2 \ge -2 - CK = -2 + C\Sigma \ge -2$, since $C\Sigma \ge 0$.

□

Remark 39 In case $k = 8$, when these 9 points are in strictly semi-general position, each fiber in this fibration is irreducible. The boundary $\overline{\mathcal{S}'}_{\Sigma,8} \backslash \mathcal{S}^0_{\Sigma,8}$ consists of fibrations which have reducible singular fibers of affine ADE-type, by Kodaira's classification [2] (see also [15,16] for Kodaira's original work).

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Appendix: A short review of Kac–Moody Lie algebras

In this appendix, we recall some results on Kac–Moody Lie algebras [14,24]. A matrix $A_{n\times n} \in Mat(n, \mathbb{Z})$ is called decomposable, if, after reordering the indices (i.e. a permutation of its rows and the same permutation of the columns), A decomposes into a nontrivial direct sum, that is, a block diagonal matrix $diag(A_1, A_2)$. A is called symmetrizable, if there is a rational positive definite diagonal matrix $D = diag(d_1, \dots, d_n)$, such that $B := DA$ is a symmetric matrix; furthermore, if B is positive (semi-)definite, A is called positive (semi-)definite.

Definition 40 A matrix $A \in Mat(n, \mathbb{Z})$ is called a Cartan matrix if it satisfies the following conditions:

(i) $a_{ii} = 2, i = 1, \dots, n$;
(ii) $a_{ij} \leq 0$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$, for $i \neq j$;
(iii) A is indecomposable and positive definite.

A is called a generalized Cartan matrix if it satisfies the conditions (i) and (ii).
A is called of affine type if it is a generalized Cartan matrix and it satisfies the following extra condition:

(iii') A is indecomposable and positive semi-definite of rank $n-1$ (or equivalently, of corank 1).

Once we are given a (generalized) Cartan matrix A, we can construct a Lie algebra $\mathfrak{g}(A)$ using generators and generating relations (Serre's relations).

Let $A = (a_{ij})_{i,j=1}^n$ be a (generalized) Cartan matrix. Let $\mathfrak{h}$ be a complex vector space of dimension n with basis $h_i, i = 1, \dots, n$. Let $e_i, f_i, i = 1, \dots, n$ be $2n$ elements. Then the *Kac–Moody Lie algebra* $\mathfrak{g}(A)$ associated to A is defined by the $3n$ generators $h_i, e_i, f_i, i = 1, \dots, n$ and Serre's *defining relations*:

(i) $[h, h'] = 0, h, h' \in \mathfrak{h}$;
(ii) $[e_i, f_j] = \delta_{ij} h_i, i, j = 1, \dots, n$;
(iii) $[h_i, e_j] = a_{ij} e_j, [h_i, f_j] = -a_{ij} f_j$;
(iv) $(ade_i)^{1-a_{ij}} e_j = 0, (adf_i)^{1-a_{ij}} f_j = 0$ if $i \neq j$.

When A is a Cartan matrix, the algebra defined as above is a simple Lie algebra (of finite dimension). In general we will obtain an infinite dimensional Lie algebra. When A is of affine type, we will obtain a Lie algebra of affine type. Simple Lie algebras are classified completely by the Dynkin diagrams (or equivalently, by the root systems). And Kac–Moody Lie algebras of affine type are also classified completely by their Dynkin diagrams (or their root systems). There are three types in the case of affine type: affine-1 type (untwisted type), affine-2 and affine-3 types (twisted type). In our case, only algebras of affine-1 (untwisted) type (more precisely, their derived algebras) are considered. Lie algebras of affine-1 type are isomorphic to 1-toroidal Lie algebras, which are the central extensions of loop algebras. This isomorphism provides a concrete realization for all untwisted affine Lie algebras.

Definition 41 Let $\dot{\mathfrak{g}}$ be a complex simple Lie algebra. The loop algebra $\mathcal{L}(\dot{\mathfrak{g}})$ associated to $\dot{\mathfrak{g}}$ is defined as

$$\mathcal{L}(\dot{\mathfrak{g}}) := \mathbb{C}[t, t^{-1}] \otimes \dot{\mathfrak{g}},$$

where $\mathbb{C}[t, t^{-1}]$ is the Laurent polynomial ring in t. The Lie bracket $[\,,\,]_0$ is defined by

$$[P \otimes x, Q \otimes y]_0 = PQ[x, y]\ (P, Q \in \mathbb{C}[t, t^{-1}];\ x, y \in \dot{\mathfrak{g}}).$$

And the 1-toroidal algebra $\mathcal{T}(\dot{\mathfrak{g}})$ associated to $\dot{\mathfrak{g}}$ is defined to be a one-dimensional central extension of $\mathcal{L}(\dot{\mathfrak{g}})$.

Note that a *one-dimensional central extension* of $\mathcal{L}(\dot{\mathfrak{g}})$ is uniquely determined by a *2-cocycle* on $\mathcal{L}(\dot{\mathfrak{g}})$, that is, by definition, a bilinear $\mathbb{C}$-valued function ψ which satisfies the following two conditions:

(Co i) $\psi(a, b) = -\psi(b, a), a, b \in \mathcal{L}(\dot{\mathfrak{g}})$,
(Co ii) $\psi([a, b], c) + \psi([b, c], a) + \psi([c, a], b) = 0, a, b, c \in \mathcal{L}(\dot{\mathfrak{g}})$.

In our case, we take the 1-toroidal Lie algebra to be

$$\mathcal{T}(\dot{\mathfrak{g}}) := \mathcal{L}(\dot{\mathfrak{g}}) \oplus \mathbb{C}\langle c\rangle,$$

where the bracket is defined by

$$[a + \lambda c, b + \mu c] := [a, b]_0 + \psi(a, b)c.$$

Here we take the 2-cocycle $\psi(a, b) = (da, b)_0, d = t\frac{d}{dt}$ and $(t^m x, t^n y)_0 = \delta_{m+n,0}(x, y)$, where (x, y) is the Killing form on $\dot{\mathfrak{g}}$. That is

$$\psi(ut^m x, vt^n y) = m\delta_{m+n,0}uv(x, y),$$

where $u, v \in \mathbb{C}, m, n \in \mathbb{Z}$, and $x, y \in \dot{\mathfrak{g}}$.

Proposition 42 *Every untwisted affine Lie algebra is isomorphic to a 1-toroidal Lie algebra defined as above.*

Before establishing the isomorphism, we first recall the structure of the untwisted affine Lie algebra $\mathfrak{g}(A)$, where $A \in Mat(n + 1, \mathbb{Z})$ is a generalized Cartan matrix of affine type.

In this case, the Dynkin diagram of $\mathfrak{g}(A)$ is the affine Dynkin diagram of a simple Lie algebra $\dot{\mathfrak{g}}$. Denote by $\dot{\Delta}$ the root system of $\dot{\mathfrak{g}}$ with simple roots $\alpha_1, \ldots, \alpha_n$ and the highest root $-\alpha_0$, such that $\langle\alpha_i, \alpha_j\rangle = a_{ij}, 1 \leq i, j \leq n$ and $\langle\alpha_0, \alpha_j\rangle = -a_{n+1,j}, \langle\alpha_i, \alpha_0\rangle = -a_{i,n+1}$. Let $\dot{\mathfrak{h}}$ be the Cartan subalgebra of $\dot{\mathfrak{g}}$. Let α_{n+1} be linearly independent of $\alpha_1, \ldots, \alpha_n$, such that $\langle\alpha_{n+1}, \alpha_j\rangle = \langle\alpha_0, \alpha_j\rangle, \langle\alpha_i, \alpha_{n+1}\rangle = \langle\alpha_i, \alpha_0\rangle$

and $\langle \alpha_{n+1}, \alpha_{n+1} \rangle = \langle \alpha_0, \alpha_0 \rangle$. Let $\delta = \alpha_{n+1} - \alpha_0$ be the null root, which is an eigenvector with zero eigenvalue of A. Then the root system of $\mathfrak{g}$ is

$$\Delta = \{\alpha + n\delta | \alpha \in \dot{\Delta}, n \in \mathbb{Z}\} \cup \{n\delta | n \neq 0, n \in \mathbb{Z}\}.$$

Here $\Delta^{re} := \{\alpha + n\delta | \alpha \in \dot{\Delta}, n \in \mathbb{Z}\}$ is called the real root system, and $\Delta^{im} := \{n\delta | n \neq 0, n \in \mathbb{Z}\}$ is called the imaginary root system. The Weyl group $W(\mathfrak{g})$ is the group generated by the reflections $s_{\alpha_i}, i = 1, \ldots, n+1$. And there is a natural gradation, similar to the triangular decomposition in the finite case, of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{h} \oplus \oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha$ is of dimension 1 for $\alpha \in \Delta^{re}$ and of dimension n for $\alpha \in \Delta^{im}$, and $\mathfrak{h} = \dot{\mathfrak{h}} \oplus \mathbb{C}\langle c \rangle$ is the Cartan subalgebra of $\mathfrak{g}$.

The isomorphism between the untwisted affine Lie algebra $\mathfrak{g}$ and the 1-toroidal Lie algebra $\mathcal{T}(\dot{\mathfrak{g}})$ is the following:

$$\begin{cases} \mathfrak{g}_{\alpha + j\delta} \to t^j \otimes \dot{\mathfrak{g}}_\alpha, \\ \quad \mathfrak{g}_{j\delta} \to t^j \otimes \dot{\mathfrak{h}}. \end{cases}$$

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