SYZ transformations for toric varieties

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1 Mirror symmetry

In physics, the string theory on a space X can be loosely regarded as the quantum mechanics on the space of loops in X. This is because a closed string is simply a loop in X. There are two different sectors of string theory, called the *A-model* and *B-model*. Mathematically, A- and B-models correspond to symplectic and complex geometries respectively. Mirror symmetry asserts that these two seemingly very different geometries are equivalent to each other, but on different manifolds. This is very surprising from the mathematical point of view because, apparently, symplectic geometry is quite linear in nature while complex geometry is rather nonlinear. It turns out that mirror symmetry works only after we include quantum corrections to the A-model to make it more nonlinear.

In the beginning, mirror symmetry is only studied for Calabi-Yau manifolds. Recall that a $2n$-dimensional Riemannian manifold (X, g) is called a *Calabi-Yau* manifold if its holonomy group is a subgroup of $SU(n)$. This is equivalent to X being a Kähler manifold with zero Ricci curvature, at least when X is compact and simply connected. Recall that a Hermitian complex structure J on a Riemannian manifold X is Kähler if the two form ω defined by $\omega(u, v) = g(Ju, v)$ is closed $d\omega = 0$. In particular, (X, ω) is a symplectic manifold. The celebrated result of Yau [32, 33] says that when X is a compact Kähler manifold, then X admits a Calabi-Yau structure if and only if its first Chern class $c_1(X)$ is zero. Yau proved this by solving a complex Monge-Ampère equation which corresponds to finding a Kähler metric so that the holomorphic volume form $\Omega \in \Omega^{n,0}(X)$ has constant length at every point. For example, $\Omega = dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$ on $X = \mathbb{C}^n$ is a holomorphic volume form with constant length. We remark that the complex structure on X is determined completely by Ω.

The mirror symmetry conjecture roughly says that there are pairs of Calabi-Yau manifolds X and Y of the same dimension such that

(i) the symplectic geometry of (X, ω_X) is equivalent to the complex geometry of (Y, Ω_Y), and

(ii) the complex geometry of (X, Ω_X) is equivalent to the symplectic geometry of (Y, ω_Y).

The first successful application of mirror symmetry is the prediction by Candelas-de la Ossa-Green-Parkes [4] of the number of rational curves on the quintic Calabi-Yau hypersurface

$$X = \left\{ z_0^5 + z_1^5 + z_2^5 + z_3^5 + z_4^5 + t z_0 z_1 z_2 z_3 z_4 = 0 \right\} \subset \mathbb{P}^4.$$

In this case its mirror manifold Y, called the *mirror quintic*, can be constructed as (the resolution of singularities of) the quotient of X by a finite Abelian group $(\mathbb{Z}_5)^3$. It was observed that the Hodge diamonds of X and Y are *mirror* to each other, that is

$$\begin{aligned} h^{1,1}(X) &= 1 = h^{2,1}(Y), \\ h^{2,1}(X) &= 101 = h^{1,1}(Y). \end{aligned}$$

Here $h^{p,q}(X) = \dim H^{p,q}(X)$ and $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$ is the Hodge decomposition.

Recall that $[\omega_X] \in H^{1,1}(X)$ and $[\Omega_X] \in H^{3,0}(X) \simeq \mathbb{C}$. Since $H^{1,0}(X) = H^{2,0}(X) = 0$ by the Kodaira vanishing theorem, we have $H^{1,1}(X) = H^2(X, \mathbb{C})$ which parametrizes the first order deformations of the (complexified) symplectic structure ω_X on X. On the other hand, small deformations of complex structures on X are determined by variations of $H^{3,0}(X)$ inside $H^3(X, \mathbb{C})$. Thus $H^{2,1}(X)$ parametrizes the first order deformations of the complex structure Ω_X on X, by Griffiths transversality. The same applies to Y.

The cup product on X defines a cubic form $\mathcal{Y}_X^{A,cl}$ on $H^{1,1}(X)$: for any $\alpha_i \in H^{1,1}(X)$ $(i = 1, 2, 3)$,

$$\mathcal{Y}_X^{A,cl}(\alpha_1, \alpha_2, \alpha_3) := \int_X \alpha_1 \cup \alpha_2 \cup \alpha_3.$$

This classical structure needs to be modified to the *A-Yukawa coupling*

$$\mathcal{Y}_X^A : \bigotimes^3 H^{1,1}(X) \to \mathbb{C}$$

by including *quantum corrections*, i.e. genus zero three pointed Gromov-Witten invariants $GW_{0,3,\beta}^X(\alpha_1, \alpha_2, \alpha_3)$ which roughly speaking counts the number of holomorphic spheres representing a homology class $\beta \in H_2(X, \mathbb{Z})$ in X which meet the cycles $PD[\alpha_1], PD[\alpha_2], PD[\alpha_3]$, where $PD[\alpha]$ stands for the Poincarè dual of α. Indeed, the moduli space $\mathcal{M}_{\text{sympl}}(X)$ of symplectic structures on X is locally modeled on $H^2(X)$, which is too linear in nature, and it becomes more nonlinear by incorporating quantum corrections to the cubic form.

On the mirror quintic Calabi-Yau threefold Y, the moduli space $\mathcal{M}_{cpx}(Y)$ of complex structures on Y can be described by periods and this problem can be solved via Picard-Fuchs equations. It determines the *B-Yukawa coupling*

$$\mathcal{Y}_Y^B : \bigotimes^3 H^{2,1}(Y) \to \mathbb{C}.$$

The most degenerating complex structure, called the *large complex structure limit* (LCSL), which can be characterized in term of maximally unipotent monodromy for the periods, is given by $z_0 z_1 z_2 z_3 z_4 = 0$. Furthermore, using monodromy of periods, one obtains canonical flat coordinates near the LCSL limit. By matching this flat coordinate structure on $\mathcal{M}_{cpx}(Y)$ with the linear structure on $\mathcal{M}_{sympl}(X)$ given by $H^2(X)$, one obtain a *mirror map* f^{mirror} identifying these spaces. From the duality prediction in physics, this should identify $\mathcal{Y}_X^A$ with $\mathcal{Y}_Y^B$, and in particular the Gromov-Witten invariants (or number of rational curves) of X can be obtained by certain classical computations involving the complex geometry of its mirror Y.

These invariants are important in algebraic geometry, especially in enumerative problems, and they are usually very difficult to determine. That is why such a prediction coming from mirror symmetry has excited many mathematicians. This prediction has finally been proven mathematically using localization methods by Givental [15] and Lian-Liu-Yau [28]. This is called the *mirror theorem*.

When the mirror map is expanded around the LCSL point, Lian-Yau [29] show that its coefficients have integrality property! It has been suspected that this mysterious integrality should count something. As we will explain soon, for toric Calabi-Yau threefolds, like $K_{\mathbb{P}^2}$, they count holomorphic disks with Lagrangian boundary conditions.

Symplectic Geometry (A-model)	Complex Geometry (B-model)
Linear + quantum corrections	Nonlinear
A-Yukawa coupling on $H^{1,1}$	B-Yukawa coupling on $H^{2,1}$
A-cycles: (special) Lagrangian submanifolds + flat bundles	B-cycles: complex submanifolds + (Yang-Mills) holomorphic bundles
Fukaya category of Lagrangian submanifolds	derived category of coherent sheaves

Kontsevich [22] formulates mirror symmetry as an equivalence between categories. More precisely, Kontsevich's *homological mirror symmetry* (HMS) conjecture says that the (derived) Fukaya category of Lagrangians in X is equivalent to the derived category of coherent sheaves on Y, and vice versa.

The important question remains as to why such an amazing duality exists. Is there an explicit transformation that interchanges these two kinds of geometries?

2 SYZ transformations

In 1996, Strominger-Yau-Zaslow (SYZ) [31] has a ground breaking proposal which says that mirror symmetry is a form of Fourier transformation, called *T-duality*. Namely, when X and Y are mirror Calabi-Yau manifolds, then

(i) X should admit a special Lagrangian T^n-fibration and Y is the dual torus fibration, and

(ii) there is a fiberwise Fourier transformation which interchanges the symplectic geometry (resp. complex geometry) of X with the complex geometry (resp. symplectic geometry) of Y.

Let us give a heuristic reasoning for the SYZ proposal: The complex manifold Y is trivially the moduli space of certain B-cycles in Y, namely points in Y. As mirror symmetry should identify B-cycles in Y with A-cycles in X, at least in the LCSL, Y is also the moduli space of certain special Lagrangian submanifolds S in X coupled with flat $U(1)$-bundles. As these B-cycles, namely points, swipe over Y exactly once, we expect the same is true for the corresponding A-cycles in X, thus giving a special Lagrangian T^n-fibration on X. Different A-cycles with the same underlying special Lagrangian T^n-fiber T in X corresponds to different flat $U(1)$-bundles over T and they form the *dual* torus T^* to T. Hence, the corresponding B-cycles in Y, which are points in Y, form a submanifold T^* in Y. By varying the T^n-fibers in X, we obtain a dual torus fibration on Y.

Next, the total space Y is another obvious B-cycle in Y, which meets every point B-cycle in Y exactly once. Thus, the corresponding A-cycle in X will be a special Lagrangian section to the above T^n-fibration and this section should be considered as the zero section for the torus fibration. Similarly, the mirror to a holomorphic line bundle over Y is a general special Lagrangian section in X. We call this the *SYZ transformation*. In the holomorphic setting, the analog is the Fourier-Mukai transformation.

This SYZ transformation is much easier to study when there are no singular fibers, say in the semi-flat case. However, we shall emphasize that there is no way one could avoid singular Lagrangian fibers for any compact Calabi-Yau manifolds. In fact, the investigation of the effect of these singular fibers is probably the most important part of this subject. In the toric Calabi-Yau manifolds case, we will explain below how they are related to the SYZ construction of mirror Calabi-Yau manifolds.

The SYZ proposal has been tested successfully [26] in the semi-flat setting where the Calabi-Yau metric is flat along fibers. In this setting, the SYZ transformation can be constructed very explicitly as quantum corrections are absent. Let us illustrate this in the following example: Recall that the base space B for any Lagrangian T^n-fibration has a natural affine structure. For simplicity, let us assume that B is an open subset in $\mathbb{R}^n$ with coordinates $x^1, \cdots, x^n$. Then the symplectic manifold X is given as $X = T^*B/\Lambda^*$, where Λ is a fiberwise lattice in the tangent bundle TB, with coordinates $x^1, \cdots, x^n, y_1, \cdots, y_n$ and canonical symplectic form $\omega_X = \sum_j dx^j \wedge dy_j$. The complex manifold Y is given by $Y = TB/\Lambda$ with coordinates $x^1, \cdots, x^n, y^1, \cdots, y^n$ and the canonical complex structure on Y has complex coordinate $z^1, \cdots, z^n$ with $z^j = x^j + iy^j$. Recall that a complex structure is also determined by its holomorphic volume form, which is $\Omega_Y = \prod_j \left(dx^j + idy^j\right)$ in this case.

By direct computations, we have

$$\Omega_Y = \int_{T^*} e^{\omega_X} \cdot e^{i\sum_j dy^j \wedge dy_j},$$
$$e^{\omega_X} = \int_T \Omega_Y \cdot e^{-i\sum_j dy^j \wedge dy_j}.$$

This can be naturally interpreted as a Fourier-Mukai transformation as $e^{i\sum_j dy^j \wedge dy_j}$ is the Chern character form for a universal connection on the fiberwise Poincare bundle $\mathcal{P}$ on $X \times_B Y$, that is formally, $\Omega_Y = \mathcal{F}(e^{\omega_X})$ and $e^{\omega_X} = \mathcal{F}(\Omega_Y)$ where $\mathcal{F}(-) = \pi_{2*}(\pi_1^*(-) \otimes \mathcal{P})$, where π_j is the projection of the j^{th}-factor.

Let us look at the Fourier(-Mukai) transformation on a single torus $T = V/\Lambda$ more closely. The above $\mathcal{F} : \Omega^0(T) \to \Omega^n(T^*)$ is given by $\mathcal{F}(\phi) = \int_T \phi(y^1, \cdots, y^n) e^{i\sum_j dy^j \wedge dy_j}$, while the usual Fourier transformation gives a function $\hat{\phi} : \Lambda^* \to \mathbb{C}$ on the dual lattice Λ^* by

$$\hat{\phi}(m_1, ..., m_n) = \int_T \phi(y^1, \cdots, y^n) e^{i\sum_j m_j y^j} dy^1 \wedge \cdots \wedge dy^n.$$

It is natural to combine both the Fourier-Mukai transformation and Fourier transformation to define $\mathcal{F}$ which transforms differential forms on $T \times \Lambda$ to those on $T^* \times \Lambda^*$. Notice that $T \times \Lambda$ is the space of geodesic (or affine) loops in T inside the loop space of T, i.e. $T \times \Lambda = \mathcal{L}_{\min} T \subset \mathcal{L}T$, and the loop space certainly plays important roles in string theory. We are going to describe how such a transformation $\mathcal{F}^{SYZ}$, called *SYZ transformation*, interchanges symplectic and complex geometries, with quantum corrections included, in the toric case.

3 Fano toric varieties

Let X_Δ be a Fano toric variety. Mirror symmetry has an extension to non-Calabi-Yau manifolds like X_Δ and the mirror is given by $Y = (\mathbb{C}^\times)^n$ together with a function $W : (\mathbb{C}^\times)^n \to \mathbb{C}$ called the *superpotential*. For instance, when $X_\Delta = \mathbb{P}^n$, we have $W = z_1 + \ldots + z_n + \frac{q}{z_1 \ldots z_n}$.

To study the mirror symmetry for X_Δ from the SYZ viewpoint, we consider the open dense subset $X \simeq (\mathbb{C}^\times)^n$ of X_Δ which is the union of Lagrangian torus fibers of the moment map. See Figure 1 below for a visualization of the moment map for $X_\Delta = \mathbb{P}^2$. Symplectically, we write $X = T^*B/\Lambda^*$ and we denote by $\tilde{X} = X \times \Lambda^* \subset \mathcal{L}X_\Delta$ the space of fiberwise geodesics/affine loops in X_Δ. On $\tilde{X}$, we consider $\tilde{\omega}_X = \omega_X + \Psi$, where Ψ is a generating function which counts holomorphic disks with boundaries lying on fiber Lagrangians (see Cho-Oh [10] for details). In [8], we show that

1. The SYZ transformation carries the corrected symplectic structure on $\tilde{X}$ to the holomorphic volume form of the pair (Y, W):

$$\mathcal{F}^{SYZ}\left(e^{\omega_X + \Psi}\right) = e^W \Omega.$$

2. $\mathcal{F}^{SYZ} : QH^*(X_\Delta) \to Jac(W)$ gives an isomorphism between the quantum cohomology of X_Δ with the Jacobian ring of the superpotential W.

Figure 1: The moment map for $\mathbb{P}^2$.

This is proven by describing holomorphic spheres as suitable gluing of holomorphic disks. More precisely, we pass to the tropical limit (for an introduction to tropical geometry, see for example Mikhalkin [30]) and observe that a tropical curve can be seen as a gluing of tropical disks. For instance, a line $\mathbb{P}^1$ in $\mathbb{P}^2$ is obtained as a gluing of three disks tropically (see Figure 2 below). This observation was later generalized and used by Gross [19] in his study of mirror symmetry for the big quantum cohomology of $\mathbb{P}^2$ via tropical geometry.

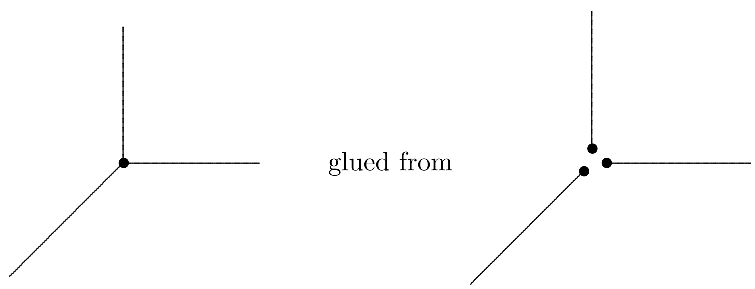

Figure 2: A tropical curve as a gluing of tropical disks.

In fact, the first part of the above result can be generalized to *any* compact toric manifold by appropriately defining the function Ψ as a *virtual* counting of holomorphic disks (see Fukaya-Oh-Ohta-Ono [12] for details). The second part has been generalized to toric manifolds with $c_1 \geq 0$ and $\dim X_\Delta < 3$ by Chan-Lau [6]; while the general case follows from a recent result of Fukaya-Oh-Ohta-Ono [14].

4 Toric Calabi-Yau

Next we look at toric Calabi-Yau manifolds. These manifolds are necessarily noncompact and a typical example is the total space of the canonical line bundle over a projective space, i.e. $X_\Delta = K_{\mathbb{P}^{n-1}}$. Instead of the toric fibration, we consider another Lagrangian fibration on X_Δ which is constructed by Gross [18] and Goldstein [16] independently. The affine structure on the base B is the upper half space but there are interior singular points lying on a hyperplane called the *wall*. Figure 3 depicts the fibration when $X = K_{\mathbb{P}^1}$.

As a point moves across the wall, the (virtual) number of holomophic disks bounded by the corresponding Lagrangian fiber jumps. This is called the *wall-*

Figure 3: The Lagrangian fibration on $K_{\mathbb{P}^1}$ constructed by Gross and Goldstein. The discriminant locus consists of the boundary $\mathbb{R} \times \{-K_2\}$ and two interior points $\{(r_1, 0), (r_2, 0)\}$. The wall is the x-axis $\mathbb{R} \times \{0\}$.

crossing phenomenon, as have been studied by Auroux [1, 2]. An instance of wall-crossing when $X = K_{\mathbb{P}^1}$ is drawn in Figure 4.

Figure 4: Wall-crossing phenomenon for $X = K_{\mathbb{P}^1}$. Below the wall a Lagrangian fiber bounds essentially only one holomorphic disk. As the image of the Lagrangian fiber moves upward across the wall, another family of holomorphic disks comes up.

Because of the presence of the wall in B, one needs to apply SYZ dual fibration construction on each connected component in the complement of the wall, with quantum corrections included. Then the wall-crossing formula let us glue the resulting pieces together to obtain a complex manifold Y (see Chan-Lau-Leung [7] for details). For instance, in the case of $X = K_{\mathbb{P}^2}$, we have

$$Y = \left\{ (z, w, u, v) \in \left(\mathbb{C}^\times\right)^2 \times \mathbb{C}^2 : uv = h\left(q\right) + z + w + \frac{q}{zw} \right\},$$

which belongs to the mirror family as constructed by Hori-Iqbal-Vafa [21] from physical considerations. Furthermore, this SYZ mirror construction is more precise in the sense that it tells us exactly what complex structure on Y is corresponding to any given symplectic structure on X. This is because the coefficient $h(q)$ is expressed entirely and explicitly in terms of the Kähler parameters and counting of holomorphic disks in X.

On the other hand, the SYZ construction naturally gives us a map from the symplectic moduli space of X to the complex moduli space of Y. We call this the *SYZ map* f^{SYZ}. In [7], we show that the SYZ map coincides with the mirror map f^{mirror} for certain toric Calabi-Yau threefolds of the form K_Z. This explains the mysterious integrality property of the mirror map: its coefficients come from the counting of holomorphic disks with Lagrangian fibers boundary conditions.

The main step in proving these results is to compute certain *open Gromov-Witten* invariants in an *obstructed* situation. For instance, in the $K_{\mathbb{P}^2}$ case, we need to compute the number $n_{\beta+kl}$ of such holomorphic disks in $K_{\mathbb{P}^2}$ representing the relative homology class $\beta + kl$, where $l \in H_2\left(\mathbb{CP}^2, \mathbb{Z}\right)$ is the hyperplane class, $k \in \mathbb{N}$ and $\beta \in \pi_2\left(K_{\mathbb{P}^2}, F\right)$ is the class of an irreducible holomorphic disk with Maslov index two which has non-trivial intersection with the zero section $\mathbb{P}^2 \subset K_{\mathbb{P}^2}$. Our computation show that they are given by

$$n_{\beta+kl} = 1, -2, 5, -32, 286, -3038, \ldots$$

for $k = 0, 1, 2, 3, 4, 5, \ldots$ respectively. From this, we have

$$h(q) = 1 - 2q + 5q^2 - 32q^3 + 286q^4 - 3038q^5 + \ldots$$

and conclude that the SYZ map coincides with the mirror map. Our method also works for some other toric Calabi-Yau threefolds such as $K_{\mathbb{P}^1 \times \mathbb{P}^1}$. This also explains why the mirror map for the resolved conifold $O_{\mathbb{P}^1}(-1, -1)$ is trivial.

We remark that the relevance of the work of Graber-Zaslow [17] to the relationship between the series of $h(q)$ and canonical coordinates was first mentioned in Remark 5.1 of the paper [20] by Gross and Siebert. They were also the first to observe that the coefficients of the above series have geometric meanings; namely, they show that these coefficients can be obtained by imposing the "normalization" condition for slabs, which is a condition necessary to run their program and construct toric degenerations of Calabi-Yau manifolds.

In dimension two, Lau-Leung-Wu [25] show that this method works for all toric Calabi-Yau surfaces. In particular, the SYZ map f^{SYZ} coincides with the mirror map f^{mirror}, which is also compatible with the mirror construction by twistor rotations. Again the main ingredient is to compute open Gromov-Witten invariants in a certain obstructed situation.

We remark that in our earlier study of SYZ transformations for semi-flat Calabi-Yau manifolds, singular Lagrangian fibers and hence quantum corrections are absent. Then in the toric Calabi-Yau case, there are singular Lagrangian fibers, so we need to include quantum corrections and study the wall-crossing phenomenon. For general Calabi-Yau manifolds, one will in addition need to consider *scattering* phenomenon (as studied by Kontsevich-Soibelman [23] and Gross-Siebert [20]) in the study of SYZ transformations.

5 Computing open Gromov-Witten invariants

We now discuss our approach to compute open Gromov-Witten invariants. These invariants count the number of 1-marked Maslov index two holomorphic disks in a toric variety X_Δ with boundaries lying on a toric Lagrangian fiber. The corresponding generating function is the superpotential W for the mirror. The expected dimension of the moduli space of such disks is always zero.

When X_Δ is Fano, i.e. $c_1(X_\Delta) > 0$, Cho-Oh [10] show that there is exactly one such disk corresponding to each toric divisor. For instance, when $X_\Delta = \mathbb{P}^1$ with north and south poles being the two toric divisors and the equator being the Lagrangian fiber, the upper and lower hemispheres are the two holomorphic disks.

When X_Δ is not Fano, the moduli space of these disks could have positive dimension, thus we are in an obstructed situation. For the Hirzebruch surface $\mathbb{F}_2$, the superpotential W is computed by Fukaya-Oh-Ohta-Ono in [13] using their big machinery [11] and by Auroux [2] using wall-crossing and degeneration techniques (which works for $\mathbb{F}_3$ as well). In this case, beside those standard disks given by Cho-Oh, there is an extra bubbling disk whose bubble is the (-2)-curve in $\mathbb{F}_2$ (see Figure 5).

Figure 5: The bubbling disk in $\mathbb{F}_2$, which is a union of a holomorphic disk together with the holomorphic sphere D_4 .

Chan-Lau [6] compute W for all toric surfaces with $c_1 \geq 0$. For these surfaces, every toric divisor has self-intersection number at least -2. Holomorphic disks with bubbles can only occur along a chain of (-2)-curves. By a result of Chan [5] (and its generalization by Lau-Leung-Wu [24]), we can identify this open Gromov-Witten invariant with a certain 1-pointed *closed* Gromov-Witten invariant where the irreducible disk component is being completed to a holomorphic sphere. These (obstructed) closed Gromov-Witten invariants have been previously computed by Bryan-Leung [3] and hence we obtain W. As a corollary, we get an explicit presentation for the quantum cohomology of any toric surface with $c_1 \geq 0$.

For instance, let X be the toric surface defined by the following fan and polytope (see Figure 6 below).

Figure 6: The fan Σ and the polytope P defining X. The numbers beside the divisors indicate their self-intersection numbers.

Then the superpotential W is given by

$$\begin{aligned} W =&(1+q_1)z_1 + z_2 + \frac{q_1 q_2 q_3^2 q_4^3}{z_1 z_2} + (1+q_2+q_2 q_3)\frac{q_1 q_3 q_4^2}{z_2} \\ &+ (1+q_3+q_2 q_3)\frac{q_1 q_4 z_1}{z_2} + \frac{q_1 z_1^2}{z_2}, \end{aligned}$$

where $q_l = \exp(-t_l)$, $l = 1, \ldots, 4$, are the Kähler parameters on X.

Next we want to compute W for $K_{\mathbb{P}^2}$. Again there are unobstructed holomorphic disks as given by Cho-Oh. The obstructed case is the computation of the invariant $n_{\beta+kl}$ for $k > 0$. Using the result of Chan [5] again, we can identify it with a closed Gromov-Witten invariant:

$$n_{\beta+kl} = GW_{0,1,f+kl}\left(X_1\right),$$

where $X_1 = \mathbb{P}_{\mathbb{P}^2}\left(K_{\mathbb{P}^2} \oplus O_{\mathbb{P}^2}\right)$ and f is the fiber class of the $\mathbb{P}^1$-bundle $X_1 \to \mathbb{P}^2$. By blowing up X_1 at a vertex in the infinity section of X_1 to obtain X_2, we can get rid of the marked point condition, i.e. we have

$$GW_{0,1,f+kl}\left(X_1\right) = GW_{0,0,l_2+kl}\left(X_2\right),$$

where l_2 is the strict transform of the fiber in X_1 passing through the blown-up point.

Notice that l_2 is a $(-1,-1)$-curve and we can flop it to obtain another space X_3. Now, by the result of Li-Ruan [27], we can express the invariant $GW_{0,0,l_2+kl}(X_2)$ in terms of Gromov-Witten invariants for X_3. More precisely, when we perform the flop, we have blown up $\mathbb{P}^2$ at a point to give $\mathbb{F}_1$ and the flop of l_2 becomes the (-1)-curve l_3 inside this $\mathbb{F}_1$. Indeed, $X_3 = \mathbb{P}_{\mathbb{F}_1}(K_{\mathbb{F}_1} \oplus O_{\mathbb{F}_1})$ and we are computing $GW_{0,0,kl-e}(X_3)$, where e is the exceptional curve coming from the blow-up $\mathbb{F}_1 \to \mathbb{P}^2$. This is the same as a local invariant for the local Calabi-Yau threefold $K_{\mathbb{F}_1}$ which has been computed previously by Chiang-Klemm-Yau-Zaslow [9] using localization methods.

Hence, we can determine W for $K_{\mathbb{P}^2}$ completely and verify that the SYZ map coincides with the mirror map in this and other similar cases. The whole strategy is illustrated by Figure 7 below.

Figure 7: The procedures to compute the open Gromov-Witten invariants of $K_{\mathbb{P}^2}$.

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References

[1] D. Auroux, *Mirror symmetry and T-duality in the complement of an anticanonical divisor.* J. Gökova Geom. Topol. GGT 1 (2007), 51–91.

[2] D. Auroux, *Special Lagrangian fibrations, wall-crossing, and mirror symmetry.* Surveys in differential geometry. Vol. XIII. Geometry, analysis, and algebraic geometry: forty years of the Journal of Differential Geometry, 1–47, Surv. Differ. Geom., 13, Int. Press, Somerville, MA, 2009.

[3] J. Bryan and N.C. Leung, *The enumerative geometry of K3 surfaces and modular forms.* J. Amer. Math. Soc. 13 (2000), no. 2, 371–410.

[4] P. Candelas, X. de la Ossa, P. Green and L. Parkes, *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory.* Nuclear Phys. B 359 (1991), no. 1, 21–74.

[5] K. Chan, *A formula equating open and closed Gromov-Witten invariants and its applications to mirror symmetry.* arXiv:1006.3827.

[6] K. Chan and S.-C. Lau, *Open Gromov-Witten invariants and superpotentials for semi-Fano toric surfaces.* arXiv:1010.5287.

[7] K. Chan, S.-C. Lau and N.C. Leung, *SYZ mirror symmetry for toric Calabi-Yau manifolds.* arXiv:1006.3830.

[8] K. Chan and N.C. Leung, *Mirror symmetry for toric Fano manifolds via SYZ transformations.* Adv. Math. 223 (2010), no. 3, 797–839.

[9] T.-M. Chiang, A. Klemm, S.-T. Yau and E. Zaslow, *Local mirror symmetry: calculations and interpretations.* Adv. Theor. Math. Phys. 3 (1999), no. 3, 495–565.

[10] C.-H. Cho and Y.-G. Oh, *Floer cohomology and disc instantons of Lagrangian torus fibers in Fano toric manifolds.* Asian J. Math. 10 (2006), no. 4, 773–814.

[11] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian intersection Floer theory: anomaly and obstruction.* AMS/IP Studies in Advanced Mathematics, 46. American Mathematical Society, Providence, RI; International Press, Somerville, MA, 2009.

[12] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory on compact toric manifolds. I.* Duke Math. J. 151 (2010), no. 1, 23–174.

[13] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Toric degeneration and nondisplaceable Lagrangian tori in* $S^2 \times S^2$. arXiv:1002.1660.

[14] K. Fukaya, Y.-G. Oh, H. Ohta and K. Ono, *Lagrangian Floer theory and mirror symmetry on compact toric manifolds.* arXiv:1009.1648.

[15] A. Givental, *Equivariant Gromov-Witten invariants.* Internat. Math. Res. Notices 1996, no. 13, 613–663.

[16] E. Goldstein, *Calibrated fibrations on noncompact manifolds via group actions.* Duke Math. J. 110 (2001), no. 2, 309–343.

[17] T. Graber and E. Zaslow, *Open-string Gromov-Witten invariants: calculations and a mirror theorem*, Orbifolds in mathematics and physics (Madison, WI, 2001), Contemp. Math., vol. 310, Amer. Math. Soc., Providence, RI, 2002, pp. 107-121.

[18] M. Gross, *Examples of special Lagrangian fibrations.* Symplectic geometry and mirror symmetry (Seoul, 2000), 81–109, World Sci. Publ., River Edge, NJ, 2001.

[19] M. Gross, *Mirror symmetry for* $\mathbb{P}^2$ *and tropical geometry.* Adv. Math. 224 (2010), no. 1, 169–245.

[20] M. Gross and B. Siebert, *From real affine geometry to complex geometry.* arXiv:math/0703822.

[21] K. Hori, A. Iqbal and C. Vafa, *D-branes and mirror symmetry.* arXiv:hep-th/0005247.

[22] M. Kontsevich, *Homological algebra of mirror symmetry.* Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994), 120–139, Birkhäuser, Basel, 1995.

[23] M. Kontsevich and Y. Soibelman, *Affine structures and non-Archimedean analytic spaces.* The unity of mathematics, 321–385, Progr. Math., 244, Birkhäuser Boston, Boston, MA, 2006.

[24] S.-C. Lau, N.C. Leung and B. Wu, *A relation for Gromov-Witten invariants of local Calabi-Yau threefolds.* arXiv:1006.3828.

[25] S.-C. Lau, N.C. Leung and B. Wu, *Mirror maps equal SYZ maps for toric Calabi-Yau surfaces.* arXiv:1008.4753.

[26] N.C. Leung, S.-T. Yau and E. Zaslow, *From special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai transform.* Adv. Theor. Math. Phys. 4 (2000), no. 6, 1319–1341.

[27] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds.* Invent. Math. 145 (2001), no. 1, 151–218.

[28] B. Lian, K. Liu and S.-T. Yau, *Mirror principle. I.* Asian J. Math. 1 (1997), no. 4, 729–763.

[29] B. Lian and S.-T. Yau, *Arithmetic properties of mirror map and quantum coupling.* Comm. Math. Phys. 176 (1996), no. 1, 163–191.

[30] G. Mikhalkin, *Tropical geometry and its applications.* International Congress of Mathematicians. Vol. II, 827–852, Eur. Math. Soc., Zürich, 2006.

[31] A. Strominger, S.-T. Yau and E. Zaslow, *Mirror symmetry is T-duality.* Nuclear Phys. B 479 (1996), no. 1-2, 243–259.

[32] S.-T. Yau, *Calabi's conjecture and some new results in algebraic geometry.* Proc. Nat. Acad. Sci. U.S.A. 74 (1977), no. 5, 1798–1799.

[33] S.-T. Yau, *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I.* Comm. Pure Appl. Math. 31 (1978), no. 3, 339–411.