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GROMOV-WITTEN INVARIANTS FOR G/B AND PONTRYAGIN PRODUCT FOR ΩK

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ABSTRACT. We give an explicit formula for (T-equivariant) 3-pointed genus zero Gromov-Witten invariants for G/B. We derive it by finding an explicit formula for the Pontryagin product on the equivariant homology of the based loop group ΩK.

1. INTRODUCTION

A flag variety G/B is the quotient of a simply-connected simple complex Lie group by its Borel subgroup and it plays very important roles in many different branches of mathematics. There are natural Schubert cycles inside G/B. The corresponding Schubert cocycles σ^u form a basis of the cohomology ring $H^*(G/B)$. In terms of this basis, the structure coefficients $N_{u,v}^w$ of the intersection product,

$$\sigma^u \cdot \sigma^v = \sum_w N_{u,v}^w \sigma^w,$$

are called *Schubert structure constants*, which is a direct generalization of the Littlewood-Richardson coefficients for complex Grassmannians. When

$$G = SL(n+1, \mathbb{C}),$$

the coefficients $N_{u,v}^w$ count suitable Young tableaus (see e.g. [10]) or honeycombs [19], [20]. An explicit formula for $N_{u,v}^w$ in all cases are given by Kostant and Kumar [21] by considering Kac-Moody groups and an effective algorithm is obtained by Duan [6] via topological methods. Note that a ring presentation of $H^*(G/B, \mathbb{C})$ was given much earlier by Borel [2] in terms of Chern classes of universal bundles over $G/B = K/T$, where K is a maximal compact Lie subgroup of G and $T = K \cap B$ is a maximal torus of K.

The (small) quantum cohomology ring of G/B, or more generally of any symplectic manifold, was introduced by the physicist Vafa [38] and it is a deformation of the ring structure on $H^*(G/B)$ by incorporating genus zero Gromov-Witten invariants of G/B into the intersection product. As complex vector spaces, the quantum cohomology ring $QH^*(G/B)$ is isomorphic to $H^*(G/B) \otimes \mathbb{C}[\mathbf{q}]$ with $\mathbf{q}_\lambda = q_1^{a_1} \cdots q_n^{a_n}$ for $\lambda = (a_1, \cdots, a_n) \in H_2(G/B, \mathbb{Z})$. The structure coefficients $N_{u,v}^{w,\lambda}$ of the quantum product,

$$\sigma^u \star \sigma^v = \sum_{w,\lambda} N_{u,v}^{w,\lambda} \mathbf{q}_\lambda \sigma^w,$$

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are called *quantum Schubert structure constants*. As we will see in section 6.3, $N_{u,v}^{w,\lambda} = I_{0,3,\lambda}(\sigma^u, \sigma^v, \sigma^{\omega_0 w})$ is the 3-pointed genus zero Gromov-Witten invariant for $\sigma^u, \sigma^v, \sigma^{\omega_0 w} \in H^*(G/B)$ by the definition of the quantum product $\sigma^u \star \sigma^v$. We will use the terminology "quantum Schubert structure constants" instead of "Gromov-Witten invariants" for G/B throughout this paper, in analog with the classical Schubert structure constants.

Because of the lack of functoriality, the study of the quantum cohomology ring of G/B, or more generally partial flag varieties G/P, is a challenging problem. A presentation of the ring structure on $QH^*(G/B)$ is given by Kim [18] in terms of Toda lattice for the Langlands dual Lie group. There have been many studies of $QH^*(G/P)$ in special cases including complex Grassmannians, partial flag varieties of type A, isotropic Grassmannians and two exceptional minuscule homogeneous varieties (see e.g. [3], [4], [22], [23] and [5], respectively, and the excellent survey [9]). Nevertheless, the quantum Schubert structure constants had only been computed explicitly for very few cases, such as complex Grassmannians and complete flag varieties of type A.

In this article, we give an explicit formula for the (equivariant) quantum Schubert structure constants of the quantum cohomology ring $QH^*(G/B)$ (for partial flag varieties G/P, see [29]). We should note that an algorithm to determine the equivariant quantum Schubert structure constants[1] was obtained earlier by Mihalcea [33] and he used it to find a characterization of the torus-equivariant quantum cohomology $QH_T^*(G/P)$. To describe the formula, we consider the affine Weyl group $W_{\rm af} = W \ltimes Q^\vee$, which is the semidirect product of the Weyl group W and the coroot lattice $Q^\vee$. For any $x, y \in W_{\rm af}$, we define $c_{x,[y]}$ and $d_{x,[y]}$ combinatorially, which are rational functions in simple roots α_i. In particular, for $x = ut_A, y = vt_A$ and $z = wt_{2A+\lambda}$ with $A = -12n(n+1)\sum_{i=1}^n w_i^\vee$ a sum of fundamental coweights $w_i^\vee$, the rational function $\sum_{\lambda_1,\lambda_2\in Q^\vee} c_{x,[t_{\lambda_1}]}c_{y,[t_{\lambda_2}]}d_{z,[t_{\lambda_1+\lambda_2}]}$ will be shown to be a constant, provided that $\langle\lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w)$, where ρ is the summation of fundamental weights w_i. Furthermore, this number coincides with $N_{u,v}^{w,\lambda}$ as stated in our main theorem.

Main Theorem. *Let $u, v, w \in W$, $\lambda \in Q^\vee, \lambda \succcurlyeq 0$. Let $A = -12n(n+1)\sum_{i=1}^n w_i^\vee$. The quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ for G/B is given by*

$$N_{u,v}^{w,\lambda} = \sum_{\lambda_1,\lambda_2\in Q^\vee} c_{ut_A,[t_{\lambda_1}]}c_{vt_A,[t_{\lambda_2}]}d_{wt_{2A+\lambda},[t_{\lambda_1+\lambda_2}]},$$

provided that $\langle\lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w)$ and zero otherwise.

The above summation does make sense, since there are in fact only finitely many nonzero terms involved. Indeed, the summation over the infinite set $Q^\vee$ can be simplified to the finite set $\Gamma \times W$ with $\Gamma = \{(\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 \succcurlyeq A, \lambda_1 + \lambda_2 \preccurlyeq 2A + \lambda, \lambda_1 \text{ and } \lambda_2 \text{ are anti-dominant elements in } Q^\vee\}$, and we obtain

$$N_{u,v}^{w,\lambda} = \sum_{(\lambda_1,\lambda_2,v_1)\in\Gamma\times W} c_{ut_A,[v_1t_{\lambda_1}]}c_{vt_A,[v_1t_{\lambda_2}]}d_{wt_{2A+\lambda},[v_1t_{\lambda_1+\lambda_2}]}.$$

[1]Explicitly, the equivariant (quantum) Schubert structure constants are homogeneous polynomials.

Quantum Schubert structure constants for G/P can be identified with certain quantum Schubert structure constants for G/B via the Peterson-Woodward comparison formula [39]; the corresponding formula and its applications are discussed in [29].

When v is a simple reflection, the equivariant quantum product $\sigma^u \star \sigma^v$ can be given explicitly by the equivariant quantum Chevalley formula. This formula was originally stated by Peterson in his unpublished lecture notes [35] and has recently been proved by Mihalcea [33]. In [33], Mihalcea also showed that the multiplication in $QH_T^*(G/B)$ is determined by the equivariant quantum Chevalley formula together with a few other natural properties (see e.g. Proposition 4.18). As a consequence, a recursive algorithm to determine $N_{u,v}^{w,\lambda}$ was given in [33]. However, an explicit formula is still lacking.

In [35] Peterson already stated that $QH_T^*(G/B)$ is ring isomorphic to $H_*^T(\Omega K)$ after localization; this is called Peterson's Theorem. Here ΩK is the based loop group of the maximal compact subgroup K of G and the *Pontryagin product* defines a ring structure on its (Borel-Moore) homology group $H_*^T(\Omega K)$. In Peterson's notes [35], the powerful tool of the nil-Hecke ring of Kostant-Kumar [21] was used heavily. Lam and his coauthors had done many important works along this direction, such as [25], [28], [26] and [27]. The proofs of Peterson's Theorem in [35] are incomplete, and in [26], Lam and Shimozono proved this result, with the help of Peterson's j-isomorphism.

The homology $H_*^T(\Omega K)$ is an associative algebra over $S = H_T^*(\mathrm{pt})$ and it has an additive S-basis given by Schubert homology classes $\{\mathfrak{S}_x \mid x \in W_{\mathrm{af}}^-\}$, where W_{af}^- is the set of minimal length representatives of cosets in W_{af}/W. We obtain the following explicit formula for the Pontryagin product of Schubert classes in $H_*^T(\Omega K)$, based on well-known localization formulas due to Arabia [1] for affine flag manifolds.

Theorem 3.3. *For any Schubert classes $\mathfrak{S}_x$ and $\mathfrak{S}_y$ in $H_*^T(\Omega K)$, the structure coefficients for their Pontryagin product*

$$\mathfrak{S}_x\mathfrak{S}_y = \sum_{z \in W_{af}^-} b_{x,y}^z \mathfrak{S}_z$$

are given by

$$b_{x,y}^z = \sum_{\lambda,\mu \in Q^\vee} c_{x,[t_\lambda]} c_{y,[t_\mu]} d_{z,[t_{\lambda+\mu}]}.$$

In the present paper, we give an alternative proof of Peterson's Theorem, in the sense that we find elementary proofs of the following two formulas of Peterson-Lam-Shimozono [26] on the Pontryagin product of certain Schubert classes, by analyzing the combinatorial nature of the summation in the formula of $b_{x,y}^z$:

(i) *For any $wt_\lambda, t_\mu \in W_{\mathrm{af}}^-$, one has $\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu} = \mathfrak{S}_{wt_{\lambda+\mu}}$.*

(ii) *For any $\sigma_i t_\lambda, ut_\mu \in W_{\mathrm{af}}^-$ with $\sigma_i = \sigma_{\alpha_i}$, $i \in I$, one has*

$$\mathfrak{S}_{\sigma_i t_\lambda}\mathfrak{S}_{ut_\mu} = (u(w_i) - w_i)\mathfrak{S}_{ut_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_1} \langle\gamma^\vee, w_i\rangle \mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_2} \langle\gamma^\vee, w_i\rangle \mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}},$$

where $\Gamma_1 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1\}$ and $\Gamma_2 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1 - \langle\gamma^\vee, 2\rho\rangle\}$.

Indeed, Lam and Shimozono noticed that combining the above formulas with the criterion of Mihalcea gives a proof of Peterson's Theorem. This in turn shows that any structure constant of $QH_T^*(G/B)$ coincides with certain structure constants of $H_*^T(\Omega K)$, which yields our Main Theorem. In particular, there is a choice of certain $b_{x,y}^z$'s which coincide with the same $N_{u,v}^{w,\lambda}$. For instance, we can choose one such A as in the Main Theorem to make a certain choice $(x,y,z)=(ut_A, vt_A, wt_{2A+\lambda})$. In many cases, we can replace it by a *smaller* one (see section 4.3 for more details on the choices). As a consequence, there are only a few nonzero terms in the summation for $N_{u,v}^{w,\lambda}$ in many cases. For instance for $G=SL(3,\mathbb{C})$ with $u=v=s_1s_2s_1, w=s_1s_2$ and $\lambda=\theta^\vee$, where θ is the highest root (see section 5.1 for more details on the notation), it suffices to take $A=-\theta^\vee$ and the summation for $N_{u,v}^{w,\lambda}$ in fact contains one term only, namely $N_{u,v}^{w,\lambda}=c_{s_0,[s_0]}^2 d_{s_2s_0,[s_1s_2s_1t_{-2\theta^\vee}]}$, where $c_{s_0,[s_0]}=(-1)^1\frac{1}{s_0(\alpha_0)}\big|_{\alpha_0=-\theta}=-\frac{1}{\theta}$ and $d_{s_2s_0,[s_1s_2s_1t_{-2\theta^\vee}]}=d_{s_2s_0,[s_0s_1s_2s_1s_0]}=s_0s_1(\alpha_2)s_0s_1s_2s_1(\alpha_0)\big|_{\alpha_0=-\theta}=\theta^2$ by definition. Hence, $N_{u,v}^{w,\lambda}=(-\frac{1}{\theta})^2\cdot\theta^2=1$. This coefficient can also be determined by Mihalcea's algorithm, but our formula is more effective. To show the computational power of our formula, we will compute some nontrivial coefficients for the higher rank group $Spin(7,\mathbb{C})$.

There could be an alternative way to determine our structure coefficients by finding polynomial representatives for Schubert classes. For instance, this approach has been used by Fomin, Gelfand and Postnikov for complete flag varieties of type A [8]. The work of Magyar [31] could be relevant for general cases. See also [7].

This paper is organized as follows. In section 2, we set up the notation that will be used throughout this article and review some well-known facts on the theory of Kac-Moody algebras and groups. In section 3, we define the important quantities $c_{x,[y]}$, $d_{x,[y]}$ and derive an explicit formula for the Pontryagin product on $H_*^T(\Omega K)$. In section 4, we analyze our formula and prove our Main Theorem. In section 5, we give examples to demonstrate the effectiveness of our formula. The proofs of some propositions stated in section 4 are given in the the appendix.

2. Notation

2.1. **Notation.** We introduce the notation used throughout the paper.

- G: a simply-connected simple complex Lie group of rank n.
- B,H: B is a Borel subgroup of G; H is a maximal torus of G contained in B.
- K: a maximal compact subgroup of G.
- T: $T=K\cap H$ is a maximal torus in K.
- $\mathfrak{g},\mathfrak{h}$: $\mathfrak{g}=\mathrm{Lie}(G)$; $\mathfrak{h}=\mathrm{Lie}(H)$.
- I,I_{af}: $I=\{1,\cdots,n\}$; $I_{\mathrm{af}}=\{0,1,\cdots,n\}$.
- R,Δ: R is the root system of $(\mathfrak{g},\mathfrak{h})$; $\Delta=\{\alpha_i\mid i\in I\}$ is a basis of simple roots.
- R^+: $R^+=R\cap\bigoplus_{i\in I}\mathbb{Z}_{\geq0}\alpha_i$ is the set of the positive roots; $R=\big(-R^+\big)\bigsqcup R^+$.
- $\alpha_i^\vee$,$Q^\vee$: $\{\alpha_i^\vee\mid i\in I\}$ are the simple coroots; $Q^\vee=\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$ is the coroot lattice.
- $\tilde{Q}^\vee$: $\tilde{Q}^\vee=\{\mu\in Q^\vee\mid\langle\mu,\alpha_i\rangle\leq0,\ i\in I\}$ is the set of anti-dominant elements.
- w_i,ρ: $\{w_i\mid i\in I\}$ are the fundamental weights; $\rho=\frac{1}{2}\sum_{\beta\in R^+}\beta\ \big(=\sum_{i\in I}w_i\big)$.
- $w_i^\vee$: $\{w_i^\vee\mid i\in I\}$ are the fundamental coweights.
- W: $W=\langle\sigma_{\alpha_i}\ :\ i\in I\rangle$ is the Weyl group of $(\mathfrak{g},\mathfrak{h})$.
- θ,ω_0: θ is the highest (long) root of R; ω_0 is the longest element in W.
- $\mathfrak{g}_{\mathrm{af}}$: the (untwisted) affine Kac-Moody algebra associated to $\mathfrak{g}$.
- $\mathfrak{h}_{\mathrm{af}}$: Cartan subalgebra of $\mathfrak{g}_{\mathrm{af}}$.

α_0, δ: α_0 is the affine simple root; $\delta = \alpha_0 + \theta$ is the null root.
$R^+_{\rm re}$: $R^+_{\rm re} = \{\alpha + m\delta \mid \alpha \in R, m \in \mathbb{Z}^+\} \cup R^+$ is the set of positive real roots.
$\mathcal{S}, Y$: $\mathcal{S} = \{\sigma_{\alpha_i} \mid i \in I_{\rm af}\}$; $Y \subset \Delta$ is a subset.
$W_{\rm af}$: the Weyl group of $\mathfrak{g}_{\rm af}$; $W_{\rm af} = \langle \sigma_{\alpha_i} : i \in I_{\rm af}\rangle$.
$W_{{\rm af},Y}$: the subgroup of $W_{\rm af}$ generated by $\{\sigma_\alpha \mid \alpha \in Y\}$.
$W^Y_{\rm af}$: the subset $\{x \in W_{\rm af} \mid \ell(x) \leq \ell(y), \forall y \in xW_{{\rm af},Y}\}$ of $W_{\rm af}$.
$\mathcal{G}$: the Kac-Moody group associated to the Kac-Moody algebra $\mathfrak{g}_{\rm af}$.
$\mathcal{B}$: the standard Borel subgroup of $\mathcal{G}$.
$\mathcal{P}_Y$: the standard parabolic subgroup of $\mathcal{G}$ associated to Y; $\mathcal{P}_Y \supset \mathcal{B}$.
$W^-_{\rm af}$: $W^-_{\rm af} = W^\Delta_{\rm af}$.
$\mathcal{P}_0$: $\mathcal{P}_0 = \mathcal{P}_\Delta$.
LK: $LK = \{f : \mathbb{S}^1 \to K \mid f \text{ is smooth}\}$.
ΩK: $\Omega K = \{f \in LK \mid f(1_{\mathbb{S}^1}) = 1_K\}$.
$S, \hat{S}$: $S = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$; $\hat{S} = \mathbb{Q}[\alpha_0, \alpha_1, \cdots, \alpha_n]$.
q_λ: $q_\lambda = q_1^{a_1} \cdots q_n^{a_n}$ for $\lambda = \sum_{i=1}^n a_i\alpha_i^\vee \in Q^\vee$.
σ_i, σ_β: $\sigma_i = \sigma_{\alpha_i}$ is a simple reflection; σ_β is a reflection for $\beta \in R^+_{\rm re} \sqcup \big(-R^+_{\rm re}\big)$.
σ_u, σ^u: Schubert classes for G/B, where $u \in W$; $\sigma_u \in H_{2\ell(u)}(G/B, \mathbb{Z})$ and $\sigma^u \in H^{2\ell(u)}(G/B, \mathbb{Z})$ are defined in section 6.3.
$\mathfrak{S}_x$: Schubert homology classes for $\mathcal{G}/\mathcal{P}_Y$ as defined in section 6.4; $\mathfrak{S}_x \in H_{2\ell(x)}(\mathcal{G}/\mathcal{P}_Y, \mathbb{Z})$.
$\mathfrak{S}^x$: Schubert cohomology classes for $\mathcal{G}/\mathcal{P}_Y$ as defined in section 6.4; $\mathfrak{S}^x \in H^{2\ell(x)}(\mathcal{G}/\mathcal{P}_Y, \mathbb{Z})$.
$c_{x,[y]}$: defined in section 3.1 for any $x, y \in W^-_{\rm af}$.
$d_{x,[y]}$: defined in section 3.1 for any $x, y \in W^-_{\rm af}$.
$c'_{x,y}$: $c'_{x,y} = c_{x,y}\big|_{\alpha_0 = -\theta}$ with $c_{x,y}$ defined in section 3.1.
Γ_1: $\Gamma_1(u) = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1\}$, or simply $\Gamma_1 = \Gamma_1(u)$.
Γ_2: $\Gamma_2(u) = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1 - \langle\gamma^\vee, 2\rho\rangle\}$, or simply $\Gamma_2 = \Gamma_2(u)$.

2.2. **Some more explanations.** See [17] and [24] for the meaning of the notation as in section 2.1 as well as the theory of Kac-Moody algebras and groups.

The fundamental weights $\{w_i \mid i \in I\}$ are the dual basis to the simple coroots $\{\alpha_i^\vee \mid i \in I\}$ with respect to the natural pairing $\langle\cdot,\cdot\rangle : \mathfrak{h} \times \mathfrak{h}^* \to \mathbb{C}$. The simple reflections $\{\sigma_i = \sigma_{\alpha_i} \mid i \in I\}$ act on $\mathfrak{h}$ by $\sigma_i(\lambda) = \lambda - \langle\lambda, \alpha_i\rangle\alpha_i^\vee$ for $\lambda \in \mathfrak{h}$. Therefore the Weyl group W, which is generated by the simple reflections, acts on $\mathfrak{h}$ and $\mathfrak{h}^*$ naturally. Note that $R = W \cdot \Delta$. For any $\gamma \in R$, $\gamma = w(\alpha_i)$ for some $w \in W$ and $i \in I$. We can well define $\gamma^\vee = w(\alpha_i^\vee)$, which is independent of the expressions of γ.

The Weyl group $W_{\rm af}$ of $\mathfrak{g}_{\rm af}$ is in fact an affine group, $W_{\rm af} = W \ltimes Q^\vee$, where we denote t_λ[2] as the image of $\lambda \in Q^\vee$ in $W_{\rm af}$ (by abusing notation). To be more precise, one has $\sigma_\beta = \sigma_\alpha t_{m\alpha^\vee}$ for $\beta = \alpha + m\delta \in R_{\rm re} = \big(-R^+_{\rm re}\big) \sqcup R^+_{\rm re}$. In particular, $\sigma_{\alpha_0} = \sigma_\theta t_{-\theta^\vee}$. Given $w \in W, \lambda \in Q^\vee, \gamma \in \bigoplus_{i\in I}\mathbb{Z}\alpha_i$ and $m \in \mathbb{Z}$, we have $t_{w\cdot\lambda} = wt_\lambda w^{-1}$ and the following action:

$$wt_\lambda \cdot (\gamma + m\delta) = w \cdot \gamma + (m - \langle\lambda, \gamma\rangle)\delta.$$

Since $\big(W_{\rm af}, \mathcal{S}\big)$ is a Coxeter system, we can define the length function $\ell : W_{\rm af} \to \mathbb{Z}_{\geq 0}$ and the Bruhat order $(W_{\rm af}, \preccurlyeq)$ (see e.g. [16]). We use the notation

$$x = [\sigma_{\beta_1} \cdots \sigma_{\beta_r}]_{\rm red}$$

[2] The notation t_λ is used instead of $t_{\nu(\lambda)}$ as in chapter 6 of [17].

whenever $(\sigma_{\beta_1}, \cdots, \sigma_{\beta_r})$ is a reduced decomposition of $x \in W_{\rm af}$; that is, $r = \ell(x)$, $x = \sigma_{\beta_1} \cdots \sigma_{\beta_r}$ and β_i's are simple roots. (It is possible that $\beta_i = \beta_j$ for $i \neq j$.) This notation will also be used throughout this article.

Explicitly, the affine Kac-Moody group $\mathcal{G}$ is realized as a central extension by $\mathbb{C}^*$ of the loop group consisting of the $\mathbb{C}((t))$-rational points $G(\mathbb{C}((t)))$ of G extended by one dimensional complex torus. For each subset $Y \subset \Delta$, there is a standard parabolic subgroup $P_Y \subset \mathcal{G}$ corresponding to Y. In particular, $\mathcal{B} = \mathcal{P}_\emptyset$ and we denote $\mathcal{P}_0 = \mathcal{P}_\Delta$. For our purpose of studying the generalized flag varieties $\mathcal{G}/\mathcal{B}$ and $\mathcal{G}/\mathcal{P}_0$, the group $\mathcal{G}$ can be taken simply to be $\mathcal{G} = G(\mathbb{C}((t)))$. That is, $\mathcal{G} = \mathrm{Mor}(\mathbb{C}^*, G)$. As a consequence, $\mathcal{P}_0 = G(\mathbb{C}[[t]]) = \mathrm{Mor}(\mathbb{C}, G)$ and $\mathcal{B} = \{f \in \mathcal{P}_0 \mid f(0) \in B\}$.

In the present paper, we only consider the following two cases: $Y = \emptyset$ and $Y = \Delta$. Note that $W_{\rm af,\emptyset} = \{1\}$, $W_{\rm af}^{\emptyset} = W_{\rm af}$ and $W_{\rm af,\Delta} = W$. We denote $W_{\rm af}^- = W_{\rm af}^{\Delta}$.

3. Pontryagin product on equivariant homology of ΩK

The T-equivariant (Borel-Moore) homology $H_*^T(\Omega K)$ of based loop group ΩK is a module over $S = H_T^*(\mathrm{pt}) = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ with an S-basis of Schubert classes $\{\mathfrak{S}_x \mid x \in W_{\rm af}^-\}$, where $W_{\rm af}^-$ is the set of minimal length representatives of cosets in $W_{\rm af}/W$. T acts on ΩK by pointwise conjugation. The Pontryagin product $\Omega K \times \Omega K \to \Omega K$, given by $(f \cdot g)(t) = f(t) \cdot g(t)$, is associative and T-equivariant. Therefore, it induces a product map $H_*^T(\Omega K) \otimes H_*^T(\Omega K) \to H_*^T(\Omega K)$, making $H_*^T(\Omega K)$ an associative S-algebra. The structure constants $b_{x,y}^z \in S$ are defined by

$$\mathfrak{S}_x \mathfrak{S}_y = \sum\nolimits_{z \in W_{\rm af}^-} b_{x,y}^z \mathfrak{S}_z$$

for $x, y \in W_{\rm af}^-$. The main result of this section is Theorem 3.3, giving an explicit formula for the Pontryagin product as $b_{x,y}^z = \sum_{\lambda,\mu \in Q^\vee} c_{x,[t_\lambda]} c_{y,[t_\mu]} d_{z,[t_{\lambda+\mu}]}$ in which the summation is in fact only over finitely many nonzero terms and $c_{x,[y]}, d_{x,[y]}$ are defined combinatorially as below. Due to Peterson's Theorem which was proved by Lam and Shimozono, these structure coefficients $b_{x,y}^z$ correspond to quantum Schubert structure constants for the equivariant quantum cohomology $QH_T^*(G/B)$.

3.1. Definitions and properties of $c_{x,[y]}$ and $d_{x,[y]}$.

Definition 3.1. For any $x, y \in W_{\rm af}$, we define the homogeneous rational function $c_{x,y} = c_{x,y}(\alpha_0, \cdots, \alpha_n) \in \mathbb{Q}[\alpha_0^{\pm 1}, \alpha_1^{\pm 1}, \cdots, \alpha_n^{\pm 1}]$ as follows. Let $x = [\sigma_{\beta_1} \cdots \sigma_{\beta_m}]_{\rm red}$. If $y \not\preccurlyeq x$, then $c_{x,y} = 0$, and if $y \preccurlyeq x$, then

$$c_{x,y} := (-1)^m \sum \left(\sigma_{\beta_1}^{\varepsilon_1}(\beta_1) \sigma_{\beta_1}^{\varepsilon_1} \sigma_{\beta_2}^{\varepsilon_2}(\beta_2) \cdots \sigma_{\beta_1}^{\varepsilon_1} \cdots \sigma_{\beta_m}^{\varepsilon_m}(\beta_m)\right)^{-1},$$

where the summation runs over all $(\varepsilon_1, \cdots, \varepsilon_m) \in \{0,1\}^m$ satisfying $\sigma_{\beta_1}^{\varepsilon_1} \cdots \sigma_{\beta_m}^{\varepsilon_m} = y$.

We define $c_{x,[y]} \in \mathbb{Q}[\alpha_1^{\pm 1}, \cdots, \alpha_n^{\pm 1}]$ as follows:

$$c_{x,[y]} := \sum\nolimits_{z \in yW} c_{x,z}\big|_{\alpha_0 = -\theta} = \sum\nolimits_{z \in yW} c_{x,z}(-\theta, \alpha_1, \cdots, \alpha_n).$$

Let γ_k denote the (positive real) root $\sigma_{\beta_1} \cdots \sigma_{\beta_{k-1}}(\beta_k)$. We define the homogeneous polynomial $d_{y,x} = d_{y,x}(\alpha_0, \alpha_1, \cdots, \alpha_n) \in \mathbb{Q}[\alpha_0, \cdots, \alpha_n]$ as follows.

If $y \not\preccurlyeq x$, then $d_{y,x} = 0$; if $y = 1$, then $d_{y,x} = 1$; if $y \preccurlyeq x$ and $y \neq 1$, then

$$d_{y,x} := \sum \gamma_{i_1} \cdots \gamma_{i_r},$$

where the summation runs over all subsequences $(i_1, \cdots, i_r)$ of $(1, \cdots, m)$ such that $y = [\sigma_{\beta_{i_1}} \cdots \sigma_{\beta_{i_r}}]_{\rm red}$.

We define $d_{y,[x]} \in \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ as follows:

$$d_{y,[x]} := d_{y,x}\big|_{\alpha_0=-\theta} = d_{y,x}(-\theta, \alpha_1, \cdots, \alpha_n).$$

Note that for any $y, y' \in W_{\rm af}$ with $yW = y'W$, one has $c_{x,[y]} = c_{x,[y']}$. In addition, one has $d_{x,[y]} = d_{x,[y']}$ provided $x \in W_{\rm af}^-$ (following from Lemma 4.14).

Proposition 3.2 ([21]; see also chapter 11 of [24]). *$c_{x,y}$ and $d_{y,x}$ are well-defined, independent of the choices of reduced decompositions of x. The transpose of $\big(c_{x,y}\big)$ is the inverse of the matrix $\big(d_{x,y}\big)$ in the following sense:*

$$\sum\nolimits_{z\in W_{\rm af}} c_{x,z}d_{y,z} = \delta_{x,y} = \sum\nolimits_{z\in W_{\rm af}} c_{z,x}d_{z,y}, \quad \textit{for any} \ \ x, y \in W_{\rm af}.$$

Note that both summations in the above proposition contain only finitely many nonzero terms.

3.2. **Explicit formula for Pontryagin product on $H_*^T(\Omega K)$.** Because of the homotopy-equivalence between $\mathcal{G}/\mathcal{P}_0$ and ΩK, we interchange the notation $\mathcal{G}/\mathcal{P}_0$ and ΩK freely. Let $\hat{T}_{\mathbb{C}}$ denote the standard maximal torus of $\mathcal{G}$ with maximal compact subtorus $\hat{T}$. The $\hat{T}$-equivariant cohomology $H^*_{\hat{T}}(\mathcal{G}/\mathcal{P}_0)$ is an $\hat{S}$-algebra with a basis of Schubert classes $\{\hat{\mathfrak{S}}^x \mid x \in W_{\rm af}^-\}$, where $\hat{S} = H^*_{\hat{T}}(\mathrm{pt}) = \mathbb{Q}[\alpha_0, \alpha_1, \cdots, \alpha_n]$. Note that $T \subset \hat{T}$ is a subtorus. The T-equivariant cohomology $H^*_T(\mathcal{G}/\mathcal{P}_0)$ is an S-algebra with a basis of Schubert classes $\{\mathfrak{S}^x \mid x \in W_{\rm af}^-\}$, where $S = H^*_T(\mathrm{pt}) = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$. Furthermore, one has the evaluation maps $\mathrm{ev} : H^*_{\hat{T}}(\mathcal{G}/\mathcal{P}_0) \to H^*_T(\mathcal{G}/\mathcal{P}_0)$ and $ev : \hat{S} \to S$ such that $\mathrm{ev}(f\hat{\mathfrak{S}}^x) = ev(f)\mathfrak{S}^x$, where $f = f(\alpha_0, \alpha_1, \cdots, \alpha_n) \in \hat{S}$ and $ev(f) = f(-\theta, \alpha_1, \cdots, \alpha_n) \in S$. (See section 6.4 of the appendix for more details on the above descriptions.)

The T-equivariant homology $H_*^T(\mathcal{G}/\mathcal{P}_0)$ is the submodule of $\mathrm{Hom}_S(H^*_T(\mathcal{G}/\mathcal{P}_0), S)$ spanned by the equivariant Schubert homology classes $\{\mathfrak{S}_x \mid x \in W_{\rm af}^-\}$,[3] which for any $x, y \in W_{\rm af}^-$ satisfy $\langle \mathfrak{S}_x, \mathfrak{S}^y \rangle = \delta_{x,y}$ with respect to the natural pairing.

The adjoint action of T on K induces a canonical action on ΩK by pointwise conjugation; that is, $(t \cdot f)(s) := t \cdot f(s) \cdot t^{-1}$ for any $t \in T$ and $f \in \Omega K$. The group multiplication $K \times K \to K$ induces a so-called Pontryagin product $\Omega K \times \Omega K \to \Omega K$ by pointwise multiplication; that is, $(f \cdot g)(s) = f(s) \cdot g(s)$ for any $f, g \in \Omega K$. The Pontryagin product is obviously associative and T-equivariant. Therefore it induces $H_*^T(\Omega K) \otimes H_*^T(\Omega K) \to H_*^T(\Omega K)$, which is also called the Pontryagin product. As a consequence, $H_*^T(\Omega K)$ is an associative S-algebra (see [35], [25]), and therefore the structure coefficients $b_{x,y}^z$ for the Pontryagin product

$$\mathfrak{S}_x\mathfrak{S}_y = \sum\nolimits_{z\in W_{\rm af}^-} b_{x,y}^z \mathfrak{S}_z$$

for $x, y \in W_{\rm af}^-$ are polynomials in S. Now we state the main result of this section as follows.

Theorem 3.3. *For any $x, y, z \in W_{\rm af}^-$, the structure coefficient $b_{x,y}^z$ for $H_*^T(\Omega K)$ is given by*

$$b_{x,y}^z = \sum_{\lambda,\mu\in Q^\vee} c_{x,[t_\lambda]}c_{y,[t_\mu]}d_{z,[t_{\lambda+\mu}]}.$$

[3]We should note that we are using the equivariant Borel-Moore homology (see e.g. [13]).

In particular, $b^z_{x,y} = b^z_{y,x}$ for all z, which implies $\mathfrak{S}_x\mathfrak{S}_y = \mathfrak{S}_y\mathfrak{S}_x$. Furthermore, $c_{x,[t_\lambda]} \neq 0$ only if there exists $w \in t_\lambda W$ such that $c_{x,w} \neq 0$, which holds only if $w \preccurlyeq x$. In particular, there are only finitely many nonzero terms of $c_{x,[t_\lambda]}$ and $c_{y,[t_\mu]}$ once x and y are fixed. Hence, the above summation for $b^z_{x,y}$ does make sense.

Note that $\mathcal{P}_0 = \mathcal{P}_\Delta$ and $W^-_{\mathrm{af}} = W^\Delta_{\mathrm{af}}$. By replacing Δ with a general subset $Y \subset \Delta$, $H^T_*(\mathcal{G}/\mathcal{P}_Y)$ and $\mathfrak{S}^x_Y$ can be defined in a similar manner. To distinguish these with the case of our main interest $\mathcal{G}/\mathcal{P}_0$, we denote $\mathfrak{S}^x_0, \mathfrak{S}^0_x$ for the case $Y = \emptyset$ (note that $\mathcal{P}_\emptyset = \mathcal{B}$). These notions can be extended to $\hat{T}$-equivariant (co)homology for $\mathcal{G}/\mathcal{P}_Y$ for a larger $\hat{T}$-action. The corresponding Schubert classes are denoted by "$\hat{\mathfrak{S}}$" instead of "$\mathfrak{S}$".

Definition 3.4. Given $x \in W_{\mathrm{af}}$, we define the element ψ^Y_x in $\mathrm{Hom}_S(H^*_T(\mathcal{G}/\mathcal{P}_Y), S)$ to be the canonical morphism $\psi^Y_x := (\iota^Y_x)^* : H^*_T(\mathcal{G}/\mathcal{P}_Y) \to H^*_T(\mathrm{pt}) = S$, where ι^Y_x is the T-equivariant map $\iota^Y_x : \mathrm{pt} \to \mathcal{G}/\mathcal{P}_Y$ given by $\mathrm{pt} \mapsto x\mathcal{P}_Y$.

We define $\hat{\psi}^Y_x$ to be the element $\hat{\psi}^Y_x = (\iota^Y_x)^*$ in $\mathrm{Hom}_{\hat{S}}(H^*_{\hat{T}}(\mathcal{G}/\mathcal{P}_Y), \hat{S})$ by considering the action by the larger group $\hat{T}$.

Since we consider the cases $Y = \Delta$ and $Y = \emptyset$ only, we simply denote

$$\psi_x = \psi^\Delta_x; \quad \hat{\psi}_x = \hat{\psi}^\Delta_x; \quad \psi^0_x = \psi^\emptyset_x; \quad \hat{\psi}^0_x = \hat{\psi}^\emptyset_x.$$

The relation between $\hat{\mathfrak{S}}^0_x$ and $\hat{\psi}^0_y$ is expressed in the following lemma.

Lemma 3.5 (Proposition 3.3.1 of [1][4])**.** *For any* $x \in W_{\mathrm{af}}$, $\hat{\mathfrak{S}}^0_x = \sum_{y \in W_{\mathrm{af}}} c_{x,y}\hat{\psi}^0_y$.

Note that one has $\psi_x = \psi_y$ whenever $xW = yW$ (following the definition) and that the canonical map $\pi_* : H^T_*(\mathcal{G}/\mathcal{B}) \to H^T_*(\mathcal{G}/\mathcal{P}_0)$, induced by the natural projection $\pi : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_0$, is given by (see e.g. Lemma 11.3.3 of [24])

$$\pi_*(\mathfrak{S}^0_x) = \begin{cases} 0, & \text{if } x \notin W^-_{\mathrm{af}}, \\ \mathfrak{S}_x, & \text{if } x \in W^-_{\mathrm{af}}. \end{cases}$$

Proposition 3.6. (i) *For any* $x \in W_{\mathrm{af}}$, $\psi^0_x = \sum_{y \in W_{\mathrm{af}}} d_{y,[x]}\mathfrak{S}^0_y$ *in* $H^T_*(\mathcal{G}/\mathcal{B})$.

(ii) *For any* $x \in W^-_{\mathrm{af}}$, $\psi_x = \sum_{y \in W^-_{\mathrm{af}}} d_{y,[x]}\mathfrak{S}_y$ *in* $H^T_*(\mathcal{G}/\mathcal{P}_0)$.

(iii) *For any* $x \in W^-_{\mathrm{af}}$, $\mathfrak{S}_x = \sum_{t \in Q^\vee} c_{x,[t]}\psi_t$.

Proof. (i) It follows from Proposition 3.2 and Lemma 3.5 that

$$\hat{\psi}^0_x = \sum_{z \in W_{\mathrm{af}}} \delta_{x,z}\hat{\psi}^0_z = \sum_{z \in W_{\mathrm{af}}}\sum_{y \in W_{\mathrm{af}}} d_{y,x}c_{y,z}\hat{\psi}^0_z = \sum_{y \in W_{\mathrm{af}}} d_{y,x}\hat{\mathfrak{S}}^0_y.$$

Note that $\iota^\emptyset_x$ is both T-equivariant and $\hat{T}$-equivariant and that $\mathrm{id}_{\mathrm{pt}}$, $\mathrm{id}_{\mathcal{G}/\mathcal{B}}$ are morphisms preserving the T-action. Hence, the identity $\mathrm{id}_{\mathcal{G}/\mathcal{B}} \circ \iota^\emptyset_x = \iota^\emptyset_x \circ \mathrm{id}_{\mathrm{pt}}$ induces $\psi^0_x \circ \mathrm{ev} = ev \circ \hat{\psi}^0_x : H^*_{\hat{T}}(\mathcal{G}/\mathcal{B}) \to S$. Therefore,

$$\psi^0_x(\mathfrak{S}^y_0) = \psi^0_x \circ \mathrm{ev}(\hat{\mathfrak{S}}^y_0) = ev \circ \hat{\psi}^0_x(\hat{\mathfrak{S}}^y_0) = ev(d_{y,x}) = d_{y,[x]}.$$

Hence, $\psi^0_x = \sum_{y \in W_{\mathrm{af}}} d_{y,[x]}\mathfrak{S}^0_y$, where the summation contains only finitely many nonzero terms since $d_{y,[x]} = 0$ whenever $y \not\preccurlyeq x$. Thus $\psi^0_x \in H^T_*(\mathcal{G}/\mathcal{B})$.

[4]The terminologies used in [1] and the present paper can be identified as follows: $\mathcal{L}_x = \hat{\mathfrak{S}}^0_x$ and $\Theta(\mu)(y) = \hat{\psi}^0_y(\mu)$ for $\mu \in H^*_{\hat{T}}(\mathcal{G}/\mathcal{B})$.

(ii) Note that ι_x^Δ, $\iota_x^\emptyset$ and π are all T-equivariant maps. Hence, the identity $\iota_x^\Delta = \pi \circ \iota_x^\emptyset$ induces $\psi_x = \psi_x^0 \circ \pi^* : H_T^*(\mathcal{G}/\mathcal{P}_0) \to S$. Therefore,

$$\psi_x = \pi_*(\psi_x^0) = \pi_*(\sum\nolimits_{y\in W_{\rm af}} d_{y,[x]}\mathfrak{S}_y^0) = \sum\nolimits_{y\in W_{\rm af}^-} d_{y,[x]}\mathfrak{S}_y \in H_*^T(\mathcal{G}/\mathcal{P}_0),$$

noting that the summation contains only finitely many nonzero terms.

(iii) Denote $c'_{x,y} = c_{x,y}\big|_{\alpha_0=-\theta}$. It follows from (ii) and Proposition 3.2 that

$$\mathfrak{S}_x = \sum_{y\in W_{\rm af}} \delta_{x,y}\mathfrak{S}_y = \sum_{y\in W_{\rm af}}\sum_{z\in W_{\rm af}} c'_{x,z}d_{y,[z]}\mathfrak{S}_y = \sum_{z\in W_{\rm af}} c'_{x,z}\psi_z.$$

Note that $\psi_z = \psi_y$ whenever $z\in yW$ and that each coset yW has a unique representative of translation in $Q^\vee \cong W_{\rm af}/W$. Hence for any $x \in W_{\rm af}^-$,

$$\mathfrak{S}_x = \sum_{z\in W_{\rm af}} c'_{x,z}\psi_z = \sum_{t\in Q^\vee}\big(\sum_{z\in tW} c'_{x,z}\big)\psi_t = \sum_{t\in Q^\vee} c_{x,[t]}\psi_t.$$

□

The above proposition was essentially contained in Peterson's notes [35].

The Pontryagin product gives an associative S-algebra structure on the equivariant homology $H_*^T(\Omega K)$. The following proposition was stated in [35] by Peterson. We learned the following proof from Thomas Lam.

Proposition 3.7. *For any $\lambda,\mu\in Q^\vee$, the Pontryagin product of ψ_{t_λ} and ψ_{t_μ} in $H_*^T(\Omega K)$ is given by $\psi_{t_\lambda}\psi_{t_\mu} = \psi_{t_\lambda t_\mu} = \psi_{t_{\lambda+\mu}}$.*

Proof. Identify $\lambda\in Q^\vee$ with the cocharacter $\lambda : \mathbb{S}^1 \to T$, which gives a point $\lambda : \mathbb{S}^1 \to K$ in ΩK. These are the T-fixed points of ΩK. Note that ψ_t is the map $H_T^*(\Omega K) \to H_T^*(\mathrm{pt})$ induced by the map $\mathrm{pt} \to \Omega K$ with image t. Thus $\psi_{t_\lambda}\psi_{t_\mu}$ is the map $H_T^*(\Omega K) \longrightarrow H_T^*(\mathrm{pt})$ induced by the following composition of maps:

$$\mathrm{pt} \longrightarrow \Omega K\times\Omega K \longrightarrow \Omega K, \quad \text{which is given by } \mathrm{pt} \mapsto (t_\lambda, t_\mu) \mapsto t_\lambda t_\mu.$$

Note that pointwise multiplication on the group takes the loops $\lambda : \mathbb{S}^1 \to T$, $\mu : \mathbb{S}^1 \to T$ to the loop $(\lambda+\mu) : \mathbb{S}^1 \to T$. Thus $\psi_{t_\lambda}\psi_{t_\mu} = \psi_{t_\lambda t_\mu} = \psi_{t_{\lambda+\mu}}$. □

Now we can easily derive the proof of Theorem 3.3.

Proof of Theorem 3.3. It follows from Proposition 3.6 and Proposition 3.7 that

$$\begin{aligned}\mathfrak{S}_x\mathfrak{S}_y &= \big(\sum_{\lambda\in Q^\vee} c_{x,[t_\lambda]}\psi_{t_\lambda}\big)\big(\sum_{\mu\in Q^\vee} c_{y,[t_\mu]}\psi_{t_\mu}\big)\\ &= \sum\nolimits_{\lambda,\mu\in Q^\vee} c_{x,[t_\lambda]}c_{y,[t_\mu]}\psi_{t_\lambda}\psi_{t_\mu}\\ &= \sum\nolimits_{\lambda,\mu\in Q^\vee} c_{x,[t_\lambda]}c_{y,[t_\mu]}\psi_{t_{\lambda+\mu}}\\ &= \sum_{\lambda,\mu\in Q^\vee, z\in W_{\rm af}^-} c_{x,[t_\lambda]}c_{y,[t_\mu]}d_{z,[t_{\lambda+\mu}]}\mathfrak{S}_z.\end{aligned}$$

Hence, $b_{x,y}^z = \sum_{\lambda,\mu\in Q^\vee} c_{x,[t_\lambda]}c_{y,[t_\mu]}d_{z,[t_{\lambda+\mu}]}$. □

Remark 3.8. The equivariant Schubert structure constants for the equivariant cohomology of the based loop group ΩK can also be expressed in terms of $c_{x,[y]}$ and $d_{x,[y]}$. (See section 6.4 of the appendix for more details.)

Note that $c_{x,[t_\lambda]}, c_{y,[t_\mu]}$ and $d_{z,[t_{\lambda+\mu}]}$ are homogeneous rational functions of degree $-\ell(x), -\ell(y)$ and $\ell(z)$, respectively. Since $b_{x,y}^z$ is a polynomial, we obtain the following corollary.

Corollary 3.9. *Let $x, y, z \in W_{\rm af}^-$; one has $b_{x,y}^z = 0$ unless $\ell(z) \geq \ell(x) + \ell(y)$. Furthermore if $\ell(z) = \ell(x) + \ell(y)$, then the rational function $b_{x,y}^z$ is a constant.*

4. Explicit formula for (equivariant) quantum cohomology of G/B

In this section, we give an alternative proof of Peterson's Theorem proved by Lam and Shimozono, and derive the following combinatorial formula for quantum Schubert structure constants $N_{u,v}^{w,\lambda}$, which covers the Main Theorem as stated in the introduction.

Theorem 4.1. *For any $u, v, w \in W$, $\lambda \in Q^\vee$ with $\lambda \succcurlyeq 0$, the quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ for G/B is given as follows.*

Denote $A = -12n(n+1)\sum_{i\in I} w_i^\vee$ (which is in fact a regular and anti-dominant element in $Q^\vee$). Let $x = ut_A, y = vt_A$ and $z = wt_{2A+\lambda}$.

1. *If $\langle \lambda, 2\rho\rangle \neq \ell(u) + \ell(v) - \ell(w)$, then $N_{u,v}^{w,\lambda} = 0$.*
2. *If $\langle \lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w)$, then*
   - (a) *The rational function $\sum_{\lambda_1,\lambda_2\in Q^\vee} c_{x,[t_{\lambda_1}]}c_{y,[t_{\lambda_2}]}d_{z,[t_{\lambda_1+\lambda_2}]}$, which belongs to $\mathbb{Q}[\alpha_1^{\pm 1}, \cdots, \alpha_n^{\pm 1}]$, is in fact a constant.*
   - (b) *The quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ is given by*

$$N_{u,v}^{w,\lambda} = \sum_{\lambda_1,\lambda_2\in Q^\vee} c_{x,[t_{\lambda_1}]}c_{y,[t_{\lambda_2}]}d_{z,[t_{\lambda_1+\lambda_2}]}.$$

Furthermore, one has (by simplifying the summation)

$$N_{u,v}^{w,\lambda} = \sum_{(\lambda_1,\lambda_2,v_1)\in\Gamma\times W} c_{x,[v_1t_{\lambda_1}]}c_{y,[v_1t_{\lambda_2}]}d_{z,[v_1t_{\lambda_1+\lambda_2}]},$$

where $\Gamma = \{(\lambda_1, \lambda_2) \mid \lambda_1, \lambda_2 \succcurlyeq A, \lambda_1 + \lambda_2 \preccurlyeq 2A + \lambda, \lambda_1, \lambda_2 \in \tilde{Q}^\vee\}$.

The above theorem is in fact a direct consequence of Peterson's Theorem and Theorem 3.3, except for its last statement. As is shown in the above theorem, our formula is a combinatorial formula, other than a recursive algorithm. Although rational functions in equivariant parameters are involved in our formula, we can take their valuations at a special point so that the formula can be easily programmed.

In section 4.1, we calculate certain structure coefficients $b_{x,y}^z$ for the Pontryagin product on $H_*^T(\Omega K)$. These calculations allow us to re-establish the equivalence between the torus-equivariant quantum cohomology of G/B and the torus-equivariant homology of ΩK after localization, which is explained in section 4.2. Finally, in section 4.3, we prove Theorem 4.1.

4.1. **Calculations for structure coefficients $b_{x,y}^z$.** In this subsection, we analyze the summation in the formula for the structure coefficients $b_{x,y}^z$. We obtained several useful formulas, including the following two as the main results of this subsection. The proofs are elementary and combinatorial in nature. These two formulas have been stated in [35] and proved by Lam-Shimozono [26] by applying a different method using the nil-Hecke ring and the Peterson j-isomorphism.

Proposition 4.2. *For any $wt_\lambda, t_\mu \in W_{\rm af}^-$, one has $\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu} = \mathfrak{S}_{wt_{\lambda+\mu}}$.*

Proposition 4.3. *Let $\sigma_i t_\lambda, ut_\mu \in W_{\rm af}^-$, where $\sigma_i = \sigma_{\alpha_i}$ with $i \in I$. Then one has* $\mathfrak{S}_{\sigma_i t_\lambda}\mathfrak{S}_{ut_\mu} = (u(w_i) - w_i)\mathfrak{S}_{ut_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_1}\langle\gamma^\vee, w_i\rangle\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu}} + \sum_{\gamma\in\Gamma_2}\langle\gamma^\vee, w_i\rangle\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}}$,[5] *where Γ_1 and Γ_2 are as defined in section* 2.1.

Since the proofs of all the lemmas in this subsection only involve simple arguments, we leave them to the appendix (section 6.1).

Each element $x \in W_{\rm af} = W \ltimes Q^\vee$ can be written as $x = wt_\lambda$ for unique $w \in W$ and $t_\lambda \in Q^\vee$. Recall that for any $\mu \in Q^\vee$, μ is called regular if and only if $\langle\lambda, \alpha_i\rangle \neq 0$ for all $i \in I$, and μ is called anti-dominant if and only if $\langle\mu, \alpha_i\rangle \leq 0$ for all $i \in I$. We denote by $\tilde{Q}^\vee$ the set of anti-dominant elements in $Q^\vee$. The minimal length representatives in $W_{\rm af}^-$ are characterized as follows.

Lemma 4.4 (see e.g. [26])**.** *Let $\lambda \in Q^\vee$ and $w \in W$. Then $wt_\lambda \in W_{\rm af}^-$ if and only if $\lambda \in \tilde{Q}^\vee$, and if $\langle\lambda, \alpha_i\rangle = 0$, then $w(\alpha_i) \succcurlyeq 0$. When this happens, $\ell(wt_\lambda) = \ell(t_\lambda) - \ell(w)$. Furthermore, $\ell(t_\mu) = \langle w(\mu), -2\rho\rangle$ whenever $\mu \in Q^\vee$ satisfies $w(\mu) \in \tilde{Q}^\vee$.*

Recall that for any $x, y \in W_{\rm af}$, $y \preccurlyeq x$ with respect to the Bruhat order $(W_{\rm af}, \preccurlyeq)$ if and only if y has an induced expression from a given reduced decomposition of x. That is, if $x = [\sigma_{\beta_1}\cdots\sigma_{\beta_r}]_{\rm red}$, then $y = \sigma_{\beta_{k_1}}\cdots\sigma_{\beta_{k_s}}$ for some subsequence $(k_1, \cdots, k_s)$. As shown in the next proposition, the Bruhat order among certain elements becomes quite simple. This result, which was mentioned explicitly by Lusztig in section 2 of [30], is proved by Stembridge as a special case of Theorem 4.10 of [37].

Proposition 4.5. *For any $\lambda, \mu \in \tilde{Q}^\vee$, $t_\mu \preccurlyeq t_\lambda$ if and only if $\lambda \preccurlyeq \mu$; that is, $\mu - \lambda = \sum_{i\in I} a_i\alpha_i^\vee$ with $a_i \geq 0$ for each $i \in I$.*

Corollary 4.6. *Let $t_\lambda, wt_\mu \in W_{\rm af}^-$. Then $wt_\mu \preccurlyeq t_\lambda$ if and only if $\lambda \preccurlyeq \mu$.*

Proof. We use induction on $\ell(w)$ and leave the details to section 6.1. □

Lemma 4.7. *Suppose $\lambda \in Q^\vee$ satisfies $\langle\lambda, \alpha_i\rangle \leq -2$ for each $i \in I$.Then $t_\lambda \in W_{\rm af}^-$ admits a reduced decomposition of the form $t_\lambda = \omega_0\sigma_0 u_1\sigma_0\cdots u_r\sigma_0$, where $u_j \in W$ for all j. That is, we can write $t_\lambda = [\sigma_{\beta_1}\cdots\sigma_{\beta_m}]_{red}$ such that $\beta_{\ell+1} = \beta_m = \alpha_0$ and $\omega_0 = [\sigma_{\beta_1}\cdots\sigma_{\beta_\ell}]_{red}$. Furthermore for any $w \in W$, there is a subsequence $1 \leq i_1 < \cdots < i_k \leq m$ such that $wt_\lambda = \sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}$, and any such subsequence must satisfy $(i_{a+1}, \cdots, i_k) = (\ell+1, \cdots, m)$ for some $0 \leq a < k$.*

Denote $[z]$ the coset zW for $z \in W_{\rm af}$. Each coset $[z]$ contains a unique element $m_{[z]} \in W_{\rm af}^-$ of minimum length and a unique element of translation $t_{[z]} \in Q^\vee$. Note that if $m_{[z]} = v_1 t_{\lambda_1}$, then $t_{[z]} = t_{v_1(\lambda_1)} = v_1 t_{\lambda_1} v_1^{-1}$.

Definition 4.8. The length of a coset $[z]$ is defined to be $\ell([z]) = \ell(m_{[z]})$. Let $x \in W_{\rm af}^-$. We define (i) $x \preccurlyeq [z]$ if $x \preccurlyeq m_{[z]}$ and (ii) $[z] \preccurlyeq x$ if $m_{[z]} \preccurlyeq x$.

Lemma 4.9. *Let $x = wt_\lambda \in W_{\rm af}^-$ and $z \in W_{\rm af}$. If $x \preccurlyeq z$, then $x \preccurlyeq [z]$. If $y = ut_\mu \in W_{\rm af}^-$, then $c_{x,[y]} = c_{x,y}\big|_{\alpha_0=-\theta}$ whenever either $\lambda = \mu$ with λ regular or $\ell(x) = \ell(y) + 1$ holds.*

[5]The coefficient $\langle\gamma^\vee, w_i\rangle$ is always equal to zero if either $\gamma \in \Gamma_1$ and $u\sigma_\gamma t_{\lambda+\mu} \notin W_{\rm af}^-$ or $\gamma \in \Gamma_2$ and $u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee} \notin W_{\rm af}^-$.

For any $x, y \in W_{\rm af}$, we denote $c'_{x,y} = c_{x,y}\big|_{\alpha_0=-\theta}$. To prove Proposition 4.2, we analyze the effective summation first. In this case, it turns out that the summation for the product contains at most one nonzero term as follows.

Proposition 4.10. *Let $wt_\lambda, t_\mu \in W_{\rm af}^-$. If $\langle \mu, \alpha_i\rangle < -\ell(\omega_0)$ for all $i \in I$, then*

$$\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu} = c'_{wt_\lambda, wt_\lambda} c'_{t_\mu, wt_\mu} d_{wt_{\lambda+\mu},[wt_{\lambda+\mu}w^{-1}]}\mathfrak{S}_{wt_{\lambda+\mu}}.$$

Proof. By Theorem 3.3, $\mathfrak{S}_x\mathfrak{S}_y = \sum_z b_{x,y}^z \mathfrak{S}_z = \sum_z \big(\sum_{t_1,t_2} c_{x,[t_1]}c_{y,[t_2]}d_{z,[t_1t_2]}\big)\mathfrak{S}_z$. Note that $c_{x,[t_1]} = 0$ unless $x \succcurlyeq [t_1]$, $c_{y,[t_2]} = 0$ unless $y \succcurlyeq [t_2]$, and $d_{z,[t_1t_2]} = 0$ unless $z \preccurlyeq t_1t_2$, which implies $z \preccurlyeq [t_1t_2]$ by Lemma 4.9. Combining with Corollary 3.9, we conclude that the effective summation for $\mathfrak{S}_x\mathfrak{S}_y$ is only over those (z, t_1, t_2)'s in $W_{\rm af}^- \times Q^\vee \times Q^\vee$ satisfying $\ell(z) \geq \ell(x)+\ell(y)$, $x \succcurlyeq [t_1]$, $y \succcurlyeq [t_2]$ and $z \preccurlyeq t_1t_2$. Then we claim $[t_1] = [wt_\lambda]$, $[t_2] = [wt_\mu]$ and $z = wt_{\lambda+\mu}$. Consequently, $[t_1t_2] = [wt_{\lambda+\mu}w^{-1}]$. Thus the statement follows by using Lemma 4.9 again.

It remains to show our claim. Indeed, we note $\ell(z) \leq \ell([t_1t_2])$ and denote $v_jt_{\lambda_j} = m_{[t_j]}$ for $j = 1, 2$. Then $v_2t_{\lambda_2} = m_{[t_2]} \preccurlyeq t_\mu$, and consequently $\mu \preccurlyeq \lambda_2$ by Corollary 4.6. Therefore, $\lambda_2 = \mu + \kappa$ with $\kappa = \sum_{i\in I} c_i\alpha_i^\vee$, in which $c_i \geq 0$ for each $i \in I = \{1, \cdots, n\}$. If $\sum_{i\in I} 2c_i > \ell(\omega_0)$, then $\ell(t_{\lambda_2}) = \langle \mu + \kappa, -2\rho\rangle = \ell(t_\mu) - \sum_{i\in I} 2c_i < \ell(t_\mu) - \ell(\omega_0)$, which deduces a contradiction as follows:

$$\begin{aligned}\ell(x) + \ell(y) \leq \ell([t_1t_2]) = \ell([v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}]) &\leq \ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2})\\ &\leq \ell(v_1t_{\lambda_1}) + \ell(v_1^{-1}v_2) + \ell(t_{\lambda_2})\\ &\leq \ell(x) + \ell(\omega_0) + \ell(t_{\lambda_2})\\ &< \ell(x) + \ell(t_\mu) = \ell(x) + \ell(y).\end{aligned}$$

If $\sum_{i\in I} 2c_i \leq \ell(\omega_0)$, then for each $j \in I$ one has $\langle \lambda_2, \alpha_j\rangle = \langle \mu + \sum_{i\in I} c_i\alpha_i^\vee, \alpha_j\rangle < -\ell(\omega_0) + 2c_j \leq 0$. Hence, $\lambda_2 \in \tilde{Q}^\vee$ is regular. Therefore $v_1^{-1}v_2t_{\lambda_2} \in W_{\rm af}^-$ and

$$\begin{aligned}\ell(x) + \ell(y) \leq \ell(z) \leq \ell([t_1t_2]) = \ell([v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}]) &\leq \ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2})\\ &\leq \ell(v_1t_{\lambda_1}) + \ell(v_1^{-1}v_2t_{\lambda_2})\\ &\leq \ell(x) + \ell(t_{\lambda_2}) - \ell(v_1^{-1}v_2)\\ &\leq \ell(x) + \ell(t_\mu) - 0 = \ell(x) + \ell(y).\end{aligned}$$

Hence, all inequalities are indeed equalities. Thus $\ell(z) = \ell([t_1t_2])$, $\ell(v_1^{-1}v_2) = 0$, $v_1t_{\lambda_1} = x = wt_\lambda$ and $\lambda_2 = \mu$. Hence, $z = m_{[t_1t_2]} = wt_{\lambda+\mu}$ and $v_1 = v_2 = w$. □

To prove Proposition 4.2, we need to compute the only coefficient as in the above proposition. The following lemma is useful for calculations of certain structure constants.

Lemma 4.11 (Lemma 11.1.22 of [24])**.** *For $v, u \in W$, $\dfrac{d_{v,u}}{\prod_{\beta\in R^+}\beta} = u\big(c_{v^{-1}\omega_0, u^{-1}\omega_0}\big)$.*

Notation 4.12. Suppose that $\langle \lambda, \alpha_i\rangle \leq -2$ and $\langle \mu, \alpha_i\rangle \leq -2$ for all $i \in I$. Let $m = \ell(t_\lambda)$ and $p = \ell(t_\mu)$. Because of Lemma 4.7, we can take reduced expressions $t_\lambda = [\sigma_{\beta_1}\cdots\sigma_{\beta_m}]_{\rm red}$ and $t_\mu = [\sigma_{\beta_{m+1}}\cdots\sigma_{\beta_{m+p}}]_{\rm red}$ such that $\beta_{r+1} = \beta_{m+r+1} = \alpha_0$ and $\omega_0 = [\sigma_{\beta_1}\cdots\sigma_{\beta_r}]_{\rm red} = [\sigma_{\beta_{m+1}}\cdots\sigma_{\beta_{m+r}}]_{\rm red}$, where $r = \ell(\omega_0)$. Note that $t_{\lambda+\mu} = [\sigma_{\beta_1}\sigma_{\beta_2}\cdots\sigma_{\beta_{m+p}}]_{\rm red}$. Denote $H_i := \sigma_{\beta_1}\cdots\sigma_{\beta_{i-1}}(\beta_j)$ for $1 \leq i \leq m+p$, and denote $\tilde{H}_j := \sigma_{\beta_{m+1}}\cdots\sigma_{\beta_{m+j-1}}(\beta_{m+j})$ for $1 \leq j \leq p$. Clearly, $H_{m+j} = t_\lambda(\tilde{H}_j)$.

Proposition 4.13. *Suppose $\langle\lambda,\alpha_i\rangle\le -2$ for all $i\in I$. Using notation 4.12, we have $c_{vt_\lambda,ut_\lambda}=\dfrac{u(d_{v^{-1},u^{-1}})}{\prod_{j=1}^m u(H_j)}$ for any $v,u\in W$.*

Proof. Due to Lemma 4.7, we can write $vt_\lambda=[\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}\sigma_{\beta_{r+1}}\cdots\sigma_{\beta_m}]_{\rm red}$ with $(i_1,\cdots,i_k)$ a subsequence of $(1,\cdots,r)$ and $v\omega_0=[\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}]_{\rm red}$. Furthermore, if $ut_\lambda=\sigma_{\beta_{i_1}}^{\varepsilon_{i_1}}\cdots\sigma_{\beta_{i_k}}^{\varepsilon_{i_k}}\sigma_{\beta_{r+1}}^{\varepsilon_{r+1}}\cdots\sigma_{\beta_m}^{\varepsilon_m}$ for $(\varepsilon_{i_1},\cdots,\varepsilon_{i_k},\varepsilon_{r+1},\cdots,\varepsilon_m)\in\{0,1\}^{m-r+k}$, then $\varepsilon_{r+1}=\cdots=\varepsilon_m=1$ and $\sigma_{\beta_{i_1}}^{\varepsilon_{i_1}}\cdots\sigma_{\beta_{i_k}}^{\varepsilon_{i_k}}=u\omega_0$. As a consequence, we have $c_{vt_\lambda,ut_\lambda}=c_{v\omega_0,u\omega_0}\cdot\frac{1}{\prod_{j=r+1}^m u(H_j)}$ by definition. Note that $\{H_1,\cdots,H_r\}$ are all the positive roots which are mapped to negative roots under ω_0^{-1} (see e.g. Lemma 6.1). Thus $\prod_{i=1}^r H_i=\prod_{\beta\in R^+}\beta$. Hence, the statement follows by Lemma 4.11. □

Lemma 4.14. *For any $x,y\in W_{\rm af}^-$ and any $w\in W$, one has $d_{x,yw}=d_{x,y}$.*

Proof of Proposition 4.2. We first assume $\langle\mu,\alpha_i\rangle<-\ell(\omega_0)$ for each $i\in I$. Note $t_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta}=\tilde H_j\big|_{\alpha_0=-\theta}$. By definition, $d_{wt_{\lambda+\mu},wt_{\lambda+\mu}}=d_{wt_\lambda,wt_\lambda}\prod_{j=1}^p wt_\lambda(\tilde H_j)$, $d_{1,w^{-1}}=1$, and $c_{wt_\lambda,wt_\lambda}d_{wt_\lambda,wt_\lambda}=1$. Thus by Lemma 4.14 and Proposition 4.13,

$$\begin{aligned}&c'_{wt_\lambda,wt_\lambda}c'_{t_\mu,wt_\mu}d_{wt_{\lambda+\mu},[wt_{\lambda+\mu}w^{-1}]}\\&\quad=c_{wt_\lambda,wt_\lambda}\cdot\frac{w(d_{1,w^{-1}})}{\prod_{j=1}^p w(\tilde H_j)}\cdot d_{wt_\lambda,wt_\lambda}\prod_{j=1}^p wt_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta}\\&\quad=\frac{\prod_{j=1}^p wt_\lambda(\tilde H_j)}{\prod_{j=1}^p w(\tilde H_j)}\Big|_{\alpha_0=-\theta}=1.\end{aligned}$$

Hence, we have $\mathfrak{S}_{wt_\lambda}\mathfrak{S}_{t_\mu}=\mathfrak{S}_{wt_{\lambda+\mu}}$, by Proposition 4.10.

In general, we take $\kappa\in\tilde Q^\vee$ such that $\langle\kappa,\alpha_i\rangle<-\ell(\omega_0)$ and $\langle\kappa+\mu,\alpha_i\rangle<-\ell(\omega_0)$ for each $i\in I$. Denote $x=wt_\lambda, y=t_\mu$. Because of the associativity of the product,

$$\mathfrak{S}_{xt_{\mu+\kappa}}=\mathfrak{S}_x\mathfrak{S}_{t_{\mu+\kappa}}=\mathfrak{S}_x(\mathfrak{S}_y\mathfrak{S}_{t_\kappa})=(\mathfrak{S}_x\mathfrak{S}_y)\mathfrak{S}_{t_\kappa}=\sum\nolimits_z b_{x,y}^z\mathfrak{S}_z\mathfrak{S}_{t_\kappa}=\sum\nolimits_z b_{x,y}^z\mathfrak{S}_{zt_\kappa}.$$

Hence, $b_{x,y}^z=0$ if $zt_\kappa\neq xt_{\mu+\kappa}$ and $b_{x,y}^z=1$ if $zt_\kappa=xt_{\mu+\kappa}$, that is, $z=wt_{\lambda+\mu}$. □

With similar arguments, we obtain the following two propositions. Recall that for $u\in W$,

$$\Gamma_1=\{\gamma\in R^+\mid\ell(u\sigma_\gamma)=\ell(u)+1\},\Gamma_2=\{\gamma\in R^+\mid\ell(u\sigma_\gamma)=\ell(u)+1-\langle\gamma^\vee,2\rho\rangle\}.$$

Proposition 4.15. *Let $x,y\in W_{\rm af}^-$ with $x=\sigma_it_\lambda$ and $y=ut_\mu$, where $\sigma_i=\sigma_{\alpha_i}$ for some $i\in I$. Suppose $\langle\lambda,\alpha_j\rangle<-\ell(\omega_0)$ and $\langle\mu,\alpha_j\rangle<-\ell(\omega_0)$ for all $j\in I$. Then*

$$\mathfrak{S}_x\mathfrak{S}_y=c'_{x,ut_\lambda}c'_{y,y}d_{ut_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]}\mathfrak{S}_{ut_{\lambda+\mu}}+\sum_{\gamma\in\Gamma_1}A_\gamma\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu}}+\sum_{\gamma\in\Gamma_2}B_\gamma\mathfrak{S}_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}},$$

where

$$\begin{aligned}A_\gamma&=c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]}+c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_\mu}d_{u\sigma_\gamma t_{\lambda+\mu},[u\sigma_\gamma t_{\lambda+\mu}\sigma_\gamma u^{-1}]},\\B_\gamma&=c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[ut_{\lambda+\mu}u^{-1}]}\\&\quad+c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_{\mu+\gamma^\vee}}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}\sigma_\gamma u^{-1}]}.\end{aligned}$$

Proof. See the appendix (section 6.2). □

Proposition 4.16. *Suppose that $\langle\lambda,\alpha_j\rangle<-\ell(\omega_0)$ and $\langle\mu,\alpha_j\rangle<-\ell(\omega_0)$ for all $j\in I$. Let $u,w\in W$ and $\sigma_i=\sigma_{\alpha_i}$ with $i\in I$. Then we have $d_{\sigma_i,u}=w_i-u(w_i)$. Using notation* 4.12*, we have $w(d_{w^{-1},w^{-1}})\cdot d_{wt_{\lambda+\mu},wt_{\lambda+\mu}}=\prod_{j=1}^{m+p}w(H_j)$. Furthermore,*

(1) $d_{u\sigma_\gamma t_{\lambda+\mu},ut_{\lambda+\mu}}=\dfrac{1}{u(\gamma)}\cdot d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}$, *for any* $\gamma\in\Gamma_1$.

(2) $d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},ut_{\lambda+\mu}}=\dfrac{1}{u(\gamma+\delta)}\cdot d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}$, *for any* $\gamma\in\Gamma_2$.

(3) $c_{ut_\mu,u\sigma_\gamma t_{\mu+\gamma^\vee}}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}}=-\dfrac{\prod_{j=1}^m u\sigma_\gamma t_{\mu+\gamma^\vee}(H_j)}{u(\gamma+\delta)}$, *for any* $\gamma\in\Gamma_2$.

Proof. See the appendix (section 6.2). □

Proof of Proposition 4.3. Denote $x=\sigma_i t_\lambda$ and $y=ut_\mu$. We first assume $\langle\lambda,\alpha_k\rangle<-\ell(\omega_0)$ and $\langle\mu,\alpha_k\rangle<-\ell(\omega_0)$ and for each $k\in I$. Note that $t_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta}=\tilde H_j\big|_{\alpha_0=-\theta}$. It follows from Lemma 4.14, Proposition 4.13 and Proposition 4.16 that

$$\begin{aligned}c'_{x,ut_\lambda}c'_{y,y}d_{ut_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]}&=\frac{u(d_{\sigma_i,u^{-1}})}{\prod_{j=1}^m u(H_j)}\cdot\frac{u(d_{u^{-1},u^{-1}})}{\prod_{j=1}^p u(\tilde H_j)}\cdot d_{ut_\lambda,ut_\lambda}\prod_{j=1}^p ut_\lambda(\tilde H_j)\big|_{\alpha_0=-\theta}\\&=\frac{u(w_i-u^{-1}(w_i))}{\prod_{j=1}^m u(H_j)}\cdot\frac{\prod_{j=1}^m u(H_j)}{\prod_{j=1}^p u(\tilde H_j)}\cdot\prod_{j=1}^p u(\tilde H_j)\big|_{\alpha_0=-\theta}\\&=u(w_i)-w_i.\end{aligned}$$

For $\gamma\in\Gamma_1$, one has

$$\begin{aligned}&c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu},[ut_{\lambda+\mu}u^{-1}]}+c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_\mu}d_{u\sigma_\gamma t_{\lambda+\mu},[u\sigma_\gamma t_{\lambda+\mu}\sigma_\gamma u^{-1}]}\\=&c_{x,ut_\lambda}c_{y,y}\frac{d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}}{u(\gamma)}+\frac{u\sigma_\gamma(d_{\sigma_i,\sigma_\gamma u^{-1}})}{\prod_{j=1}^m u\sigma_\gamma(H_j)}\cdot\frac{u\sigma_\gamma(d_{u^{-1},\sigma_\gamma u^{-1}})}{\prod_{j=1}^p u\sigma_\gamma(\tilde H_j)}d_{u\sigma_\gamma t_{\lambda+\mu},u\sigma_\gamma t_{\lambda+\mu}}\big|_{\alpha_0=-\theta}\\=&\frac{u(w_i)-w_i}{u(\gamma)}+\frac{u\sigma_\gamma(w_i)-w_i}{\prod_{j=1}^{m+p}u\sigma_\gamma(H_j)}\cdot u\sigma_\gamma(\frac{d_{\sigma_\gamma u^{-1},\sigma_\gamma u^{-1}}}{\gamma})d_{u\sigma_\gamma t_{\lambda+\mu},u\sigma_\gamma t_{\lambda+\mu}}\big|_{\alpha_0=-\theta}\\=&\frac{u(w_i)-w_i}{u(\gamma)}-\frac{u\sigma_\gamma(w_i)-w_i}{\prod_{j=1}^{m+p}u\sigma_\gamma(H_j)}\cdot\frac{\prod_{j=1}^{m+p}u\sigma_\gamma(H_j)}{u(\gamma)}\big|_{\alpha_0=-\theta}\\=&\frac{u(w_i-\sigma_\gamma(w_i))}{u(\gamma)}=\langle\gamma^\vee,w_i\rangle.\end{aligned}$$

For $\gamma\in\Gamma_2$, one has

$$\begin{aligned}&c'_{x,ut_\lambda}c'_{y,y}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[ut_{\lambda+\mu}u^{-1}]}+c'_{x,u\sigma_\gamma t_\lambda}c'_{y,u\sigma_\gamma t_{\mu+\gamma^\vee}}d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},[u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}\sigma_\gamma u^{-1}]}\\=&c_{x,ut_\lambda}c_{y,y}\frac{d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}}{u(\gamma+\delta)}+\frac{u\sigma_\gamma(d_{\sigma_i,\sigma_\gamma u^{-1}})}{\prod_{j=1}^m u\sigma_\gamma(H_j)}\cdot\frac{-\prod_{j=1}^m u\sigma_\gamma t_{\mu+\gamma^\vee}(H_j)}{u(\gamma+\delta)}\big|_{\alpha_0=-\theta}\\=&\frac{u(w_i)-w_i}{u(\gamma)}-\frac{u\sigma_\gamma(w_i)-w_i}{\prod_{j=1}^m u\sigma_\gamma(H_j)}\cdot\frac{\prod_{j=1}^m u\sigma_\gamma(H_j)}{u(\gamma)}\big|_{\alpha_0=-\theta}\\=&\frac{u(w_i-\sigma_\gamma(w_i))}{u(\gamma)}=\langle\gamma^\vee,w_i\rangle.\end{aligned}$$

Hence, the statement holds.

For general cases, the statement follows from Proposition 4.2 and the associativity and the commutativity of the Pontryagin product. □

4.2. **Equivalence between $QH_T^*(G/B)$ and $H_*^T(\Omega K)$.** In his lecture notes [35] D. Peterson stated that there is an isomorphism between the torus-equivariant quantum cohomology of G/B and the torus-equivariant homology of ΩK after localization. However, the proofs in these notes are incomplete, and in [26], Lam and Shimozono proved this result, using some of Peterson's original approach together with Mihalcea's criterion. For completeness, we describe the literature of this equivalence.

The T-equivariant quantum cohomology $QH_T^*(G/B)$ is a torus-equivariant extension of the quantum cohomology ring $QH^*(G/B)$. (See section 6.3 for more details.) It is a commutative ring and has an $S[\mathbf{q}]$-basis of Schubert classes σ^u with $S[\mathbf{q}] = \mathbb{Q}[\alpha_1, \cdots, \alpha_n, q_1, \cdots, q_n]$:

$$\sigma^u \star_T \sigma^v = \sum_{w \in W, \lambda \in Q^\vee} \tilde{N}_{u,v}^{w,\lambda} q_\lambda \sigma^w, \quad \text{where } \tilde{N}_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}(\alpha) \in S = \mathbb{Q}[\alpha_1, \cdots, \alpha_n].$$

When $\lambda = 0$, $\tilde{N}_{u,v}^{w,\lambda}$ is equivalent to the corresponding equivariant Schubert structure constant. The evaluation $\tilde{N}_{u,v}^{w,\lambda}|_{\alpha_1 = \cdots = \alpha_n = 0}$ equals the quantum Schubert structure constant $N_{u,v}^{w,\lambda}$. A direct calculation of a general $N_{u,v}^{w,\lambda}$ can be rather difficult. However, if v is a simple reflection, then the following equivariant quantum Chevalley formula holds, which was originally stated by Peterson in [35] and has been proved by Mihalcea in [33].

Proposition 4.17 (Equivariant quantum Chevalley formula)**.** *Let $u \in W$ and $s_i = \sigma_{\alpha_i}$ with $i \in I$. Then in $QH_T^*(G/B)$ one has*

$$\sigma^{s_i} \star_T \sigma^u = (w_i - u(w_i))\sigma^u + \sum_{\gamma \in \Gamma_1} \langle \gamma^\vee, w_i \rangle \sigma^{u\sigma_\gamma} + \sum_{\gamma \in \Gamma_2} \langle \gamma^\vee, w_i \rangle q_{\gamma^\vee} \sigma^{u\sigma_\gamma},$$

where $\Gamma_1 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1\}$ and $\Gamma_2 = \{\gamma \in R^+ \mid \ell(u\sigma_\gamma) = \ell(u) + 1 - \langle \gamma^\vee, 2\rho \rangle\}$.

By evaluating at $w_i = 0$, the quantum Chevalley formula (see [12]) is recovered.

Furthermore, the equivariant quantum Chevalley formula completely determines the multiplication in $QH_T^*(G/B)$. That is the following Mihalcea's criterion, a special case ($P = B$) which is only stated here.

Proposition 4.18 (Mihalcea's criterion; see Theorem 2 of [33])**.** *Denote $\mathbb{Q}[\alpha, \mathbf{q}] = \mathbb{Q}[\alpha_1, \cdots, \alpha_n, q_1, \cdots, q_n]$ and $\mathbb{Q}[\alpha^{\pm 1}, \mathbf{q}] = \mathbb{Q}[\alpha_1^{\pm 1}, \cdots, \alpha_n^{\pm 1}, q_1, \cdots, q_n]$. Let $\mathcal{A} = \bigoplus_{u \in W} \mathbb{Q}[\alpha^{\pm 1}, \mathbf{q}]\sigma^u$ be any $\mathbb{Q}[\alpha^{\pm 1}, \mathbf{q}]$-algebra with the product written as $\sigma^u * \sigma^v = \sum_{w,\lambda} C_{u,v}^{w,\lambda} q_\lambda \sigma^w$, where $\lambda \succcurlyeq 0$. Suppose the structure coefficients $C_{u,v}^{w,\lambda}$ satisfy the following:*

1. *(homogeneity) $C_{u,v}^{w,\lambda} \in \mathbb{Q}[\alpha^{\pm 1}]$ is a homogeneous rational function of degree*
$$\deg(C_{u,v}^{w,\lambda}) = \ell(u) + \ell(v) - \ell(w) - \langle \lambda, 2\rho \rangle, \text{ whenever } C_{u,v}^{w,\lambda} \neq 0;$$
2. *(multiplication by unit)* $C_{id,v}^{w,\lambda} = \begin{cases} 1, & \text{if } \lambda = 0 \text{ and } w = v, \\ 0, & \text{otherwise}; \end{cases}$
3. *(commutativity) $\sigma^u * \sigma^v = \sigma^v * \sigma^u$ for any $u, v \in W$;*
4. *(associativity) $(\sigma^{s_i} * \sigma^u) * \sigma^v = \sigma^{s_i} * (\sigma^u * \sigma^v)$ for any $u, v \in W$ and any simple reflection $s_i \in W$;*

(5) *(equivariant quantum Chevalley formula) for any $u \in W$ and any simple reflection $s_i \in W$, the product of $\sigma^{s_i} * \sigma^u$ is given by*

$$\sigma^{s_i} * \sigma^u = (w_i - u(w_i))\sigma^u + \sum_{\gamma\in\Gamma_1} \langle \gamma^\vee, w_i\rangle \sigma^{u\sigma_\gamma} + \sum_{\gamma\in\Gamma_2} \langle \gamma^\vee, w_i\rangle q_{\gamma^\vee}\sigma^{u\sigma_\gamma}.$$

Then for any u, v, w, λ, we have

$$C_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}.$$

In particular, $C_{u,v}^{w,\lambda} = 0$ if $\deg(C_{u,v}^{w,\lambda}) < 0$, and $(\bigoplus_{u\in W} \mathbb{Q}[\alpha, \mathbf{q}]\sigma^u, *)$ is canonically isomorphic to $QH_T^*(G/B)$ as $\mathbb{Q}[\alpha, \mathbf{q}]$-algebras.

Remark 4.19. As shown in [34], the equivariant Schubert structure constants are in fact nonnegative combinations of monomials in the negative simple roots. Therefore, Mihalcea chose the negative simple roots instead of the positive ones for positivity reasons. For the same reason, we define the new algebra $(H_*^T(\Omega K), \bullet)$ below. As a consequence, the canonical isomorphism after localization between $(H_*^T(\Omega K), \cdot)$ and $QH_T^*(G/B)$ looks even more natural.

Define a new product $\bullet$ on $H_*^T(\Omega K)$ as follows:

$$\mathfrak{S}_x \bullet \mathfrak{S}_y = \sum\nolimits_{z\in W_{\text{af}}^-} \tilde{b}_{x,y}^z \mathfrak{S}_z, \text{ where } \tilde{b}_{x,y}^z = (-1)^{\ell(z)-\ell(x)-\ell(y)} b_{x,y}^z.$$

Note that $b_{x,y}^z$ is a homogeneous polynomial of degree $\ell(z) - \ell(x) - \ell(y)$. Thus $(H_*^T(\Omega K), \bullet)$ is canonically isomorphic to $(H_*^T(\Omega K), \cdot)$ as S-algebras. Immediately, it follows from the definition of $\bullet$ and Proposition 4.2 that $\mathfrak{S}_x \bullet \mathfrak{S}_{t_\mu} = \mathfrak{S}_{xt_\mu}$ for any $x, t_\mu \in W_{\text{af}}^-$. As a consequence, $\{\mathfrak{S}_t \mid t \in \tilde{Q}^\vee\}$ is a multiplicatively closed set without zero divisors. It is easy to show that the following map φ is an S-module isomorphism.

$$\varphi: \ H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee] \longrightarrow QH_T^*(G/B)[\mathbf{q}^{-1}],$$
$$\mathfrak{S}_{wt_\lambda} \bullet \mathfrak{S}_{t_\mu}^{-1} \longmapsto q_{\lambda-\mu}\sigma^w,$$

where $QH_T^*(G/B)[\mathbf{q}^{-1}] = QH_T^*(G/B)[q_i^{-1}|i \in I]$. As a consequence, the algebra $H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee]$ has an S-basis $\{\varphi^{-1}(q_\lambda\sigma^w) \mid \lambda \in Q^\vee, w \in W\}$. Therefore for any $A, B \in H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee]$, $A \bullet B = \sum_{w,\lambda} C_{A,B}^{w,\lambda}\varphi^{-1}(q_\lambda\sigma^w)$. Thanks to Mihalcea's criterion, it becomes routine to give a proof of Peterson's Theorem as below, as was done by Lam and Shimozono [26].

Theorem 4.20. (i) *$\varphi : H_*^T(\Omega K)[\mathfrak{S}_t^{-1} \mid t \in \tilde{Q}^\vee] \longrightarrow QH_T^*(G/B)[\mathbf{q}^{-1}]$ is an S-algebra isomorphism.*

(ii) *Let $u, v, w \in W$ and $\lambda \in Q^\vee$. Take $\eta, \kappa, \mu \in \tilde{Q}^\vee$ such that $x = ut_\eta, y = vt_\kappa, z = wt_\mu$ lie in W_{af}^- and $\lambda = \mu - \eta - \kappa$. Then we have*

$$\tilde{N}_{u,v}^{w,\lambda} = \tilde{b}_{x,y}^z.$$

Remark 4.21. The assumption "$\lambda \succcurlyeq 0$" in Mihalcea's criterion becomes obvious by using Proposition 4.24.

In section 4.1, we have given elementary proofs of Proposition 4.2 and Proposition 4.3. Since both of these two propositions play very important roles in the proof made by Lam and Shimozono besides Mihalcea's criterion, we give an alternative proof of Peterson's Theorem in this sense.

4.3. **Proof of Theorem 4.1.** Denote by $v_i t_{\lambda_i} = m_{[t_i]} \in W_{\mathrm{af}}^-$ the minimal length representative in the coset $[t_i] = t_i W$ as before, where $t_i \in Q^\vee, i = 1, 2$. Note that for any $x \in W_{\mathrm{af}}^-$ one has that $c_{x,[t_i]} = c_{x,[v_i t_{\lambda_i}]}$ by definition and that $d_{x,t_1t_2} = d_{x,v_1 t_{\lambda_1} v_1^{-1} v_2 t_{\lambda_2} v_2^{-1}} = d_{x, v_2 t_{v_2^{-1} v_1(\lambda_1)+\lambda_2}}$ following from Lemma 4.14. Therefore for any $x, y, z \in W_{\mathrm{af}}^-$,

$$b_{x,y}^z = \sum_{t_1,t_2 \in Q^\vee} c_{x,[t_1]} c_{y,[t_2]} d_{z,[t_1t_2]} = \sum_{v_1 t_{\lambda_1}, v_2 t_{\lambda_2} \in W_{\mathrm{af}}^-} c_{x,[v_1 t_{\lambda_1}]} c_{y,[v_2 t_{\lambda_2}]} d_{z,[v_2 t_{v_2^{-1} v_1(\lambda_1)+\lambda_2}]}.$$

The following lemma is contained in Lemma 13.2.A of [15].

Lemma 4.22. *For any $\lambda \in Q^\vee$, there exists a unique $\lambda' \in \tilde{Q}^\vee$ and some $w \in W$ such that $\lambda' = w(\lambda)$. Furthermore, $\lambda' \preccurlyeq \lambda$.*

Lemma 4.23. *Let $\lambda_1, \lambda_2 \in \tilde{Q}^\vee$ with $\langle \lambda_j, \alpha_i \rangle \leq -2\ell(\omega_0)$ for all $j \in \{1, 2\}$ and $i \in I$. Let $v_1, v_2 \in W$. If $v_1 \neq v_2$, then $\ell(t_{v_2^{-1} v_1(\lambda_1)+\lambda_2}) \leq \langle \lambda_1 + \lambda_2, -2\rho \rangle - 4\ell(\omega_0)$.*

Proof. Take $w \in W$ such that $w(v_2^{-1} v_1(\lambda_1) + \lambda_2) \in \tilde{Q}^\vee$. By Lemma 4.22, one has $w v_2^{-1} v_1(\lambda_1) = \lambda_1 + \mu_1$ with $\mu_1 \succcurlyeq 0$. If $w \neq 1$, then $w = [\sigma_{\beta_1} \cdots \sigma_{\beta_k}]_{\mathrm{red}}$ with $k \geq 1$. Note that $w(\lambda_2) = \lambda_2 - \sum_{j=1}^k \langle \lambda_2, \beta_j \rangle \gamma_j^\vee$, where $\gamma_j = \sigma_{\beta_1} \cdots \sigma_{\beta_{j-1}}(\beta_j) \in R^+$. Therefore, it follows from Lemma 4.4 that

$$\begin{aligned} \ell(t_{v_2^{-1} v_1(\lambda_1)+\lambda_2}) &= \langle \lambda_1 + \mu_1 + w(\lambda_2), -2\rho \rangle \\ &= \langle \lambda_1 + \lambda_2 + \mu_1, -2\rho \rangle + \sum\nolimits_{j=1}^k \langle \lambda_2, \beta_j \rangle \langle \gamma_j^\vee, 2\rho \rangle \\ &= \langle \lambda_1 + \lambda_2, -2\rho \rangle - \langle \mu_1, 2\rho \rangle + \langle \lambda_2, \beta_1 \rangle \cdot 2 + \sum_{j=2}^k \langle \lambda_2, \beta_j \rangle \langle \gamma_j^\vee, 2\rho \rangle \\ &\leq \langle \lambda_1 + \lambda_2, -2\rho \rangle - 0 - 2\ell(\omega_0) \cdot 2 + 0. \end{aligned}$$

If $w = 1$, then $w(v_2^{-1} v_1(\lambda_1) + \lambda_2) = \lambda_2 + w'(\lambda_1)$ with $w' = v_2^{-1} v_1 \neq 1$. With the same argument as above, one has $\ell(t_{v_2^{-1} v_1(\lambda_1)+\lambda_2}) \leq \langle \lambda_1 + \lambda_2, -2\rho \rangle - 4\ell(\omega_0)$. $\square$

Proposition 4.24. *Let $x, y \in W_{\mathrm{af}}^-$ with $x = u t_\eta, y = v t_\kappa$. If $\langle \eta, \alpha_i \rangle \leq -5\ell(\omega_0)$ and $\langle \kappa, \alpha_i \rangle \leq -5\ell(\omega_0)$ for each $i \in I$, then we have*

$$\mathfrak{S}_x \mathfrak{S}_y = \sum_{\substack{w t_\mu \in W_{\mathrm{af}}^- \\ \ell(w t_\mu) \geq \ell(x)+\ell(y)}} \sum_{\substack{v_1 \in W, \lambda_1, \lambda_2 \in \tilde{Q}^\vee \\ \lambda_1 \succcurlyeq \eta, \lambda_2 \succcurlyeq \kappa, \lambda_1+\lambda_2 \preccurlyeq \mu}} c_{x,[v_1 t_{\lambda_1}]} c_{y,[v_1 t_{\lambda_2}]} d_{w t_\mu, [v_1 t_{\lambda_1+\lambda_2}]} \mathfrak{S}_{w t_\mu}.$$

Proof. Note that $\mathfrak{S}_x \mathfrak{S}_y = \sum_{z \in W_{\mathrm{af}}^-; \ell(z) \geq \ell(x)+\ell(y)} b_{x,y}^z \mathfrak{S}_z$ by Corollary 3.9. Now let $z = w t_\mu \in W_{\mathrm{af}}^-$ with $\ell(z) \geq \ell(x) + \ell(y) = \langle \eta + \kappa, -2\rho \rangle - \ell(u) - \ell(v)$.

Note that $c_{x,[v_1 t_{\lambda_1}]} \neq 0$ only if $v_1 t_{\lambda_1} \preccurlyeq u t_\eta \preccurlyeq t_\eta$, which implies $\lambda_1 \succcurlyeq \eta$, and $c_{y,[v_2 t_{\lambda_2}]} \neq 0$ only if $v_2 t_{\lambda_2} \preccurlyeq v t_\kappa \preccurlyeq t_\kappa$, which implies $\lambda_2 \succcurlyeq \kappa$. Hence, $\lambda_1 = \eta + \lambda_3 = \eta + \sum_{i \in I} a_i \alpha_i^\vee$, $\lambda_2 = \kappa + \lambda_4 = \kappa + \sum_{i \in I} b_i \alpha_i^\vee$ with $a_i, b_i \geq 0$ for each $i \in I$. Note that $d_{z,[v_2 t_{v_2^{-1} v_1(\lambda_1)+\lambda_2}]} \neq 0$ only if $z \preccurlyeq v_2 t_{v_2^{-1} v_1(\lambda_1)+\lambda_2}$, in particular, only if

$\ell(v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) \geq \ell(z) \geq \ell(t_{\eta+\kappa}) - \ell(u) - \ell(v) \geq \ell(t_{\eta+\kappa}) - 2\ell(\omega_0)$. Furthermore,

$$\begin{aligned}\ell(v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) &= \ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2})\\ &\leq \ell(v_1t_{\lambda_1}) + \ell(v_1^{-1}v_2) + \ell(t_{\lambda_2})\\ &= \langle\lambda_1, -2\rho\rangle - \ell(v_1) + \ell(v_1^{-1}v_2) + \langle\lambda_2, -2\rho\rangle\\ &= \langle\eta+\kappa, -2\rho\rangle - 2\sum\nolimits_{i\in I}(a_i+b_i)\langle\alpha_i^\vee, \rho\rangle - \ell(v_1) + \ell(v_1^{-1}v_2)\\ &\leq \ell(t_{\eta+\kappa}) - 2\sum\nolimits_{i\in I}(a_i+b_i) - 0 + \ell(\omega_0).\end{aligned}$$

Hence, $2\sum_{i\in I}(a_i+b_i) \leq 3\ell(\omega_0)$. In particular, $0 \leq 2a_i \leq 3\ell(\omega_0), 0 \leq 2b_i \leq 3\ell(\omega_0)$, $\langle\lambda_1, \alpha_i\rangle = \langle\eta, \alpha_i\rangle + a_i\langle\alpha_i^\vee, \alpha_i\rangle + \sum_{j\neq i} a_j\langle\alpha_j^\vee, \alpha_i\rangle \leq -5\ell(\omega_0) + 3\ell(\omega_0) + 0 = -2\ell(\omega_0)$ and $\langle\lambda_2, \alpha_i\rangle \leq -2\ell(\omega_0)$ for each $i \in I$. Therefore, $v_1 = v_2$; if $v_1 \neq v_2$, then we deduce a contradiction by noting the following inequality from Lemma 4.23:

$$\ell(v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}) \leq \ell(v_2) + \ell(t_{\lambda_1+\lambda_2}) - 4\ell(\omega_0) \leq \ell(\omega_0) + \ell(t_{\eta+\kappa}) - 4\ell(\omega_0).$$

So far, we have shown that the effective summation for $z = wt_\mu$ runs over those elements $v_1t_{\lambda_1}, v_1t_{\lambda_2} \in W_{\rm af}^-$ with $\lambda_1 \succcurlyeq \eta$ and $\lambda_2 \succcurlyeq \kappa$. Note that $\lambda_1, \lambda_2 \in \tilde{Q}^\vee$ are regular; then $v_1t_{\lambda_i} \in W_{\rm af}^-$ for any $v_1 \in W$. Note $d_{z,[v_1t_{\lambda_1+\lambda_2}]} \neq 0$ only if $wt_\mu \preccurlyeq v_1t_{\lambda_1+\lambda_2} \preccurlyeq t_{\lambda_1+\lambda_2}$, which implies $\lambda_1+\lambda_2 \preccurlyeq \mu$. Thus the statement follows. □

Due to Peterson's Theorem (Theorem 4.20) and Theorem 3.3, in fact, we obtain an explicit formula for all the equivariant quantum Schubert structure constants $\tilde{N}_{u,v}^{w,\lambda}$. Namely, we just need to find $\eta, \kappa, \mu \in \tilde{Q}^\vee$ such that $x = ut_\eta, y = vt_\kappa, z = wt_\mu$ lie in $W_{\rm af}^-$ and $\lambda = \mu - \eta - \kappa$, and then compute $\tilde{N}_{u,v}^{w,\lambda} = (-1)^{\ell(z)-\ell(x)-\ell(y)}b_{x,y}^z$. In particular, we obtain a formula for quantum Schubert structure constants $N_{u,v}^{w,\lambda}$ by taking the nonequivariant limit $(\alpha_1, \cdots, \alpha_n) \to (0, \cdots, 0)$ at the origin. Although $c_{x,[t_1]}, c_{y,[t_2]}$ are rational functions, the summation $b_{x,y}^z = \sum_{t_1,t_2} c_{x,[t_1]}, c_{y,[t_2]}d_{z,[t_1t_2]}$ turns out to be a polynomial in α_i so that the nonequivariant limit does exist, which will equal 0 if the degree does not match. Furthermore, if the degree matches, then $b_{x,y}^z$ turns out to be a constant function (Corollary 3.9) so that we can take a nonequivariant limit at any point. In order to compute $N_{u,v}^{w,\lambda}$ in practice (by hand or by computer), we would like to choose η, κ such that $\ell(x), \ell(y)$ are as small as possible. However, in order to give a neat formula, we would like to choose $A = \eta = \kappa$ that satisfies the assumption of Proposition 4.24. For instance, we have chosen one such A in Theorem 4.1, in which the remark "which is in fact a regular and anti-dominant element in $Q^\vee$" ensures that $ut_A, vt_A, wt_{2A+\lambda}$ do lie in $W_{\rm af}^-$.

Proof of Theorem 4.1. Note that $N_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}\big|_{\alpha_1=\cdots=\alpha_n=0}$, the evaluation of the equivariant quantum Schubert structure constant $\tilde{N}_{u,v}^{w,\lambda} \in \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$ at the origin $(\alpha_1, \cdots \alpha_n) = (0, \cdots, 0)$.

It follows from Theorem 4.20 that $\tilde{N}_{u,v}^{w,\lambda} = (-1)^{\ell(z)-\ell(x)-\ell(y)}b_{x,y}^z$ for $x, y, z \in W_{\rm af}^-$ with $x = ut_\eta, y = vt_\kappa, z = wt_\mu$ and $\lambda = \mu - \eta - \kappa$. Note that $b_{x,y}^z$ is a homogeneous (rational) polynomial of degree $\ell(z) - \ell(x) - \ell(y) = \ell(t_\mu) - \ell(w) - (\ell(t_\eta) - \ell(u)) - (\ell(t_\kappa) - \ell(v)) = \ell(u) + \ell(v) - \ell(w) - \langle\lambda, 2\rho\rangle$.

(1) If $\langle\lambda, 2\rho\rangle > \ell(u) + \ell(v) - \ell(w)$, then it follows from Corollary 3.9 that $b_{x,y}^z = 0$, and therefore $N_{u,v}^{w,\lambda} = 0$. If $\langle\lambda, 2\rho\rangle < \ell(u) + \ell(v) - \ell(w)$, then $b_{x,y}^z$ is a homogeneous polynomial of positive degree $\ell(z) - \ell(x) - \ell(y) > 0$. The evaluation of $b_{x,y}^z$ at the origin is 0, and therefore $N_{u,v}^{w,\lambda} = 0$.

(2) If $\langle\lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w)$, then $b^z_{x,y}$ is a constant polynomial. In particular,

$$N^{w,\lambda}_{u,v} = \tilde{N}^{w,\lambda}_{u,v}\big|_{\alpha_1=\cdots=\alpha_n=0} = b^z_{x,y}\big|_{\alpha_1=\cdots=\alpha_n=0} = b^z_{x,y}\big|_{\alpha_1=\cdots=\alpha_n=1}.$$

Take $\eta, \kappa, \mu \in \tilde{Q}^\vee$ such that $x, y, z \in W^-_{\mathrm{af}}$ with $x = ut_\eta, y = vt_\kappa, z = wt_\mu$ and $\lambda = \mu - \eta - \kappa$. This can be done as follows.

The possible determinant of the Cartan matrix $\big(\langle\alpha_i^\vee, \alpha_j\rangle\big)$ is $1, 2, 3, 4$ and $n+1$ (see e.g. section 13 of [15]). As a consequence, the element $A = -12n(n+1)\sum_{i\in I} w_i^\vee$ is in the coroot lattice $Q^\vee$. Furthermore, $A \in \tilde{Q}^\vee$ and $\langle A, \alpha_i\rangle = -12n(n+1) < -5|R^+| = -5\ell(\omega_0)$ (see e.g. [15]). Note that $\lambda = \sum_{i\in I} a_i\alpha_i^\vee \succcurlyeq 0$ and $\langle\lambda, \alpha_i\rangle \le 2a_i \le \langle\lambda, 2\rho\rangle = \ell(u) + \ell(v) - \ell(w) \le 2\ell(\omega_0)$. Thus $\langle 2A + \lambda, \alpha_i\rangle < 0$ for each $i \in I$. Let $x = ut_A, y = vt_A$ and $z = wt_{2A+\lambda}$. Then $x, y, z \in W^-_{\mathrm{af}}$ and $\lambda = (2A + \lambda) - A - A$.

Hence, the first formula follows from Theorem 3.3 and Theorem 4.20, and the second formula immediately follows from Proposition 4.24. □

The following vanishing criterion is also a consequence of Peterson's Theorem.

Proposition 4.25. *For any $u, v, w \in W$ and $\lambda \in Q^\vee$ with $\lambda \succcurlyeq 0$, we take $\eta, \kappa \in \tilde{Q}^\vee$ such that $x = ut_\eta$ and $y = vt_\kappa$ lie in W^-_{af} and we denote $\mu = \eta+\kappa+\lambda$. If $wt_\mu \notin W^-_{\mathrm{af}}$, then the equivariant quantum Schubert structure constant $\tilde{N}^{w,\lambda}_{u,v}$ vanishes.*

Proof. Denote $d = \ell(u) + \ell(v) - \ell(w) - \langle\lambda, 2\rho\rangle$. Take $M \in \mathbb{N}$ with $12(n+1)|M$ and $M \gg 0$ such that $B = -M\sum_{i\in I} w_i^\vee \in \tilde{Q}^\vee$, which does exist (following the proof of Theorem 4.1), and $\mu + 2B \in \tilde{Q}^\vee$ is regular. Then $xt_B, yt_B, wt_{\mu+2B} \in W^-_{\mathrm{af}}$. Therefore it follows from Theorem 4.20 that $\tilde{N}^{w,\lambda}_{u,v} = (-1)^d b^{wt_{\mu+2B}}_{xt_B,yt_B}$. On the other hand, it follows from Proposition 4.2 that

$$\mathfrak{S}_{xt_B}\mathfrak{S}_{yt_B} = \mathfrak{S}_{t_{2B}}\mathfrak{S}_x\mathfrak{S}_y = \mathfrak{S}_{t_{2B}}\sum\nolimits_{z\in W^-_{\mathrm{af}}} b^z_{x,y}\mathfrak{S}_z = \sum\nolimits_{z\in W^-_{\mathrm{af}}} b^z_{x,y}\mathfrak{S}_{zt_{2B}}.$$

Therefore for $zt_{2B} \in W^-_{\mathrm{af}}$, $b^{zt_{2B}}_{xt_B,yt_B} \neq 0$ only if $z \in W^-_{\mathrm{af}}$. Hence, $\tilde{N}^{w,\lambda}_{u,v} = 0$. □

As we will see in section 5.1, Proposition 4.25 is useful when we need to compute the quantum Schubert structure constants for G/B by hand when the rank of G is not too large.

5. Examples

In this section, we give two examples to demonstrate the effectiveness of our formula. To make the procedure precise, the first example is simple and includes some more explanations.

5.1. **Type A_2.** $G = SL(3, \mathbb{C})$; $B \subset G$ consists of upper triangular matrices in G. In this case, $X = G/B = \{V_1 \leqslant V_2 \leqslant \mathbb{C}^3 \mid \dim_{\mathbb{C}} V_i = i, i = 1, 2\}$.

$\Delta = \{\alpha_1, \alpha_2\}, R^+ = \{\alpha_1, \alpha_2, \theta = \alpha_1 + \alpha_2\}$. Denote $s_i = \sigma_{\alpha_i}$; then one has $W = \{1, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\} \cong S_3$. $\sigma^{s_1s_2} \star \sigma^{s_1s_2}$, $\sigma^{s_1s_2} \star \sigma^{s_2s_1}$, $\sigma^{s_2s_1} \star \sigma^{s_2s_1}$, $\sigma^{s_1s_2} \star \sigma^{s_1s_2s_1}$, $\sigma^{s_2s_1} \star \sigma^{s_1s_2s_1}$ and $\sigma^{s_1s_2s_1} \star \sigma^{s_1s_2s_1} \in QH^*(X)$ are the only products that are not given by the quantum Chevalley formula directly. As an application of our theorems, we compute one of them in detail as follows.

General discussion. *For $u,v \in W$ with $\ell(u) \geq 2$ and $\ell(v) \geq 2$, $\sigma^u \star \sigma^v = \sum_{w,\lambda} N_{u,v}^{w,\lambda} q_\lambda \sigma^w$. Note that $N_{u,v}^{w,\lambda} = 0$ unless $\lambda = a_1\alpha_1^\vee + a_2\alpha_2^\vee \succcurlyeq 0$ and $2(a_1+a_2) = \langle \lambda, 2\rho\rangle = \ell(u)+\ell(v)-\ell(w) \geq 4-\ell(w) \geq 1$, in which case $q_\lambda = q_1^{a_1}q_2^{a_2}$. Note that $-\theta^\vee = -\alpha_1^\vee - \alpha_2^\vee \in \tilde{Q}^\vee$ is regular. Therefore, $x = ut_{-\theta^\vee}, y = vt_{-\theta^\vee} \in W_{af}^-$. Let $z = wt_{-2\theta^\vee+\lambda}$. By Proposition 4.25, Theorem 4.20 and Theorem 4.1, we have the following:*

(i) *If $z \notin W_{af}^-$ or $\ell(z) \neq \ell(x)+\ell(y)$, then $N_{u,v}^{w,\lambda} = 0$.*
(ii) *If $z \in W_{af}^-$, thensIn*

$$N_{u,v}^{w,\lambda} = b_{x,y}^z = \sum_{t_1,t_2\in Q^\vee} c_{x,[t_1]}c_{y,[t_2]}d_{z,[t_1t_2]} = \sum c_{x,[v_1t_{\lambda_1}]}c_{y,[v_2t_{\lambda_2}]}d_{z,[v_2t_{v_2^{-1}v_1(\lambda_1)+\lambda_2}]},$$

where the effective summation runs over those $v_it_{\lambda_i} = m_{[t_i]} \in W_{af}^-$ satisfying $v_1t_{\lambda_1} \preccurlyeq x$ and $v_2t_{\lambda_2} \preccurlyeq y$.

Furthermore if $x \neq 1$, then $d_{z,[t_2]} = 0$ as $\ell(z) > \ell(y) \geq \ell([t_2])$. In particular, if $x, y \neq 1$, then we do not need to consider the case $v_it_{\lambda_i} = 1$.

Calculation for the case $u = s_1s_2$ and $v = s_1s_2s_1$. In this case, $x = s_1s_2t_{-\theta^\vee} = s_2s_0$ and $y = s_1s_2s_1t_{-\theta^\vee} = s_0$. $\lambda = a_1\alpha_1^\vee + a_2\alpha_2^\vee$ with $a_1, a_2 \geq 0$ and $2(a_1+a_2) = \ell(u)+\ell(v)-\ell(w) = 5-\ell(w)$. Hence, $(a_1,a_2) = (1,1), (2,0), (0,2), (1,0)$ or $(0,1)$.

If $(a_1,a_2) = (2,0)$, then $z \notin W_{\rm af}^-$ by noting $-2\theta^\vee+\lambda = 2\alpha_2^\vee \notin \tilde{Q}^\vee$. If $(a_1,a_2) = (1,0)$, then $\lambda = \alpha_1$, $\ell(w) = 3$ and $w = s_1s_2s_1$. Since $\langle -2\theta^\vee + \alpha_1^\vee, \alpha_1\rangle = 0$ while $w(\alpha_1) = -\alpha_2 \notin R^+$, $z \notin W_{\rm af}^-$. Similarly, we can show $z \notin W_{\rm af}^-$ if $(a_1,a_2) = (0,2)$ or $(0,1)$. Hence, $N_{u,v}^{w,\lambda} = 0$ unless $(a_1,a_2) = (1,1)$.

If $(a_1,a_2) = (1,1)$, then $\ell(w) = 5-2(1+1) = 1$, and therefore $w = s_1$ or s_2. Hence, $\sigma^u \star \sigma^v = C_1q_1q_2\sigma^{s_1} + C_2q_1q_2\sigma^{s_2}$ for some real numbers C_1 and C_2.

Note that $1 \neq v_1t_{\lambda_1} \preccurlyeq x = s_2s_0$ with $v_1t_{\lambda_1} \in W_{\rm af}^-$ implies that $v_1t_{\lambda_1} = s_0$ or s_2s_0. $1 \neq v_2t_{\lambda_2} \preccurlyeq y = s_0$ with $v_2t_{\lambda_2} \in W_{\rm af}^-$ implies that $v_2t_{\lambda_2} = s_0 = y$. Hence,

$$\begin{aligned} b_{x,y}^z &= c_{x,[s_0]}c_{y,[y]}d_{z,[\sigma_\theta t_{-\theta^\vee-\theta^\vee}]} + c_{x,[s_2s_0]}c_{y,[y]}d_{z,[\sigma_\theta t_{\sigma_\theta s_1s_2(-\theta^\vee)-\theta^\vee}]} \\ &= c'_{s_2s_0,s_0}c'_{s_0,s_0}d_{z,[\sigma_\theta t_{-2\theta^\vee}]} + c'_{s_2s_0,s_2s_0}c'_{s_0,s_0}d_{z,[\sigma_\theta t_{-\alpha_2^\vee-\theta^\vee}]}. \end{aligned}$$

Note that

$$\begin{aligned} c'_{s_0,s_0} &= (-1)^1 \cdot \frac{1}{s_0(\alpha_0)}\big|_{\alpha_0=-\theta} = -\frac{1}{\theta}, \\ c'_{s_2s_0,s_0} &= (-1)^2 \cdot \frac{1}{\alpha_2 s_0(\alpha_0)}\big|_{\alpha_0=-\theta} = \frac{1}{\alpha_2\theta}, \\ c'_{s_2s_0,s_2s_0} &= (-1)^2 \frac{1}{s_2(\alpha_2)s_2s_0(\alpha_0)}\big|_{\alpha_0=-\theta} = \frac{1}{\alpha_2 s_2(\alpha_0)}\big|_{\alpha_0=-\theta} = -\frac{1}{\alpha_2\alpha_1}. \end{aligned}$$

Note that $\sigma_\theta t_{-2\theta^\vee} = [s_0s_2s_1s_2s_0]_{\rm red}$ and $\sigma_\theta t_{-\alpha_2^\vee-\theta^\vee} = [s_0s_2s_1s_0s_1]_{\rm red}$.

Now for $w = s_1$ and $\lambda = \theta^\vee$, we have $z = s_1t_{-\theta^\vee} = [s_2s_1s_0]_{\rm red}$. Therefore

$$\begin{aligned} d_{z,[\sigma_\theta t_{-2\theta^\vee}]} &= d_{s_2s_1s_0,[s_0s_2s_1s_2s_0]} = s_0(\alpha_2)s_0s_2(\alpha_1)s_0s_2s_1s_2(\alpha_0)\big|_{\alpha_0=-\theta} = -\alpha_1\theta^2, \\ d_{z,[\sigma_\theta t_{-\alpha_2^\vee-\theta^\vee}]} &= d_{s_2s_1s_0,[s_0s_2s_1s_0s_1]} = s_0(\alpha_2)s_0s_2(\alpha_1)s_0s_2s_1(\alpha_0)\big|_{\alpha_0=-\theta} = -\alpha_1^2\theta. \end{aligned}$$

$$\text{Hence,}\quad C_1 = b_{x,y}^z = \frac{1}{\alpha_2\theta}\cdot\frac{-1}{\theta}\cdot(-\alpha_1\theta^2) + \frac{-1}{\alpha_2\alpha_1}\frac{-1}{\theta}\cdot(-\alpha_1^2\theta) = \frac{\alpha_1}{\alpha_2} + (-\frac{\alpha_1}{\alpha_2}) = 0.$$

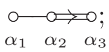
Now for $w = s_2$ and $\lambda = \theta^\vee$, we have $z = s_2 t_{-\theta^\vee} = [s_1 s_2 s_0]_{\rm red}$. Note that $d_{z,[\sigma_\theta t_{-2\theta^\vee}]} = d_{s_1s_2s_0,[s_0s_2s_1s_2s_0]} = s_0s_2(\alpha_1)s_0s_2s_1(\alpha_2)s_0s_2s_1s_2(\alpha_0)\big|_{\alpha_0=-\theta} = -\alpha_2\theta^2$ and that $d_{z,[\sigma_\theta t_{-\alpha_2^\vee-\theta^\vee}]} = d_{s_1s_2s_0,[s_0s_2s_1s_0s_1]} = 0$ as $s_1s_2s_0 \not\preccurlyeq s_0s_2s_1s_0s_1$. Therefore, $C_2 = b^z_{x,y} = \frac{1}{\alpha_2\theta}\cdot\frac{-1}{\theta}\cdot(-\alpha_2\theta^2) = 1$.

Hence,

$$\sigma^{s_1s_2} \star \sigma^{s_1s_2s_1} = q_1q_2\sigma^{s_2}.$$

Similarly, we can compute quantum products for the remaining cases.

5.2. **Type** B_3**.** $G = Spin(7,\mathbb{C})$; $X = G/B = \{V_1 \leqslant V_2 \leqslant V_3 \leqslant \mathbb{C}^7 \mid \dim_{\mathbb{C}} V_i = i, (V_i, V_i) = 0, i = 1,2,3\}$, where $(\cdot,\cdot)$ is a quadratic form on $\mathbb{C}^7$. See e.g. [15] for



; $\theta = \alpha_1 + 2\alpha_2 + 2\alpha_3 = s_2s_3s_2(\alpha_1)$, $\theta^\vee = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$, $|R^+| = 9$.

W is generated by simple reflections $\{s_1, s_2, s_3\}$, $|W| = 48$, $\sigma_\theta = [s_2s_3s_2s_1s_2s_3s_2]_{\rm red}$.

Calculation for $\sigma^u \star \sigma^v$**, where** $u = s_1s_2s_3s_1s_2$ **and** $v = s_3s_1s_2s_3s_1s_2$**.** Note that $\langle -\theta^\vee, \alpha_2\rangle = -1 < 0$, $\langle -\theta^\vee, \alpha_i\rangle = 0$ while $u(\alpha_i) \succcurlyeq 0$ and $v(\alpha_i) \succcurlyeq 0$ for $i = 1,3$. Hence, it suffices to take $A = -\theta^\vee$. Then one has $x = ut_{-\theta^\vee} = [s_3s_2s_0]_{\rm red}$ and $y = vt_{-\theta^\vee} = [s_2s_0]_{\rm red}$. Note that $u(-\theta^\vee) = \alpha_1^\vee + \alpha_2^\vee$ and $v(-\theta^\vee) = \alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee$. Denote m_t as the minimal length representative in the coset tW for $t \in Q^\vee$. Then one has

$$b^z_{x,y} = \sum c_{x,[v_1t_{\lambda_1}]}c_{y,[v_2t_{\lambda_2}]}d_{z,[m_{t_1t_2}]},$$

where $v_it_{\lambda_i} = m_{t_i} \in W^-_{\rm af}$ and the effective summation runs over those satisfying $1 \neq v_1t_{\lambda_1} \preccurlyeq x$ and $1 \neq v_2t_{\lambda_2} \preccurlyeq y$. Explicitly, the possible nonzero terms are listed in the following table, where we denote $\eta = -2\alpha_1^\vee - 3\alpha_2^\vee - 2\alpha_3^\vee$, $\kappa = -2\alpha_1^\vee - 2\alpha_2^\vee - \alpha_3^\vee$.

$([v_1t_{\lambda_1}],[v_2t_{\lambda_2}])$	$t_1\cdot t_2$	$m_{t_1t_2}$	$[m_{t_1t_2}]_{\rm red}$
$([s_0],[s_0])$	$t_{\theta^\vee+\theta^\vee}$	$\sigma_\theta t_{-2\theta^\vee}$	$s_0s_2s_3s_2s_1s_2s_3s_2s_0$
$([s_0],[s_2s_0])$	$t_{\theta^\vee+v(-\theta^\vee)}$	$s_1s_2s_3s_2s_1s_3s_2s_3t_\eta$	$s_0s_2s_3s_1s_2s_0$
$([s_2s_0],[s_0])$	$t_{v(-\theta^\vee)+\theta^\vee}$	$s_1s_2s_3s_2s_1s_3s_2s_3t_\eta$	$s_0s_2s_3s_1s_2s_0$
$([s_2s_0],[s_2s_0])$	$t_{v(-\theta^\vee)+v(-\theta^\vee)}$	$vt_{-2\theta^\vee}$	$s_2s_0s_2s_3s_2s_1s_2s_3s_2s_0$
$([s_2s_1s_0],[s_0])$	$t_{u(-\theta^\vee)+\theta^\vee}$	$s_1s_2s_1s_3s_2s_1s_3t_\eta$	$s_0s_3s_2s_3s_1s_2s_0$
$([s_2s_1s_0],[s_2s_0])$	$t_{u(-\theta^\vee)+v(-\theta^\vee)}$	$s_1s_2s_3s_2s_1t_\kappa$	$s_0s_2s_3s_2s_0$

Note that $s_3s_0 = s_0s_3 \in s_0W$. By definition,

$$c_{x,[s_0]} = c'_{s_3s_2s_0,s_0} + c'_{s_3s_2s_0,s_3s_0} = \Big(\frac{(-1)^3}{\alpha_3\alpha_2s_0(\alpha_0)} + \frac{(-1)^3}{s_3(\alpha_3)s_3(\alpha_2)s_3s_0(\alpha_0)}\Big)\Big|_{\alpha_0=-\theta}$$
$$= \frac{-2}{\alpha_2\theta(\alpha_2+2\alpha_3)},$$
$$c_{x,[s_2s_0]} = c'_{s_3s_2s_0,s_2s_0} = \frac{(-1)^3}{\alpha_3s_2(\alpha_2)s_2s_0(\alpha_0)}\Big|_{\alpha_0=-\theta} = \frac{1}{\alpha_2\alpha_3(\alpha_1+\alpha_2+2\alpha_3)},$$
$$c_{x,[s_3s_2s_0]} = c'_{s_3s_2s_0,s_3s_2s_0} = \frac{(-1)^3}{s_3(\alpha_3)s_3s_2(\alpha_2)s_3s_2s_0(\alpha_0)}\Big|_{\alpha_0=-\theta}$$
$$= \frac{-1}{\alpha_3(\alpha_2+2\alpha_3)(\alpha_1+\alpha_2)},$$

$$c_{y,[s_0]} = c'_{s_2s_0,s_0} = \frac{(-1)^2}{\alpha_2 s_0(\alpha_0)}\Big|_{\alpha_0=-\theta} = \frac{1}{\alpha_2\theta},$$
$$c_{y,g[s_2s_0]} = c'_{s_2s_0,s_2s_0} = \frac{(-1)^2}{s_2(\alpha_2)s_2s_0(\alpha_0)}\Big|_{\alpha_0=-\theta} = \frac{-1}{\alpha_2(\alpha_1+\alpha_2+2\alpha_3)}.$$

Let $z = wt_{-2\theta^\vee+\lambda}$. Then $N_{u,v}^{w,\lambda} \neq 0$ only if $z \in W_{\mathrm{af}}^-$ and $\ell(z) = \ell(x) + \ell(y) = 5$. Note that the only possibilities are $z = s_2s_3s_1s_2s_0$, $s_1s_2s_3s_2s_0$ or $s_0s_2s_3s_2s_0$.

For $z = s_1s_2s_3s_2s_0 = wt_{-2\theta^\vee+\lambda}$, we have $w = s_2s_3s_2$ and $\lambda = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$. Note that $d_{z,[t_1t_2]} \neq 0$ only if $z \preccurlyeq m_{t_1t_2}$. Thus only $d_{z,[\sigma_\theta t_{-2\theta^\vee}]}$ and $d_{z,[vt_{-2\theta^\vee}]}$ are nonzero. Furthermore, we have

$$d_{z,[\sigma_\theta t_{-2\theta^\vee}]} = -\alpha_2\theta^2(\alpha_2+\alpha_3)(\alpha_2+2\alpha_3),$$
$$d_{z,[vt_{-2\theta^\vee}]} = \alpha_2\alpha_3(\alpha_2+2\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2.$$

Hence,

$$\begin{aligned} b_{x,y}^z &= c_{x,[s_0]}c_{y,[s_0]}d_{z,[\sigma_\theta t_{-2\theta^\vee}]} + c_{x,[s_2s_0]}c_{y,[s_2s_0]}d_{z,[vt_{-2\theta^\vee}]} \\ &= \frac{2\alpha_2\theta^2(\alpha_2+\alpha_3)(\alpha_2+2\alpha_3)}{\alpha_2\theta(\alpha_2+2\alpha_3)\cdot\alpha_2\theta} - \frac{\alpha_2\alpha_3(\alpha_2+2\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2}{\alpha_2\alpha_3(\alpha_1+\alpha_2+2\alpha_3)\cdot\alpha_2(\alpha_1+\alpha_2+2\alpha_3)} \\ &= \frac{2(\alpha_2+\alpha_3)}{\alpha_2} - \frac{\alpha_2+2\alpha_3}{\alpha_2} = 1. \end{aligned}$$

For $z = s_2s_3s_1s_2s_0 = wt_{-2\theta^\vee+\lambda}$, we have $w = s_3s_1s_2$ and $\lambda = \alpha_1^\vee + 2\alpha_2^\vee + \alpha_3^\vee$. Then,

$$\begin{aligned} d_{z,[t_{2\theta^\vee}]} &= -2\theta^3(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3), \\ d_{z,[t_{\theta^\vee+v(-\theta^\vee)}]} &= -\theta(\alpha_2+\alpha_3)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2, \\ d_{z,[t_{v(-2\theta^\vee)}]} &= -\alpha_3(\alpha_1+\alpha_2+2\alpha_3)^2(2\alpha_1^2+2\alpha_1\alpha_2+\alpha_2^2+6\alpha_1\alpha_3 \\ &\qquad + 4\alpha_2\alpha_3+4\alpha_3^2), \\ d_{z,[t_{u(-\theta^\vee)+\theta^\vee}]} &= -\alpha_2\theta(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+\alpha_3), \\ d_{z,[t_{u(-\theta^\vee)+v(-\theta^\vee)}]} &= 0. \end{aligned}$$

Substituting them in the summation for $b_{x,y}^z$ and simplifying, we obtain $b_{x,y}^z = 1$.

For $z = s_0s_2s_3s_2s_0 = wt_{-2\theta^\vee+\lambda}$, we have $w = s_1s_2s_3s_2s_1$ and $\lambda = 2\alpha_2^\vee + \alpha_3^\vee$. Then,

$$\begin{aligned} d_{z,[t_{2\theta^\vee}]} &= -\theta^3(\alpha_1^2+3\alpha_1\alpha_2+3\alpha_2^2+3\alpha_1\alpha_3+6\alpha_2\alpha_3+2\alpha_3^2), \\ d_{z,[t_{\theta^\vee+v(-\theta^\vee)}]} &= -\theta^2(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+2\alpha_3)^2, \\ d_{z,[t_{v(-2\theta^\vee)}]} &= -(\alpha_1+\alpha_2+2\alpha_3)^2(\alpha_1^2+2\alpha_1\alpha_2+3\alpha_1\alpha_3+\alpha_2\alpha_3+2\alpha_3^2), \\ d_{z,[t_{u(-\theta^\vee)+\theta^\vee}]} &= -\theta^2(\alpha_1+\alpha_2)^2(\alpha_1+\alpha_2+\alpha_3), \\ d_{z,[t_{u(-\theta^\vee)+v(-\theta^\vee)}]} &= -\alpha_1\theta(\alpha_1+\alpha_2)(\alpha_1+\alpha_2+\alpha_3)(\alpha_1+\alpha_2+2\alpha_3). \end{aligned}$$

Substituting them in the summation for $b_{x,y}^z$ and simplifying, we obtain $b_{x,y}^z = 1$.

Hence, we obtain the following:

$$\sigma^{s_1s_2s_3s_1s_2} \star \sigma^{s_3s_1s_2s_3s_1s_2} = q_1q_2^2q_3(\sigma^{s_2s_3s_2} + \sigma^{s_3s_1s_2}) + q_2^2q_3\sigma^{s_1s_2s_3s_2s_1}.$$

6. Appendix

6.1. **Proofs of the lemmas in section 4.1 and Corollary 4.6.** In this subsection, we first prove all the lemmas in section 4.1 after reviewing some basic facts on the affine Weyl group (as a Coxeter group). Then we give the proofs of all the lemmas in section 4.1 as well as Corollary 4.6.

Recall that $\mathcal{S} = \{\sigma_i \mid i \in I_{\mathrm{af}}\}$. Denote $\mathcal{T} = \{x\sigma_i x^{-1} \mid x \in W_{\mathrm{af}}, \sigma_i \in \mathcal{S}\} = \{\sigma_\gamma \mid \gamma \in R^+_{\mathrm{re}}\}$. Let $x, x' \in W_{\mathrm{af}}$. We say x covers x', denoted by $x' \to x$ or $x' \overset{\sigma_\gamma}{\to} x$, if there exists some $\sigma_\gamma \in \mathcal{T}$ such that $x = \sigma_\gamma x'$ and $\ell(x) = \ell(x') + 1$. We say $x' \preccurlyeq x$ with respect to the Bruhat order $(W_{\mathrm{af}}, \preccurlyeq)$ if there exists a chain $x' = x_1 \to x_2 \to \cdots \to x_k = x$. We list some well-known facts (from [14] and [16]) for the Coxeter system $(W_{\mathrm{af}}, \mathcal{S})$ as follows.

Lemma 6.1. *Let $x, y \in W_{\mathrm{af}}$ with $x = [\sigma_{\beta_1} \cdots \sigma_{\beta_r}]_{\mathrm{red}}$. Denote $\gamma_k = \sigma_{\beta_1} \cdots \sigma_{\beta_{k-1}}(\beta_k)$.*

(a) *If $y \to x$, then there exists a unique $j, 1 \le j \le r$, such that $x = \sigma_{\gamma_j} y$ and $y = [\sigma_{\beta_1} \cdots \sigma_{\beta_{j-1}} \sigma_{\beta_{j+1}} \cdots \sigma_{\beta_r}]_{\mathrm{red}}$.*

(b) *If $y \preccurlyeq x$, then $y = [\sigma_{\beta_{k_1}} \cdots \sigma_{\beta_{k_s}}]_{\mathrm{red}}$ for some subsequence $(k_1, \cdots, k_s)$, which we call an induced reduced decomposition of y from $x = [\sigma_{\beta_1} \cdots \sigma_{\beta_r}]_{\mathrm{red}}$. In particular, if $y \preccurlyeq x$ and $\ell(x) = \ell(y) + 1$, then $y \overset{\sigma_{\gamma_j}}{\to} x$ for a unique j.*

(c) (Lifting Property) *Let $\sigma_i \in \mathcal{S}$. Suppose $\ell(\sigma_i x) > \ell(x)$ and $\ell(\sigma_i y) < \ell(y)$; then the following are equivalent:*

$$\text{(i) } \sigma_i x \preccurlyeq y; \qquad \text{(ii) } x \preccurlyeq y; \qquad \text{(iii) } x \preccurlyeq \sigma_i y.$$

(d) $\{\gamma_1, \cdots, \gamma_r\} = \{\gamma \in R^+_{\mathrm{re}} \mid x^{-1}(\gamma) \in -R^+_{\mathrm{re}}\}, \qquad \ell(xy) \le \ell(x) + \ell(y).$

(e) *Let $\gamma \in R^+_{\mathrm{re}}$. Then $\ell(\sigma_\gamma x) \le \ell(x) \Longleftrightarrow x^{-1}(\gamma) \in -R^+_{\mathrm{re}}$.*

Lemma 6.2 (see e.g. [32])**.** *For any $\gamma \in R^+$, $\ell(\sigma_\gamma) \le \langle \gamma^\vee, 2\rho\rangle - 1$.*

The following lemma is on the property of the longest element ω_0 in W.

Lemma 6.3. $\omega_0(-\theta) = \theta$.

Proof. (We learned the proof from Victor Reiner.) The highest root $\theta_0 \in R^+$ is characterized among all the positive roots by the property that $\langle \theta_0, \alpha_i^\vee\rangle \ge 0$ for all α_i. Note that $\omega_0(-\theta)$ is a positive root. Furthermore for any simple root α_i, $\ell(\sigma_{\omega_0(\alpha_i)}) = \ell(\omega_0\sigma_i\omega_0) = \ell(\omega_0) - \ell(\sigma_i\omega_0) = \ell(\omega_0) - (\ell(\omega_0) - \ell(\sigma_i)) = 1$, which implies that $\omega_0(\alpha_i) = -\alpha_j$ for some $j \in I$. Note that $\omega_0 = \omega_0^{-1}$. Hence for $\theta_0 = \omega_0(-\theta)$, $\langle \theta_0, \alpha_i^\vee\rangle = \langle \omega_0(-\theta), \alpha_i^\vee\rangle = \langle \theta, -\omega_0(\alpha_i)^\vee\rangle = \langle \theta, \alpha_j^\vee\rangle \ge 0$. Thus $\theta_0 = \theta$. □

The next lemma is a consequence of Lemma 4.4.

Lemma 6.4. *Suppose $\lambda \in Q^\vee$ is anti-dominant and regular. Then for any $w, u \in W$, we have* (i) *$wt_\lambda \preccurlyeq t_\lambda$;* (ii) *$wt_\lambda u \preccurlyeq t_\lambda$ implies $u = 1$.*

Proof. Write $w = [\sigma_{\beta_1} \cdots \sigma_{\beta_r}]_{\mathrm{red}}$. Denote $x_j = \sigma_{\beta_j} \cdots \sigma_{\beta_r} t_\lambda$ for each $1 \le j \le r$. Since $\lambda \in \tilde{Q}^\vee$ is regular, $x_j \in W^-_{\mathrm{af}}$ for each $1 \le j \le r+1$, where we denote $x_{r+1} = t_\lambda$.

(i) $\ell(x_{j+1}) = \ell(t_\lambda) - \ell(\sigma_{\beta_{j+1}} \cdots \sigma_{\beta_r}) = \ell(t_\lambda) - (r-j)$. Note that $x_{j+1} = \sigma_{\beta_j} x_j$ and $\ell(x_{j+1}) = \ell(x_j) + 1$. Thus $x_j \preccurlyeq x_{j+1}$ for $1 \le j \le r$. Hence, $wt_\lambda = x_1 \preccurlyeq x_{r+1} = t_\lambda$.

(ii) Note that $x_j \in W_{\mathrm{af}}^-$ for $1 \le j \le r+1$. Hence, $\ell(x_j u) = \ell(x_j) + \ell(u)$, $\ell(\sigma_{\beta_j} x_j u) = \ell(x_{j+1}u) > \ell(x_j u)$ and $\ell(\sigma_{\beta_j} x_{r+1}) = \ell(x_{r+1}) - 1 < \ell(x_{r+1})$. Therefore if $x_j u \preccurlyeq x_{r+1}$, then $x_{j+1}u = \sigma_{\beta_j} x_j u \preccurlyeq x_{r+1}$ by Lemma 6.1. Hence, $x_1 u = wt_\lambda u \preccurlyeq t_\lambda = x_{r+1}$ implies $t_\lambda u = x_{r+1}u \preccurlyeq x_{r+1}$, by induction on j. Thus $\ell(u) = 0$ and $u = 1$. □

Proof of Lemma 4.7. By Lemma 6.3, $t_\lambda = t_{-\theta^\vee} t_{\lambda+\theta^\vee} = \omega_0\sigma_\theta t_{-\theta^\vee}\sigma_\theta\omega_0 t_{\lambda+\theta^\vee}$. Note that for each $i \in I$, $\langle \lambda+\theta^\vee, \alpha_i\rangle \le -2 + \langle \theta^\vee, \alpha_i\rangle \le -2+1 = -1$. Hence, $\lambda + \theta^\vee \in \tilde{Q}^\vee$ is regular and we have $\sigma_\theta\omega_0 t_{\lambda+\theta^\vee} \in W_{\mathrm{af}}^-$. Hence, any reduced decomposition of $\sigma_\theta\omega_0 t_{\lambda+\theta^\vee}$ must be of the form $u_1\sigma_0\cdots u_r\sigma_0$. Furthermore,

$$\begin{aligned}
\ell(t_\lambda) &\le \ell(\omega_0\sigma_\theta t_{-\theta^\vee}) + \ell(\sigma_\theta\omega_0 t_{\lambda+\theta^\vee})\\
&= \ell(\omega_0\sigma_0) + \ell(t_{\lambda+\theta^\vee}) - \ell(\sigma_\theta\omega_0)\\
&= \ell(\omega_0) + 1 + \langle \lambda+\theta^\vee, -2\rho\rangle - (\ell(\omega_0) - \ell(\sigma_\theta))\\
&= 1 + \ell(t_\lambda) - \langle \theta^\vee, 2\rho\rangle + \ell(\sigma_\theta)\\
&\le 1 + \ell(t_\lambda) - \langle \theta^\vee, 2\rho\rangle + \langle \theta^\vee, 2\rho\rangle - 1 = \ell(t_\lambda).
\end{aligned}$$

Hence, all inequalities are indeed equalities. Thus t_λ admits a reduced decomposition of the form $\omega_0\sigma_0 u_1\sigma_0\cdots u_r\sigma_0$.

For any $w \in W$, we have $wt_\lambda \preccurlyeq t_\lambda$ by Lemma 6.4. Thus there exists a subsequence $(i_1, \cdots, i_k)$, as required. Furthermore, any such sequence gives an expression of wt_λ of the form $wt_\lambda = u_0'\sigma_0 u_1'\sigma_0\cdots u_r'\sigma_0$ with $u_0' \preccurlyeq \omega_0$ and $u_j' \preccurlyeq u_j$ for each $1 \le j \le r$. If $u_j' \ne u_j$ for some $1 \le j \le r$, then $\ell(u_j') \le \ell(u_j) - 1$. Note that $t_\lambda = \omega_0\sigma_0 u_1\sigma_0\cdots u_r\sigma_0 = w^{-1}u_0'\sigma_0 u_1'\sigma_0\cdots u_r'\sigma_0$. Thus

$$\begin{aligned}
\ell(\omega_0) + r + 1 + \sum\nolimits_{1\le k\le r}\ell(u_k) &= \ell(\omega_0\sigma_0 u_1\sigma_0\cdots u_r\sigma_0)\\
&= \ell(w^{-1}u_0'\sigma_0 u_1'\sigma_0\cdots u_r'\sigma_0)\\
&\le \ell(w^{-1}u_0') + r + 1 + \sum\nolimits_{1\le k\le r}\ell(u_k')\\
&\le \ell(\omega_0) + r + 1 + \ell(u_j) - 1 + \sum\nolimits_{k\ne j}\ell(u_k)\\
&= \ell(\omega_0) + r + \sum\nolimits_{1\le k\le r}\ell(u_k).
\end{aligned}$$

This is a contradiction. Hence, the statement follows. That is, induced decomposition(s) of wt_λ must be of the form $u_0\sigma_0 u_1\sigma_0\cdots u_r\sigma_0$ (in which $u_0 = w\omega_0 = \sigma_{i_1}\cdots\sigma_{i_a}$). □

Proof of Lemma 4.9. Write $m_{[z]} = ut_\mu \in W_{\mathrm{af}}^-$; then $z = ut_\mu w$ for some $w \in W$. Let $w = [\sigma_{\beta_1}\cdots\sigma_{\beta_k}]_{\mathrm{red}}$ and denote $\tilde{z} = ut_\mu\sigma_{\beta_1}\cdots\sigma_{\beta_{k-1}}$. Note that $\ell(\sigma_{\beta_k}x^{-1}) = \ell(x^{-1}) + 1 > \ell(x^{-1})$ and $\ell(\sigma_{\beta_k}z^{-1}) = \ell(z^{-1}) - 1 < \ell(z^{-1})$. Since $x \preccurlyeq z$, $x^{-1} \preccurlyeq z^{-1}$ and therefore $x^{-1} \preccurlyeq \sigma_{\beta_k}z^{-1} = \tilde{z}^{-1}$ by (c) of Lemma 6.1. Hence, $x \preccurlyeq \tilde{z}$ with $\ell(\tilde{z}) = \ell(z) - 1$. Hence, the first half of the statement holds by induction on $\ell(w)$.

To prove the second half, we recall that $c_{x,[y]} = \sum_{\tilde{y}\in yW} c'_{x,\tilde{y}}$. Given $\tilde{y} \in yW$, we have $\tilde{y} = ut_\lambda v$ for some $v \in W$. Note that $c'_{x,\tilde{y}} \ne 0$ only if $\tilde{y} \preccurlyeq x$. Suppose $\lambda = \mu$

and λ is regular; then $ut_\lambda v \preccurlyeq wt_\lambda \preccurlyeq t_\lambda$ implies $v = 1$ by Lemma 6.4. Suppose $\ell(x) = \ell(y) + 1$. If $\tilde{y} \neq y$, then $\ell(\tilde{y}) > \ell(y)$, and therefore $\tilde{y} = x$ follows from $\tilde{y} \preccurlyeq x$, which contradicts to the uniqueness of the minimal length representative in each coset. Hence, $c_{x,[y]} = c'_{x,y}$ if either of the two assumptions holds. □

Proof of Lemma 4.14. Let $y = [\sigma_{\beta_1} \cdots \sigma_{\beta_k}]_{\rm red}$ and $w = [\sigma_{\beta_{k+1}} \cdots \sigma_{\beta_{k+s}}]_{\rm red}$. Then $\beta_{k+1}, \cdots, \beta_{k+s} \in \Delta$, $\beta_k = \alpha_0$ and $yw = [\sigma_{\beta_1} \cdots \sigma_{\beta_{k+s}}]_{\rm red}$. For any subsequence $J = (j_1, \cdots, j_a)$ of $(1, \cdots, k+s)$, we denote $\sigma_J = \sigma_{\beta_{j_1}} \cdots \sigma_{\beta_{j_a}}$. Suppose $x = [\sigma_J]_{\rm red}$; then J must be a subsequence of $(1, \cdots, k)$. Therefore, $d_{x,yw} = d_{x,y}$ by definition. Indeed, if $J = J_1 \sqcup J_2$ with $J_2 \subset (k+1, \cdots, k+s)$ nonempty and $J_1 \subset (1, \cdots, k)$, then $\sigma_{J_2} \in W$ and $\ell(x(\sigma_{J_2})^{-1}) = \ell(\sigma_{J_1}\sigma_{J_2}(\sigma_{J_2})^{-1}) = \ell(\sigma_{J_1}) \leq |J_1| < |J| = \ell(x)$, which contradicts the fact that x is of minimal length in the coset xW. □

Proof of Corollary 4.6. We can assume λ, μ to be regular. (Otherwise, we take any regular $\tau \in \tilde{Q}^\vee$ and consider $\lambda + \tau, \mu + \tau$.) We claim that $wt_\mu \preccurlyeq t_\lambda \Longleftrightarrow t_\mu \preccurlyeq t_\lambda$. Hence, the statement follows from Proposition 4.5 immediately.

Indeed, we have $wt_\mu \preccurlyeq t_\mu$ by Lemma 6.4. Suppose $t_\mu \preccurlyeq t_\lambda$; then we have $wt_\mu \preccurlyeq t_\lambda$. Suppose $wt_\mu \preccurlyeq t_\lambda$. Write $w = [\sigma_{\beta_1} \cdots \sigma_{\beta_k}]_{\rm red}$. Note that $\ell(\sigma_{\beta_1} wt_\mu) = \ell(t_\mu) - \ell(w) + 1 > \ell(wt_\mu)$ and $\ell(\sigma_{\beta_1} t_\lambda) = \ell(t_\lambda) - 1 < \ell(t_\lambda)$. Hence, $\tilde{w}t_\mu \preccurlyeq t_\lambda$ holds by Lemma 6.1, where $\tilde{w} = \sigma_{\beta_1} w$ with $\ell(\tilde{w}) = \ell(w) - 1$. Thus we can deduce $t_\mu \preccurlyeq t_\lambda$ by induction on $\ell(w)$. □

6.2. **Proofs of Proposition 4.15 and Proposition 4.16.**

Lemma 6.5. *Let $x, y, z \in W_{\rm af}^-$ with $x = \sigma_i t_\lambda$ and $y = ut_\mu$, where $i \in I$. Let $t_j \in Q^\vee$ and denote $v_j t_{\lambda_j} = m_{[t_j]}$, $j = 1, 2$. Suppose $x \succcurlyeq [t_1], y \succcurlyeq [t_2], z \preccurlyeq [t_1 t_2]$ and $\ell(z) \geq \ell(x) + \ell(y)$. Then only the following two possibilities can happen:*
Case A: $\ell([t_1 t_2]) = \ell(x) + \ell(y) + 1$; **Case B:** $\ell([t_1 t_2]) = \ell(z) = \ell(x) + \ell(y)$.

Proof. Note that $z \preccurlyeq [t_1 t_2]$ implies $\ell(z) \leq \ell([t_1 t_2])$. Therefore,

$$\begin{aligned}
\ell(x) + \ell(y) \leq \ell([t_1 t_2]) &= \ell([v_1 t_{\lambda_1} v_1^{-1} v_2 t_{\lambda_2}]) \\
&\leq \ell(v_1 t_{\lambda_1} v_1^{-1} v_2 t_{\lambda_2}) \\
&\leq \ell(v_1 t_{\lambda_1}) + \ell(v_1^{-1}) + \ell(v_2 t_{\lambda_2}) \\
&= \ell(t_{\lambda_1}) + \ell(v_2 t_{\lambda_2}) \\
&\leq \ell(t_\lambda) + \ell(y) = \ell(x) + 1 + \ell(y).
\end{aligned}$$

Hence, only two cases (**Case A** or **Case B**) are possible. □

Lemma 6.6. *Under the same assumptions as in Lemma 6.5, we assume λ is regular. If **Case A** occurs, then $v_1 t_{\lambda_1} = ut_\lambda$ and $v_2 t_{\lambda_2} = ut_\mu$. Furthermore, only one of the following three possibilities can happen:*

a) $z = ut_{\lambda+\mu}$;
b) *there exists $\gamma \in \Gamma_1$ such that $z = u\sigma_\gamma t_{\lambda+\mu}$;*
c) *there exists $\gamma \in \Gamma_2$ such that $z = u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}$.*

Proof. Since **Case A** holds, it follows from the proof of Lemma 6.5 that $\lambda_1 = \lambda$ and $v_2 t_{\lambda_2} = ut_\mu$. Furthermore,

$$\begin{aligned}\ell(x)+\ell(y)+1 = \ell([t_1t_2]) = \ell([t_2t_1]) &= \ell([ut_\mu u^{-1}v_1t_\lambda])\\ &\le \ell(ut_\mu u^{-1}v_1t_\lambda)\\ &\le \ell(ut_\mu)+\ell(u^{-1}v_1t_\lambda)\\ &= \ell(y)+\ell(t_\lambda)-\ell(u^{-1}v_1)\\ &= \ell(y)+\ell(x)+1-\ell(u^{-1}v_1).\end{aligned}$$

Hence $\ell(u^{-1}v_1)=0$, and therefore $v_1 = u$. Hence, $[t_1t_2] = [ut_\lambda u^{-1}ut_\mu] = [ut_{\lambda+\mu}]$.

Note that $\ell(x)+\ell(y)\le \ell(z)\le \ell(x)+\ell(y)+1 = \ell([ut_{\lambda+\mu}]) = \ell(ut_{\lambda+\mu})$.

If $\ell(z) = \ell(x)+\ell(y)+1 = \ell(ut_{\lambda+\mu})$, then the condition $z \preccurlyeq [ut_{\lambda+\mu}]$ implies that $z = ut_{\lambda+\mu}$. This is just case a).

If $\ell(z) = \ell(x)+\ell(y) = \ell(ut_{\lambda+\mu})-1$, then the condition $z\preccurlyeq [ut_{\lambda+\mu}]$ implies that $z \xrightarrow{\sigma_{\gamma+m\delta}} ut_{\lambda+\mu}$ for some $\gamma+m\delta\in R^+_{\rm re}$. Note that $m\ge 0$ and that $z=\sigma_{\gamma+m\delta}ut_{\lambda+\mu} = \sigma_\gamma ut_{mu^{-1}(\gamma)^\vee+\lambda+\mu}\in W^-_{\rm af}$. Since $\ell(\sigma_{\gamma+m\delta}ut_{\lambda+\mu}) = \ell(z) < \ell(ut_{\lambda+\mu})$, it follows from Lemma 6.1 that $(ut_{\lambda+\mu})^{-1}(\gamma+m\delta) = u^{-1}(\gamma)+\big(m+\langle\lambda+\mu, u^{-1}(\gamma)\rangle\big)\delta\in -R^+_{\rm re}$. Hence, $m+\langle \lambda+\mu, u^{-1}(\gamma)\rangle \le 0$. Since $\lambda+\mu\in\tilde Q^\vee$ and $m\ge 0$, we must have $u^{-1}(\gamma)\in R^+$. Therefore,

$$\begin{aligned}\ell(t_{\lambda+\mu})-\ell(u)-1 &= \ell(z)\\ &= \langle mu^{-1}(\gamma)^\vee+\lambda+\mu, -2\rho\rangle - \ell(\sigma_\gamma u)\\ &= \ell(t_{\lambda+\mu}) - m\langle u^{-1}(\gamma)^\vee, 2\rho\rangle - \ell(u\sigma_{u^{-1}\gamma})\\ &\le \ell(t_{\lambda+\mu}) - m\langle u^{-1}(\gamma)^\vee, 2\rho\rangle - \ell(u) + \ell(\sigma_{u^{-1}(\gamma)})\\ &\le \ell(t_{\lambda+\mu}) - m\langle u^{-1}(\gamma)^\vee, 2\rho\rangle - \ell(u) + \langle u^{-1}(\gamma)^\vee, 2\rho\rangle - 1.\end{aligned}$$

Hence, $(m-1)\langle u^{-1}(\gamma)^\vee, 2\rho\rangle\le 0$. Since $u^{-1}(\gamma)\in R^+$, $\langle u^{-1}(\gamma)^\vee, 2\rho\rangle >0$. Therefore, $0\le m\le 1$; that is, $m=0$ or 1. Denote $\tilde\gamma = u^{-1}(\gamma)$. Note that $\tilde\gamma\in R^+$.

If $m=0$, then $\tilde\gamma\in R^+, z = u\sigma_{\tilde\gamma}t_{\lambda+\mu}$ and $\ell(u\sigma_{\tilde\gamma}) = \ell(u)+1$. This is just case b).

If $m=1$, then $\tilde\gamma\in R^+, z = u\sigma_{\tilde\gamma}t_{\lambda+\mu+\tilde\gamma^\vee}$ and $\ell(u\sigma_{\tilde\gamma}) = \ell(u)+1-\langle\tilde\gamma^\vee, 2\rho\rangle$. This is just case c). □

Lemma 6.7. *Under the same assumptions as in Lemma* 6.5*, we assume that* $\langle\lambda,\alpha_j\rangle < -\ell(\omega_0)$ *and* $\langle\mu,\alpha_j\rangle < -\ell(\omega_0)$ *for all* $j\in I$. *If* **Case B** *occurs, then only one of the following two possibilities can happen:*

a) *there exists* $\gamma\in\Gamma_1$ *such that* $v_1t_{\lambda_1} = u\sigma_\gamma t_\lambda$, $v_2t_{\lambda_2} = u\sigma_\gamma t_\mu$, $z = u\sigma_\gamma t_{\lambda+\mu}$;

b) *there exists* $\gamma\in\Gamma_2$ *such that* $v_1t_{\lambda_1} = u\sigma_\gamma t_\lambda$, $v_2t_{\lambda_2} = u\sigma_\gamma t_{\mu+\gamma^\vee}$, $z = u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}$.

Proof. Note that we have $z=[t_1t_2]$ in this case. Since $v_1t_{\lambda_1}\preccurlyeq\sigma_i t_\lambda\preccurlyeq t_\lambda$, $\lambda\preccurlyeq\lambda_1$ by Corollary 4.6. Hence, $\ell(t_{\lambda_1}) = \ell(t_\lambda)-2M$ for some $M\ge 0$. Therefore,

$$\begin{aligned}\ell(x)+\ell(y) = \ell([t_1t_2]) &= \ell([v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2}])\\ &\le \ell(v_1t_{\lambda_1}v_1^{-1}v_2t_{\lambda_2})\\ &\le \ell(v_1t_{\lambda_1})+\ell(v_1^{-1})+\ell(v_2t_{\lambda_2})\\ &= \ell(t_{\lambda_1})+\ell(v_2t_{\lambda_2})\\ &\le \ell(t_{\lambda_1})+\ell(y) = \ell(x)+1-2M+\ell(y).\end{aligned}$$

Hence, $M = 0$, $\lambda_1 = \lambda$ and $\quad \ell(y) \geq \ell(v_2 t_{\lambda_2}) \geq \ell(x) + \ell(y) - \ell(t_{\lambda_1}) = \ell(y) - 1.$

Hence, there are only the following two possibilities.

Case (i). $\ell(v_2 t_{\lambda_2}) = \ell(y)$, which implies that $v_2 t_{\lambda_2} = y = ut_\mu$.

Case (ii). $\ell(v_2 t_{\lambda_2}) = \ell(y) - 1$.

Due to Lemma 6.9 below, Case (i) is impossible. It remains to discuss Case (ii). In this case,

$$\begin{aligned}\ell(x) + \ell(y) = \ell([t_2 t_1]) &= \ell([v_2 t_{\lambda_2} v_2^{-1} v_1 t_\lambda]) \\ &\leq \ell(v_2 t_{\lambda_2} v_2^{-1} v_1 t_\lambda) \\ &\leq \ell(v_2 t_{\lambda_2}) + \ell(v_2^{-1} v_1 t_\lambda) \\ &= \ell(y) - 1 + \ell(t_\lambda) - \ell(v_2^{-1} v_1) = \ell(y) + \ell(x) - \ell(v_2^{-1} v_1).\end{aligned}$$

Hence $\ell(v_2^{-1} v_1) = 0$, and therefore $v_1 = v_2$.

Since $\ell(v_2 t_{\lambda_2}) = \ell(y) - 1$ and $v_2 t_{\lambda_2} \preccurlyeq y$, there exists $\gamma + m\delta \in R^+_{\rm re}$ such that $v_2 t_{\lambda_2} = \sigma_{\gamma + m\delta} ut_\mu$. With the same discussion as in the proof of Lemma 6.6, we have $\tilde{\gamma} = u^{-1}(\gamma) \in R^+$ and $v_2 t_{\lambda_2} = u\sigma_{\tilde{\gamma}} t_{\mu + m\tilde{\gamma}^\vee} \in W_{\rm af}^-$. Hence, $v_1 = v_2 = u\sigma_{\tilde{\gamma}}$ and $z = u\sigma_{\tilde{\gamma}} t_{\lambda + \mu + m\tilde{\gamma}^\vee} \in W_{\rm af}^-$. With the same argument as in the proof of Lemma 6.6 again, we can deduce that either $m = 0$ or $m = 1$.

If $m = 0$, then we have $\tilde{\gamma} \in R^+$ such that case a) holds; that is,

$$v_1 t_{\lambda_1} = u\sigma_{\tilde{\gamma}} t_\lambda, \quad v_2 t_{\lambda_2} = u\sigma_{\tilde{\gamma}} t_\mu, \quad z = u\sigma_{\tilde{\gamma}} t_{\lambda+\mu}, \qquad \ell(u\sigma_{\tilde{\gamma}}) = \ell(u) + 1.$$

If $m = 1$, then we have $\tilde{\gamma} \in R^+$ such that $v_1 t_{\lambda_1} = u\sigma_{\tilde{\gamma}} t_\lambda$, $v_2 t_{\lambda_2} = u\sigma_{\tilde{\gamma}} t_{\mu + \tilde{\gamma}^\vee}$, $z = u\sigma_{\tilde{\gamma}} t_{\lambda + \mu + \tilde{\gamma}^\vee}$ and $\ell(u\sigma_{\tilde{\gamma}}) = \ell(u) + 1 - \langle \tilde{\gamma}^\vee, 2\rho\rangle$. That is, case b) holds. □

Remark 6.8. The condition "$\langle \mu, \alpha_j\rangle < -\ell(\omega_0)$ for all $j \in I$" does imply that $u\sigma_\gamma t_{\mu+\gamma^\vee}, u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee} \in W_{\rm af}^-$ whenever $\gamma \in \Gamma_2$ and $\lambda \in \tilde{Q}^\vee$.

Indeed, the statement can be checked directly for the case $|I| = n = 1, 2$. For any $\gamma \in R^+$, write $\gamma^\vee = \sum_i a_i \alpha_i^\vee$. Note that $\ell(\omega_0) = |R^+| \geq 9$ and $a_i \leq 4$ if $n = 3, 4$, and that $\ell(\omega_0) > 12$ and $a_i \leq 6$ if $n \geq 5$ (see e.g. page 66 of [15]). Hence, $\mu + \gamma^\vee \in \tilde{Q}^\vee$ is regular if $n \geq 3$. In particular, the statement holds.

Lemma 6.9. Case (i) *in the proof of Lemma* 6.7 *can never occur.*

Proof. Assume Case (i) holds. Then we have $\lambda_1 = \lambda, v_2 t_{\lambda_2} = ut_\mu$ and

$$\begin{aligned}\ell(x) + \ell(y) = \ell([t_2 t_1]) &= \ell([ut_\mu u^{-1} v_1 t_\lambda]) \\ &\leq \ell(ut_\mu u^{-1} v_1 t_\lambda) \\ &\leq \ell(ut_\mu) + \ell(u^{-1} v_1 t_\lambda) \\ &= \ell(y) + \ell(t_\lambda) - \ell(u^{-1} v_1) = \ell(y) + \ell(x) + 1 - \ell(u^{-1} v_1).\end{aligned}$$

Therefore, we have either $\ell(u^{-1} v_1) = 0$ or $\ell(u^{-1} v_1) = 1$.

For the former case, we have $v_1 = u$, and therefore $\ell(x) + \ell(y) = \ell([t_1t_2]) = \ell(ut_{\lambda+\mu}) = \ell(x) + \ell(y) + 1$. This is a contradiction.

For the latter case, $v_1 = u\sigma_j$ for some j. If $\langle\lambda, \alpha_j\rangle \le \langle\mu, \alpha_j\rangle$, then $\lambda + \mu - \langle\mu, \alpha_j\rangle\alpha_j^\vee \in \tilde{Q}^\vee$. Note that the integer $\langle\mu, \alpha_j\rangle < -\ell(\omega_0)$. Therefore

$$\begin{aligned}
\ell(x) + \ell(y) = \ell([t_2t_1]) &= \ell([ut_\mu u^{-1}u\sigma_j t_\lambda]) \\
&\le \ell(ut_\mu\sigma_j t_\lambda) \\
&= \ell(u\sigma_j t_{\mu-\langle\mu,\alpha_j\rangle\alpha_j^\vee+\lambda}) \\
&\le \ell(u\sigma_j) + \ell(t_{\mu-\langle\mu,\alpha_j\rangle\alpha_j^\vee+\lambda}) \\
&= \ell(u\sigma_j) + \langle\mu - \langle\mu, \alpha_j\rangle\alpha_j^\vee + \lambda, -2\rho\rangle \\
&= \ell(u\sigma_j) + \ell(t_{\lambda+\mu}) + 2\langle\mu, \alpha_j\rangle \\
&\le \ell(\omega_0) + \ell(x) + 1 + \ell(y) + \ell(u) - 2\ell(\omega_0) - 2 < \ell(x) + \ell(y).
\end{aligned}$$

If $\langle\lambda, \alpha_j\rangle > \langle\mu, \alpha_j\rangle$, then $\lambda + \mu - \langle\lambda, \alpha_j\rangle\alpha_j^\vee \in \tilde{Q}^\vee$. Therefore,

$$\begin{aligned}
\ell(x) + \ell(y) = \ell([t_1t_2]) &= \ell([u\sigma_j t_\lambda\sigma_j u^{-1}ut_\mu]) \\
&\le \ell(u\sigma_j t_\lambda\sigma_j t_\mu) \\
&= \ell(ut_{\lambda-\langle\lambda,\alpha_j\rangle\alpha_j^\vee+\mu}) \\
&\le \ell(u) + \ell(t_{\lambda-\langle\lambda,\alpha_j\rangle\alpha_j^\vee+\mu}) \\
&= \ell(u) + \langle\lambda - \langle\lambda, \alpha_j\rangle\alpha_j^\vee + \mu, -2\rho\rangle \\
&= \ell(u) + \ell(t_{\lambda+\mu}) + 2\langle\lambda, \alpha_j\rangle \\
&\le \ell(\omega_0) + \ell(x) + 1 + \ell(y) + \ell(u) - 2\ell(\omega_0) - 2 < \ell(x) + \ell(y).
\end{aligned}$$

Both cases deduce contradictions. Hence, Case (i) is impossible. □

Proof of Proposition 4.15. Note that $\mathfrak{S}_x\mathfrak{S}_y = \sum_{z\in W_{\text{af}}^-}\sum_{t_1,t_2} c_{x,[t_1]}c_{y,[t_2]}d_{z,[t_1t_2]}\mathfrak{S}_z$, where the only nonzero terms are those satisfying $x \succcurlyeq [t_1], y \succcurlyeq [t_2], z \preccurlyeq [t_1t_2]$ and $\ell(z) \ge \ell(x) + \ell(y)$. Therefore, our result follows from Lemma 6.6, Lemma 6.7 and Lemma 4.9 immediately. □

Proof of Proposition 4.16. $d_{\sigma_i,u} = w_i - u(w_i)$ holds by expanding the right side (with respect to a reduced expression of u) and comparing both sides. It follows from the definition, Lemma 6.1 and Lemma 4.7 that $d_{w,w} = \prod_{\substack{\gamma\in R^+\\ w^{-1}(\gamma)\in -R^+}} \gamma$ and $d_{wt_\lambda,wt_\lambda} = d_{w\omega_0,w\omega_0}\cdot\prod_{j=r+1}^m w(H_j) = \Big(\prod_{\substack{\gamma\in R^+\\ \omega_0 w^{-1}(\gamma)\in -R^+}} \gamma\Big)\cdot\prod_{j=r+1}^m w(H_j)$. Note that ω_0 is an involution that maps $-R^+$ to R^+, that $\{\gamma \in R^+ \mid w(\gamma) \in R^+\}$ is w-invariant and that $\prod_{i=1}^r H_i = \prod_{\beta\in R^+}\beta$. Hence,

$$\begin{aligned} w(d_{w^{-1},w^{-1}})\cdot d_{wt_{\lambda+\mu},wt_{\lambda+\mu}} &= w(\prod_{\substack{\gamma\in R^+\\ w(\gamma)\in -R^+}}\gamma)\Big(\prod_{\substack{\gamma\in R^+\\ w^{-1}(\gamma)\in R^+}}\gamma\Big)\cdot\prod_{j=r+1}^{m+p} w(H_j)\\ &= w(\prod_{\substack{\gamma\in R^+\\ w(\gamma)\in -R^+}}\gamma)\cdot w(\prod_{\substack{\gamma\in R^+\\ w(\gamma)\in R^+}}\gamma)\cdot\prod_{j=r+1}^{m+p} w(H_j)\\ &= w(\prod_{\gamma\in R^+}\gamma)\cdot\prod_{j=r+1}^{m+p} w(H_j)\\ &= w(\prod_{j=1}^{r} H_j)\cdot\prod_{j=r+1}^{m+p} w(H_j) = \prod_{j=1}^{m+p} w(H_j). \end{aligned}$$

Let $ut_\mu = [\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_k}}]_{\rm red}$, $ut_{\lambda+\mu} = [\sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_s}}]_{\rm red}$ $(k<s)$ and denote $\gamma_j = \sigma_{\beta_{i_1}}\cdots\sigma_{\beta_{i_{j-1}}}(\beta_{i_j})$. Note that $u\sigma_\gamma t_{\mu+\gamma^\vee}, u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}\in W^-_{\rm af}$ whenever $\gamma\in\Gamma_2$, by Remark 6.8. Therefore, for $\gamma\in\Gamma_2$ we have $\ell(u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}) = \ell(ut_{\lambda+\mu}) - 1$ and $\ell(u\sigma_\gamma t_{\mu+\gamma^\vee}) = \ell(ut_\mu) - 1$. Note that $u(\gamma+\delta) = u(\gamma)+\delta\in R^+_{\rm re}$ and $u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee} = \sigma_{u(\gamma+\delta)}ut_{\lambda+\mu}$. Hence, there is a unique $1\le j\le s$ such that $u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee} = [\sigma_{\beta_{i_1}}\cdots\widehat{\sigma_{\beta_{i_j}}}\cdots\sigma_{\beta_{i_s}}]_{\rm red}$ and $\gamma_j = u(\gamma+\delta)$ by Lemma 6.1. As a consequence, $d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},ut_{\lambda+\mu}} = \frac{1}{\gamma_j}\prod_{a=1}^s\gamma_a = \frac{1}{u(\gamma+\delta)}d_{ut_{\lambda+\mu},ut_{\lambda+\mu}}$. Hence, (2) holds. Similarly, (1) also holds.

With the same argument as above, there is a unique $1\le j\le k$ such that $\gamma_j = u(\gamma+\delta)$ and $u\sigma_\gamma t_{\mu+\gamma^\vee} = \sigma_{\beta_{i_1}}\cdots\widehat{\sigma_{\beta_{i_j}}}\cdots\sigma_{\beta_{i_k}}$. Denote $(a_1,\cdots,a_{k-1}) = (i_1,\cdots,\hat{i}_j,\cdots,i_k)$ and denote $\tilde\gamma_b = \sigma_{\beta_{a_1}}\cdots\sigma_{\beta_{a_{b-1}}}(\beta_{a_b})$. Immediately, we have $c_{ut_\mu,u\sigma_\gamma t_{\mu+\gamma^\vee}} = -(\gamma_j\prod_{b=1}^{k-1}\tilde\gamma_b)^{-1}$ and $d_{u\sigma_\gamma t_{\mu+\gamma^\vee},u\sigma_\gamma t_{\mu+\gamma^\vee}} = \prod_{b=1}^{k-1}\tilde\gamma_b$ by definition. Therefore, (3) also holds by the following observation:

$$d_{u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee},u\sigma_\gamma t_{\lambda+\mu+\gamma^\vee}} = d_{u\sigma_\gamma t_{\mu+\gamma^\vee},u\sigma_\gamma t_{\mu+\gamma^\vee}}\cdot\prod_{j=1}^m u\sigma_\gamma t_{\mu+\gamma^\vee}(H_j). \qquad \square$$

6.3. **Equivariant quantum cohomology of $X = G/B$.** The Lie group G possesses a so-called Bruhat decomposition $G = \bigsqcup_{w\in W} B\dot{w}B$, labelled by elements in the Weyl group W. It induces a decomposition of $X := G/B$ into Schubert cells: $X = \bigsqcup_{w\in W} B\dot{w}B/B$, in which $B\dot{w}B/B\cong\mathbb{C}^{\ell(w)}$. The closures $X_w := \overline{B\dot{w}B/B}$ are called Schubert varieties in X. Let σ_w denote the image of the fundamental class $[X_w]$ under the canonical map $H_*(X_w,\mathbb{Z})\to H_*(X,\mathbb{Z})$. Then $\sigma_w\in H_{2\ell(w)}(X,\mathbb{Z})$ and these Schubert homology classes σ_w form an additive basis of $H_*(X,\mathbb{Z})$. The cohomology group $H^*(X,\mathbb{Z})$ also has an additive basis of Schubert cohomology classes σ^w such that $\langle\sigma_u,\sigma^v\rangle = \delta_{u,v}$ for any $u,v\in W$. If we write $g^{u,v} = \int_{[X]}\sigma^u\cup\sigma^v$, then the matrix $\left(g^{u,v}\right)$ is invertible with its inverse denoted as $\left(g_{u,v}\right) = \left(g^{u,v}\right)^{-1}$.

For each $i\in I$, we denote $s_i = \sigma_{\alpha_i}$ and introduce a formal variable q_i. Identify $H_2(X,\mathbb{Z}) = \bigoplus_{i\in I}\mathbb{Z}\sigma_{s_i}$ with $Q^\vee$ via $\beta = \sum_i d_i\sigma_{s_i}\mapsto\lambda_\beta = \sum_i d_i\alpha_i^\vee$. Denote $q_{\lambda_\beta} = q^\beta = \prod_{i\in I} q_i^{d_i}$.

Let $\overline{\mathcal{M}}_{0,m}(X,\beta)$ be the Kontsevich moduli space of stable maps of degree β of m-pointed genus 0 curves into X (see [11]). Let ${\rm ev}_i$ denote the i-th canonical evaluation map ${\rm ev}_i : \overline{\mathcal{M}}_{0,m}(X,\beta)\to X$ given by ${\rm ev}_i([f : C\to X; p_1,\cdots,p_m]) = f(p_i)$. The genus zero Gromov-Witten invariant for $\gamma_1, \cdots, \gamma_m \in H^*(X) = H^*(X, \mathbb{Q})$ is defined as

$$I_{0,m,\beta}(\gamma_1, \cdots, \gamma_m) = \int_{\overline{\mathcal{M}}_{0,m}(X,\beta)} \mathrm{ev}_1^*(\gamma_1) \cup \cdots \cup \mathrm{ev}_m^*(\gamma_m).$$

The (small) quantum product for $a, b \in H^*(X)$ is a deformation of the cup product defined as follows:

$$a \star b = \sum_{u,v \in W; \beta \in H_2(X,\mathbb{Z})} I_{0,3,\beta}(a, b, \sigma^u) g_{u,v} \sigma^v q^\beta.$$

The $\mathbb{Q}[\mathbf{q}]$-module $H^*(X)[\mathbf{q}] := H^*(X) \otimes \mathbb{Q}[\mathbf{q}]$ equipped with $\star$ is called the small quantum cohomology ring of X and denoted as $QH^*(X)$. Therefore, the same Schubert classes $\sigma^u = \sigma^u \otimes 1$ form a basis for $QH^*(X)$ over $\mathbb{Q}[\mathbf{q}]$, and we write

$$\sigma^u \star \sigma^v = \sum_{w \in W, \lambda \in Q^\vee} N_{u,v}^{w,\lambda} q_\lambda \sigma^w.$$

The coefficients $N_{u,v}^{w,\lambda}$ are called the *quantum Schubert structure constants.* In fact, $\sum_{v_1 \in W} g_{v_1,w} \sigma^{v_1} = \sigma^{\omega_0 w}$ (see e.g. [12]). Compared with the original definition of quantum product, the quantum Schubert structure constant $N_{u,v}^{w,\lambda}$ is exactly equal to the (3-pointed genus zero) Gromov-Witten invariant $I_{0,3,\lambda}(\sigma^u, \sigma^v, \sigma^{\omega_0 w})$. When $\lambda = 0$, they give the classical Schubert structure constants for $H^*(X)$. The T-action on X induces an action on the moduli space $\overline{\mathcal{M}}_{0,3}(X, \beta)$ given by $t \cdot (f : C \to X; p_1, p_2, p_3) = (f_t : C \to X; p_1, p_2, p_3)$, where $f_t(x) := t \cdot f(x)$. The evaluation maps ev_i are T-equivariant. We use the same notation σ^u to denote the equivariant Schubert class in $H_T^*(X)$. The equivariant Gromov-Witten invariant is defined as $I_{0,3,\beta}^T(\sigma^u, \sigma^v, \sigma^w) = \pi_*^T(\mathrm{ev}_1^T(\sigma^u) \cdot \mathrm{ev}_2^T(\sigma^v) \cdot \mathrm{ev}_3^T(\sigma^w))$, where π_*^T is the equivariant Gysin push forward. As a consequence, the equivariant (small) quantum product $\star_T$ is defined (see e.g. [33]). The equivariant quantum cohomology ring $QH_T^*(X) = (H^*(X)[\alpha, \mathbf{q}], \star_T)$ is commutative and associative, which has an $S[\mathbf{q}]$-basis of Schubert classes with $S[\mathbf{q}] = \mathbb{Q}[\alpha_1, \cdots, \alpha_n, q_1, \cdots, q_n]$. Furthermore,

$$\sigma^u \star_T \sigma^v = \sum_{w \in W, \lambda \in Q^\vee} \tilde{N}_{u,v}^{w,\lambda} q_\lambda \sigma^w, \quad \text{where } \tilde{N}_{u,v}^{w,\lambda} = \tilde{N}_{u,v}^{w,\lambda}(\alpha) \in S = \mathbb{Q}[\alpha_1, \cdots, \alpha_n].$$

6.4. **Equivariant cohomology of ΩK.** The affine Kac-Moody group $\mathcal{G}$ possesses a Bruhat decomposition $\bigsqcup_{x \in W_{\mathrm{af}}} \mathcal{B} x \mathcal{B}$, where the canonical identification $W_{\mathrm{af}} \cong N(\hat{T}_{\mathbb{C}})/\hat{T}_{\mathbb{C}}$ is used. Here $\hat{T}_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{Z}}(\hat{\mathfrak{h}}_{\mathbb{Z}}^*, \mathbb{C}^*)$ denotes the standard maximal torus of $\mathcal{G}$, in which $\hat{\mathfrak{h}}_{\mathbb{Z}}$ is the integral form of $\hat{\mathfrak{h}} = \mathfrak{h}_{\mathrm{af}}$ (see e.g. chapter 6 of [24]). The Bruhat decomposition of $\mathcal{G}$ induces a decomposition of $\mathcal{G}/\mathcal{P}_Y$ into Schubert cells: $\mathcal{G}/\mathcal{P}_Y = \bigsqcup_{x \in W_{\mathrm{af}}^Y} \mathcal{B} x \mathcal{P}_Y/\mathcal{P}_Y$. Schubert varieties are the closures of $\mathcal{B} x \mathcal{P}_Y/\mathcal{P}_Y$ in $\mathcal{G}/\mathcal{P}_Y$. Let $\mathfrak{S}_x^Y$ denote the image of the fundamental class $[\overline{\mathcal{B} x \mathcal{P}_Y/\mathcal{P}_Y}]$ under the canonical map $H_*(\overline{\mathcal{B} x \mathcal{P}_Y/\mathcal{P}_Y}) \to H_*(\mathcal{G}/\mathcal{P}_Y)$. Then $H_*(\mathcal{G}/\mathcal{P}_Y, \mathbb{Z})$ has an additive basis of Schubert homology classes $\{\mathfrak{S}_x^Y \mid x \in W_{\mathrm{af}}^Y\}$. We denote $\mathfrak{S}_x = \mathfrak{S}_x^Y$ wherever there is no confusion. Similarly, the cohomology group $H^*(\mathcal{G}/\mathcal{P}_Y, \mathbb{Z})$ has an additive basis of Schubert cohomology classes $\{\mathfrak{S}^x \mid x \in W_{\mathrm{af}}^Y\}$, where $\langle \mathfrak{S}_x, \mathfrak{S}^y \rangle = \delta_{x,y}$ with respect to the natural pairing.

The standard maximal torus $\hat{T}_{\mathbb{C}}$ of $\mathcal{G}$ has complex dimension $n+2$ with maximal compact subtorus $\hat{T} = \mathrm{Hom}_{\mathbb{Z}}(\hat{\mathfrak{h}}_{\mathbb{Z}}^*, \mathbb{S}^1)$. With respect to the natural action of $\hat{T}_{\mathbb{C}}$ on $\mathcal{G}/\mathcal{B}$, we consider the equivariant cohomology $H_{\hat{T}}^*(\mathcal{G}/\mathcal{B})$, which is an $\hat{S}$-module with $\hat{S} = S[\hat{\mathfrak{h}}_{\mathbb{Z}}^*] = H^*_{\hat{T}}(\mathrm{pt})$. Note that the one dimensional subtorus $\mathbb{C}^*$, which comes from the central extension, acts on $\mathcal{G}/\mathcal{B}$ trivially. As a consequence, the equivariant Schubert structure constants are polynomials in $\mathbb{Q}[\delta, \alpha_1, \cdots, \alpha_n] \subset \hat{S}$ only. Since we are concerned with the nontrivial part of the $\hat{T}_{\mathbb{C}}$-action only, we denote $\hat{S} = \mathbb{Q}[\delta, \alpha_1, \cdots, \alpha_n] = \mathbb{Q}[\alpha_0, \alpha_1, \cdots, \alpha_n]$ by abuse of notation. $H^*_{\hat{T}}(\mathcal{G}/\mathcal{B})$ is an $\hat{S}$-module spanned by the basis of equivariant Schubert classes (see e.g. [24] for concrete definitions), which we also denote as $\{\mathfrak{S}^x \mid x \in W_{\mathrm{af}}\}$ simply. Via the embedding $\pi^* : H^*_{\hat{T}}(\mathcal{G}/\mathcal{P}_Y) \hookrightarrow H^*_{\hat{T}}(\mathcal{G}/\mathcal{B})$ induced by the natural projection $\pi : \mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_Y$, the equivariant cohomology $H^*_{\hat{T}}(\mathcal{G}/\mathcal{P}_Y)$ is also an $\hat{S}$-module spanned by the basis of equivariant Schubert classes $\{\mathfrak{S}^x \mid x \in W^Y_{\mathrm{af}}\}$. As a consequence, equivariant Schubert structure constants for $\mathcal{G}/\mathcal{P}_Y$ are covered by equivariant Schubert structure constants for $\mathcal{G}/\mathcal{B}$.

Remark 6.10. For $\mathfrak{S}^x, \mathfrak{S}^y \in H^*_{\hat{T}}(\mathcal{G}/\mathcal{B})$, $\mathfrak{S}^x\mathfrak{S}^y = \sum_{z\in W_{\mathrm{af}}} p^z_{x,y}\mathfrak{S}^z$. The equivariant Schubert structure constant $p^z_{x,y}$ is a polynomial in $\hat{S}$. In terms of a combination of rational functions, one has $p^z_{x,y} = \sum_{v\in W_{\mathrm{af}}} d_{x,v}d_{y,v}c_{z,v}$ (see e.g. chapter 11 of [24]).

Let $L_{\mathrm{an}}K = \{f \in \mathcal{G} \mid f(\mathbb{S}^1) \subset K\}$ and $\Omega_{\mathrm{an}}K = \{f \in L_{\mathrm{an}}K \mid f(1_{\mathbb{S}^1}) = 1_K\}$. Note that each $f \in \mathcal{G}$ can be written as $f(t) = f_K(t) \cdot f_P(t)$ for some unique $f_K \in \Omega_{\mathrm{an}}K$ and $f_P \in \mathcal{P}_0$. Therefore we can realize $\mathcal{G}/\mathcal{P}_0$ as $\Omega_{\mathrm{an}}K$, which is homotopy-equivalent to ΩK, via the ($L_{\mathrm{an}}K$-equivariant) homeomorphism $\mathcal{G}/\mathcal{P}_0 \to \Omega_{\mathrm{an}}K$ (see [36] and the references therein for more details). Since we are concerned with properties at the level of (co)homology only, we do not distinguish between $\Omega_{\mathrm{an}}K$ and ΩK. The Bruhat decomposition of $\mathcal{G}/\mathcal{P}_0$ readily gives a Bruhat decomposition of ΩK. As a consequence and by abusing notation, we know that $H_*(\Omega K, \mathbb{Z})$ (resp. $H^*(\Omega K, \mathbb{Z})$) has an additive $\mathbb{Z}$-basis of Schubert (co)homology classes $\{\mathfrak{S}_x(\text{resp. } \mathfrak{S}^x) \mid x \in W^-_{\mathrm{af}}\}$.

The (nontrivial part of the) $\hat{T}$-action on $\mathcal{G}/\mathcal{P}_0$ corresponds to the natural $\mathbb{S}^1 \times T$ action on ΩK, which consists of the rotation action of $\mathbb{S}^1$ on ΩK and the action of T on ΩK by pointwise conjugation. By considering the T-action only, we obtain the evaluation maps $\mathrm{ev} : H^*_{\hat{T}}(\mathcal{G}/\mathcal{P}_0) \to H^*_T(\mathcal{G}/\mathcal{P}_0)$ and $ev : \hat{S} = H^*_{\hat{T}}(\mathrm{pt}) \to H^*_T(\mathrm{pt}) = S$, where the T-equivariant cohomology $H^*_T(\mathcal{G}/\mathcal{P}_0)$ is an S-module with $S = \mathbb{Q}[\alpha_1, \cdots, \alpha_n]$. The image of the null root $\delta = \alpha_0 + \theta$ in S is 0. More precisely, we have $H^*_{\hat{T}}(\mathcal{G}/\mathcal{P}_0) = \mathrm{Span}_{\hat{S}}\{\hat{\mathfrak{S}}^x \mid x \in W^-_{\mathrm{af}}\}$ and $H^*_T(\mathcal{G}/\mathcal{P}_0) = \mathrm{Span}_S\{\mathfrak{S}^x \mid x \in W^-_{\mathrm{af}}\}$. Let $f = f(\alpha_0, \alpha_1, \cdots, \alpha_n) \in \hat{S}$, then we have $ev(f) = f(-\theta, \alpha_1, \cdots, \alpha_n) \in S$ and $\mathrm{ev}(f\hat{\mathfrak{S}}^x) = ev(f)\mathfrak{S}^x$.

Remark 6.11. $\mathfrak{S}^x\mathfrak{S}^y = \sum_{z\in W^-_{\mathrm{af}}} \tilde{p}^z_{x,y}\mathfrak{S}^z$. The T-equivariant Schubert structure constant $\tilde{p}^z_{x,y}$ is a polynomial in S. It follows from Remark 6.10 and Lemma 4.14 that $\tilde{p}^z_{x,y} = \sum_{v\in W^-_{\mathrm{af}}} d_{x,[v]}d_{y,[v]}c_{z,[v]}$ as a combination of rational functions.

Remark 6.12. The T-equivariant Schubert structure constant $p^w_{u,v}$ for G/B can also be expressed in terms of $c_{u,v}$ and $d_{u,v}$. The polynomial $p^w_{u,v}$ is given by $p^w_{u,v} = \sum_{v_1\in W} d_{u,v_1}d_{v,v_1}c_{w,v_1}$ as a combination of rational functions (see e.g. [24]).

Remark 6.13. Because of the natural S-module isomorphism $H^*_T(\Omega K \times \Omega K) \cong H^*_T(\Omega K)\otimes_S H^*_T(\Omega K)$, the Pontryagin product $\Omega K \times \Omega K \to \Omega K$, which is associative and T-equivariant, induces a coassociative coproduct

$$H^*_T(\Omega K) \to H^*_T(\Omega K) \otimes H^*_T(\Omega K).$$

In particular, this gives an alternative definition of the T-equivariant homology of $\mathcal{G}/\mathcal{P}_0$, which doesn't use Borel-Moore homology. Indeed, define $H_*^T(\mathcal{G}/\mathcal{P}_0)$ to be the submodule of $\mathrm{Hom}_S(H_T^*(\mathcal{G}/\mathcal{P}_0), S)$ spanned by those $\mathfrak{S}_x \in \mathrm{Hom}_S(H_T^*(\mathcal{G}/\mathcal{P}_0), S)$ which for any $x, y \in W_{\mathrm{af}}^-$ satisfy $\langle \mathfrak{S}_x, \mathfrak{S}^y \rangle = \delta_{x,y}$ with respect to the natural pairing. Then the product of $H_*^T(\mathcal{G}/\mathcal{P}_0)$, induced from the coproduct of $H_T^*(\mathcal{G}/\mathcal{P}_0) = H_T^*(\Omega K)$, makes $H_*^T(\mathcal{G}/\mathcal{P}_0)$ an S-algebra. Note that the elements $\mathfrak{S}_x$ coincide with the integration operators $\mathcal{L}_w$ defined in [1], Prop. 2.5.1, so Arabia's localization formula can be applied. Thus the whole proof of the formula for structure coefficients $b_{x,y}^z$ still goes through.

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