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The SYZ Proposal

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1 Pre-SYZ

In the beginning of the nineties, there is a ground-breaking event in mathematics, namely the birth of the mirror symmetry.

Greene, a postdoc of Yau, and Plesser proposed [7] that the super string theory compactified on a Calabi-Yau threefold X should admit a dual theory which is compactified on a different Calabi-Yau threefold Y. The equivalence of these two theories is called mirror symmetry, and Y is the mirror manifold to X and vice versa.

The simplest Calabi-Yau threefold is the zero locus of a degree five homogeneous polynomial f in $\mathbb{CP}^4$, the *quintic Calabi-Yau* threefold. For example if we take

$$f(z_0, z_1, ..., z_4) = z_0^5 + z_1^5 + \cdots + z_4^5 + \psi(z_0 z_1 \cdots z_4)$$

then $X = \{f = 0\}$ is a smooth Calabi-Yau threefold, called the *Fermat Calabi-Yau threefold,* provided that ψ is any complex number not equal to one.

Candelas et al [2] did a highly nontrivial calculation of this equivalence for the Fermat Calabi-Yau threefold and showed physically that the number of rational curves of any degree in X can be read off explicitly from the periods of Y. This is an astonishing discovery as it relates two very different but equally important subjects in algebraic geometry, namely the enumerative geometry of X and the variation of complex structures of Y.

In string theory, each Calabi-Yau manifold X determines two twisted theories, called the *A-model* and *B-model*, and the mirror symmetry between X and Y interchanges these two models between them. From the mathematical perspective, A-model is about the symplectic geometry and B-model is about the complex geometry.

$$\begin{matrix}\text{A-model on } X \\ \text{(symplectic geometry)}\end{matrix} \xleftrightarrow[\text{mirror symmetry}]{} \begin{matrix}\text{B-model on } Y \\ \text{(complex geometry)}\end{matrix}$$

In 1994 Kontsevich [16] proposed a much more precise conjecture on this duality between symplectic and complex geometries, called the *homological mirror*

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symmetry (HMS): If X and Y are mirror manifolds to each other, then the Fukaya-Floer category of Lagrangian intersections in X is equivalent to the bounded derived category of coherent sheaves on Y. There are also generalizations of this duality for Fano manifolds and general type manifolds.

2 The birth of SYZ

Ever since the work of Candelas et al, there had been much work on verifying various surprising predictions coming out from mirror symmetry. This includes searching the right definitions of the Gromov-Witten invariants and the Fukaya-Floer category; developing Bott localization techniques and use them to compute Gromov-Witten invariants; using toric geometry to construct mirror manifolds and so on. Nevertheless, it is still a complete mystery as to why such a powerful duality exists.

In 1997, Strominger, Yau and Zaslow proposed a resolution in their groundbreaking paper [24]. They conjectured that (i) both X and Y should admit *special Lagrangian* torus fibrations with sections in the large volume/complex structure limit;

$$\begin{array}{ccc} T & \underset{\longleftrightarrow}{\text{dual tori}} & T^* \\ \downarrow & & \downarrow \\ X & & Y \\ \downarrow & & \downarrow \\ B & & B^* \end{array}$$

(ii) they are dual torus fibrations to each other; (iii) a fiberwise Fourier-Mukai transformation along fibers interchanges the symplectic (resp. complex) geometry on X with the complex (resp. symplectic) geometry on Y.

This surprises everyone as it says that the mysterious duality is simply a Fourier transformation! The quantum corrections, for instance the Gromov-Witten invariants, come from the higher Fourier modes. The challenging task is to understand the behavior of the Calabi-Yau metrics near the large complex structure limit.

The SYZ is a very elegant conjecture. This is both very simple and very deep, and its importances in the development of geometry is hard to be overstated. On the one hand, the SYZ conjecture is the guiding light for all our effort in trying to understand the mirror symmetry which transforms the symplectic geometry to complex geometry and vice versa. This includes works of Gross, Kontsevich, Siebert, Vafa, Yau, Zaslow and many others. On the other hand, it has led to vivid developments of other branches of mathematics, including the calibrated geometry of special Lagrangian submanifolds by Schoen's school and the affine geometry with singularities.

A brief reasoning behind SYZ is as follows: From physical considerations, branes in B-model are complex submanifolds, or more precisely complex of coherent sheaves up to derived equivalences, and branes in A-model are special Lagrangian submanifolds coupled with unitary flat bundles. As mirror symmetry interchanges the complex geometry of Y with the symplectic geometry of X, their moduli spaces of branes should be identified as well, at least at the large complex structure limit where quantum corrections had been suppressed. Since Y can be regarded as the moduli space of points which are complex submanifolds, Y should also be the moduli space of certain A-branes in X. Furthermore the underlying Lagrangian submanifolds of these A-branes should cover X everywhere once, just like what points in Y did. Then we can argue via deformation theory to show that X should admit a special Lagrangian torus fibration

$$T \to X \xrightarrow{\pi} B.$$

When we consider the complex submanifold which is Y itself, the moduli space is a single point and the corresponding A-brane in X would be a special Lagrangian section to π.

Next, given any torus fiber T in X, its dual torus T^* parametrizes flat $U(1)$-bundles over T, namely A-branes in X with support T. Under mirror symmetry, this T^* also parametrizes corresponding B-branes in Y, which are points in Y. Thus T^* is a subspace in Y and therefore Y also has a torus fibration by such T^*'s.

$$T^* \to Y \xrightarrow{\pi} B^*.$$

One can further argue that these two are dual special Lagrangian torus fibrations, at least in the large complex structure limit where quantum corrections have been suppressed.

Besides giving dual fibrations on mirror manifolds X and Y, we have a *transformation* between special Lagrangian fibers in X with zero dimensional complex submanifolds in Y, namely points. This is a special case of a fiberwise Fourier-Mukai transformation. For more general special Lagrangian submanifolds in X, say a section to the above fibration, then the intersection point of it with any fiber T would determine a flat $U(1)$ connection on T^* because $(T^*)^* = T$. By patching them for various fibers T, we obtain a $U(1)$ connection on the whole manifold Y. One expects that this determines a holomorphic line bundle on Y which is the mirror to the section in X. This was verified in [20] in the semiflat case. We call this transformation between the symplectic geometry of X and the complex geometry of Y the *SYZ mirror transformation.*

In order to complete the SYZ proposal, we still have much work to do. This includes describing the Calabi-Yau metrics near the large complex structure limit point; constructing special Lagrangian fibrations and understanding their singularity structures; studying affine geometry with singularities; developing a geometric theory of Fourier transformation with quantum corrections and so on.

3 The growing up of SYZ

The SYZ proposal has led to exciting developments of several branches of mathematics. We are going to explain some of these aspects.

3.1 Special Lagrangian geometry

Special Lagrangian submanifolds coupled with unitary flat bundles are branes in A-model of string theory. They are mirror to coherent sheaves as branes in B-model. Let us recall its definition. Let us denote the Kähler form and the holomorphic volume form on a complex n-dimensional Calabi-Yau manifold X as $\omega \in \Omega^{1,1}(X)$ and $\Omega \in \Omega^{n,0}(X)$ respectively. A middle dimensional submanifold L in X is *Lagrangian* if $\omega|_L = 0$ and such an L is *special* if $\operatorname{Im}\Omega|_L = 0$. This is equivalent to $\operatorname{Re}\Omega|_L = vol_L$, that is L is calibrated by $\operatorname{Re}\Omega|_L$. From the calibration theory developed by Harvey and Lawson, such manifolds are absolute minimum for the volume functional, just like complex submanifolds in Kähler manifolds.

These geometric objects becomes of central importance in mathematics because of the SYZ conjecture. Many examples were constructed using cohomogeneity one method by Joyce, using singular perturbation method by Butscher, Lee, Haskins, Kapouleas and others, and using other methods by Bryant and Haskin. There are also many investigations of their properties, including the studies of their deformation theory by McLean and later by Schoen's school, their moduli spaces by Hitchin, their existence and regularity problem by Schoen and Wolfson using variational approach and many other aspects of the geometry of special Lagrangian submanifolds.

The mirror symmetry conjecture predicts that (special) Lagrangian submanifolds should behave like (Hermitian Yang-Mills) holomorphic vector bundles, modulo quantum effects. Thomas and Yau [25] formulated a very interesting conjecture on the existence of special Lagrangian submanifolds which is the *mirror* of the important theorem of Donaldson, Uhlenbeck and Yau which says that there is a unique Hermitian Yang-Mills connection on any stable holomorphic vector bundle.

3.2 Special Lagrangian fibrations

SYZ conjecture predicts that mirror Calabi-Yau manifolds should admit dual torus fibrations whose fibers are special Lagrangian submanifolds, possibly with singularities.

Lagrangian fibrations is an important notion in symplectic geometry as real polarizations, as well as in dynamical system as completely integrable systems. Their smooth fibers admit canonical integral affine structures and therefore they must be tori in the compact situation. Toric varieties $\mathbb{P}_\Delta$ are examples of symplectic manifolds with Lagrangian fibrations in which the fibers are orbits of an Hamiltonian torus action and the base is a convex polytope Δ. The simplest compact toric varieties are certainly complex projective spaces $\mathbb{CP}^{n+1}$.

A complex hypersurface $X = \{f = 0\}$ in $\mathbb{CP}^{n+1}$ is Calabi-Yau if and only if $\deg f = n + 2$. The most singular ones is when X is a union of coordinate hyperplanes in $\mathbb{CP}^{n+1}$ and this is called the *large complex structure limit* (LCSL). At this most singular limit, X inherits a torus fibration from the toric structure on $\mathbb{CP}^{n+1}$. Thus one can try to perturb this to obtain Lagrangian fibration structures on nearby smooth Calabi-Yau manifolds. This approach was carried out by Gross, Mikhalkin, Ruan and Zharkov.

This approach can be generalized to Calabi-Yau hypersurfaces X in any Fano toric variety $\mathbb{P}_\Delta$. Furthermore, their mirror manifolds Y are Calabi-Yau hypersurfaces in another Fano toric variety $\mathbb{P}_\nabla$ whose defining polytope is the polar dual to Δ. Thus we can see that the Lagrangian fibration structures on X and Y should be given by dual tori, at least away from singular fibers.

However the question of whether one can make the Lagrangian fibrations on X special is a much more delicate question as we do not understand the behavior of the Calabi-Yau metrics, whose existences are asserted by the celebrated theorem of Yau [26]. It is possible that only *approximate* special Lagrangian fibrations exists for Calabi-Yau *threefolds* near the large complex structure limit as it was indicated by Joyce in his work on special Lagrangian fibrations on generic almost Calabi-Yau threefolds.

The situation is quite different for Calabi-Yau *twofolds*, namely K3 surfaces, or more generally for hyperkähler manifolds. In this case, the Calabi-Yau metric on X is Kähler with respect to three complex structures I, J and K. When X admits a J-holomorphic Lagrangian fibration, then this fibration is a special Lagrangian fibration with respect to the Kähler metric ω_I, as well as ω_K. Furthermore, SYZ also predicts that mirror symmetry is merely a twistor rotation from I to K in this case. For K3 surfaces, there are plenty of elliptic fibrations and they are automatically complex Lagrangian fibrations because of their low dimension. Furthermore Gross and Wilson [13] described the Calabi-Yau metrics for generic elliptic K3 surfaces by using the singular perturbation method. They used model metrics which were constructed by Greene, Shapere, Vafa and Yau [6] away from singular fibers and by Ooguri and Vafa [21] near singular fibers.

The topology and geometry of special Lagrangian fibrations are studied by Gross [8][9][10] and Goldstein [5].

3.3 Affine geometry

A well-known fact from integrable system says that given any compact Lagrangian fibration

$$T \to X \xrightarrow{\pi} B,$$

its base space B admits a canonical integral affine structure, possibly with *singularities*. This affine structure will dictate the Calabi-Yau geometry at the large complex structure limit.

Outside the preimage of the singular set of B, the total space X is given by the quotient of the cotangent bundle T^*B by a lattice subbundle symplectically. In order to understand the A-model on X, we need to be able to describe rational curves and holomorphic disks on X in terms of the affine structure on B. There has been much progress on this by the work of Fukaya, Kontsevich-Soibelman, Siebert and Gross and others. Here *tropical geometry* plays an important role.

The tangent bundle TB of any affine manifold B admits a canonical complex structure away from its singularities. Kontsevich and Soibelman [17], Gross and Siebert [12] described how to deform this complex structure at the large complex structure limit to nearby complex structures. Recall that the physical calculations of Candelas et al [2] showed that the variation of their Hodge structures should determine the Gromov-Witten invariants of rational curves of the mirror manifold. This important formula was later proven by Givental, Lian, Liu and Yau via a clever computation of Gromov-Witten invariants using localization method. The above program will eventually give a mathematical explanation of this phenomenon.

The first step in understanding the behavior of Calabi-Yau metrics on Calabi-Yau threefolds near large complex structure limits is to construct affine analogs of Calabi-Yau metrics on the affine manifold B. Notice that the affine structure on B is singular and the nontrivial part of the singular set $Sing\,(B)$ is locally given by a planar Y-vertex in $\mathbb{R}^3$. Therefore, such an affine Calabi-Yau metric on B should also be singular along $Sing\,(B)$ with appropriate monodromy around it. Using the scaling symmetry of the Y-vertex, Loftin, Yau and Zaslow [21] had constructed an affine Calabi-Yau metric around it by using PDE methods in two dimension. However we are still pursuing one with monodromy predicted by the mirror symmetry.

3.4 SYZ transformation

Recall that the SYZ conjecture says that the duality between mirror manifolds is a Fourier-Mukai transformation along dual special Lagrangian torus fibrations. Soon we realized that we also need to couple the fiberwise Fourier transformation with a Legendre transformation on the affine manifolds which are the bases for the dual fibrations. This SYZ transformation was generalized to the mirror symmetry for *local* Calabi-Yau manifolds by Leung and Vafa in [19]. In [15], Hori and Vafa gave a *physical proof* of the mirror symmetry using the SYZ proposal.

On the mathematical side, Leung, Yau and Zaslow [20], [18] used the SYZ transformation to verify various correspondences between symplectic geometry and complex geometry between *semi-flat* Calabi-Yau manifolds. In this situation, there is no quantum corrections from instantons, namely rational curves or holomorphic disks. To include quantum corrections in the SYZ transformation for Calabi-Yau manifolds is a much more difficult problem. However in the Fano case, there are recent results on applying the SYZ transformation with quantum corrections by Auroux [1], Chan and Leung [3] and Fang [4].

4 Future of SYZ

SYZ has generated a huge amount of research activities in mathematics, as well as in physics. This includes geometry of special Lagrangain submanifolds, and more generally calibrated submanifolds, Lagrangian fibrations, SYZ Fourier transformations, tropical geometry, affine geometry and so on. Such an elegant proposal will continue to be the driving force for the rapid development in geometry and physics. And it will eventually lead to a satisfactory solution to the mirror symmetry conjecture.

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