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Jae-Hyouk Lee, Naichung Conan Leung

Grassmannians of symplectic subspaces

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Abstract. We study the geometry of the Grassmannians of symplectic subspaces in a symplectic vector space. We construct symplectic twistor spaces by the symplectic quotient construction and use them to describe the symplectic geometry of the symplectic Grassmannians.

1. Introduction

On a $2n$-dimensional real vector space V, the set of all $2k$-dimensional linear subspaces $S \subset V$ is the real Grassmannian $Gr\,(2k, 2n)$. When V is equipped with a linear complex structure, the complex Grassmannian $Gr_{\mathbb{C}}\,(k, n)$ parametrizes those subspaces S which are complex. $Gr_{\mathbb{C}}\,(k, n)$ carries a natural complex structure, which plays important roles in complex geometry.

Throughout this paper, V is a symplectic vector space of dimension $2n$ with symplectic form ω_V, or simply ω. We define the *symplectic Grassmannian* $Gr^{Sp}\,(2k, 2n)$ to be the set of all $2k$-dimensional linear subspaces in V which are symplectic. The objective of this paper is to study the geometry of $Gr^{Sp}\,(2k, 2n)$.

First we have an identification

$$Gr^{Sp}\,(2k, 2n) \simeq \frac{Sp\,(2n, \mathbb{R})}{Sp\,(2k, \mathbb{R})\,Sp\,(2n-2k, \mathbb{R})},$$

and natural inclusions

$$Gr_{\mathbb{C}}\,(k, n) \subset Gr^{Sp}\,(2k, 2n) \subset Gr\,(2k, 2n)\,,$$

or equivalently,

$$\frac{U\,(n)}{U\,(k)\,U\,(n-k)} \subset \frac{Sp\,(2n, \mathbb{R})}{Sp\,(2k, \mathbb{R})\,Sp\,(2n-2k, \mathbb{R})} \subset \frac{O\,(2n)}{O\,(2k)\,O\,(2n-2k)}.$$

J.-H. Lee (✉): Korea Institute for Advanced Study, Seoul, Korea.
e-mail: jaehyoukl@kias.re.kr

J.-H. Lee: Department of Mathematics, Ewha Womans University, 11-1 Daehyun-Dong, Seodaenum-gu, 120-750 Seoul, Korea. e-mail: jaehyoukl@ewha.ac.kr

N. C. Leung: Department of Mathematics, The Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, NT, Hong Kong.
e-mail: leung@math.cuhk.edu.hk

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The inclusion $Gr^{Sp}(2k, 2n) \subset Gr(2k, 2n)$ is the complement of a hyperplane in $\Lambda^{2k}V$, and the inclusion $Gr_{\mathbb{C}}(k, n) \subset Gr^{Sp}(2k, 2n)$ is the intersection with $\Lambda^{k,k}_{\mathbb{R}}V$, and topologically it is a deformation retract. In order to describe the symplectic geometry of $Gr^{Sp}(2k, 2n)$, we parametrize symplectic subspaces by symplectic homomorphisms $\phi \in Hom(S, V)$, i.e. $\phi^*\omega_V = \omega_S$. We show that this set is the inverse image of J_S under a moment map

$$\mu : Hom(S, V) \to o(2k)^*,$$

and the symplectic quotient $Hom(S, V)//O(2k)$ is the set of symplectic subspaces in V together with compatible complex structures on them. We call this the *symplectic half twistor Grassmannian*, denoted $\mathcal{T}(2k, 2n)$. It is the total space of a fiber bundle

$$\frac{Sp(2k, \mathbb{R})}{U(k)} \to \mathcal{T}(2k, 2n) \to Gr^{Sp}(2k, 2n),$$

and every fiber is a symplectic manifold in $\mathcal{T}(2k, 2n)$. We show that such a symplectic fiber bundle structure is sufficient to have naturally defined notions of symplectic submanifolds and Lagrangian submanifolds in $Gr^{Sp}(2k, 2n)$ even though the base manifold $Gr^{Sp}(2k, 2n)$ does not carry a symplectic structure. Then we construct symplectic and Lagrangian submanifolds in $Gr^{Sp}(2k, 2n)$ using symplectic and isotropic/coisotropic linear subspaces in V.

In order to describe the Kähler geometry in this setting, we introduce the *symplectic twistor Grassmannian* $\hat{\mathcal{T}}(2k, 2n)$ which is the set of triples (S, J_S, J_{S^ω}) with S a symplectic subspace in V and J_S (resp. J_{S^ω}) a compatible complex structure on S (resp. S^ω). The symplectic twistor Grassmannian is also the total space of a symplectic fiber bundle

$$\frac{Sp(2k, \mathbb{R})}{U(k)} \times \frac{Sp(2n-2k, \mathbb{R})}{U(n-k)} \to \hat{\mathcal{T}}(2k, 2n) \to Gr^{Sp}(2k, 2n).$$

In addition, it is the total space of another symplectic fiber bundle given by

$$Gr_{\mathbb{C}}(k, n) \longrightarrow \hat{\mathcal{T}}(2k, 2n) \xrightarrow{\beta} \frac{Sp(2n, \mathbb{R})}{U(n)},$$

and this latter bundle structure gives $\hat{\mathcal{T}}(2k, 2n)$ its natural Kähler structure. Note that $Sp(2n, \mathbb{R})/U(n)$ is the set of all compatible complex structures on V and thus we are using all complex structures on (V, ω_V) to define one on $\hat{\mathcal{T}}(2k, 2n)$. Remark that other symplectic twistor theories with different motivations can be found in the researches of Albuquerque, Reznikov et al.

The organization of this article is as follows: In Sect. 2, we introduce the symplectic Grassmannians and describe their basic properties. We also review basics of symplectic geometry that we need. In Sect. 3, we define the symplectic (half) twistor Grassmannians and construct natural symplectic structures on them using the symplectic quotient construction. We also prove basic properties of them. In Sect. 4, we use the symplectic fiber bundle structures on symplectic (half) twistor Grassmannians to define the symplectic geometry on $Gr^{Sp}(2k, 2n)$ and we construct symplectic submanifolds and Lagrangian submanifolds in $Gr^{Sp}(2k, 2n)$ as special kinds of Schubert cycles. We also analyze the Kähler geometry of the symplectic twistor Grassmannians.

2. Symplectic Grassmannians

2.1. Definition and basic properties

In this subsection, we define symplectic Grassmannians and discuss their basic properties and their relationships with real and complex Grassmannians. Recall that V always denotes a $2n$-dimensional symplectic vector space with symplectic form ω.

Definition 1. Given any $k \leq n$, the set of all $2k$-dimensional symplectic linear subspaces in V is called the *symplectic Grassmannian, and it is denoted as* $Gr^{Sp}(2k, V)$, or simply $Gr^{Sp}(2k, 2n)$.

Since a subspace being symplectic is an open condition, the symplectic Grassmannian has a natural topology coming from its being an open subset of the real Grassmannian $Gr(2k, 2n)$ of all $2k$-dimensional real linear subspaces in V. Recall that $Gr(2k, 2n) \simeq O(2n)/O(2k)\,O(2n-2k)$. To see this, we pick a metric g on V to obtain an orthogonal decomposition $V = P \oplus P^{\perp}$ for each subspace P in V. Here $P^{\perp} = \{v \in V : g(u, v) = 0 \text{ for any } u \in P\}$ is the g-orthogonal complement to P. The orthonormal bases of P and $P^{\perp}$ give one on V and therefore $Gr(2k, 2n)$ is a homogeneous space of $O(2n)$ with the isotropy subgroup the product of orthogonal groups of P and $P^{\perp}$. For the same reason, if we equip a complex vector space (V, J) with a Hermitian metric, then the *complex Grassmannian* $Gr_{\mathbb{C}}(k, n)$ can be identified with $U(n)/U(k)\,U(n-k)$. With respect to any compatible symplectic structure ω, every complex subspace is a symplectic subspace in V. Many of our later constructions will require a choice of a complex structure or a metric structure which is compatible with ω.

Definition 2. Let V be an even dimensional vector space. (1) A Hermitian structure on V is a triple (ω, J, g) with ω a symplectic structure, J a complex structure and g a metric structure on V satisfying

$$\omega(u, v) = g(Ju, v)$$

for any $u, v \in V$ and J is g-orthogonal, i.e. $g(Ju, Jv) = g(u, v)$. (2) Among symplectic, complex and metric structures, any two of these structures on V are called compatible with each other if the relation $\omega(u, v) = g(Ju, v)$ defines the third structure and together they make V into a Hermitian vector space.

Remark. The space of all compatible complex structures on (V, ω) is the Siegel upper half space $Sp(2n, \mathbb{R})/U(n)$.

There is also a canonical identification between $Gr^{Sp}(2k, 2n)$ and the homogeneous space $Sp(2n, \mathbb{R}) / Sp(2k, \mathbb{R})\, Sp(2n-2k, \mathbb{R})$, and the canonical inclusions

$$Gr_{\mathbb{C}}(k, n) \subset Gr^{Sp}(2k, 2n) \subset Gr(2k, 2n)$$

correspond to the following inclusions of homogeneous spaces

$$\frac{U(n)}{U(k)\,U(n-k)} \subset \frac{Sp(2n, \mathbb{R})}{Sp(2k, \mathbb{R})\,Sp(2n-2k, \mathbb{R})} \subset \frac{O(2n)}{O(2k)\,O(2n-2k)}.$$

In order to describe $Gr^{Sp}(2k, 2n)$, we recall the following notions.

Definition 3. Let P be any linear subspace in (V, ω).

(i) The symplectic complement of P is

$$P^{\omega} := \{u \in V \mid \omega(u, v) = 0 \text{ for all } v \in P\}.$$

(ii) The *null space of* P is $N(P) := P \cap P^{\omega}$ and its dimension is called the nullity $n(P) := \dim N(P)$.

(iii) The symplectic rank, or simply rank, of P is $r(P) := \max\left\{r \in \mathbb{N} : (\omega|_P)^r \neq 0\right\}$.

Symplectic complements satisfy the following basic properties:

(i) $\left(P^{\omega}\right)^{\omega} = P$, (ii) $(P \cap Q)^{\omega} = P^{\omega} + Q^{\omega}$ and (iii) $P \subset Q$ iff $P^{\omega} \supset Q^{\omega}$.

The null space $N(P)$ is the largest subset in P where the restriction of ω vanishes and $P / (P^{\omega} \cap P)$ thereby has an induced symplectic structure, called the *symplectic reduction* of P. The rank is half of the maximal possible dimension among symplectic subspaces in P and it satisfies $\dim P = 2 \cdot r(P) + n(P)$, or equivalently $r(P) = \frac{1}{2} \dim P / (P^{\omega} \cap P)$.

In particular, the following statements are equivalent: (1) P is a symplectic subspace in V; (2) $n(P) = 0$; (3) $\dim P = 2 \cdot r(P)$ and (4) $V = P \oplus P^{\omega}$.

Proposition 4. *There is a natural identification*

$$Gr^{Sp}(2k, 2n) \simeq \frac{Sp(2n, \mathbb{R})}{Sp(2k, \mathbb{R})\,Sp(2n-2k, \mathbb{R})}.$$

Proof. We apply the same arguments as in the cases of real and complex Grassmannians together with the above property (4), $V = P \oplus P^{\omega}$ for any symplectic subspace P, to obtain the identification. □

Before we proceed, let us recall definitions and basic properties of other important notions in linear symplectic geometry: (1) A linear subspace S of V is called *symplectic* if the restriction of ω to S defines a symplectic structure on S, or equivalently $n(S) = 0$. In particular, S must be of even dimension. While symplectic subspaces are those subspaces where the restriction of ω are the most nondegenerate, isotropic/coisotropic/Lagrangian subspaces are those subspaces where the restriction of ω are the most degenerate. (2) A subspace P is called *isotropic* (resp. *coisotropic*) if $P \subset P^{\omega}$ (resp. $P^{\omega} \subset P$), and this condition is equivalent to $\omega|_P = 0$; namely $P = N(P)$ (resp. $P^{\omega} = N(P)$). A subspace which is both isotropic and coisotropic is called *Lagrangian*. We also have the following characterizations of coisotropic subspaces.

Lemma 5. *Let P be any linear subspace in (V, ω). We have*

(i) $r(P) + n(P) \leq n$ *and equality holds precisely when P is coisotropic.*
(ii) *Suppose* $\dim P = n + k$*, then* $\omega^k|_P \neq 0$ *and* $\omega^{k+1}|_P = 0$ *if and only if P is coisotropic.*

It is well-known that the symplectic complement of a symplectic (resp. isotropic) subspace in (V, ω) is symplectic (resp. coisotropic). In fact, the same is true for orthogonal complements.

Lemma 6. *Let P be any linear subspace in a Hermitian vector space (V, ω, J, g). We have*

$$JN(P) = N\left(P^{\perp}\right).$$

Proof. The compatibility condition $\omega(u, v) = g(Ju, v)$ implies that

$$(JP)^{\omega} = J\left(P^{\omega}\right) = P^{\perp}.$$

Taking the symplectic complement we get $JP = \left(P^{\perp}\right)^{\omega}$. Therefore,

$$\begin{aligned} JN(P) &= J\left(P \cap P^{\omega}\right) = JP \cap J\left(P^{\omega}\right) \\ &= \left(P^{\perp}\right)^{\omega} \cap P^{\perp} \\ &= N\left(P^{\perp}\right). \end{aligned}$$

Hence the result. □

As a corollary, we obtain

Corollary 7. *Let P be any linear subspace in a Hermitian vector space (V, ω, J, g). Then P is symplectic iff $P^{\perp}$ is symplectic. Also, a subspace P is coisotropic iff $P^{\perp}$ is isotropic.*

Proof. By the above lemma and $J^2 = -id$, we have $n(P) = n(P^{\perp})$. The first assertion follows from P being symplectic iff $n(P) = 0$.

For any subspace P in V, by the previous lemma,

$$\dim V = \dim P + \dim P^{\perp} = 2\left(r(P) + n(P) + r\left(P^{\perp}\right)\right).$$

And if P is coisotropic, $2(r(P) + n(P)) = \dim V$ and we have $r\left(P^{\perp}\right) = 0$ (i.e. $\omega|_{P^{\perp}} = 0$). Therefore $P^{\perp}$ is isotropic and vice versa. □

Thus we could restrict our attention to only those subspaces in V which are at most half dimensional. For the rest of this section, we assume that $2k \leq n$.

2.2. The inclusion $Gr^{Sp}(2k, 2n) \subset Gr(2k, 2n)$

Since $Sp(2n, \mathbb{R})$ acts on $V \simeq \mathbb{R}^{2n}$, it also acts on the set of its linear subspaces $Gr(2k, 2n)$. It is a well-known fact that two subspaces in V are in the same $Sp(2n, \mathbb{R})$-orbit if and only if they have the same rank. We denote the orbit of rank r subspaces as $\mathcal{O}_r$ and we have a disjoint union decomposition

$$Gr(2k, 2n) = \coprod_{r=0}^{k} \mathcal{O}_r.$$

Clearly $\mathcal{O}_k = Gr^{Sp}(2k, 2n)$ is the unique open orbit in $Gr(2k, 2n)$ and the orbit closure $\overline{\mathcal{O}_l} = \coprod_{r=0}^{l} \mathcal{O}_r$ for any $l \leq k$. The unique closed orbit $\mathcal{O}_0$ consists of isotropic subspaces in V and we have an identification

$$\mathcal{O}_0 \simeq \frac{U(n)}{U(n-2k)\, O(2k)}.$$

By a counting argument, we can show that $\mathcal{O}_r$ in $Gr(2k, 2n)$ has codimension $(k-r)(2(k-r)-1)$. In particular the complement of $Gr^{Sp}(2k, 2n)$ is a hypersurface in $Gr(2k, 2n)$.

Given any $[S] \in Gr(2k, 2n)$, we choose any basis $v_1, v_2, \ldots, v_{2k}$ of S. The element $v_1 \wedge v_2 \wedge \cdots \wedge v_{2k} \in \bigwedge^{2k} V$ is independent of the choice of the basis, up to a nonzero constant multiple. Thus we have a natural inclusion

$$Gr(2k, 2n) \subset \mathbb{P}\left(\bigwedge^{2k} V\right)$$

and the image consists of nonzero decomposable elements in $\bigwedge^{2k} V$.

If we choose a metric g on V, then the set of decomposable elements in the unit sphere in $\Lambda^{2k} V$ can be naturally identified with a double cover of $Gr(2k, 2n)$, the Grassmannian $Gr^{+}(2k, 2n)$ of *oriented* linear subspaces in V.

$$Gr^{+}(2k, 2n) \simeq \frac{SO(2n)}{SO(2k)\, SO(2n-2k)}.$$

For a $2n$-dimensional symplectic vector space (V, ω), the top form $\omega^n / n!$ defines a canonical orientation on V and it gives a natural isomorphism Φ,

$$\Phi : \bigwedge^{2k} V \overset{\simeq}{\to} \bigwedge^{2n-2k} V^*$$

defined as follow: For any $\alpha \in \bigwedge^{2k} V$, the linear functional

$$\Phi(\alpha) \in \bigwedge^{2n-2k} V^* = Hom\left(\bigwedge^{2n-2k} V, \mathbb{R}\right)$$

is given by

$$\Phi(\alpha)(\beta) = \frac{\omega^n}{n!}(\alpha \wedge \beta),$$

for any $\beta \in \bigwedge^{2n-2k} V$. For any integer l between 1 and k, we define a linear homomorphism,

$$\begin{aligned} F_l : \bigwedge^{2k} V &\to \bigwedge^{2(n-k+l)} V^* \\ F_l(\alpha) &= \Phi(\alpha) \wedge \omega^l / l!. \end{aligned}$$

We restrict the function $F_k : \bigwedge^{2k} V \to \bigwedge^{2n} V^*$ to $Gr^+(2k, 2n)$ and identify $\bigwedge^{2n} V^*$ with $\mathbb{R}$ via $\omega^n/n!$,

$$F_k : Gr^+(2k, 2n) \to \mathbb{R}.$$

Lemma 8. *The function F_k can be written as follow,*

$$F_k([S]) = \frac{1}{k!} \frac{(\omega|_S)^k}{Vol_S}$$

where Vol_S is the volume form on S induced from g.

Proof. Given any volume element of S, $0 \neq \nu_S \in \Lambda^{2k} S$, there is a unique volume element $\nu_{S^\perp} \in \Lambda^{2n-2k} S^\perp$ which satisfies

$$\frac{\omega^n}{n!} \left(\nu_S \wedge \nu_{S^\perp}\right) = 1.$$

The following homomorphism identifies volume elements in V and in S,

$$\begin{aligned} \Psi : \Lambda^{2n} V^* &\xrightarrow{\simeq} \Lambda^{2k} S^* \\ \Psi(\alpha) &= \left(\nu_{S^\perp} \lrcorner \alpha\right)|_S. \end{aligned}$$

Recall that $F_k : \Lambda^{2k} V \to \Lambda^{2n} V^*$ is given by

$$F_k(\beta) = \left(\beta \lrcorner \frac{\omega^n}{n!}\right) \wedge \frac{\omega^k}{k!}.$$

We have

$$\begin{aligned} \Psi \circ F_k(\nu_S) &= \Psi\left(\left(\nu_S \lrcorner \frac{\omega^n}{n!}\right) \wedge \frac{\omega^k}{k!}\right) \\ &= \Psi\left(\nu_{S^\perp} \wedge \frac{\omega^k}{k!}\right) \\ &= \frac{\omega^k}{k!}|_S. \end{aligned}$$

Hence the lemma. □

Recall that a two-form η on S is non-degenerate if and only if $\eta^k \neq 0$. In particular, S is a symplectic subspace in V if and only if $F_k([S]) \neq 0$. Hence we have the following corollary.

Corollary 9. *The natural inclusion $Gr^{Sp}(2k, 2n) \subset Gr(2k, 2n)$ is the complement of the hyperplane $\{F_k = 0\}$ in $\bigwedge^{2k} V$.*

In general, every orbit closure of the $Sp(2n,\mathbb{R})$-action on $Gr(2k,2n)$ is the intersection of $Gr(2k,2n)$ with a linear subspace in $\Lambda^{2k}V$.

Proposition 10. *We have*

$$\overline{\mathcal{O}_{l-1}} = Gr(2k,2n) \cap \{F_l = 0\} \subset \bigwedge^{2k} V$$

for any $l \leq k$.

2.3. *The inclusion* $Gr_{\mathbb{C}}(k,n) \subset Gr^{Sp}(2k,2n)$

Recall that $Gr^{Sp}(2k,2n) \subset Gr(2k,2n)$ is the complement of the hyperplane $\{F_k = 0\}$ in $\bigwedge^{2k} V$. Using calibration theory [3], we have $\operatorname{Im} F_k = [-1,1]$ and $F_k^{-1}(1) = Gr_{\mathbb{C}}(k,n)$. To see this, we notice that given any orthonormal oriented vectors u and v in V,

$$\omega(u,v) = g(Ju,v) \leq |Ju|\,|v| = 1$$

and equality holds if and only if $v = Ju$; namely, *span* (u,v) is a complex line in (V,J). Furthermore, a generalization of this is the Wirtinger inequality which is

$$\frac{1}{k!}\omega^k(\varsigma) \leq 1 \quad \text{for all } \varsigma \in Gr(2k,2n) \subset \Lambda^{2k} V,$$

and equality holds if and only if ς is a complex subspace, i.e. $\varsigma \in Gr_{\mathbb{C}}(k,n)$. Here $\omega^k(\varsigma)$ denotes

$$\omega^k(e_1 \wedge e_2 ... \wedge e_{2k})$$

where $e_1, e_2, ..., e_{2k}$ is an oriented orthonormal basis of ς. Equivalently, $\omega^k/k!$ is a calibration with contact set $Gr_{\mathbb{C}}(k,n)$. From the fact that linear subspaces calibrated by $\omega^k/k!$ are complex subspaces, $F^{-1}(\{1,-1\}) = Gr_{\mathbb{C}}(k,n) \cup Gr_{\mathbb{C}}^{-}(k,n)$, where $Gr_{\mathbb{C}}^{-}(k,n)$ is the complex Grassmannian whose elements have the reverse orientation of elements in $Gr_{\mathbb{C}}(k,n)$, i.e.

$$Gr_{\mathbb{C}}^{-}(k,n) = -Gr_{\mathbb{C}}(k,n) \subset \Lambda^{2k} V.$$

In particular $Gr^{Sp}(2k,2n)$ has two connected components. Combining with the earlier result, we have the following proposition.

Proposition 11. *Let* (V,ω,J,g) *be a* $2n$*-dimensional Hermitian vector space. The inclusions in the following sequence*

$$Gr_{\mathbb{C}}(k,n) \subset Gr^{Sp}(2k,2n) \subset Gr(2k,2n)$$

correspond to the restriction of

$$\{F_k = 1\} \subset \{F_k \neq 0\} \subset \Lambda^{2k} V$$

to $Gr(2k,2n)$.

We can also describe $Gr_{\mathbb{C}}(k,n)$ as the intersection of $Gr(2k,2n)$ with a linear subspace in $\Lambda^{2k}V$. In fact, we only need to use the complex structure on V for this purpose. Let us recall the (p,q)-decomposition of tensors: Eigenvalues of J on V are $\pm i$ and we denote its eigenspace decomposition as

$$V \otimes \mathbb{C} = V^{1,0} \oplus V^{0,1} \text{ with } V^{0,1} = \overline{V^{1,0}}.$$

By taking wedge products, we have

$$\Lambda^l V \otimes \mathbb{C} = \textstyle\sum_{p+q=l} \Lambda^{p,q} V \quad \text{with } V^{q,p} = \overline{V^{p,q}}.$$

Here

$$\Lambda^{p,q} V = \Lambda^p V^{1,0} \otimes_{\mathbb{C}} \Lambda^q V^{0,1}.$$

We also write

$$\Lambda^{k,k}_{\mathbb{R}} V = \Lambda^{k,k} V \cap \Lambda^{2k} V.$$

Then we have the following characterization of $Gr_{\mathbb{C}}(k,n)$ inside $Gr(2k,2n)$.

Proposition 12. *Given any complex vector space* (V,J)*, we have*

$$Gr_{\mathbb{C}}(k,n) = Gr(2k,2n) \cap \Lambda^{k,k}_{\mathbb{R}} V,$$

inside $\Lambda^{2k}V$.

Proof. We extend J to $\Lambda^l V \otimes \mathbb{C}$ by the following formula,

$$J(v_1 \wedge \cdots \wedge v_l) = (Jv_1) \wedge \cdots \wedge (Jv_l).$$

The action of J on $\Lambda^{p,q}V$ is given by multiplication by i^{p-q}. In particular, J is identity on $\Lambda^{k,k}V$.

Let S be any $2k$-dimensional real linear subspace in V. We pick any basis $v_1,\ldots,v_{2k}$ of S and write

$$\nu_S = v_1 \wedge \cdots \wedge v_{2k} \in \Lambda^{2k} V.$$

Then a vector $v \in V$ belongs to S if and only if $v \wedge \nu_S = 0$.

Now we suppose $[S] \in Gr(2k,2n) \cap \Lambda^{k,k}_{\mathbb{R}} V$, i.e. $\nu_S \in \Lambda^{k,k}V$. For any $e \in S$, we have $e \wedge \nu_S = 0$. By applying J, we have

$$0 = J(e \wedge \nu_S) = Je \wedge J\nu_S = Je \wedge \nu_S.$$

This implies that $Je \in S$. That is, S is a complex linear subspace in V. Conversely, if S is a complex subspace in V, then we find a complex coordinate system $z_1,\ldots,z_n$ on V with S being the $\mathbb{C}$-span of $z_1,\ldots,z_k$. By writing $z_j = x_j + iy_j$ for each j, we have

$$\begin{aligned}\nu_S &= x_1 \wedge y_1 \wedge \cdots \wedge x_k \wedge y_k \\ &= \left(\frac{i}{2}\right)^k z_1 \wedge \bar{z}_1 \wedge \cdots \wedge z_k \wedge \bar{z}_k \in \Lambda^{k,k} V.\end{aligned}$$

Hence the proposition. □

In the following theorem, we show that $Gr_{\mathbb{C}}(k,n) \cup Gr_{\mathbb{C}}^{-}(k,n)$ is actually a strong deformation retract of $Gr^{Sp}(2k,2n)$. In particular, they are homotopically equivalent to each other.

Theorem 13. *Let (V,ω,J,g) be a $2n$-dimensional Hermitian vector space. Then the complex Grassmannian $Gr_{\mathbb{C}}(k,n)$ is a strong deformation retract of a connected component of the symplectic Grassmannian $Gr^{Sp}(2k,2n)$.*

Proof. First, we define a natural basis on S for each $[S]$ in $Gr^{Sp}(2k,2n)$. For each $\xi \in Gr(2,S)$, consider

$$\omega(\xi) = g(Ju,v)$$

where u and v is an oriented orthonormal basis of ξ. There is a maximizing oriented orthonormal pair of vectors u_1 and v_1 in S satisfying $0 < g(Ju_1,v_1) \le 1$. Note that $g(Ju_1,v_1)$ is nonzero because S is symplectic. Furthermore, we observe that for each unit vector a in $S \cap \mathit{span}\{u_1,v_1\}^{\perp}$, the function

$$f(\theta) := g(Ju_1, \cos\theta v_1 + \sin\theta a)$$

has a maximum value at $\theta = 0$ and hence the derivative of f at $\theta = 0$ vanishes.

$$0 = f'(0) = g(Ju_1,a),$$

i.e. $a \perp Ju_1$ and similarly, $a \perp Jv_1$. This implies that a is perpendicular to $\mathit{span}\{u_1,v_1,Ju_1,Jv_1\}$ and a maximizing pair (u_1,v_1) is isolated. Furthermore, $S \cap \mathit{span}\{u_1,v_1\}^{\perp}$ is symplectic. Therefore, we can repeat the above process with S being replaced by $S \cap \mathit{span}\{u_1,v_1\}^{\perp}$ and we obtain an ordered orthonormal basis $\{u_1,v_1,...,u_k,v_k\}$ of S such that

$$g(Ju_1,v_1) \ge g(Ju_2,v_2) \ge ... \ge g(Ju_k,v_k) > 0,$$

and $S = \mathit{span}\{u_1,v_1\} \oplus \cdots \oplus \mathit{span}\{u_k,v_k\}$.

Now, we deform the symplectic subspace S to a complex subspace: For each $0 \le t \le 1$ and $i = 1,...,k$, define

$$U_i(t) := \cos\left(\frac{\pi}{2}t\right) u_i + \sin\left(\frac{\pi}{2}t\right) \frac{v_i + Ju_i}{\sqrt{g(v_i+Ju_i, v_i+Ju_i)}}$$
$$V_i(t) := \cos\left(\frac{\pi}{2}t\right) v_i + \sin\left(\frac{\pi}{2}t\right) \frac{-u_i + Jv_i}{\sqrt{g(-u_i+Jv_i, -u_i+Jv_i)}}$$

and consider $S(t) := \mathit{span}\{U_1(t), V_1(t), ..., U_k(t), V_k(t)\}$. Here, one can check that (i) $S(t)$ is symplectic for each t, (ii) $S(0) = S$ and (iii) $S(1)$ is a complex subspace in V. In order to obtain a deformation retract, we need to show that the path $S(t)$ is independent of the choice of the basis on S. First we can replace each $\{u_j,v_j\}$ by any oriented orthonormal vectors in the 2-dimensional subspace. But this will not change the span of $U_j(t)$ and $V_j(t)$. Second, when

$g\left(Ju_j, v_j\right) = g\left(Ju_{j+1}, v_{j+1}\right)$, then we may have another order of the subspaces spanned by $\left\{u_j, v_j\right\}$ but this will not change $S(t)$ neither. Thus the family

$$\begin{aligned} Gr^{Sp}(2k, 2n) \times [0, 1] &\longrightarrow Gr^{Sp}(2k, 2n) \\ (S, t) &\mapsto S(t). \end{aligned}$$

can be easily seen to be a strong deformation retract from $Gr^{Sp}(2k, 2n)$ to $Gr_{\mathbb{C}}(k, n)$. □

If we vary the complex structure J on V, then the corresponding $Gr_{\mathbb{C}}(k, n)$ will move inside $Gr^{Sp}(2k, 2n)$ and cover the whole symplectic Grassmannian (see Sect. 3).

Remark. When $k = 1$, $Gr(2, 2n)$ has a natural symplectic structure. Indeed it is a complex projective quadric hypersurface

$$Gr(2, 2n) = \left\{z_1^2 + z_2^2 + \cdots + z_{2n}^2 = 0\right\} \subset \mathbb{CP}^{2n-1} = \mathbb{P}(V \otimes \mathbb{C}).$$

To see this, given any 2-dimensional subspace $S \subset V \simeq \mathbb{R}^{2n}$, we choose any orthonormal basis e_1, e_2 of S. We write $e_1 = (x_1, ..., x_{2n})$ and $e_2 = (y_1, ..., y_{2n})$. Defines $z_j = x_j + iy_j$. Then $(z_1, ..., z_{2n}) \in V \otimes \mathbb{C}$ lies in the unit sphere S^{4n-1} and satisfies the equation $\sum z_j^2 = 0$. Furthermore, its projection to $\mathbb{P}(V \otimes \mathbb{C})$ is independent of the choice of the orthonormal basis on S.

Recall that the symplectic Grassmannian $Gr^{Sp}(2, 2n)$ is the complement of $\{F_1 = 0\}$ in $Gr(2, 2n)$. In this case, the function F_1 can be extended to a homogeneous function on $\mathbb{C}^{2n}$, namely $F_1(z_1, ..., z_{2n}) = -\operatorname{Im}\left(\sum_{j=1}^n z_{2j-1}\bar{z}_{2j}\right)$.

2.4. *A nondegenerate two-form on* $Gr^{Sp}(2k, 2n)$

Recall that a nondegenerate two-form on a manifold is called a *symplectic form* if it is closed. We will construct a two-form on $Gr(2k, 2n)$ which is nondegenerate exactly on $Gr^{Sp}(2k, 2n)$. Even though this form is not closed, it does have many Lagrangian submanifolds.

In the next section, we construct the symplectic half twistor Grassmannian $\mathcal{T}(2k, 2n)$ and show that it does have a natural symplectic form. We will use it to describe the symplectic geometry on $Gr^{Sp}(2k, 2n)$ in Sect. 4. Before we do this, let us recall a natural symplectic structure on the space of all *compact* symplectic submanifolds in any given symplectic manifold.

2.4.1. Symplectic knot spaces Let (M, ω) be a $2n$-dimensional symplectic manifold and let $K_\Sigma(M)$, or simply $K(M)$, be the space of $2k$-dimensional submanifolds in M which are diffeomorphic to a fixed *compact* manifold Σ. It is called a higher dimensional knot space of M. The open subset $K^{Sp}(M) \subset K(M)$ consisting of those submanifolds which are symplectic is called a *symplectic knot space* of M (see [4]).

For any $[S] \in K(M)$ and for any tangent vectors $u, v \in T_{[S]}K(M) = \Gamma\left(S, N_{S/M}\right)$, we define

$$\Omega(u, v) = \int_S \iota_{u\wedge v} \frac{\omega^{k+1}}{(k+1)!}.$$

Then Ω defines a closed two-form on $K(M)$ which is non-degenerate precisely on $K^{Sp}(M)$, provided that $k \leq n-2$. When $k = n-1$, Ω is non-degenerate on the whole $K(M)$. Suppose C is a $(n+k)$-dimensional submanifold in M, then C is a coisotropic submanifold in M if and only if $K(C) \cap K^{Sp}(M)$ is a Lagrangian submanifold in $K^{Sp}(M)$. More discussions on the symplectic geometry of $K^{Sp}(M)$ can be found in [4].

2.4.2. A nondegenerate 2-form on $Gr^{Sp}(2k, 2n)$ The above symplectic form Ω cannot be defined on $Gr^{Sp}(2k, 2n)$ as $\mathbb{R}^{2k}$ is noncompact. In order to find a nondegenerate two-form on $Gr^{Sp}(2k, 2n)$, we first recall that given any element $[S]$ in the Grassmannian $Gr(2k, 2n)$, its tangent space can be identified with $Hom(S, V/S)$, or equivalently $S^* \otimes V/S$. Therefore

$$\begin{aligned}\wedge^2 T_{[S]}Gr(2k, 2n) &= \wedge^2\left(S^* \otimes V/S\right)\\ &= \wedge^2 S^* \otimes Sym^2 V/S + Sym^2 S^* \otimes \wedge^2 V/S.\end{aligned}$$

Here $Sym^2 S^*$ is the second symmetric tensor power of S^*. Therefore, using a symplectic structure ω and a metric g on V, we can define a 2-form Ω^g on $Gr(2k, 2n)$ as follow: Given any $[S] \in Gr(2k, 2n)$,

$$\Omega^g_S : \wedge^2\left(S^* \otimes V/S\right) \to \mathbb{R}$$

is the composition of the following maps

$$\wedge^2\left(S^* \otimes V/S\right) \xrightarrow{\alpha} \wedge^2 V/S \xrightarrow{\hat{\omega}_S} \wedge^{2k} S^* \xrightarrow{\beta} \mathbb{R}.$$

Here α is the inner product of the components in S^* using the induced metric on S from g on V; $\hat{\omega}_S$ is given by

$$\hat{\omega}_S(v \wedge w) := \frac{\iota_{(v\wedge w)}\omega^{k+1}\mid_S}{(k+1)!},$$

where v and w are vectors in V/S, and β is the division by Vol_S, which is the volume form on S of the metric tensor. Note that $\hat{\omega}_S$ is well-defined since for v or w in S, $\iota_{(v\wedge w)}\omega^{k+1}\mid_S$ cannot produce a top degree form on S. Also, if g is chosen to be compatible with ω and S is a complex subspace, then Vol_S is $(\omega\mid_S)^k/k!$ and

$$\Omega^g_S(a, b) := \frac{\iota_{g^*|_{S^*}(a\wedge b)}\omega^{k+1}\mid_S}{(k+1)!\,Vol_S}.$$

Note that the definition of $\hat{\omega}_S$ does not depend on g_S. In the following proposition, we show that this form is non-degenerate exactly on $Gr^{Sp}(2k, 2n)$.

Proposition 14. *For* $0 \leq k \leq n-2$ *and* $[S] \in Gr\,(2k, 2n)$*,* $\hat{\omega}_S$ *is nondegenerate if and only if* S *is symplectic, i.e.* $[S]$ *is an element in* $Gr^{Sp}\,(2k, 2n)$*.*

Proof. Assume $\hat{\omega}_S$ is nondegenerate. Suppose S is not a symplectic subspace in V, i.e. S is a $2k$-dimensional subspace with rank less than k. In general, a subspace P with $r\,(P) < k$ has

$$\dim P = 2r\,(P) + n\,(P) \leq n + k - 1$$

since $r\,(P) + n\,(P) \leq n$, and equality holds when P is a coisotropic subspace with $r\,(P) = k - 1$. For the above S, it is easy to see there is a $(n + k - 1)$-dimensional coisotropic subspace C containing S. Since $k < n - 1$, S is a proper subspace of C and we can take a vector v in $C \backslash S$. Observe that $span\,\{v\} + S$ still has rank less than k because $r(C) < k$. We also denote v as an element of C/S. Now, for any w in V/S,

$$\begin{aligned}(k+1)!\hat{\omega}_S\,(v \wedge w) &= \iota_{(v\wedge w)}\omega^{k+1}\mid_S \\ &= \omega\,(v \wedge w)\,\omega^k \mid_S + \iota_v\omega^k \wedge \iota_w\omega\mid_S \\ &= 0,\end{aligned}$$

for $r\,(S) < k$ and $r\,(span\,\{v\} + S) < k$. Therefore, $\hat{\omega}_S$ is degenerate.

Conversely, assume S is a symplectic subspace. Then, its symplectic complement S^ω is also symplectic and $V = S \oplus S^\omega$. Furthermore, the symplectic structure can be decomposed into

$$\omega = \omega\mid_S + \omega\mid_{S^\omega}.$$

Observe that S^ω and V/S are isomorphic. We consider

$$\begin{array}{ccc}\wedge^2 V/S & \overset{\hat{\omega}_S}{\rightarrow} & \wedge^{2k}S^* \\ \downarrow & & \downarrow \\ \wedge^2 S^\omega & \overset{\omega|_{S^\omega}}{\rightarrow} & \mathbb{R}\end{array}$$

where the vertical map at right is given by $\alpha/\left(\omega^k/k!\mid_S\right)$ for α in $\wedge^{2k}S^*$. Note these two vertical maps are isomorphisms and it is easy to see that the diagram commutes, because for any v in S^ω, $\iota_v\omega\mid_S = 0$. Since $\omega\mid_{S^\omega}$ is nondegenerate, $\hat{\omega}_S$ is also non-degenerate. $\square$

As a corollary, Ω^g is a nondegenerate two-form on $Gr^{Sp}\,(2k, 2n)$, provided that $k \leq n-2$. When $k = n-1$, Ω^g is nondegenerate on the whole Grassmannian $Gr\,(2n-2, 2n)$. This is because $\omega^{k+1} = \omega^n$ is a nonvanishing top degree form on V.

3. Symplectic twistor Grassmannians

In this section we introduce the notion of a symplectic half twistor Grassmannian $\mathcal{T}(2k, 2n)$ by considering all compatible linear complex structures on each $[S] \in Gr^{Sp}(2k, 2n)$. This is analogous to the definition of the twistor space $\mathcal{T}_M$ for any oriented Riemannian four manifold M where we consider all compatible linear complex structures on each tangent space of M. Since the space of all compatible linear complex structures on $\mathbb{R}^4$ is diffeomorphic to $SO(4)/U(2) \cong S^2$, we have the following fiber bundle

$$S^2 \to \mathcal{T}_M \to M.$$

The total space $\mathcal{T}_M$ admits a canonical almost complex structure in which every fiber is a pseudo-holomorphic curve in $\mathcal{T}_M$. Furthermore this almost complex structure is integrable if and only if M is a self-dual manifold. In this situation, the conformal geometry of M can be described in terms of the complex geometry of $\mathcal{T}_M$.

We will construct a canonical symplectic structure on $\mathcal{T}(2k, 2n)$ by using the symplectic quotient construction in the next subsection. Furthermore the symplectic geometry on $\mathcal{T}(2k, 2n)$ will be used to define a *symplectic geometry* on $Gr^{Sp}(2k, 2n)$.

Definition 15. Given any $k \leq n$, (i) the symplectic half twistor Grassmannian $\mathcal{T}(2k, V)$, or simply $\mathcal{T}(2k, 2n)$, is the set of pairs (S, J_S) where S is a $2k$-dimensional symplectic linear subspace in V and J_S is a linear complex structure on S compatible with the symplectic form $\omega|_S$.

(ii) The symplectic twistor Grassmannian $\hat{\mathcal{T}}(2k, V)$, or simply $\hat{\mathcal{T}}(2k, 2n)$, is the set of pairs (S, J_S, J_{S^ω}) where S is a $2k$-dimensional linear symplectic subspace in V and J_S (resp. J_{S^ω}) is a linear complex structure on S compatible with the symplectic form $\omega|_S$ (resp. $\omega|_{S^\omega}$).

Since the map which sends a symplectic subspace S to its symplectic complement S^ω gives a natural identification

$$Gr^{Sp}(2k, 2n) \cong Gr^{Sp}(2n - 2k, 2n),$$

we have

$$\hat{\mathcal{T}}(2k, 2n) = \mathcal{T}(2k, 2n) \underset{Gr^{Sp}(2k,2n)}{\times} \mathcal{T}(2n - 2k, 2n).$$

Recall that the space of all compatible linear complex structures on $\mathbb{C}^k$ is the Hermitian symmetric space $Sp(2k, \mathbb{R})/U(k)$ and therefore we have a fiber bundle

$$Sp(2k, \mathbb{R})/U(k) \to \mathcal{T}(2k, 2n) \to Gr^{Sp}(2k, 2n).$$

When $k = n$, we have $\mathcal{T}(2n, 2n) = Sp(2n, \mathbb{R})/U(n)$ because $Gr^{Sp}(2n, 2n)$ is a single point. Given any triple (S, J_S, J_{S^ω}) in $\hat{\mathcal{T}}(2k, 2n)$, $J_S \oplus J_{S^\omega}$ defines a compatible complex structure on $S \oplus S^\omega = V$, in which both S and S^ω are complex subspaces. Thus $\hat{\mathcal{T}}(2k, 2n)$ is a homogenous space of $Sp(2n, \mathbb{R})$. The isotropy subgroup consists of unitary transformations of S and S^ω. Similarly, $\mathcal{T}(2k, 2n)$

is a homogenous space of $Sp\,(2n, \mathbb{R})$ with isotropy subgroup consists of unitary transformations of S and symplectic transformations of S^{ω}. Thus we have obtained the following proposition.

Proposition 16. *There are natural identifications*

$$\mathcal{T}\,(2k, 2n) = \frac{Sp\,(2n, \mathbb{R})}{U\,(k)\,Sp\,(2n-2k, \mathbb{R})},$$

and

$$\hat{\mathcal{T}}\,(2k, 2n) = \frac{Sp\,(2n, \mathbb{R})}{U\,(k)\,U\,(n-k)}.$$

Moreover, the fiber bundle

$$Sp\,(2k, \mathbb{R})\,/\,U\,(k) \to \mathcal{T}\,(2k, 2n) \to Gr^{Sp}\,(2k, 2n)\,,$$

can be identified with the following bundle of homogenous spaces,

$$\frac{Sp\,(2k, \mathbb{R})}{U\,(k)} \to \frac{Sp\,(2n, \mathbb{R})}{U\,(k)\,Sp\,(2n-2k, \mathbb{R})} \to \frac{Sp\,(2n, \mathbb{R})}{Sp\,(2k, \mathbb{R})\,Sp\,(2n-2k, \mathbb{R})},$$

which comes from the following commutative diagram of injective group homomorphisms

$$\begin{array}{ccc} Sp\,(2k, \mathbb{R}) & \to & Sp\,(2n, \mathbb{R}) \\ \uparrow & & \uparrow \\ U\,(k) & \to & U\,(k)\,Sp\,(2n-2k, \mathbb{R}) \end{array}.$$

Similar arguments also give the following

Proposition 17. *For any $1 \leq k \leq n$, the symplectic twistor Grassmannian $\hat{\mathcal{T}}\,(2k, 2n)$ is the total space of the following two fiber bundles*

$$\frac{Sp\,(2k, \mathbb{R})}{U\,(k)} \times \frac{Sp\,(2n-2k, \mathbb{R})}{U\,(n-k)} \longrightarrow \hat{\mathcal{T}}\,(2k, 2n) \xrightarrow{\alpha} Gr^{Sp}\,(2k, 2n)\,,$$

and

$$Gr_{\mathbb{C}}\,(k, n) \longrightarrow \hat{\mathcal{T}}\,(2k, 2n) \xrightarrow{\beta} \frac{Sp\,(2n, \mathbb{R})}{U\,(n)}.$$

Given any compatible complex structure J on (V, ω), i.e. $[J] \in Sp\,(2n, \mathbb{R})\,/\,U\,(n)$, the fiber $\beta^{-1}\,(J)$ consists of all J-complex subspaces in (V, J) and α embeds $\beta^{-1}\,(J)$ into the space of symplectic Grassmannian,

$$Gr_{\mathbb{C}}\,(k, n) \subset Gr^{Sp}\,(2k, 2n)\,.$$

Given any symplectic subspace in (V, ω), $[S] \in Gr^{Sp}\,(2k, 2n)$, the fiber $\alpha^{-1}\,(S)$ consists of all ω-compatible complex structures on V which are diagonal with respect to the decomposition $V = S \oplus S^{\omega}$ and we have the following natural embedding given by the restriction of β to $\alpha^{-1}\,(S)$,

$$\frac{Sp\,(2k, \mathbb{R})}{U\,(k)} \times \frac{Sp\,(2n-2k, \mathbb{R})}{U\,(n-k)} \subset \frac{Sp\,(2n, \mathbb{R})}{U\,(n)}.$$

Indeed the above two fiber bundles arise from successive quotients of the inclusions (i) $U(k)\,U(n-k) \subset Sp(2k,\mathbb{R})\,Sp(2n-2k,\mathbb{R}) \subset Sp(2n,\mathbb{R})$ which gives α and (ii) $U(k)\,U(n-k) \subset U(n) \subset Sp(2n,\mathbb{R})$ which gives β.

Corollary 18. *(i) Every symplectic subspace S in V is a J-complex subspace for some compatible complex structure J, i.e.*

$$Gr^{Sp}(2k,2n) = \bigsqcup_{J \in Sp(2n,\mathbb{R})/U(n)} Gr_{\mathbb{C}}(k,(V,J)).$$

(ii) The set of all such J's for a given $[S] \in Gr^{Sp}(2k,2n)$ is isomorphic to $\frac{Sp(2k,\mathbb{R})}{U(k)} \times \frac{Sp(2n-2k,\mathbb{R})}{U(n-k)}$.

3.1. $\mathcal{T}(2k,2n)$ as a symplectic quotient

In this subsection, we define a natural symplectic structure on $\mathcal{T}(2k,2n)$ by the symplectic quotient construction. Let us briefly recall the symplectic quotient construction (see e.g. [1]): Suppose (M,ω) is a symplectic manifold with a Hamiltonian action by a Lie group G and with a moment map,

$$\mu : M \to (LieG)^{*}.$$

That is, μ is equivariant with respect to the G-action on M and the coadjoint action on the dual Lie algebra $(LieG)^{*}$, and for any $X \in LieG$, the function $\mu_X : M \to \mathbb{R}$ defined by $\mu_X(x) = (\mu(x))(X)$ satisfies

$$d\mu_X = \iota_{\tilde{X}}\omega$$

where $\tilde{X}$ is the vector field on M generated by the infinitesimal action of X. Choose any element $J \in (LieG)^{*}$, let G_J be the isotropy subgroup of J in G, i.e.

$$G_J = \left\{ g \in G : Ad_g^{*}(J) = 0 \right\}.$$

If the action of G_J on $\mu^{-1}(J)$ is regular, then the quotient space $\mu^{-1}(J)/G_J$ is a manifold and ω induces a symplectic structure on $\mu^{-1}(J)/G_J$. The resulting symplectic manifold together with its induced symplectic form $\omega_{M//G}$ is called the *symplectic quotient of* M by G, denoted $M//G$.

3.1.1. Symplectic quotient of $Hom(S,V)$ by $O(S)$ Given any real vector spaces S and V of dimension $2k$ and $2n$ respectively, we consider the linear space

$$M = Hom(S,V) = S^{*} \otimes V.$$

The space of (linear) 2-forms on M is

$$\bigwedge^2 M^{*} = \left(\bigwedge^2 S \otimes Sym^2 V^{*}\right) \oplus \left(Sym^2 S \otimes \bigwedge^2 V^{*}\right).$$

It is easy to check the following lemma by linear algebra arguments.

Lemma 19. *(i) Given any $g_S \in Sym^2 S$ and $\omega_V \in \bigwedge^2 V^*$, the two-form $\omega_M = g_S \otimes \omega_V$ is a symplectic form on M if and only if both g_S and ω_V are non-degenerate. (ii) Given any $\omega_S \in \bigwedge^2 S$ and $g_V \in Sym^2 V^*$, the two-form $\omega_M = \omega_S \otimes g_V$ is a symplectic form on M if and only if both ω_S and g_V are non-degenerate.*

Now we choose a positive definite inner product g_S on S and a symplectic form ω_V on V. Since g_S is non-degenerate, it defines a nondegenerate element $g_S^* \in Sym^2 S$ and we have a linear symplectic form on M given by

$$\omega_M = g_S^* \otimes \omega_V .$$

By construction, ω_M is invariant under the natural Lie group action of $O(S, g_S) \times Sp(V, \omega_V) \simeq O(2k) \times Sp(2n, \mathbb{R})$. In this article, we only study the $O(2k)$-action and the $Sp(2n, \mathbb{R})$-action will be studied in [5].

Proposition 20. *Let $M = Hom(S, V)$ be as above. Then the symplectic $O(2k)$-action on (M, ω_M) is Hamiltonian with moment map*

$$\mu : M \to \mathrm{o}(2k)^*$$

given by

$$\mu(f) = \iota_{g_S}\left(f^* \omega_V\right)$$

for any $f \in M$. Here $\iota_{g_S} : \Lambda^2 S^* \to \mathrm{o}(2k)^*$ is the natural map given by the metric g_S.

Proof. Choose any bases s_i for S and v_α for V. We write their corresponding dual bases as s^i and v^α. Thus the metric g_S on S and the symplectic form ω_V on V can be written explicitly as

$$g_S = g_{ij} s^i s^j \text{ and } \omega_V = \omega_{\alpha\beta} v^\alpha v^\beta,$$

satisfying $g_{ij} = g_{ji}$ and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$.

Consider the vector space $M = Hom(S, V) \cong S^* \otimes V$. The symplectic structure ω_V on V induces one ω_M on M. We will simply denote it by ω and it is given explicitly as

$$\omega = g^{ij} \omega_{\alpha\beta} s_i s_j v^\alpha v^\beta$$

Given any $f \in M$ we write $Y \in T_f M = S^* \otimes V$ as

$$Y = Y_k^\beta s^k v_\beta .$$

Under the natural identification $o(S) \cong \Lambda^2 S$ given by the metric $g_S = g_{ij} s^i s^j$, an element $X = X_j^i s_i s^j \in o(S)$ corresponds to the two-form $X^{ij} s_i s_j = g^{jk} X_k^i s_i s_j$ satisfying $X^{ij} = -X^{ji}$.

Corresponding to the action of $O(S)$ on M, X gives rise to a vector field on M, denoted by X again. At any $f \in M$ we have

$$X_f = f_i^\alpha X_j^i s^j v_\alpha \in T_f M .$$

Thus we have

$$\omega\left(X_f, Y\right) = g^{ij}\omega_{\alpha\beta} f_l^{\alpha} X_j^l Y_i^{\beta}.$$

In terms of these local coordinates, the μ map in the proposition $\mu : M \to o(S)^* \subset S^* \otimes S$ is given explicitly as

$$\mu(f) = g^{jk}\omega_{\alpha\beta} f_i^{\alpha} f_j^{\beta} s_k s^i.$$

We want to show that μ is the moment map corresponding to the natural action of $O(S)$ on M.

For any $X \in o(S)$, we write

$$\mu_X : M \to \mathbb{R}$$

as

$$\mu_X(f) = \langle X, \mu(f)\rangle$$

where $\langle\ \rangle$ is the natural pairing between the Lie algebra $o(S)$ and its dual. So we need to verify that $\iota_X\omega = d\mu_X$. Using the above explicit coordinate system on S and V we have

$$2\mu_X(f) = g^{jk} f_j^{\beta}\omega_{\alpha\beta} f_i^{\alpha} X_k^i$$

and

$$2Y(\mu_X(f)) = g^{jk} Y_j^{\beta}\omega_{\alpha\beta} f_i^{\alpha} X_k^i + g^{jk} f_j^{\beta}\omega_{\alpha\beta} Y_i^{\alpha} X_k^i = 2\omega_{\alpha\beta} f_l^{\alpha} X^{lj} Y_j^{\beta}$$

Here we have used $g_{ij} = g_{ji}$ and $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$ and $X^{ij} = -X^{ji}$. Hence the claim. □

Next we choose a linear complex structure J_S on S which is compatible with the metric g_S. This defines a Kähler structure on S with Kähler form ω_S given by

$$g_S(J_S(s), t) = \omega_S(s, t).$$

We regard J_S as an element in $o(2k)^*$ and consider the symplectic quotient of M by $O(2k)$ at the value J_S.

Using the above description of the moment map, we have

$$\mu^{-1}(J_S) = \{f \in Hom(S, V) : f^*\omega_V = \omega_S\}.$$

In particular, f is an injective homomorphism.

By sending $f \in \mu^{-1}(J_S)$ to the symplectic subspace $f(S)$ in V, we have,

$$\begin{aligned} \pi : \mu^{-1}(J_S) &\to Gr^{Sp}(2k, 2n) \\ \pi(f) &= [f(S)]. \end{aligned}$$

It is not difficult to verify the following lemma.

Lemma 21. *π is a surjective map and each fiber $\pi^{-1}(\pi(S))$ consists of all symplectomorphisms of (S, ω_S), i.e. $Sp(2k, \mathbb{R})$.*

Recall that the symplectic quotient is the space

$$M//O(2k) = \mu^{-1}(J_S)/O(2k)_{J_S}$$

where $O(2k)_{J_S}$ is the isotropy subgroup of J_S in $O(2k)$, that is $O(2k)_{J_S} \simeq U(k)$. Thus π descends to a surjective map

$$\mu^{-1}(J_S)/U(k) \to Gr^{Sp}(2k, 2n)$$

with fibers $Sp(2k, \mathbb{R})/U(k)$ and this gives the following identification.

Proposition 22. *The symplectic quotient $M//O(2k)$ at the value $J_S \in \mathrm{o}(2k)^*$ equals $\mathcal{T}(2k, 2n)$.*

We will therefore denote the natural symplectic form on the symplectic quotient $\mathcal{T}(2k, 2n)$ as $\omega_{\mathcal{T}}$. This symplectic form is invariant under the action of $Sp(2n, \mathbb{R})$ because the symplectic form ω_M on M is invariant under $Sp(2n, \mathbb{R})$.

When $k = n$, we have $O(S) \simeq O(2n)$ action on $M = Hom(S, V) \cong S^* \otimes V$ with symplectic quotient $\mathcal{T}(2n, 2n) \simeq Sp(2n, \mathbb{R})/U(n)$ because $Gr^{Sp}(2n, 2n)$ is a single point. Indeed the symplectic structure on $\mathcal{T}(2k, 2n)$ coincides with the Hermitian Kähler form on $Sp(2n, \mathbb{R})/U(n)$ up to a constant multiple because both of them are $Sp(2n, \mathbb{R})$-invariant.

3.1.2. $\hat{\mathcal{T}}(2k, 2n)$ as a symplectic quotient We will describe $\hat{\mathcal{T}}(2k, 2n)$ as a symplectic quotient in such a way that both $\mathcal{T}(2k, 2n)$ and $\mathcal{T}(2n-2k, 2n)$ are symplectic sub-quotients in it. Recall that when both (S, g_S) and (V, ω_V) are of dimension $2n$, then the symplectic quotient of $Hom(S, V)$ at a compatible complex structure $J \in o(2n)^*$ is

$$Hom(S, V)//O(S) = \mathcal{T}(2k, 2n) \simeq Sp(2n, \mathbb{R})/U(n).$$

Now we restrict the Hamiltonian action to the smaller subgroup $O(2k) \times O(2n-2k)$ in $O(2n)$.

Theorem 23. *Suppose (S, g_S) and (V, ω_V) are metric and symplectic vector spaces of the same dimension $2n$. We fix an orthogonal decomposition $S = S_1 \oplus S_2$ with $\dim S_1 = 2k$. Then (i) the symplectic quotient of $Hom(S, V)$ by $O\left(S_1, g_S|_{S_1}\right) O\left(S_2, g_S|_{S_2}\right)$ in $O(2n) \simeq O(S, g)$ at a complex structure $J_S \in \mathrm{o}(2k)^* + \mathrm{o}(2n-2k)^*$ is given by*

$$Hom(S, V)//O(2k)\,O(2n-2k) \simeq \hat{\mathcal{T}}(2k, 2n) \simeq \frac{Sp(2n, \mathbb{R})}{U(k)\,U(n-k)}.$$

(ii) $\mathcal{T}(2k, 2n) \subset \hat{\mathcal{T}}(2k, 2n)$ is a symplectic sub-quotient, i.e.

$$\mathcal{T}(2k, 2n) = Hom(S_1, V)//O(2k)$$

and

(iii) $\mathcal{T}(2n-2k, 2n) \subset \hat{\mathcal{T}}(2k, 2n)$ is a symplectic sub-quotient, i.e.

$$\mathcal{T}(2n-2k, 2n) = Hom(S_2, V)//O(2n-2k).$$

Proof. Recall that $\mu^{-1}(J_S)$ is the set of all symplectic homomorphisms from S to V. If we identify both S and V with a fixed Hermitian vector space, say $\mathbb{C}^n$, then $\mu^{-1}(J_S) \simeq Sp(2n, \mathbb{R})$. The isotropy subgroup of J_S is

$$[O(2k) \times O(2n-2k)] \cap GL(n, \mathbb{C}) = U(k) \times U(n-k).$$

Therefore we have the identification between the symplectic quotient of $Hom(S, V)$ with $Sp(2n, \mathbb{R}) / U(k) U(n-k)$. This proves (i).

Suppose (M, ω) is a symplectic manifold with a Hamiltonian G-action with moment map $\mu : M \to (LieG)^*$. Suppose N is a symplectic submanifold in M which is invariant under the action of a subgroup H in G. Then the composition $N \hookrightarrow M \xrightarrow{\mu} (LieG)^* \to (LieH)^*$ is the moment map of the action of H on N. If we choose any $J \in (LieG)^*$ to construct the symplectic quotient $\left(M//G, \omega_{M//G}\right)$, then its image in $(LieH)^*$ will also define a symplectic quotient $\left(N//H, \omega_{N//H}\right)$. We assume that G is a compact Lie group and use an invariant metric to identify the Lie algebra with its dual. We also assume that both actions are regular at J and thus both $M//G$ and $N//H$ are smooth manifolds. We have the following easy lemma.

Lemma 24. *In the above situation, $N//H$ is a symplectic submanifold of $M//G$ and the symplectic form $\omega_{M//G}$ on $M//G$ restricts to the symplectic form $\omega_{N//H}$ on $N//H$ provided that $J \in LieH$ and every G-orbit in $\mu^{-1}(J)$ contains at most one H-orbit.*

In this situation, we have $N//H$ is a symplectic sub-quotient of $M//G$.

In part (ii) of the theorem, we take $N = Hom(S_1, \mathbb{C}^n)$ and $H = O(2k)$ and the assumptions in the above lemma hold in this case. This proves (ii) and part (iii) can also be proven in the same way. □

Remark. Suppose (M, ω) is a symplectic manifold with a Hamiltonian G-action and H is a Lie subgroup of G. For any element $J \in LieH \subset LieG$, the two symplectic quotients form a fiber bundle,

$$G_J/H_J \to M//H \to M//G.$$

In the above situation, we take $M = Hom(\mathbb{C}^n, \mathbb{C}^n)$ and the subgroup $H = O(2k) \times O(2n-2k)$ inside $G = O(2n)$, then we obtain a fiber bundle

$$\frac{U(n)}{U(k)\,U(n-k)} \to \frac{Sp(2n, \mathbb{R})}{U(k)\,U(n-k)} \to \frac{Sp(2n, \mathbb{R})}{U(n)}$$

which coincides with the previous fibration,

$$Gr_{\mathbb{C}}(k, n) \longrightarrow \hat{\mathcal{T}}(2k, 2n) \xrightarrow{\beta} \frac{Sp(2n, \mathbb{R})}{U(n)}.$$

Kähler structures on $\hat{\mathcal{T}}(2k, 2n)$ If a Hamiltonian action of K on M can be extended to a complex algebraic action of its complexification $K^{\mathbb{C}}$ on a Kähler structure on M, then a result of Ness and Kempf shows that the symplectic quotient coincides with the GIT quotient, in particular $M//K$ is a quasi-projective Kähler manifold.

In our setting, $M = Hom(S, V)$ has a natural Kähler structure if we put a Hermitian structure on (V, ω_V). We can complexify the action on M from $O(2k) = O(S, g_S)$ to its complexification $O(2k, \mathbb{C})$. However, this action is not algebraic and we cannot apply the GIT quotient construction to obtain a quasi-projective (affine) Kähler structure on $M//O(2k)$. Certainly, $Sp(2n, \mathbb{R})/U(n)$ is not an affine variety. Nevertheless, as explained to us by X.W. Wang, we could use the Kähler quotient method to obtain a Kähler structure on $\mathcal{T}(2k, 2n)$, and similarly on $\hat{\mathcal{T}}(2k, 2n)$. In this paper, we will discuss a different Kähler structure on $\hat{\mathcal{T}}(2k, 2n)$. To see this, we consider the principal bundle

$$U(n) \to Sp(2n, \mathbb{R}) \to Sp(2n, \mathbb{R})/U(n).$$

Its associated vector bundle $E = Sp(2n, \mathbb{R}) \times_{U(n)} \mathbb{C}^n$ with respect to the standard representation of $U(n)$ is naturally a holomorphic vector bundle over $Sp(2n, \mathbb{R})/U(n)$,

$$\mathbb{C}^n \to E \to Sp(2n, \mathbb{R})/U(n).$$

Thus the fiberwise Grassmannian bundle has a natural complex Kähler structure. This coincides with the associated bundle for the standard action of $U(n)$ on $Gr_{\mathbb{C}}(k, n) = U(n)/U(k)U(n-k)$, namely the total space of the symplectic twistor Grassmannian,

$$Gr_{\mathbb{C}}(k, n) \longrightarrow \hat{\mathcal{T}}(2k, 2n) \xrightarrow{\beta} \frac{Sp(2n, \mathbb{R})}{U(n)}.$$

Hence $\hat{\mathcal{T}}(2k, 2n)$ has a natural Kähler structure.

3.2. $Gr_{\mathbb{C}}(k, n)$ *in* $\mathcal{T}(2k, 2n)$ *and* $\hat{\mathcal{T}}(2k, 2n)$

Suppose (V, ω_V) is given a compatible complex structure J_V, i.e. V is a Hermitian vector space. The complex Grassmannian $Gr_{\mathbb{C}}(k, n)$ of all J_V-complex linear subspaces in V is naturally a subspace in $Gr^{Sp}(2k, 2n)$. Furthermore, any such subspace S inherits a complex structure $J_V|_S$ from V. Therefore the restriction of the twistor fibration $Sp(2k, \mathbb{R})/U(k) \to \mathcal{T}(2k, 2n) \to Gr^{Sp}(2k, 2n)$ to $Gr_{\mathbb{C}}(k, n)$ has a canonical section, thus giving an embedding

$$Gr_{\mathbb{C}}(k, n) \subset \mathcal{T}(2k, 2n).$$

For any complex subspace S, we have $S^{\perp} = S^{\omega}$. Therefore we have another embedding

$$Gr_{\mathbb{C}}(k, n) \overset{\simeq}{\to} Gr_{\mathbb{C}}(n-k, n) \subset \mathcal{T}(2n-2k, 2n).$$

Combining them, we obtain an embedding of $Gr_{\mathbb{C}}(k,n)$ into the symplectic twistor Grassmannian

$$Gr_{\mathbb{C}}(k,n) \subset \hat{\mathcal{T}}(2n-2k,2n) = \mathcal{T}(2k,2n) \underset{Gr^{Sp}(2k,2n)}{\times} \mathcal{T}(2n-2k,2n).$$

We are going to describe this embedding from the symplectic quotient point of view. To do this, we equip both (S, g_S) and (V, ω_V) with compatible complex structures J_S and J_V, thus making them Hermitian vector spaces. The space of *complex* linear homomorphisms

$$Hom_{\mathbb{C}}(S,V) \subset Hom(S,V)$$

is invariant under the action of the subgroup $U(S, g_S, J_S) \simeq U(k)$ in $O(2k)$. The Hermitian complex structure J_S in $o(2k)^*$ goes to the identity element in $u(k)^*$, by viewing $u(k)^*$ as the set of Hermitian symmetric matrices. The isotropy subgroup of J_S inside $U(k)$ is the whole group $U(k)$ and therefore

$$\begin{aligned} Hom_{\mathbb{C}}(S,V)//U(k) &= \left[\mu^{-1}(J_S) \cap Hom_{\mathbb{C}}(S,V)\right]/U(k) \\ &= Gr_{\mathbb{C}}(k,n). \end{aligned}$$

The last equality follows from the fact that elements in $\mu^{-1}(J_S) \cap Hom_{\mathbb{C}}(S,V)$ are complex isometries from S into V. Thus the symplectic subquotient

$$Hom_{\mathbb{C}}(S,V)//U(k) \subset Hom(S,V)//O(2k)$$

is exactly the embedding

$$Gr_{\mathbb{C}}(k,n) \subset \mathcal{T}(2k,2n)$$

we described above.

3.3. Symplectic bundle structures on $\mathcal{T}(2k,2n)$ and $\hat{\mathcal{T}}(2k,2n)$

We show that every fiber of the bundles $\mathcal{T}(2k,2n)$ and $\hat{\mathcal{T}}(2k,2n)$ over $Gr^{Sp}(2k,2n)$ is a symplectic submanifold. Such symplectic fiber bundle structures will be used to describe the symplectic geometry on $Gr^{Sp}(2k,2n)$ in the next section.

Theorem 25. *(1) We consider the fiber bundle structure on the symplectic half twistor Grassmannian $\mathcal{T}(2k,2n)$,*

$$Sp(2k,\mathbb{R})/U(k) \to \mathcal{T}(2k,2n) \xrightarrow{\pi} Gr^{Sp}(2k,2n).$$

Every fiber of π is a symplectic submanifold in $\mathcal{T}(2k,2n)$ with its induced symplectic form isomorphic to a constant multiple of the canonical Kähler form on the Hermitian symmetric space $Sp(2k,\mathbb{R})/U(k)$.

(2) We consider the fiber bundle structure on the symplectic twistor Grassmannian $\hat{\mathcal{T}}(2k,2n)$,

$$Sp(2k,\mathbb{R})/U(k)\times Sp(2n-2k,\mathbb{R})/U(n-k)\to\hat{\mathcal{T}}(2k,2n)\xrightarrow{\hat{\pi}}Gr^{Sp}(2k,2n).$$

Every fiber of $\hat{\pi}$ is a symplectic submanifold in $\hat{\mathcal{T}}(2k,2n)$ with its induced symplectic form isomorphic to a constant multiple of the canonical Kähler form on the Hermitian symmetric space $Sp(2k,\mathbb{R})/U(k)\times Sp(2n-2k,\mathbb{R})/U(n-k)$.

(3) Furthermore, the symplectic form on $\mathcal{T}(2k,2n)$ coincides with the restriction of the symplectic form on $\hat{\mathcal{T}}(2k,2n)$, i.e. $\omega_{\mathcal{T}}=\omega_{\hat{\mathcal{T}}}|_{\mathcal{T}}$.

Proof. We fix any $2k$-dimensional symplectic subspace S_0 in V, i.e. $[S_0]\in Gr^{Sp}(2k,2n)$. We can replace the symplectic quotient construction of $Hom(S,V)$ by $Hom(S,S_0)$. It is easy to see that the symplectic sub-quotient $Hom(S,S_0)//O(2k)$ of $Hom(S,V)//O(2k)$ is simply the fiber of π over $[S_0]$, because $Gr^{Sp}(2k,2k)$ is a single point. Since the induced symplectic form on $Hom(S,S_0)//O(2k)$ is invariant under the action of $Sp\left(S_0,\omega_V|_{S_0}\right)\simeq Sp(2k,\mathbb{R})$, $Hom(S,S_0)//O(2k)$ is symplectomorphic to the Hermitian symmetric space $Sp(2k,\mathbb{R})/U(k)$, possibly up to a constant multiple.

In general, suppose N is a symplectic submanifold of M which is invariant under the Hamiltonian action by the compact group K. Then $N//K=\left(\mu^{-1}(J)\cap N\right)/K$ is a symplectic submanifold of $M//K=\mu^{-1}(J)/K$ and the restriction of $\omega_{M//K}$ to $N//K$ is exactly given by $\omega_{N//K}$.

Then part (1) follows from this general discussions with $N=Hom(S,S_0)$ and $M=Hom(S,V)$. The proof of (2) is similar. We just need to replace

$$Hom(S,S_0)//O(2k)\subset Hom(S,V)//O(2k)$$

by

$$Hom(S,S_0)\times Hom\left(S',S_0^{\omega}\right)//O(2k)\,O(2n-2k)\subset Hom\left(S\oplus S',V\right)//O(2k)\,O(2n-2k)$$

with $S\oplus S'\simeq V$. Similarly, $Hom(S,V)//O(2k)$ is a symplectic subquotient of $Hom\left(S\oplus S',V\right)//O(2k)\,O(2n-2k)$ and therefore (3) is proven in the same way. □

4. Symplectic Geometry on $Gr^{Sp}(2k,2n)$

The main objects of study in symplectic geometry are Lagrangian submanifolds (see e.g. [8]). They play important roles in Floer theory, String theory and Mirror Symmetry, and they are also the objects of study in the classical theory of integrable systems. From physical considerations, Kapustin and Orlov [2] argued that coisotropic submanifolds are also important ingredients in understanding Mirror Symmetry. The corresponding mathematical aspects were developed by Oh ([6,7]). This theory for coisotropic submanifolds is also interpreted as the Lagrangian intersection theory of symplectic knot spaces in [4].

We have constructed a non-degenerated two-form Ω^g on $Gr^{Sp}(2k, 2n)$ in Sect. 2. However, this form is not closed. In this section, we first show that the base manifold of any symplectic fiber bundle inherits a natural symplectic geometry from the total space. In the case of $Gr^{Sp}(2k, 2n)$, any of the symplectic fiber bundles $\mathcal{T}(2k, 2n)$, $\mathcal{T}(2n-2k, 2n)$ and $\hat{\mathcal{T}}(2k, 2n)$ induces the same symplectic geometry on $Gr^{Sp}(2k, 2n)$. Then we construct symplectic (resp. Lagrangian) submanifolds in $Gr^{Sp}(2k, 2n)$ from symplectic (resp. coisotropic/isotropic) linear subspaces in V as sub-Grassmannians $\mathcal{G}_P$.

4.1. Symplectic geometry on symplectic fiber bundles

In this subsection, we assume that $\mathcal{T}$ is any symplectic manifold with a symplectic bundle structure in the following sense.

Definition 26. Let $(\mathcal{T}, \omega_{\mathcal{T}})$ be a symplectic manifold and $\pi : \mathcal{T} \to B$ be a smooth fiber bundle. We call this a symplectic fiber bundle if every fiber is a symplectic submanifold in $\mathcal{T}$.

A section $s : B \to \mathcal{T}$ is called a symplectic section if $s(B)$ is a symplectic submanifold in $\mathcal{T}$.

Any such section s induced a symplectic structure $s^*\omega_{\mathcal{T}}$ on B. The following linear algebra lemma will be useful. Its proof is elementary and left to the readers.

Lemma 27. *Let $\pi : \mathcal{T} \to B$ be a smooth fiber bundle with a symplectic section s. Suppose C is submanifold in B with constant rank r with respect to $s^*\omega_{\mathcal{T}}$. Then the rank of $\pi^{-1}(C)$ equals $r + (\dim \mathcal{T} - \dim B)/2$.*

As a corollary we have the following.

Corollary 28. *Let $\pi : \mathcal{T} \to B$ be a symplectic fiber bundle with a symplectic section s and let C be a submanifold in B. (1) C is a symplectic submanifold in B if and only if $\pi^{-1}(C)$ is a symplectic submanifold in $\mathcal{T}$. (2) C is a Lagrangian submanifold in B if and only if $\pi^{-1}(C)$ is a submanifold in $\mathcal{T}$ with rank $(\dim \mathcal{T} - \dim B)/2$.*

In particular, a submanifold in B being symplectic/Lagrangian or not does not depend on the choice of the section s. Indeed, there are many *local* symplectic sections on any symplectic fiber bundle and we can use them to define and describe the symplectic geometry on B. Therefore we make the following definition.

Definition 29. Given any symplectic fiber bundle $\pi : \mathcal{T} \to B$, the rank of a submanifold C in B is defined to be $r\left(\pi^{-1}(C)\right) - (\dim \mathcal{T} - \dim B)/2$. In particular, C is a symplectic submanifold in B if $r(C) = \dim C/2$, i.e. $\pi^{-1}(C)$ is a symplectic submanifold in $\mathcal{T}$, and C is a Lagrangian submanifold in B if $r(C) = 0$.

Remark. Indeed, when we consider a $2k$-dimensional symplectic submanifold X in the $2n$-dimensional symplectic vector space, the image of the Gauss map in $Gr^{Sp}(2k, 2n)$ is also symplectic.

In the next section, we will construct many such submanifolds in $B = Gr^{Sp}(2k, 2n)$. Symplectic geometry is largely characterized by the behavior of their submanifolds, for instance, the intersection theory of Lagrangian submanifolds and the theory of Lefschetz fibrations by symplectic submanifolds. Loosely speaking, the above definition defines a symplectic geometry on the base of any symplectic fiber bundle. Furthermore it is natural with respect to symplectic fiber subbundles.

Definition 30. Let $\hat{\pi} : \hat{\mathcal{T}} \to B$ and $\pi : \mathcal{T} \to B$ be symplectic fiber bundles over the same base B. We call $\mathcal{T}$ a symplectic fiber subbundle of $\hat{\mathcal{T}}$ if there is an embedding $e : \mathcal{T} \to \hat{\mathcal{T}}$ such that (i) $e^*\left(\omega_{\hat{\mathcal{T}}}\right) = \omega_{\mathcal{T}}$ and (ii) e is a smooth subbundle map from $\mathcal{T}$ to $\hat{\mathcal{T}}$.

Proposition 31. *Given a symplectic fiber bundle $\hat{\pi} : \hat{\mathcal{T}} \to B$ and a symplectic fiber subbundle $\pi : \mathcal{T} \to B$, for any submanifold C in B, the ranks of C defined using $\mathcal{T}$ and $\hat{\mathcal{T}}$ are the same.*

This proposition follows easily from lemma 27.

Remark. We can regard $\hat{\mathcal{T}}(2k, 2n)$ as a fiber product of $\mathcal{T}(2k, 2n)$ and $\mathcal{T}(2n-2k, 2n)$ over $Gr^{Sp}(2k, 2n)$ in the category of symplectic fiber bundles.

4.2. Lagrangian and Symplectic submanifolds in $Gr^{Sp}(2k, 2n)$

We have constructed three different symplectic fiber bundles over $Gr^{Sp}(2k, 2n)$ with total spaces $\mathcal{T}(2k, 2n)$, $\mathcal{T}(2n-2k, 2n)$ and $\hat{\mathcal{T}}(2k, 2n)$. Nevertheless, all of them define the same symplectic geometry on $Gr^{Sp}(2k, 2n)$: From the above discussion, the symplectic fiber bundle $\pi : \mathcal{T}(2k, 2n) \to Gr^{Sp}(2k, 2n)$ defines a symplectic geometry on $Gr^{Sp}(2k, 2n)$. By Theorem 25, π is a symplectic fiber subbundle of $\hat{\pi} : \hat{\mathcal{T}}(2k, 2n) \to Gr^{Sp}(2k, 2n)$. Then Proposition 31 shows that both $\mathcal{T}(2k, 2n)$ and $\hat{\mathcal{T}}(2k, 2n)$ define the same symplectic geometry on $Gr^{Sp}(2k, 2n)$. On the other hand,

$$\pi : \mathcal{T}(2n-2k, 2n) \to Gr^{Sp}(2n-2k, 2n) \simeq Gr^{Sp}(2k, 2n)$$

is also a subbundle of $\hat{\pi} : \hat{\mathcal{T}}(2k, 2n) \to Gr^{Sp}(2k, 2n)$ since

$$\hat{\mathcal{T}}(2k, 2n) \simeq \hat{\mathcal{T}}(2n-2k, 2n) \simeq Sp(2n, \mathbb{R}) / U(k)\, U(n-k)\,.$$

Therefore $\mathcal{T}(2n-2k, 2n)$ also defines the same symplectic geometry on $Gr^{Sp}(2k, 2n)$. Thus we have proved the following.

Proposition 32. *Let C be any submanifold in $Gr^{Sp}(2k, 2n)$, then the ranks of C defined using the symplectic fiber bundles $\mathcal{T}(2k, 2n)$, $\mathcal{T}(2n-2k, 2n)$ and $\hat{\mathcal{T}}(2k, 2n)$ are all the same.*

We have seen that, given any compatible complex structure J on (V, ω), the corresponding complex Grassmannian $Gr_{\mathbb{C}}(k, n)$ naturally lies inside $Gr^{Sp}(2k, 2n)$. As a result, complex Schubert varieties in $Gr_{\mathbb{C}}(k, n)$ are also a natural class of subspaces in $Gr^{Sp}(2k, 2n)$.

Let us recall the definition of Schubert varieties in the (real) Grassmannian $Gr(2k, 2n)$: Fix a flag in V, i.e. a sequence of linear subspaces,

$$0 \subset V_1 \subset V_2 \subset \cdots \subset V_{2n} = V,$$

with $\dim V_j = j$ and fix a sequence of integers $0 \leq n_1 \leq n_2 \leq \cdots \leq n_{2n}$. Then the Schubert variety is the set of $[S] \in Gr(2k, 2n)$ with $\dim \left(S \cap V_j\right) \geq n_j$ for every j. This construction gives many subspaces in $Gr^{Sp}(2k, 2n)$ by restriction. For simplicity, we will only discuss the following special cases.

Definition 33. Let P be an m-dimensional linear subspace in (V, ω). We define $\mathcal{G}_P(2k, 2n)$, or simply $\mathcal{G}_P$,

$$\mathcal{G}_P := \begin{cases} \left\{[S] \in Gr^{Sp}(2k, 2n) \mid S \subset P\right\} & \text{if } m \geq 2k \\ \left\{[S] \in Gr^{Sp}(2k, 2n) \mid S \supset P\right\} & \text{if } m \leq 2k \end{cases}.$$

When $m \geq 2k$, a nontrivial $\mathcal{G}_P$ is an open dense subset of $Gr(2k, m)$ and therefore $\dim \mathcal{G}_P = 2k(m - 2k)$. When $m \leq 2k$, S/P is a $(2k - m)$-dimensional subspace in V/P and therefore $\dim \mathcal{G}_P = (2k - m)(2n - 2k)$.

Lemma 34. *Suppose P is an m-dimensional linear subspace in (V, ω) with $m \geq 2k$. Then (i) the preimage of $\mathcal{G}_P$ under the map $\pi : \mathcal{T}(2k, 2n) \to Gr^{Sp}(2k, 2n)$ is*

$$\pi^{-1}(\mathcal{G}_P) = \left(\mu^{-1}(J) \cap Hom\left(\mathbb{C}^k, P\right)\right) / U(k),$$

and (ii) the preimage of $\mathcal{G}_P$ under the map $\hat{\pi} : \hat{\mathcal{T}}(2k, 2n) \to Gr^{Sp}(2k, 2n)$ is

$$\hat{\pi}^{-1}(\mathcal{G}_P) = \left(\mu^{-1}(J) \cap Hom\left(\mathbb{C}^n, P\right)\right) / U(k) \times U(n-k).$$

The proof of this lemma follows directly from the explicit descriptions of the moment maps involved. The next result is useful for us to identify symplectic and Lagrangian sub-Grassmannians inside $Gr^{Sp}(2k, 2n)$.

Theorem 35. *Suppose P is a linear subspace in (V, ω) with $m \geq 2k$. Let $n(P)$ be the nullity of P in V. Then the nullity of $\pi^{-1}(\mathcal{G}_P)$ inside $\mathcal{T}(2k, 2n)$ is equal to $2k \cdot n(P)$ at every point of $\pi^{-1}(\mathcal{G}_P)$.*

Proof. Given any $[\phi] \in \pi^{-1}(\mathcal{G}_P)$, $\phi(S)$ is a linear subspace S_0 in V representing $[S_0] \in \mathcal{G}_P$. In general, the tangent space of $\mu^{-1}(J_S)$ at ϕ consists of $\eta + \chi : S \to V$ with $\eta(S) \subset S_0$ and $\chi(S) \subset (S_0)^{\omega}$.

The symplectic form on $Hom(S, V)$ is given by $g_S^* \otimes \omega_V$. The subset $\mu^{-1}(J_S)$ consists of symplectic homomorphisms from S to V and therefore the restriction of $g_S^* \otimes \omega_V$ to $\mu^{-1}(J_S)$ can be decomposed as $\omega_1 + \omega_2$ with

$$(\omega_1 + \omega_2)(\eta_1 + \chi_1, \eta_2 + \chi_2) = \omega_1(\eta_1, \eta_2) + \omega_2(\chi_1, \chi_2).$$

Notice that ω_1 descends to give the symplectic structure along fibers $\pi^{-1}([S_0]) \simeq Sp(2k, \mathbb{R})/U(k)$ while ω_2 acts along its symplectic complement $\left(\pi^{-1}([S_0])\right)^{\omega\mathcal{T}}$. Moreover, we have

$$\omega_2 = g_S^* \otimes (\omega_V|_{S^\omega}) = g_S^* \otimes \left(\omega_{V/S}\right).$$

Now we restrict our attention to $\pi^{-1}(\mathcal{G}_P)$. Since the tangent spaces of $\pi^{-1}(\mathcal{G}_P)$ contain all (symplectic) fiber directions, we only need to determine the nullity along the horizontal directions. Namely we need to compute the nullity of $S^* \otimes (P/S)$ inside $\left(S^* \otimes (V/S), g_S^* \otimes \left(\omega_{V/S}\right)\right)$. The nullity of P/S in V/S is the same as the nullity of P in V because S is a symplectic subspace. Since S is of dimension $2k$, the nullity of $S^* \otimes (P/S)$ is $2k$ times the nullity of P. Hence our theorem. □

Corollary 36. *Suppose P is a linear subspace in (V, ω) with $m \geq 2k$, then (i) $\mathcal{G}_P$ is a symplectic submanifold in $Gr^{Sp}(2k, 2n)$ if and only if P is a symplectic subspace in V and (ii) $\mathcal{G}_P$ is a Lagrangian submanifold in $Gr^{Sp}(2k, 2n)$ if and only if P is a $(n+k)$-dimensional coisotropic subspace in V.*

Now we assume that $\dim P = m \leq 2k$. Recall that in this case, $\mathcal{G}_P = \{[S] \in Gr^{Sp}(2k, 2n) \mid S \supset P\}$ and $S \supset P$ if and only $S^\omega \subset P^\omega$. Under the identification

$$\begin{aligned} Gr^{Sp}(2k, 2n) &\overset{\simeq}{\to} Gr^{Sp}(2n-2k, 2n) \\ P &\longmapsto P^\omega, \end{aligned}$$

the sub-Grassmannian $\mathcal{G}_P$ is identified with $\mathcal{G}_{P^\omega}$ and we reduce back to our previous considerations. In particular, we have

Proposition 37. *Suppose P is a linear subspace in (V, ω), then (i) $\mathcal{G}_P$ is a symplectic submanifold in $Gr^{Sp}(2k, 2n)$ if and only if P is a symplectic subspace in V and (ii) $\mathcal{G}_P$ is a Lagrangian submanifold in $Gr^{Sp}(2k, 2n)$ if and only if P is an isotropic/coisotropic subspace in V with* $\dim P = n \pm k$.

Note: If we define symplectic geometry on $Gr^{Sp}(2k, 2n)$ with the nondegenerate 2-form defined in Sect. 2, we also obtain the same proposition.

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