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A uniform description of compact symmetric spaces as Grassmannians using the magic square

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Abstract Suppose $\mathbb{A}$ and $\mathbb{B}$ are normed division algebras, i.e. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$, we introduce and study Grassmannians of linear subspaces in $(\mathbb{A} \otimes \mathbb{B})^n$ which are complex/Lagrangian/maximal isotropic with respect to natural two tensors on $(\mathbb{A} \otimes \mathbb{B})^n$. We show that every irreducible compact symmetric space must be one of these Grassmannian spaces, possibly up to a finite cover. This gives a simple and uniform description of all compact symmetric spaces. This generalizes the Tits magic square description for simple Lie algebras to compact symmetric spaces.

1 Introduction

Riemannian symmetric spaces[1] are important model spaces in geometry. They are classified in terms of symmetric Lie algebras. Compact and noncompact (non-flat) symmetric spaces occur naturally in pairs. In this article, we restrict our attention to those of compact type. Examples of compact symmetric spaces include *Grassmannians* of k-planes in n-dimensional vector spaces over $\mathbb{R}$, $\mathbb{C}$, or even $\mathbb{H}$. They are $O(n)/O(k)O(n-k)$, $U(n)/U(k)U(n-k)$ and $Sp(n)/Sp(k)Sp(n-k)$. Symbolically, we write them as $\{\mathbb{R}^k \subset \mathbb{R}^n\}$, $\{\mathbb{C}^k \subset \mathbb{C}^n\}$ and $\{\mathbb{H}^k \subset \mathbb{H}^n\}$. There are also *Lagrangian Grassmannians* for real and complex Lagrangian subspaces in $\mathbb{C}^n$

[1] All symmetric spaces are assumed to be Riemannian, instead of pseudo-Riemannian.

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and $\mathbb{H}^n$ respectively. As symmetric spaces, they are $U(n)/O(n)$ and $Sp(n)/U(n)$ and we denote them as $\{\mathbb{R}^n \subset \mathbb{C}^n\}$ and $\{\mathbb{C}^n \subset \mathbb{H}^n\}$ respectively. Of course, compact Lie groups are also examples of compact symmetric spaces. For example $O(n) = O(n) \times O(n)/O(n)$, which can also be identified as the Grassmannian of linear subspaces in $\mathbb{R}^n \oplus \mathbb{R}^n$ which are maximally isotropic with respect to a canonical symmetric 2-tensor of type (n, n). Similar interpretations work for $U(n)$ and $Sp(n)$ by replacing $\mathbb{R}$ with $\mathbb{C}$ and $\mathbb{H}$ respectively.

Classical compact Lie groups include $SO(n)$, $SU(n)$ and $Sp(n)$. Roughly speaking, they are groups of isometries of $\mathbb{RP}^{n-1}$, $\mathbb{CP}^{n-1}$ and $\mathbb{HP}^{n-1}$ respectively. There are also five exceptional Lie groups, namely G_2, F_4, E_6, E_7 and E_8 (see e.g. [1]). If we include the largest normed division algebra $\mathbb{O}$, the octonions, also known as the Cayley numbers Ca, then we can obtain G_2 and F_4 as groups of symmetries of $\mathbb{O}$ and $\mathbb{O}^3$ respectively as well. Since $\mathbb{O}$ is not associative, $\mathbb{O}^n$ does not make sense at all. However, if $n \leq 3$ then it is possible to define the group of symmetries of $\mathbb{O}^n$ using exceptional Jordan algebras (see e.g. [15]). For the remaining groups, Freudenthal [7] and Tits [16] introduced the *magic square* to describe *every* type of compact Lie algebra in a uniform manner by considering $(\mathbb{A} \otimes \mathbb{B})^n$ with both $\mathbb{A}$ and $\mathbb{B}$ as normed division algebras. In particular, using the magic square, we can regard the Lie algebras of E_6, E_7 and E_8 roughly as the Lie algebras of infinitesimal symmetries of $(\mathbb{C} \otimes \mathbb{O})^3$, $(\mathbb{H} \otimes \mathbb{O})^3$ and $(\mathbb{O} \otimes \mathbb{O})^3$ respectively.

For compact symmetric spaces, there are many more exceptional cases given in the classification table by Cartan [6]. At first sight, it seems hard to imagine that they can all be described in a simple and uniform manner. By realizing the symmetry of the magic square as a mirror duality between complex geometry and symplectic geometry, we generalize the magic square and succeed in giving a simple and uniform description of all compact symmetric spaces. Among them, the most nontrivial ones are $E_6/Sp(4)$ and $E_7/SU(8)$, which will be described as the Grassmannians of *double Lagrangian subspaces* $\Lambda^2(\mathbb{R} \otimes \mathbb{H})^4$ in $(\mathbb{C} \otimes \mathbb{O})^3$ and $\Lambda^2(\mathbb{C} \otimes \mathbb{H})^4$ in $(\mathbb{H} \otimes \mathbb{O})^3$ respectively!

By realizing every compact symmetric space as a Grassmannian of (possibly complex/Lagrangian/maximally-isotropic) linear subspaces, our result has many applications. For instance, one can show that its open subset consisting of *spacelike* linear subspaces is precisely its *noncompact dual* symmetric space, at least in the classical cases [12]. For Hermitian symmetric spaces, this is given by the Borel embedding. Furthermore, the various boundary components, for instance the Shilov boundary, for the noncompact dual correspond to certain directions of those spacelike linear subspaces becoming lightlike.

Notice that Hermitian symmetric spaces which are *tube domains* are precisely Grassmannians of spacelike $(\mathbb{C} \otimes \mathbb{A})$-Lagrangian subspaces in $(\mathbb{H} \otimes \mathbb{A})^n$ for various normed division algebras $\mathbb{A}$ (together with $\mathbb{A} = \mathbb{R}^m$ which is related to the spin factor Jordan algebra). Our description will be useful in constructing projectively flat connections over these tube domains, generalizing the one over $Sp(2n, \mathbb{R})/U(n)$, constructed by Axelrod et al. [2], for the geometric quantization problem. Our result will give us many new insights to the intimate relationships among different symmetric spaces and to the geometries of special holonomy [14].

Table 1 Isotropic Grassmannians $G_{\mathbb{A}\mathbb{B}}(n)$

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$SO(n)$	$SU(n)$	$Sp(n)$	F_4
$\mathbb{C}$	$SU(n)$	$SU(n)^2$	$SU(2n)$	E_6
$\mathbb{H}$	$Sp(n)$	$SU(2n)$	$SO(4n)$	E_7
$\mathbb{O}$	F_4	E_6	E_7	E_8

Table 2 Grassmannians $Gr_{\mathbb{A}\mathbb{B}}(k,n)$

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\frac{O(n)}{O(k)O(n-k)}$	$\frac{U(n)}{U(k)U(n-k)}$	$\frac{Sp(n)}{Sp(k)Sp(n-k)}$	$\frac{F_4}{Spin(9)}$
$\mathbb{C}$	$\frac{U(n)}{U(k)U(n-k)}$	$\frac{U(n)^2}{U(k)^2U(n-k)^2}$	$\frac{U(2n)}{U(2k)U(2n-2k)}$	$\frac{E_6}{Spin(10)U(1)}$
$\mathbb{H}$	$\frac{Sp(n)}{Sp(k)Sp(n-k)}$	$\frac{U(2n)}{U(2k)U(2n-2k)}$	$\frac{O(4n)}{O(4k)O(4n-4k)}$	$\frac{E_7}{Spin(12)Sp(1)}$
$\mathbb{O}$	$\frac{F_4}{Spin(9)}$	$\frac{E_6}{Spin(10)U(1)}$	$\frac{E_7}{Spin(12)Sp(1)}$	$\frac{E_8}{SO(16)}$

Main result A description of all irreducible compact symmetric spaces as Grassmannians of linear subspaces of complex, Lagrangian and/or maximally-isotropic types.

First type: *Isotropic Grassmannians (i.e. Compact (semi)simple Lie groups)*

$$G_{\mathbb{A}\mathbb{B}}(n) = \left\{(\mathbb{A}\otimes\mathbb{B})^n \subset (\mathbb{A}\otimes\mathbb{B})^n \oplus (\mathbb{A}\otimes\mathbb{B})^n\right\}^{\sigma_{g'}},$$

here $\sigma_{g'}$ is an involution induced by a canonical symmetric 2-tensor on $(\mathbb{A}\otimes\mathbb{B})^n \oplus (\mathbb{A}\otimes\mathbb{B})^n$. These spaces are given in Table 1.[2]

On the Lie algebra level, it coincides with the magic square.

Recall that the compact Lie group G_2 is the automorphism group of $\mathbb{O}$, in fact it can be treated as one of this type with $n = 1$ and one of $\mathbb{A}$ and $\mathbb{B}$ is $\mathbb{O}$, the other is $\mathbb{R}$, i.e. it can be realized as $\{\mathbb{O} \subset \mathbb{O} \oplus \mathbb{O}\}^{\sigma_{g'}}$.

Second type: *Grassmannians* (Table 2)

$$Gr_{\mathbb{A}\mathbb{B}}(k,n) = \left\{(\mathbb{A}\otimes\mathbb{B})^k \subset (\mathbb{A}\otimes\mathbb{B})^n\right\}.$$

Third type: *Lagrangian Grassmannians* (Table 3)

$$LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n) = \left\{\left(\frac{\mathbb{A}}{2}\otimes\mathbb{B}\right)^n \subset (\mathbb{A}\otimes\mathbb{B})^n\right\},$$

where $\frac{\mathbb{A}}{2}$ denotes $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ when $\mathbb{A}$ is $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ respectively.

[2] For tables in this paper, n is assumed to be 3 when $\mathbb{A}$ or $\mathbb{B}$ equals $\mathbb{O}$ and possibly with the Abelian part removed for simplicity.

Table 3 Lagrangian Grassmannians $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\frac{SU(n)}{SO(n)}$	$\frac{SU(n)^2}{SU(n)}$	$\frac{SU(2n)}{Sp(n)}$	$\frac{E_6}{F_4}$
$\mathbb{H}$	$\frac{Sp(n)}{U(n)}$	$\frac{SU(2n)}{S(U(n)^2)}$	$\frac{SO(4n)}{U(2n)}$	$\frac{E_7}{E_6U(1)}$
$\mathbb{O}$	$\frac{F_4}{Sp(3)Sp(1)}$	$\frac{E_6}{SU(6)Sp(1)}$	$\frac{E_7}{Spin(12)Sp(1)}$	$\frac{E_8}{E_7Sp(1)}$

Table 4 Double Lagrangian Grassmannians $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{C}$	$\frac{SU(n)^2}{SO(n)^2}$	$\frac{SU(2n)}{SO(2n)}$	$\frac{E_6}{Sp(4)}$
$\mathbb{H}$	$\frac{SU(2n)}{SO(2n)}$	$\frac{SO(4n)}{S(O(2n)^2)}$	$\frac{E_7}{SU(8)}$
$\mathbb{O}$	$\frac{E_6}{Sp(4)}$	$\frac{E_7}{SU(8)}$	$\frac{E_8}{SO(16)}$

The compact symmetric space $G_2/Sp(1)Sp(1)$ is a Lagrangian Grassmannian for $n = 1$ with one of $\mathbb{A}$ and $\mathbb{B}$ is $\mathbb{O}$ and the other is $\mathbb{R}$, i.e. it can be realized as $\{\mathbb{H} \subset \mathbb{O}\}$.

Fourth type: *Double Lagrangian Grassmannians* (Table 4)

$$LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n) = \left\{ \left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right)^n \subset (\mathbb{A} \otimes \mathbb{B})^n \right\},$$

where $j_1 \in \mathbb{A}$ is i, j or e[3] when $\mathbb{A}$ is $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$ respectively, and $j_2 \in \mathbb{B}$ is similar.

For this fourth type of Grassmannian, we have utilized Lagrangian structures coming from both $\mathbb{A}$ and $\mathbb{B}$. We can also replace one of these Lagrangian conditions by maximal isotropic conditions and obtain *Lagrangian-isotropic Grassmannians*

$$LIGr_{\mathbb{A}\mathbb{B}}(n) = \left\{ \left(\frac{\mathbb{A}}{2} \otimes \mathbb{B}\right)^n \subset (\mathbb{A} \otimes \mathbb{B})^n \right\} \cap \left\{ \left(\mathbb{A} \otimes \frac{\mathbb{B}}{2}\right)^n \subset 2\left(\mathbb{A} \otimes \frac{\mathbb{B}}{2}\right)^n \right\}^{\sigma_{g'}},$$

where $\sigma_{g'}$ is an involutive isometry as in the first type, here g' is a proper 2-tensor induced from $\mathbb{B}$, we could also consider *double isotropic Grassmannians* in a similar manner.

For the purpose of describing compact symmetric spaces as Grassmannians, using the first four types of Grassmannians is already sufficient with only one exception. Namely, $SO(2n)/U(n)$ can be realized as a Lagrangian Grassmannian *only* when n is even. For general n we can realize $SO(2n)/U(n)$ as a Lagrangian-isotropic Grassmannian as follows,

$$\frac{SO(2n)}{U(n)} = LIGr_{\mathbb{C}\mathbb{H}}(n),$$

[3] Assume $\mathbb{O}$ is generated by i, j and e as an algebra, and $\mathbb{H} = \frac{\mathbb{O}}{2}$ is generated by i and j.

that is

$$\frac{SO\,(2n)}{U(n)} = \frac{SU\,(2n)}{Sp(n)} \cap \frac{SO\,(2n)^2}{SO\,(2n)} \subset \frac{O\,(8n)}{O\,(4n)^2}.$$

Remark 1 A *Hermitian symmetric space* is a symmetric space G/K which admits a G-invariant complex structure and this happens precisely when K has a $U(1)$-factor. They are $U(n)/U(k)U(n-k)$, $E_6/Spin(10)U(1)$, $Sp(n)/U(n)$, $SO(2n)/U(n)$, $E_7/E_6U(1)$ and $O(m+4)/O(m+2)O(2)$. The first two are Grassmannians $Gr_{\mathbb{CB}}(k,n)$ of subspaces in $(\mathbb{C}\otimes\mathbb{B})^n$ of the type $(\mathbb{C}\otimes\mathbb{B})^k$ for any normed division algebra $\mathbb{B}$. The next three are Lagrangian Grassmannians $LGr_{\mathbb{CB}}(n)$ of subspaces in $(\mathbb{H}\otimes\mathbb{B})^n$ of the type $(\mathbb{C}\otimes\mathbb{B})^n$ for any normed division algebra $\mathbb{B}$, with the exception of $SO(2n)/U(n)$ with n odd, which we have discussed above. The remaining one $O(m+4)/O(m+2)O(2)$ can also be interpreted in the same manner as follows. Recall that when $\mathbb{B}=\mathbb{O}$ we always assume that $n\leq 3$ because in these cases the Jordan algebras of symmetries of $\mathbb{B}^n$ can be defined regardless of whether $\mathbb{B}$ is associative or not. In fact when $n=2$ the Jordan algebras of symmetries of $\mathbb{B}^2$ do not even involve the algebra structures of $\mathbb{B}$ and we can take $\mathbb{B}$ to be any inner product space $\mathbb{R}^m$. This is the last family in the classification of Jordan algebras, called the *spin factor*. If we also incorporate the spin factor, then $O(m+4)/O(m+2)O(2)$ is simply $LGr_{\mathbb{CB}}(2)$ with $\mathbb{B}=\mathbb{R}^m$. Similarly $O(m+2)/O(m+1)O(1)$ can be interpreted as $LGr_{\mathbb{RB}}(2)$ with $\mathbb{B}=\mathbb{R}^m$, which is also the usual real projective space $\mathbb{RP}^{m+1}$.

Remark 2 A *quaternionic symmetric space* is a symmetric space G/K which admits a G-invariant quaternionic structure and this happens precisely when K has a $Sp(1)$-factor. They are always quaternionic Kähler manifolds and there is exactly one for each simply connected compact Lie group G. The complete list of compact irreducible quaternionic symmetric spaces is: $Sp(m+2)/Sp(m+1)Sp(1)$, $U(m+4)/U(m+2)U(2)$, $O(m+8)/O(m+4)O(4)$, $G_2/Sp(1)^2$, $F_4/Sp(3)Sp(1)$, $E_6/SU(6)Sp(1)$, $E_7/Spin(12)Sp(1)$ and $E_8/E_7Sp(1)$. All the exceptional ones are Lagrangian Grassmannians $LGr_{\mathbb{HB}}(n)$ of subspaces in $(\mathbb{O}\otimes\mathbb{B})^n$ of the type $(\mathbb{H}\otimes\mathbb{B})^n$ for any normed division algebra $\mathbb{B}$.

In the remainder of this section, we give an intuitive explanation of the origin of these four types of Grassmannian, we first recall that all normed division algebras $\mathbb{A}$ can be obtained from a *doubling* construction starting from $\mathbb{R}$, the Cayley–Dickson process: $\mathbb{R}\rightsquigarrow\mathbb{C}\rightsquigarrow\mathbb{H}\rightsquigarrow\mathbb{O}$. For any “vector space” V over $\mathbb{R}$, this doubling procedure can be viewed in two different ways: (i) complexification $V\longmapsto(TV,J)$ and (ii) symplectification $V\longmapsto(T^*V,\omega)$. The corresponding semi-simple part of Lie algebras of infinitesimal symmetries of these doublings are (i) $\mathfrak{sl}(n,\mathbb{R})\longmapsto\mathfrak{sl}(n,\mathbb{C})\longmapsto\mathfrak{sl}(n,\mathbb{H})$ and (ii) $\mathfrak{sl}(n,\mathbb{R})\longmapsto\mathfrak{sp}(2n,\mathbb{R})\longmapsto\mathfrak{sp}(2n,\mathbb{C})$. The octonion case is more complicated which we will deal with in the next section via the magic square. Combining both of them gives Table 5 of Lie algebras.

If we also require these symmetries to preserve the metric g induced from V, then they reduce to the maximal compact subalgebras and we obtain the magic square (Table 6).

Table 5 Infinitesimal symmetries of complex (respective symplectic) structures on $\mathbb{A}$ (respective $\mathbb{B}$) of $(\mathbb{A}\otimes\mathbb{B})^n$

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{H})$
$\mathbb{C}$	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{su}(n,n)$	$\mathfrak{so}^*(4n)$
$\mathbb{H}$	$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{so}(4n,\mathbb{C})$

Table 6 Infinitesimal symmetries of $(\mathbb{A}\otimes\mathbb{B})^n$ with metrics

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$
$\mathbb{R}$	$\mathfrak{so}(n)$	$\mathfrak{su}(n)$	$\mathfrak{sp}(n)$
$\mathbb{C}$	$\mathfrak{su}(n)$	$\mathfrak{su}(n)^2$	$\mathfrak{su}(2n)$
$\mathbb{H}$	$\mathfrak{sp}(n)$	$\mathfrak{su}(2n)$	$\mathfrak{so}(4n)$

The symmetry of the magic square along the diagonal is closely related to the formula

$$g(Ju,v)=\omega(u,v),$$

which says that complex structure and symplectic structure determine each other when a metric is given.

Notice that every symplectic form ω on V determines an involution σ_ω on the Grassmannian of linear subspaces $P\subset V$ by $\sigma_\omega(P)=P^{\perp_\omega}$, the symplectic complement to P. The fixed points of σ_ω are precisely Lagrangian subspaces in (V,ω). This is analogous to every complex structure J on V determines an involution σ_J on the Grassmannian by $\sigma_J(P)=JP$ whose fixed points are precisely complex linear subspaces in (V,J).

Suppose that $V\simeq(\mathbb{A}\otimes\mathbb{B})^n$ with $\mathbb{A}$, $\mathbb{B}$ normed division algebras. Then the Grassmannian of all real linear subspaces P in V which are σ_J-invariant for every J coming from either $\mathbb{A}$ or $\mathbb{B}$ is precisely our compact symmetric space of type II, $Gr_{\mathbb{A}\mathbb{B}}(k,n)$. Similarly, if we require P to be σ_ω-invariant, instead of σ_J-invariant, for one J coming from $\mathbb{A}$, then we obtain the Lagrangian Grassmannian $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$. Furthermore, if we do this to both $\mathbb{A}$ and $\mathbb{B}$, then we obtain the double Lagrangian Grassmannian $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$.

Recall that the canonical symplectic form ω on $T^*V=V\oplus V^*$ is the skew-symmetric component of a two tensor induced from the natural pairing between V and V^*. The symmetric component of this tensor is an indefinite inner product g' on $V\oplus V^*$ with signature (m,m). If we replace σ_ω by $\sigma_{g'}$ in the definition of $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$, then we obtain the list of compact Lie groups G.

In conclusion, every irreducible compact symmetric space is a Grassmannian of linear subspaces in $W\simeq(\mathbb{A}\otimes\mathbb{B})^n$ which are invariant under involutions σ_J,σ_ω or $\sigma_{g'}$.

2 Magic square from complex and symplectic point of views

Magic square. Complex simple Lie algebras are classified by Cartan. The classical ones are complexifications of $\mathfrak{so}(n)$, $\mathfrak{su}(n)$ and $\mathfrak{sp}(n)$, which are Lie algebras of isometry groups (modulo center) of $\mathbb{R}^n$, $\mathbb{C}^n$ and $\mathbb{H}^n$ respectively. They can also be identified as derivation algebras of Jordan algebras of rank n Hermitian matrices over $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ respectively. Namely $\mathfrak{so}(n) = Der H_n(\mathbb{R})$, $\mathfrak{su}(n) = Der H_n(\mathbb{C})$ and $\mathfrak{sp}(n) = Der H_n(\mathbb{H})$. The rest are five exceptional Lie algebras, $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$. To define them, we need to include the non-associative normed division algebra $\mathbb{O}$, the octonion or the Cayley number. For instance $\mathfrak{g}_2 = Der\mathbb{O}$ and $\mathfrak{f}_4 = Der H_3(\mathbb{O})$. Even though $\mathbb{O}^n$ can not be defined properly due to the non-associative nature of $\mathbb{O}$, nevertheless, the Jordan algebra $H_n(\mathbb{O})$ has a natural interpretation as an exceptional Jordan algebra when $n = 3$ (see e.g. [13]).

Tits [16] observed that we can have a uniform description of all simple Lie algebras, including $\mathfrak{e}_6$, $\mathfrak{e}_7$ and $\mathfrak{e}_8$, if we use any two normed division algebras $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ and define

$$L_3(\mathbb{A}, \mathbb{B}) = Der H_3(\mathbb{B}) \oplus \left[H_3'(\mathbb{B}) \otimes Im\mathbb{A}\right] \oplus Der(\mathbb{A}),$$

where $H_3'(\mathbb{A})$ is the subset of trace zero Hermitian matrices in $H_3(\mathbb{A})$.

As a matter of fact, $L_n(\mathbb{A}, \mathbb{B})$ can be defined for any n as long as $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. The following table is the magic square of Lie algebras for $n = 3$.

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)^2$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Freudenthal [7–10] also found a very different version to construct the magic square in about 1958.

The magic of the square is the symmetry in $\mathbb{A}$ and $\mathbb{B}$. For example $L_n(\mathbb{R}, \mathbb{R}) = Der H_n(\mathbb{R}) = \mathfrak{so}(n)$ is the space of infinitesimal isometries of $V \cong \mathbb{R}^n$. $L_n(\mathbb{R}, \mathbb{C}) = Der H_n(\mathbb{C}) = \mathfrak{su}(n)$ is the space of infinitesimal isometries of $V \otimes_{\mathbb{R}} \mathbb{C} = V \oplus V = TV$ which also preserve the natural complex structure J on TV, the tangent bundle of V. On the other hand, any symmetric matrix $A = (A_{ij}) \in H_n'(\mathbb{R}) \subseteq L_n(\mathbb{C}, \mathbb{R})$ induces a natural transformation ϕ_A on $V \oplus V^* = T^*V$ preserving the canonical symplectic structure on the cotangent bundle,

$$\omega = \sum_{i=1}^{n} dx^i \wedge dy_i,$$

where $\{x^i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ are dual coordinates in V and V^* respectively. Here $\phi_A(x^i) = x^i$ and $\phi_A(y_i) = y_i + \Sigma_j A_{ij} x^j$. Using the metric to identify V and V^*, the complex and symplectic structures on $TV = T^*V$ are related by the formula $\omega(u, v) = g(Ju, v)$. Thus we obtain the first symmetry $L_n(\mathbb{C}, \mathbb{R}) = L_n(\mathbb{R}, \mathbb{C})$ in the magic square. By repeating this doubling process, we obtain other symmetries for the classical Lie algebras in the magic square.

We will also define the noncompact Lie algebra $\mathfrak{n}_n(\mathbb{A},\mathbb{B})$, which contains $L_n(\mathbb{A},\mathbb{B})$ as its maximal compact Lie subalgebra, by not requiring its elements to preserve the inner product. This will be discussed in Sect. 2.2.

In the next section, we apply this construction to study the classification of symmetric spaces, we will show that any irreducible compact symmetric space can be identified as a Grassmannian of multi complex/Lagrangian subspaces in $(\mathbb{A}\otimes\mathbb{B})^n$.

2.1 The doubling construction

Definition 1 Given any real vector space V, we define inductively $V^{[m,l]}$ by (i) $V^{[0,0]} = V$ and (ii) $V^{[m,l]} = TV^{[m,l-1]} = T^*V^{[m-1,l]}$, together with the l canonical complex structures $\{J_1^R, \ldots, J_l^R\}$ and m canonical symplectic structures $\{\omega_1^L, \ldots, \omega_l^L\}$ defined as follows.

To explain the complex and symplectic structures we assume that V is a $2n$ dimensional real vector space together with a complex structure and a symplectic structure as follows,

$$J_1 = \sum\left(\partial_{x^i}\otimes dx^{n+i} - \partial_{x^{n+i}}\otimes dx^i\right),$$
$$\omega_1 = \sum\left(dx^i\wedge dx^{n+i}\right),$$

where $\{x^i\}(i = 1, \ldots, 2n)$ are the real coordinates in V, $\partial_{x^i} = \frac{\partial}{\partial x^i}$.

1. On TV, besides the canonical complex structure J_2, we also have a natural complex structure and a symplectic structure induced by J_1 and ω_1 given as follows, where we still use the same notations.

$$J_1 = \sum\left(\partial_{x^i}\otimes dx^{n+i} - \partial_{x^{n+i}}\otimes dx^i\right) - \sum\left(\partial_{y^i}\otimes dy^{n+i} - \partial_{y^{n+i}}\otimes dy^i\right),$$
$$J_2 = \sum\left(\partial_{x^i}\otimes dy^i - \partial_{y^i}\otimes dx^i\right),$$
$$\omega_1 = \sum\left(dx^i\wedge dx^{n+i} + dy^i\wedge dy^{n+i}\right),$$

   where $\{y^i\}$ are the fiber coordinates of $\sum y^i\partial_{x^i}$ on the tangent bundle.
2. On T^*V, besides the canonical symplectic structure ω_2, we also have a natural complex structure and a symplectic structure induced by J_1 and ω_1 given as follows, where we still use the same notations.

$$J_1 = \sum\left(\partial_{x^i}\otimes dx^{n+i} - \partial_{x^{n+i}}\otimes dx^i\right) + \sum\left(\partial_{u_i}\otimes du_{n+i} - \partial_{u_{n+i}}\otimes du_i\right),$$
$$\omega_1 = \sum\left(dx^i\wedge dx^{n+i} - du_i\wedge du_{n+i}\right),$$
$$\omega_2 = \sum\left(dx^i\wedge du_i\right),$$

   where $\{u_i\}$ are the fiber coordinates of $\sum u_i dx^i$ on the cotangent bundle.

Proposition 1 *The above complex and symplectic structures on TV and T^*V satisfy the following conditions.*

$$\begin{aligned} J_1 J_2 &= -J_2 J_1, \\ \omega_i(J_j(\cdot), J_j(\cdot)) &= \omega_i(\cdot, \cdot), \\ \iota_{\omega_1}^{-1} \circ \iota_{\omega_2} &= -\iota_{\omega_2}^{-1} \circ \iota_{\omega_1}. \end{aligned}$$

Here $\iota_\omega : V \to V^*$ is defined as $\iota_\omega(v) = v \lrcorner \omega = \omega(v, \cdot)$.

This proposition can be easily proven by simple calculations.

2.2 Noncompact magic square

Given a vector space $V = V^{[0,0]} \cong \mathbb{R}^n$, its group of automorphisms is $GL(n, \mathbb{R})$. Recall that $V^{[0,1]}$ (resp. $V^{[1,0]}$) carries a natural complex (resp. symplectic) structure and its group of automorphisms is $GL(n, \mathbb{C})$ (resp. $Sp(2n, \mathbb{R})$). If we restrict our attention only to isometries, then both $GL(n, \mathbb{C})$ and $Sp(2n, \mathbb{R})$ reduce to $U(n)$, which is also their common maximal compact subgroup. If we only look at the semi-simple parts of these reductive groups, then they become $SL(n, \mathbb{C})$, $Sp(2n, \mathbb{R})$ and $SU(n)$ respectively. The following definitions generalize these to a general $V^{[m,l]}$.

Definition 2 Given a real vector space $V \cong \mathbb{R}^n$, we denote by $N_n(m, l) \subset GL(n \cdot 2^{m+l}, \mathbb{R})$ the connected maximal semisimple subgroup of the group of all automorphisms of $V^{[m,l]}$ which preserve the canonical complex structures $\{J_1^R, \ldots, J_l^R\}$ and symplectic structures $\{\omega_1^L, \ldots, \omega_m^L\}$ on $V^{[m,l]}$. We denote the maximal compact subgroup of $N_n(m, l)$ as $G_n(m, l)$, and the maximal semisimple subgroup of $G_n(m, l)$ as $G_n^s(m, l)$.

The Lie algebras of $N_n(m, l)$ and $G_n(m, l)$ are denoted by $\mathfrak{n}_n(m, l)$ and $\mathfrak{g}_n(m, l)$ respectively.

The main result in this subsection is the following theorem.

Theorem 1 *The groups $N_n(m, l)$, $G_n(m, l)$ and $G_n^s(m, l)$ satisfy the following properties:*

1. *$N_n(0, l)/G_n(0, l)$ is the noncompact dual to $G_n^s(1, l)/G_n(0, l)$ as symmetric spaces.*
2. *$N_n(2, l)$ is the complexification of $N_n(1, l)$.*
3. *$N_n(1, l)/G_n(1, l)$ is the noncompact dual to $G_n^s(2, l)/G_n(1, l)$ as symmetric spaces.*

If we restrict our attention to $0 \leq m, l \leq 2$, then the Lie algebra of $G_n^s(m, l)$ is the same as the corresponding Lie algebra $L_n(\mathbb{A}, \mathbb{B})$ in the Tits magic square where $dim_{\mathbb{R}} \mathbb{A} = 2^m$ and $dim_{\mathbb{R}} \mathbb{B} = 2^l$. Then we are naturally led to extend our definitions to $N_3(m, 3)$, using (i), (ii) and (iii) in the above theorem. Thus we have the following table for $N_n(m, l)$.

$m\backslash l$	0	1	2	$3(n=3)$
0	$SL(n,\mathbb{R})$	$SL(n,\mathbb{C})$	$SL(n,\mathbb{H})$	$E_{6(-26)}$
1	$Sp(2n,\mathbb{R})$	$SU(n,n)$	$SO^*(4n)$	$E_{7(-25)}$
2	$Sp(2n,\mathbb{C})$	$SL(2n,\mathbb{C})$	$SO(4n,\mathbb{C})$	$E_7{}^{\mathbb{C}}$

Remark 3 Where -26 in the group $E_{6(-26)}$ is the signature of the Killing form of the noncompact Lie group E_6, the other cases are similar.

Suppose A is a linear transformation of a vector space V. We say (i) A preserves a complex structure J on V if $A \circ J = J \circ A$ and (ii) A preserves a symplectic structure ω on V if $\omega(Au, Av) = \omega(u, v)$ for any u, v in V. The following lemma expresses this condition in terms of the isomorphism ι_ω. The proof of it is a simple exercise in linear algebra.

Lemma 1 *Suppose (V, ω) is a symplectic vector space and $A \in End(V)$. Then A preserves ω if and only if $\iota_\omega \circ A = A^* \circ \iota_\omega$, where $A^* \in End(V^*)$ is the adjoint of A.*

Remark 4 Using the natural complex structure, the vector space $V^{[0,1]} = TV$ can be identified as $\mathbb{C}^n$. Similarly $V^{[0,2]} = T(TV)$ can be identified as $\mathbb{H}^n$ since the two complex structures on $V^{[0,2]}$ are anti-commutative by Proposition 1. $V^{[m,l]}$ with l complex structures and m symplectic structures can be identified as $(T^*)^m\mathbb{B}^n$ with m symplectic structures, where $\mathbb{B}$ is $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ when $l = 0, 1, 2$ respectively. Then we view any $A \in N_n(m, l)$ as a $\mathbb{B}$-linear transformation $A \in SL(n, \mathbb{B})$ preserving all ω_i's. In terms of matrices with entries in $\mathbb{B}$, this means

$$A\omega_i\, \overline{A}^t = \omega_i,$$

which is deduced from

$$\omega_i(J_j(\cdot), J_j(\cdot)) = \omega_i(\cdot, \cdot).$$

At the Lie algebra level it is

$$A\omega_i + \omega_i\, \overline{A}^t = 0,$$

where $A \in \mathfrak{sl}(n, \mathbb{B})$. From now on we identify $N_n(m, l)$ as the maximal semisimple subgroup of all $\mathbb{B}$-linear transformations of $(T^*)^m\mathbb{B}^n$ which preserve ω_i's ($i = 1, \ldots, m$).

The following three propositions prove the three properties in Theorem 1.

Proposition 2 *$N_n(0, l)/G_n(0, l)$ is the noncompact dual to $G_n^s(1, l)/G_n(0, l)$ as symmetric spaces.*

Proof Consider $V^{[1,l]}$ together with the complex structures $J_1, \ldots, J_l$ and symplectic structure ω, let g be the standard Euclidean metric. $\omega(J_i u, J_i v) = \omega(u, v)$ and $g(J_i u, J_i v) = g(u, v)$ induce $\iota_\omega \circ J_i = J_i^* \circ \iota_\omega$ and $J_i \circ \iota_g^{-1} = \iota_g^{-1} \circ J_i^*$.

We define a complex structure J by

$$\omega(u, v) = g(Ju, v),$$

so

$$J = \iota_g^{-1} \circ \iota_\omega.$$

It is easy to check that

$$J_i \circ J = J_i \circ \iota_g^{-1} \circ \iota_\omega = \iota_g^{-1} \circ J_i^* \circ \iota_\omega = \iota_g^{-1} \circ \iota_\omega \circ J_i = J \circ J_i,$$

so

$$J_i \circ (J \circ A) = (J \circ A) \circ J_i.$$

Then we have

$$SL(V^{[1,l]}, J_1, \dots, J_l, J) = SL(V^{[0,l]}, J_1, \dots, J_l)^{\mathbb{C}}.$$

For any $A \in SL(V^{[1,l]}, J_1, \dots, J_l)$, if $g(Au, Av) = g(u, v)$, then

$$\begin{aligned} AJ = JA &\Leftrightarrow \iota_g^{-1} \circ A^* \circ \iota_\omega = A \circ \iota_g^{-1} \circ \iota_\omega = \iota_g^{-1} \circ \iota_\omega \circ A \\ &\Leftrightarrow A^* \circ \iota_\omega = \iota_\omega \circ A, \end{aligned}$$

this implies

$$SL(V^{[1,l]}, J_1, \dots, J_l, J, g) = SL(V^{[1,l]}, J_1, \dots, J_l, \omega, g).$$

So

$$\begin{aligned} G_n(1, l) &= SL(V^{[1,l]}, J_1, \dots, J_l, \omega, g) \\ &= SL(V^{[1,l]}, J_1, \dots, J_l, J, g) \\ &= \text{maximal compact subgroup of } SL(V^{[1,l]}, J_1, \dots, J_l, J) \\ &= \text{maximal compact subgroup of } SL(V^{[0,l]}, J_1, \dots, J_l)^{\mathbb{C}}, \end{aligned}$$

then the result follows.

Proposition 3 *$N_n(2, l)$ is the complexification of $N_n(1, l)$.*

Proof

$$\begin{aligned} N_n(l, 2) &= SL(V^{[2,l]}, J_1, \dots, J_l, \omega_1, \omega_2) \\ &= SL(V^{[2,l]}, J_1, \dots, J_l, \omega_1, J') \\ &= SL(V^{[1,l]}, J_1, \dots, J_l, \omega_1)^{\mathbb{C}}, \end{aligned}$$

where $J' = \iota_{\omega_1}^{-1} \circ \iota_{\omega_2}$ is a complex structure by simple calculations.

Proof of the second "=" : For any $A \in SL(V^{[2,l]}, J_1, \ldots, J_l, \omega_1)$,

$$\begin{aligned}\iota_{\omega_2} \circ A = A^* \circ \iota_{\omega_2} &\Leftrightarrow A \circ \iota_{\omega_1}^{-1} \circ \iota_{\omega_2} = \iota_{\omega_1}^{-1} \circ A^* \circ \iota_{\omega_2} = \iota_{\omega_1}^{-1} \circ \iota_{\omega_2} \circ A \\ &\Leftrightarrow AJ' = J'A.\end{aligned}$$

Proof of the last "=" : We only need to check $SL(V^{[2,l]}, J_1, \ldots, J_l, \omega_1, J')$ is a complex Lie algebra, i.e. we need to check

$$\begin{aligned}J_i J' = J' J_i, &\text{ for any } i = 1, \ldots, l, \\ \omega_1(J'u, J'v) = -\omega_1(u, v).\end{aligned}$$

$\iota_{\omega_1}^{-1} \circ \iota_{\omega_2} \circ J_i = \iota_{\omega_1}^{-1} \circ J_i^* \circ \iota_{\omega_2} = J_i \circ \iota_{\omega_1}^{-1} \circ \iota_{\omega_2}$ implies $J_i J' = J' J_i$, the other is also true by simple calculations.

Proposition 4 *$N_n(1,l)/G_n(1,l)$ is the noncompact dual to $G_n^s(2,l)/G_n(1,l)$ as symmetric spaces.*

Proof This property is a Corollary of 3.

Now we compute the group $N_n(m,l)$ for $m, l = 0, 1, 2$.

1. *The case of $m = 0$.* By Lemma 4 the group $N_n(0,l)$ is $SL(n, \mathbb{B})$ obviously, where $\mathbb{B}$ is $\mathbb{R}, \mathbb{C}, \mathbb{H}$ when $l = 0, 1, 2$ respectively.
2. *The case of $m = 1, l = 0$.* The group $N_n(1,0)$ is the maximal semisimple subgroup of the group of all $\mathbb{R}$-linear transformations of $\mathbb{R}^{2n}$ which preserve the symplectic structure ω, so this is the symplectic group $Sp(2n, \mathbb{R})$.
3. *The case of $m = l = 1$.* Denote $C = \frac{\sqrt{2}}{2}\begin{pmatrix} iI & I \\ I & iI \end{pmatrix}$, where $i = \sqrt{-1}$, then $C\,\overline{C}^t = I$ and

$$\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix} = C \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \overline{C}^t.$$

For any $A \in N_n(1,1)$, we have

$$\left(CA\overline{C}^t\right)\begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}\overline{\left(CA\overline{C}^t\right)}^t = \begin{pmatrix} iI & 0 \\ 0 & -iI \end{pmatrix}.$$

This implies $N_n(1,1)$ is conjugate to

$$CN_n(1,1)\overline{C}^t = SU(n,n).$$

4. *The case of $m = 1, l = 2$.* $N_n(1,2)$ is the group of all $\mathbb{H}$-linear transformations of $\mathbb{H}^{2n}$, which preserve the symplectic structure ω, i.e.

$$\left\{A + jB \in SL(2n, \mathbb{H}) \mid (A + jB)\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}(\overline{A + jB})^t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\right\}$$

Denote $C = \frac{1}{2}\begin{pmatrix} (i+j)I & (-1-k)I \\ (-1+k)I & (i-j)I \end{pmatrix}$, then $C\,\overline{C}^t = I$ and

$$-j\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} = C\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}\overline{C}^t.$$

So the group $N_n(1,2)$ is conjugate to

$$CN_n(1,2)\overline{C}^t = \left\{A + jB \in SL(2n,\mathbb{H}) \mid (A+jB)\; jI\; (\overline{A+jB})^t = jI\right\}.$$

Using the embedding

$$\begin{aligned} SL(2n,\mathbb{H}) &\longrightarrow SL(4n,\mathbb{C}) \\ A + jB &\mapsto \begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}, \end{aligned}$$

any element $A + jB$ of $CN_n(1,2)\overline{C}^t$ corresponds to $\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix} \in SL(4n,\mathbb{C})$ and satisfies

$$\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\overline{\begin{pmatrix} A & -\overline{B} \\ B & \overline{A} \end{pmatrix}}^t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

By some elementary calculations this is equivalent to

$$\begin{cases} A\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}\overline{A}^t = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}, \\ AA^t = I, \end{cases}$$

where $A \in SL(4n,\mathbb{C})$. That is $A \in SO^*(4n)$ (see [11, p.445]).

5. *The case of* $m = 2$. By Proposition 3 the group $N_n(2,l)$ is the complexification of $N_n(1,l)$.

Remark 5 Recall in Remark 4 we view $N_n(m,l)$ as the maximal semisimple subgroup of the group of all $\mathbb{B}$-linear transformations of $(T^*)^m\mathbb{B}^n$ which preserve all ω_i's.

When $m = 2$, by introducing the complex structure $J_1^L = J' = \iota_{\omega_1}^{-1} \circ \iota_{\omega_2}$, $(T^*)^m\mathbb{B}^n$ can be identified as the $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$-linear space $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^{2n}$, hence $N_n(m,l)$ can be viewed as the maximal semisimple subgroup of the group of all $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$-linear transformations of $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^{2n}$ which preserve $\omega_m^L = \omega_m$, where ω_m is viewed as a rank $2n$ matrix with entries in $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$.

For any A in $N_n(m,l)$, A can be viewed as an element in $SL(2n, \frac{\mathbb{A}}{2} \otimes \mathbb{B})$, which satisfies

$$A\omega_m^L\tilde{A}^t = \omega_m^L,$$

where $\tilde{}$ is the conjugation with respect to the second part of the tensor product, that is the conjugation of $\mathbb{B}$.

At the Lie algebra level, for any element A in $\mathfrak{n}_n(m,l)$, A can be viewed as an element in $M_{2n}(\frac{\mathbb{A}}{2}\otimes\mathbb{B})$, the Lie algebra of rank $2n$ matrices with entries in $\frac{\mathbb{A}}{2}\otimes\mathbb{B}$, which satisfies

$$A\omega_m^L + \omega_m^L\tilde{A}^t = 0.$$

By properly choosing coordinates such that ω_m^L is of the form $\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$ as a rank $2n$ matrix with entries in $\frac{\mathbb{A}}{2}\otimes\mathbb{B}$, every element of $\mathfrak{n}_n(l,m)$ has the form

$$\begin{pmatrix} A & B \\ C & -\tilde{A}^t \end{pmatrix},$$

where $A, B, C \in M_n(\frac{\mathbb{A}}{2}\otimes\mathbb{B})$, the algebra of rank n matrices with entries in $\frac{\mathbb{A}}{2}\otimes\mathbb{B}$, and $\tilde{B}^t = B,\ \tilde{C}^t = C$.

Remark 6 If we equip $(\frac{\mathbb{A}}{2}\otimes\mathbb{B})^{2n}$ with the standard Euclidean metric g, the symplectic structure ω_m^L induces a complex structure J_m^L, then $(\frac{\mathbb{A}}{2}\otimes\mathbb{B})^{2n}$ can be identified as $(\mathbb{A}\otimes\mathbb{B})^n$, which is also a real vector space with complex structures J_L^i and J_R^j induced from $\mathbb{A}$ and $\mathbb{B}$ respectively. The maximal compact subgroup $G_n(m,l)$ of $N_n(m,l)$ can be viewed as the group of all $\mathbb{A}\otimes\mathbb{B}$-linear transformations of $(\mathbb{A}\otimes\mathbb{B})^n$ which preserve the metric g.

From now on we consider $(\mathbb{A}\otimes\mathbb{B})^n$ instead of $(\frac{\mathbb{A}}{2}\otimes\mathbb{B})^{2n}$ unless stated otherwise, and use the above formulation of $G_n(m,l)$.

3 Compact symmetric spaces

A Riemannian manifold M is a symmetric space if each point p in M is an isolated fixed point of an involutive isometry σ_p of M. The most well-known example is the real Grassmannian $Gr_{\mathbb{R}}(k,W)$, i.e. the space of all k-dimensional linear subspaces in a real vector space W. In this case, $\sigma_P(P')$ is the reflection of the k-dimensional subspace P' along P. Other symmetric spaces can be constructed using (multi-)complex and (multi-)symplectic structures on W.

Recall when $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$ and we write $dim_{\mathbb{R}}\mathbb{A} = 2^m, dim_{\mathbb{R}}\mathbb{B} = 2^l$, then $W = (\mathbb{A}\otimes\mathbb{B})^n$ can be identified as a real vector space together with complex structures J_i^L and J_j^R, where $i = 1,\ldots,m$ and $j = 1,\ldots,l$. If we equip W with the standard Euclidean metric g, then each complex structure J_i^L (respectively J_j^R) induces a symplectic structure ω_i^L (respectively ω_j^R) on W. Let $Gr_{\mathbb{R}}(W)$ denote the space of all $\mathbb{R}$-linear subspaces in W, i.e., $Gr_{\mathbb{R}}(W) = \coprod_{k=0}^{dim_{\mathbb{R}}W} Gr_{\mathbb{R}}(k,W)$.

Each complex structure J on W induces an involution σ_J on $Gr_{\mathbb{R}}(W)$, given by

$$\sigma_J : Gr_{\mathbb{R}}(W) \to Gr_{\mathbb{R}}(W),$$
$$\sigma_J(P) = J(P) = \{J(v) | v \in P\},$$

for any $\mathbb{R}$-linear subspace $P \subset W$. Its fixed point set

$$Gr_{\mathbb{R}}(W)^{\sigma_J} = \{P \in Gr_{\mathbb{R}}(W) \mid J(P) = P\}$$

is simply the union of complex Grassmannians of J-complex linear subspaces in W, i.e.

$$Gr_{\mathbb{R}}(W)^{\sigma_J} \cong \coprod_{k=0}^{N} \frac{U(N)}{U(k)U(N-k)},$$

where $2N = dim_{\mathbb{R}} W$.

Similarly, any symplectic structure ω on W induces a map σ_ω on $Gr_{\mathbb{R}}(W)$,

$$\sigma_\omega : Gr_{\mathbb{R}}(W) \to Gr_{\mathbb{R}}(W),$$
$$\sigma_\omega(P) = P^{\perp_\omega} = \{v \in W | \omega(v, w) = 0, \text{ for any } w \in P\},$$

for any $\mathbb{R}$-linear subspace $P \subset W$. Non-degeneracy of ω implies that

$$dim_{\mathbb{R}} P + dim_{\mathbb{R}} \sigma_\omega(P) = dim_{\mathbb{R}} W.$$

Together with the simple fact $P = (P^{\perp_\omega})^{\perp_\omega}$, they imply that σ_ω is also an involution, i.e. $(\sigma_\omega)^2 = id$. Observe that $\sigma_\omega(P) = P$ if and only if P is a Lagrangian subspace of W. Namely P is a half dimensional subspace such that ω vanishes on P. Hence the fixed point set of σ_ω can be identified as the Lagrangian Grassmannian, i.e.

$$Gr_{\mathbb{R}}(W)^{\sigma_\omega} \cong \frac{U(N)}{O(N)}.$$

Proposition 5 *Given any Hermitian vector space (W, g, J, ω) of real dimension $2n$, the involutions σ_J and σ_ω are isometries on $Gr_{\mathbb{R}}(2k, W)$ ($k = 1, \ldots, n$) and $Gr_{\mathbb{R}}(n, W)$ respectively.*

Proof Take an orthonormal real basis $\{e_1, e_2, \ldots, e_{2n}\}$ of W, such that $Je_{2i-1} = e_{2i}$, $Je_{2i} = -e_{2i-1}$, for $i = 1, \ldots, n$. For any $A \in O(2n)$, define

$$\sigma_J : Gr_{\mathbb{R}}(2k, W) \to Gr_{\mathbb{R}}(2k, W),$$
$$\sigma_J(KA) = KAJ,$$

where K is $O(2k)O(2n-2k)$. σ_J is well-defined because the linear subspaces generated by $\{e_1, e_2, \ldots, e_{2k}\}$ and $\{e_{2k+1}, e_{2k+2}, \ldots, e_{2n}\}$ are complex. $J \in O(2n)$ implies σ_J is an isometry with respect to the invariant metric on $Gr_{\mathbb{R}}(2k, W)$.

For σ_ω, fix an orthonormal basis such that ω, as a non-degenerate two form, is of the form

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

For any $A \in O(2n)$, define

$$\begin{aligned} \sigma_\omega : Gr_{\mathbb{R}}(n, W) &\to Gr_{\mathbb{R}}(n, W), \\ \sigma_\omega(KA) &= K\omega A\omega^{-1}, \end{aligned}$$

where $K = O(n)O(n)$. It is easy to check σ_ω is also well-defined. Using the upper n rows of A (up to a K-action) as the coefficients of basis vectors of the linear subspace corresponding to KA,

$$\omega(A(e_1, \ldots, e_{2n})^t, (e_1, \ldots, e_{2n})(\omega A\omega^{-1})^t) = A\omega(\omega A\omega^{-1})^t = \omega = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

implies that for any linear subspace P corresponding to A, $\sigma_\omega(P) = P^{\perp_\omega}$ is represented by $\omega A\omega^{-1}$, i.e. the upper n rows of $\omega A\omega^{-1}$ (up to a K-action) are the coefficients of basis vectors of the linear subspace $P^{\perp_\omega}$.

Let X, Y be vectors in the tangent space of $Gr_{\mathbb{R}}(n, W)$ at $x = KA$, $g_x(\cdot, \cdot)$ the invariant metric on $Gr_{\mathbb{R}}(n, W)$ at x. Write L_A as the left multiplication map of $Gr_{\mathbb{R}}(n, W)$, so $(L_A^{-1})_* X = A^{-1}X$ and $(L_A^{-1})_* Y = A^{-1}Y$ are tangent vectors of $Gr_{\mathbb{R}}(n, W)$ at $x_0 = K$. The isometry of σ_ω follows from

$$\begin{aligned} & g_{\sigma_\omega(x)}((\sigma_\omega)_{*,x}(X), (\sigma_\omega)_{*,x}(Y)) \\ &= \frac{1}{2} Tr\left(\left(L^{-1}_{\omega A\omega^{-1}}\right)_*(\sigma_\omega)_{*,x}(X) \cdot (L^{-1}_{\omega A\omega^{-1}})_*(\sigma_\omega)_{*,x}(Y)\right) \\ &= \frac{1}{2} Tr\left((\omega A^{-1}\omega^{-1})(\omega X\omega^{-1})(\omega A^{-1}\omega^{-1})(\omega Y\omega^{-1})\right) \\ &= \frac{1}{2} Tr(\omega A^{-1}XA^{-1}Y\omega^{-1}) \\ &= \frac{1}{2} Tr\left(\left(L_A^{-1}\right)_* X \cdot (L_A^{-1})_* Y\right) \\ &= g_x(X, Y). \end{aligned}$$

The following lemma enables us to construct many symmetric spaces from the real Grassmannian using involutive isometries induced by complex and symplectic structures on W.

Lemma 2 *Let M be a symmetric space and $\Sigma = \{\sigma_1, \sigma_2, \ldots, \sigma_s\}$ be a collection of involutive isometries, then the fixed point set*

$$M^\Sigma = \bigcap_{\sigma \in \Sigma} M^\sigma$$

with the induced Riemannian metric is again a symmetric space.

Proof Without loss of generality we may assume that M^{Σ} is connected. First it is easy to see that M^{σ} is a closed submanifold of M for any σ in Σ. If we can show that the second fundamental form B of the Levi-Civita connection D is zero, then M^{σ} is a totally geodesic submanifold, so M^{Σ} (an intersection of totally geodesic submanifolds) is also a totally geodesic submanifold. Every geodesic in M which passes through a point x in M^{Σ} with the tangent in $T_x M^{\Sigma}$ is still a geodesic in M^{Σ}, this implies that M^{Σ} is a locally symmetric space with the induced metric. For any geodesic isometry σ_x on M, where $x \in M^{\Sigma}$, $\sigma_x(M^{\Sigma}) \subset M^{\Sigma}$ (because σ_x is just reverse of the geodesic) and M^{Σ} is closed in M, so M^{Σ} is a global symmetric space.

Now we prove the second fundamental form B is zero. For any point $x \in M^{\sigma}$, we have the decomposition of vector spaces $T_x M = T_x M^{\sigma} \oplus N_x$, where N_x is the normal vector space of M^{σ} at x. σ is an isometry, so

$$D_X(Y) = \sigma_*(D_X(Y)) = D_{\sigma_* X}(\sigma_* Y),$$

where X, Y are vectors of M at x. Denote D', D'' as the connections induced by D on TM^{σ} and the normal bundle N of M^{σ} in M, and e, f as the basis of TM^{σ} and N respectively, we have

$$\begin{pmatrix} D'_e(e) & B_e(f) \\ -B^t_f(e) & D''_f(f) \end{pmatrix} = \begin{pmatrix} D'_e(e) & -B_e(f) \\ -(-B^t_f(e)) & D''_f(f) \end{pmatrix}.$$

This obviously implies $B = 0$, i.e. the second fundamental form is zero.

Back to $W = (\mathbb{A} \otimes \mathbb{B})^n$, we denote the set of its canonical complex structures induced by $\mathbb{A}$ as $\Sigma^L = \{J_1^L, \dots, J_{l-1}^L, J_l^L\}$. We also define $\hat{\Sigma}^L = \{J_1^L, \dots, J_{l-1}^L, \omega_l^L\}$ and we have Σ^R and $\hat{\Sigma}^R$ defined in a similar fashion using complex and symplectic structures induced from $\mathbb{B}$. They define involutive isometries σ's on $Gr_{\mathbb{R}}(W)$ whose fixed points correspond to complex subspaces or Lagrangian subspaces in W.

Definition 3 Let $W = (\mathbb{A} \otimes \mathbb{B})^n$, where $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$.

1. Grassmannians $Gr_{\mathbb{A}\mathbb{B}}(k, n)$: The connected component containing $(\mathbb{A} \otimes \mathbb{B})^k$ of

$$\bigcap_{I \in \Sigma^L \cup \Sigma^R} Gr_{\mathbb{R}}(k_1, W)^{\sigma_I}, \quad \text{where } k_1 = dim_{\mathbb{R}} (\mathbb{A} \otimes \mathbb{B})^k .$$

2. Lagrangian Grassmannians $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$: The connected component containing $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^n$ of (with the Abelian part taken away)

$$\bigcap_{I \in \hat{\Sigma}^L \cup \Sigma^R} Gr_{\mathbb{R}}(k_2, W)^{\sigma_I}, \quad \text{where } k_2 = dim_{\mathbb{R}} \left(\frac{\mathbb{A}}{2} \otimes \mathbb{B}\right)^n .$$

3. Double Lagrangian Grassmannians $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$: The connected component containing $(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})^n \oplus (j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2})^n$ of (with the Abelian part taken away)

$$\bigcap_{I \in \hat{\Sigma}^L \cup \hat{\Sigma}^R} Gr_{\mathbb{R}}(k_3, W)^{\sigma_I}, \quad \text{where } k_3 = dim_{\mathbb{R}} \left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1 \frac{\mathbb{A}}{2} \otimes j_2 \frac{\mathbb{B}}{2}\right)^n.$$

By Lemma 2, these Grassmannians, Lagrangian Grassmannians and double Lagrangian Grassmannians are symmetric spaces.

Remark 7 From now on we assume $\mathbb{A}$ is not $\mathbb{R}$ when we consider Lagrangian Grassmannians, similarly both $\mathbb{A}$ and $\mathbb{B}$ are not $\mathbb{R}$ when we consider double Lagrangian Grassmannians.

3.1 Grassmannians, Lagrangian Grassmannians and double Lagrangian Grassmannians

When $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $Gr_{\mathbb{A}\mathbb{B}}(k, n)$ can be viewed as the connected component containing $(\mathbb{A} \otimes \mathbb{B})^k$ of the space of all $\mathbb{A} \otimes \mathbb{B}$-linear subspaces P in $W = (\mathbb{A} \otimes \mathbb{B})^n$ with $dim_{\mathbb{A}\otimes\mathbb{B}} P = k$. Similarly $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ can be viewed as the connected component containing $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^n$ of the space of all $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$-linear subspaces P in $W = (\mathbb{A} \otimes \mathbb{B})^n$ with $dim_{\frac{\mathbb{A}}{2}\otimes\mathbb{B}} P = n$ invariant under $\sigma_{\omega_m^L}$, and $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$ can be viewed as the connected component containing $(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})^n \oplus (j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2})^n$ of the space of all $\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}$-linear subspaces P in $W = (\mathbb{A} \otimes \mathbb{B})^n$ invariant under $\sigma_{\omega_m^L}$ and $\sigma_{\omega_l^R}$ with $dim_{\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}} P = 2n$.

Proposition 6 *When $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $Gr_{\mathbb{A}\mathbb{B}}(k, n)$, $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ and $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$ are given by the spaces in Tables 2, 3 and 4 in Sect. 1.*

Proof First we prove the Grassmannian case $Gr_{\mathbb{A}\mathbb{B}}(k, n)$. Denote the invariant subspace $(\mathbb{A} \otimes \mathbb{B})^k$ by P_0. It is easy to see that for any $A \in G_n(m, l)$, $A \cdot P_0$ is also an invariant linear subspace, since A preserves all these complex structures. The isotropy subgroup of P_0 is obviously $G_k(m, l)G_{n-k}(m, l)$. The $G_n(m, l)$ orbit of P_0 is connected. Note $O(n)/O(k)O(n-k)$ is connected though $O(n)$ is not. So the $G_n(m, l)$ orbit of P_0 is contained in $Gr_{\mathbb{A}\mathbb{B}}(k, n)$. Therefore, it is enough to show that both have the same dimension.

Let

$$\begin{pmatrix} A_1, & A_2 \end{pmatrix}$$

be the coordinates of a point P in $Gr_{\mathbb{A}\mathbb{B}}(k, n)$ (as the coefficients of basis vectors of P), where A_1 and A_2 are $k \times k$ and $k \times (n-k)$ matrices with entries in $\mathbb{A} \otimes \mathbb{B}$ respectively. Without loss of generality, we may assume A_1 is an invertible matrix (i.e. P is a point near to P_0). Use a coordinate transformation, the coordinates of P can be represented by

$$\left(I, \quad A_1^{-1}A_2\right),$$

where I is the identity matrix of rank k. This implies the dimension of $Gr_{\mathbb{A}\mathbb{B}}(k, n)$ is the same as the orbit.

The Lagrangian Grassmannian case $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ is similar to Grassmannian case. Let

$$\left(I, B\right)$$

be the coordinates of a point P in $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ (as the coefficients of basis vectors of P), where B is a rank n matrix with entries in $\frac{\mathbb{A}}{2} \otimes \mathbb{B}$. We have

$$(I, B)\, \omega_m^L\, (I, \tilde{B})^t = 0,$$

this implies

$$\tilde{B}^t = B.$$

So the dimension of $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ is the same as the $G_n(m, l)$ orbit of $(\frac{\mathbb{A}}{2} \otimes \mathbb{B})^n$ (see Remark 5).

For the double Lagrangian Grassmannian case $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$, we first consider the Lie algebra $\mathfrak{k}$ of the isotropy subgroup of $P_0 = (\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})^n \oplus (j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2})^n$ in $Gr_{\mathbb{R}}(k_3, W)$. It is easy to see that $\mathfrak{k}$ is

$$A_n\left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right) \oplus A_n\left(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right).$$

1. When $m = l = 1$, $\mathfrak{k}$ is $\mathfrak{so}(n)^2$ obviously, since in this case $j_1 = j_2 = i$, hence $A_n(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2})$ is isomorphic to $A_n(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})$.
2. When $m = 1, l = 2$. Any element of $A_n(\mathbb{C} \otimes \mathbb{R}) \oplus A_n(j\mathbb{C} \otimes i\mathbb{R})$ is represented as

   $$A \cdot 1 \otimes 1 + B \cdot i \otimes 1 + C \cdot j \otimes i + D \cdot k \otimes i,$$

   where $A, B, C, D \in M_n(\mathbb{R})$. This can be identified as

   $$\begin{pmatrix} A - D & C - B \\ C + B & A + D \end{pmatrix},$$

   it is an element in $\mathfrak{so}(2n)$, so $A_n(\mathbb{C} \otimes \mathbb{R}) \oplus A_n(j\mathbb{C} \otimes i\mathbb{R})$ is isomorphic to $\mathfrak{so}(2n)$.
3. When $m = 2, l = 1$, this is the same case as $m = 1, l = 2$.
4. When $m = l = 2$, $\mathfrak{k}$ is $A_n(\mathbb{C} \otimes \mathbb{C}) \oplus A_n(j\mathbb{C} \otimes j\mathbb{C})$. $\mathbb{C} \otimes \mathbb{C}$ can be identified as

   $$\mathbb{C} \oplus \mathbb{C},$$

this implies that $\mathfrak{k}$ is isomorphic to

$$(A_n(\mathbb{C}) \oplus A_n(j \otimes j \cdot \mathbb{C})) \oplus (A_n(\mathbb{C}) \oplus A_n(j \otimes j \cdot \mathbb{C})).$$

Then any element of $(A_n(\mathbb{C}) \oplus A_n(j \otimes j \cdot \mathbb{C}))$ can be represented as

$$A + B \cdot i + C \cdot j \otimes j + D \cdot k \otimes j,$$

where $A, B, C, D \in M_n(\mathbb{R})$, then identify this to

$$\begin{pmatrix} A - D & C - B \\ C + B & A + D \end{pmatrix},$$

so $\mathfrak{k}$ is isomorphic to $\mathfrak{so}(2n)^2$.

In order to compare the dimension of $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$ with the $G_n(m,l)$ orbit of P_0, let

$$\begin{pmatrix} I & 0 & B_1 & B_2 \\ 0 & I & B_3 & B_4 \end{pmatrix}$$

be the coordinates of a point P (near to P_0, as the coefficients of basis vectors of P) corresponding to the following decomposition:

$$\left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right)^n \oplus \left(j_1\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right)^n \oplus \left(\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right)^n,$$

where $B_1, B_2, B_3, B_4 \in M_n(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})$. Since P is an invariant subspace, we have the following conditions.

$$\begin{pmatrix} I & B \end{pmatrix} \omega_m^L \begin{pmatrix} I & \tilde{B} \end{pmatrix}^t = 0,$$
$$\begin{pmatrix} I & B \end{pmatrix} \omega_l^R \begin{pmatrix} I & \bar{\tilde{B}} \end{pmatrix}^t = 0,$$

where $B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \in M_{2n}\left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right)$.

Under the above coordinates, ω_l^R, ω_m^L is represented by the following matrices:

$$\omega_l^R = \begin{pmatrix} 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \\ 0 & I & 0 & 0 \\ -I & 0 & 0 & 0 \end{pmatrix}, \quad \omega_m^L = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \\ -I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{pmatrix}.$$

Denote

$$P_1 = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},$$

so we have

$$B \cdot P_1 + P_1 \cdot \tilde{B}^t = 0, \quad B \cdot P_2 - \ P_2 \cdot \ \bar{\tilde{B}}^t = 0.$$

By simple calculations we have

$$B = \begin{pmatrix} B_1 & B_2 \\ -\bar{B}_2 & -\bar{B}_1 \end{pmatrix},$$

where $\bar{\tilde{B}}_1^t = B_1$, $\tilde{B}_2^t = B_2$.

When $m = l = 1$, the dimension is $n^2 + n$.

When $m = 1, l = 2$ ($m = 1, l = 2$ is similar), we have

$$B_1^t = B_1, \quad \bar{B}_2^t = B_2,$$

where $B_1, B_2 \in M_n(\mathbb{R} \otimes \mathbb{C}) = M_n(\mathbb{C})$. So the dimension is $n^2 + n + n^2 = 2n^2 + n$.

For $m = l = 2$, it is easy to see the dimension is $4n^2$.

Corollary 1 *When* $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, *we have*

1. *Grassmannian case:*

$$Gr_{\mathbb{A}\mathbb{B}}(k, n) = G/K,$$

where

$$Lie(G) = \mathfrak{g}_n(m, l), \quad \textit{and}$$
$$Lie(K) = \mathfrak{g}_k(m, l) \oplus \mathfrak{g}_{n-k}(m, l).$$

2. *Lagrangian Grassmannian case:*

$$LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n) = G/K,$$

where

$$Lie(G) = L_n(\mathbb{A}, \mathbb{B}), \quad \textit{and}$$
$$Lie(K) = \mathfrak{g}_n(m - 1, l).$$

3. *Double Lagrangian Grassmannian case:*

$$LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n) = G/K,$$

where

$$Lie(G) = L_n(\mathbb{A}, \mathbb{B}), \ \textit{and}$$
$$Lie(K) = A_n\left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2}\right) \oplus A_n\left(j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2}\right).$$

When $\mathbb{A}$ or/and $\mathbb{B}$ equals $\mathbb{O}$, $Gr_{\mathbb{AB}}(k,n)$ needs to be defined with more care because of the non-existence of $\mathbb{O}^n$. Nevertheless, with the help of Jordan algebras, we can make sense of the Lie algebra $L_n(\mathbb{O})$ of infinitesimal symmetries of it when $n=2$ or 3. Namely they are $Spin(9)$ and F_4 respectively. Similarly, using the magic square, we can define the corresponding Lie algebra $L_n(\mathbb{A},\mathbb{B})$ for any normed division algebras $\mathbb{A}$ and $\mathbb{B}$, provided that $n=2,3$. It is therefore natural to extend the definition of $Gr_{\mathbb{AB}}(k,n)$, $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ and $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$ to include the octonions as below:

Theorem 2 *(Definition-Theorem)*[4]*: Suppose $\mathbb{A}$ and $\mathbb{B}$ are any normed division algebras.*

1. *We define $Gr_{\mathbb{AB}}(k,n)$ to be the symmetric space G/K with*

$$\begin{aligned} Lie(G) &= L_n(\mathbb{A},\mathbb{B}) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A_n'(\mathbb{A}\otimes\mathbb{B}), \text{ and} \\ Lie(K) &= Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A_k'(\mathbb{A}\otimes\mathbb{B}) \oplus A_{n-k}'(\mathbb{A}\otimes\mathbb{B}) \oplus Im(\mathbb{A}) \oplus Im(\mathbb{B}), \end{aligned}$$

such that if $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, then $Gr_{\mathbb{AB}}(k,n)$ is the connected component of

$$\bigcap_{I\in\Sigma^L\cup\Sigma^R} Gr_{\mathbb{R}}\left(\overline{k},(\mathbb{A}\otimes\mathbb{B})^n\right)^{\sigma_I}$$

containing $(\mathbb{A}\otimes\mathbb{B})^k$. Here $\overline{k}=dim_{\mathbb{R}}(\mathbb{A}\otimes\mathbb{B})^k=k\cdot 2^{m+l}$.

2. *We define $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ to be the symmetric space G/K with*

$$\begin{aligned} Lie(G) &= L_n(\mathbb{A},\mathbb{B}) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A_n'(\mathbb{A}\otimes\mathbb{B}), \text{and} \\ Lie\,(K) &= L_n\left(\frac{\mathbb{A}}{2},\mathbb{B}\right) \oplus Im\left(\frac{\mathbb{A}}{2}\right), \end{aligned}$$

such that if $\mathbb{A},\mathbb{B}\in\{\mathbb{R},\mathbb{C},\mathbb{H}\}$, then $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ is the connected component of

$$\bigcap_{I\in\hat{\Sigma}^L\cup\Sigma^R} Gr_{\mathbb{R}}\left(\overline{k},(\mathbb{A}\otimes\mathbb{B})^n\right)^{\sigma_I}$$

containing $(\frac{\mathbb{A}}{2}\otimes\mathbb{B})^n$ (with the Abelian part taken away), where $\overline{k}=\frac{dim_{\mathbb{R}}(\mathbb{A}\otimes\mathbb{B})^n}{2}=n\cdot 2^{m+l-1}$.

3. *We define $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$ to be the symmetric space G/K with*

$$\begin{aligned} Lie(G) &= L_n(\mathbb{A},\mathbb{B}) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A_n'(\mathbb{A}\otimes\mathbb{B}), \text{and} \\ Lie(K) &= Der\left(\frac{\mathbb{A}}{2}\right) \oplus Der\left(\frac{\mathbb{B}}{2}\right) \oplus A_n\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right) \oplus A_n\left(j_1\frac{\mathbb{A}}{2}\otimes j_2\frac{\mathbb{B}}{2}\right), \end{aligned}$$

[4] Recall n is always assumed to be 3 when either $\mathbb{A}$ or $\mathbb{B}$ is $\mathbb{O}$.

such that if $\mathbb{A}, \mathbb{B} \in \{\mathbb{C}, \mathbb{H}\}$*, then* $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$ *is the connected component containing* $(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2})^n \oplus (j_1\frac{\mathbb{A}}{2} \otimes j_2\frac{\mathbb{B}}{2})^n$ *of (with the Abelian part taken away)*

$$\bigcap_{I \in \hat{\Sigma}^L \cup \hat{\Sigma}^R} Gr_{\mathbb{R}}\left(\overline{k}, (\mathbb{A} \otimes \mathbb{B})^n\right)^{\sigma_I},$$

where $\overline{k} = \frac{dim_{\mathbb{R}}(\mathbb{A}\otimes\mathbb{B})^n}{2} = n \cdot 2^{m+l-1}$.

Remark 8 Here we use Vinberg's version of the magic square [17].

Proof This follows immediately from Proposition 6 and its Corollary 1.

Remark 9 The exceptional symmetric spaces of Grassmannian type are described as projective planes over $\mathbb{A} \otimes \mathbb{B}$ (see e.g. [3]). In fact we can construct the projective planes $\mathbb{O}\mathbb{P}^2$ and $(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$ by using Jordan algebras over $\mathbb{R}$ and $\mathbb{C}$ respectively, but there is no direct geometric construction for $(\mathbb{H}\otimes\mathbb{O})\mathbb{P}^2$ and $(\mathbb{O}\otimes\mathbb{O})\mathbb{P}^2$ because $\mathbb{H}$, $\mathbb{O}$ are not fields, thus there is no Jordan algebra defined over them.

Theorem 3 *For any normed division algebras* $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$*,* $Gr_{\mathbb{A}\mathbb{B}}(k, n)$*,* $LGr_{\frac{\mathbb{A}}{2}\mathbb{B}}(n)$ *and* $LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(n)$ *are given by the spaces in Tables 2, 3 and 4 in Sect. 1, respectively.*

Remark 10 When $n = 3$, the double Lagrangian Grassmannians can also be viewed as

$$LLGr_{\frac{\mathbb{A}}{2}\frac{\mathbb{B}}{2}}(3) = \left\{\wedge^2\left(\frac{\mathbb{A}}{2}\otimes\frac{\mathbb{B}}{2}\right)^4 \subset (\mathbb{A}\otimes\mathbb{B})^3\right\}.$$

3.2 Compact simple Lie groups

Let $W = (\mathbb{A} \otimes \mathbb{B})^n$ with complex structures J_i^L, J_j^R as before, $T^*W = W \oplus W^*$ is a vector space with the induced complex structures J_i^L, J_j^R and a symplectic structure $\omega = dx \wedge dy$ (here $\{x, y\}$ denote the coordinates of W and the cotangent space respectively). But here we consider the indefinite symmetric tensor

$$g' = dx \otimes dy + dy \otimes dx$$

rather than the skew-symmetric structure ω.

As in the case of ω, g' induces an involution on $Gr_{\mathbb{R}}(T^*W)$,

$$\sigma_{g'} : Gr_{\mathbb{R}}(T^*W) \to Gr_{\mathbb{R}}(T^*W),$$
$$\sigma_{g'}(P) = P^{\perp_{g'}} = \left\{v \in T^*W | g'(v, w) = 0, \text{ for any } w \in P\right\},$$

for any real linear subspace $P \subset T^*W$.

Let N be the dimension of W over $\mathbb{R}$, for any $A \in O(2N)$, $\sigma_{g'}$ is defined as

$$\sigma_{g'}(KA) = Kg'Ag'^{-1},$$

where K is $O(N)O(N)$ and g' is the matrix $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$, where I is the identity matrix in $O(N)$. So σ_ω is also well-defined and is an isometry on $Gr_{\mathbb{R}}(N, 2N)$.

Denote the connected component containing W (with the Abelian part taken away) of the space of all linear subspaces of T^*W which are invariant under the involutive isometries $\sigma_{J_i^L}, \sigma_{J_j^R}$ and $\sigma_{g'}$ as $G_{\mathbb{A}\mathbb{B}}(n)$, where $i = 1, \ldots, m$, $j = 1, \ldots, l$.

Proposition 7 *When* $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, $G_{\mathbb{A}\mathbb{B}}(n)$ *is the space in Table* 1 *in Sect.* 1.

Remark 11 These symmetric spaces are compact (semi)simple Lie groups.

Proof Take the coordinate transformation

$$u = \frac{1}{\sqrt{2}}(x + y)$$
$$v = \frac{1}{\sqrt{2}}(x - y),$$

then the symmetric tensor is

$$g' = du^2 - dv^2,$$

and the standard metric is

$$g = dx^2 + dy^2 = du^2 + dv^2.$$

The group of all $\mathbb{A} \otimes \mathbb{B}$-linear transformations of T^*W which preserve g' and the standard metric g is isomorphic to $G_n(m, l)^2$, the isotropy subgroup of W is obviously the group of all $\mathbb{A} \otimes \mathbb{B}$-linear transformations of W, that is $G_n(m, l)$. So the $G_n(m, l)^2$ orbit of W (with the flat part removed) is the space in Table 1 in Sect. 1.

Now we prove that the orbit is the connected component which contains W. Obviously the orbit is contained in the connected component, we only need to compare their dimensions. W is characterized by $\begin{pmatrix} I & I \end{pmatrix}_{n \times 2n}$ under the coordinates $\{u, v\}$, so we consider any nearby point $\begin{pmatrix} I & B \end{pmatrix}_{n \times 2n}$, we have

$$\begin{pmatrix} I & B \end{pmatrix} g' \begin{pmatrix} I \\ \overline{B}^t \end{pmatrix} = 0,$$

where $B \in M_n(\mathbb{A} \otimes \mathbb{B})$. By simple calculation we have

$$B\overline{B}^t = I.$$

This implies that the dimension of the $G_n(m,l)^2$ orbit is the same as the dimension of the connected component $G_{\mathbb{A}\mathbb{B}}(n)$.

Corollary 2 *When* $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$*, we have*

$$G_{\mathbb{A}\mathbb{B}}(n) = G/K,$$

where

$$\begin{aligned} Lie(G) &= (\mathfrak{g}_n(m,l))^2, \text{ and} \\ Lie(K) &= \mathfrak{g}_n(m,l). \end{aligned}$$

When $\mathbb{A}$ or/and $\mathbb{B}$ equals $\mathbb{O}$, the definition for $G_{\mathbb{A}\mathbb{B}}(n)$ is naturally extended as below.

Theorem 4 *(Definition-Theorem): Suppose* $\mathbb{A}$ *and* $\mathbb{B}$ *are any normed division algebras.*

We define $G_{\mathbb{A}\mathbb{B}}(n)$ *to be the symmetric space* G/K *with*

$$\begin{aligned} Lie(G) &= L_n(\mathbb{A},\mathbb{B})^2 = (Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A_n'(\mathbb{A}\otimes\mathbb{B}))^2 \\ Lie(K) &= L_n(\mathbb{A},\mathbb{B}), \end{aligned}$$

such that if $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$*, then* $G_{\mathbb{A}\mathbb{B}}(n)$ *is the connected component containing* $(\mathbb{A}\otimes\mathbb{B})^n$ *of (with the Abelian part taken away)*

$$\bigcap_{I \in \Sigma^L \cup \Sigma^R \cup \{\sigma_{g'}\}} Gr_{\mathbb{R}}\left(\overline{k}, T^*(\mathbb{A}\otimes\mathbb{B})^n\right)^{\sigma_I},$$

where $\overline{k} = \frac{dim_{\mathbb{R}} T^*(\mathbb{A}\otimes\mathbb{B})^n}{2} = n \cdot 2^{m+l}$.

If $\mathbb{A}$ *or* $\mathbb{B}$ *equals* $\mathbb{O}$*, then we assume that* n *equals 3.*

Proof This follows immediately from Proposition 7 and its Corollary 2.

Theorem 5 *For any normed division algebras* $\mathbb{A}, \mathbb{B} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$*,* $G_{\mathbb{A}\mathbb{B}}(n)$*'s are given by the spaces in Table 1 in Sect. 1.*

Appendix

The magic square of 3×3 matrices

The following is the Tits original constructions of the magic square.

Let $\mathbb{A}$ and $\mathbb{B}$ be real composition division algebras. Then we define

$$L_3(\mathbb{A},\mathbb{B}) = Der H_3(\mathbb{A}) \oplus H_3'(\mathbb{A}) \otimes Im\mathbb{B} \oplus Der(\mathbb{B}).$$

This is a Lie algebra with Lie subalgebras $Der H_3(\mathbb{A})$ and $Der(\mathbb{B})$ when taken with the brackets

$$\begin{aligned}
([D, A \otimes x]) &= D(A) \otimes x \\
([E, A \otimes x]) &= A \otimes E(x) \\
([D, E]) &= 0 \\
([A \otimes x, B \otimes y]) &= \frac{1}{6}\langle A, B\rangle D_{x,y} + (A * B) \otimes \frac{1}{2}[x, y] - \langle x, y\rangle [L_A, L_B]
\end{aligned}$$

with $D \in Der H_3(\mathbb{A})$; $A, B \in H_3'(\mathbb{A})$; $x, y \in Im\mathbb{B}$ and $E \in Der(\mathbb{B})$. We explain these operations as follows. $\langle A, B\rangle$ and $\langle x, y\rangle$ denote the symmetric bilinear forms on $H_3(\mathbb{A})$ and $\mathbb{B}$ respectively, given by

$$\begin{aligned}
\langle A, B\rangle &= Re(tr(A \cdot B)) = 2Re(tr(AB)) \\
\langle x, y\rangle &= \frac{1}{2}(|x + y|^2 - |x|^2 - |y|^2) = Re(x\overline{y})
\end{aligned}$$

The derivation $D_{x,y}$ is defined as

$$D_{x,y} = [L_x, L_y] + [L_x, R_y] + [R_x, R_y] \in Der(\mathbb{B}).$$

Finally $(A * B)$ is the traceless part of the Jordan product of A and B,

$$A * B = A \cdot B - \frac{1}{3} tr(A \cdot B).$$

Tits [16] showed that this gives a unified construction leading to the so-called magic square of Lie algebras of 3×3 matrices as follows.

$\mathbb{A} \backslash \mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(3)$	$\mathfrak{su}(3)$	$\mathfrak{sp}(3)$	$\mathfrak{f}_4$
$\mathbb{C}$	$\mathfrak{su}(3)$	$\mathfrak{su}(3)^2$	$\mathfrak{su}(6)$	$\mathfrak{e}_6$
$\mathbb{H}$	$\mathfrak{sp}(3)$	$\mathfrak{su}(6)$	$\mathfrak{so}(12)$	$\mathfrak{e}_7$
$\mathbb{O}$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$

Remark 12 There are several versions of the magic square:

1. The Vinberg version [17]

$$L_3(\mathbb{A}, \mathbb{B}) = Der(\mathbb{A}) \oplus Der(\mathbb{B}) \oplus A_3'(\mathbb{A} \otimes \mathbb{B}),$$

where $A_n'(\mathbb{A} \otimes \mathbb{B})$ is the trace zero skew Hermitian matrix of rank n with entries in $\mathbb{A} \otimes \mathbb{B}$;

2. The Barton–Sudbery version [4,5]

$$L_3(\mathbb{A},\mathbb{B}) = \mathfrak{tri}(\mathbb{A}) \oplus \mathfrak{tri}(\mathbb{B}) \oplus (\mathbb{A}\otimes\mathbb{B})^3,$$

where $\mathfrak{tri}(\mathbb{A})$ is the triality algebra of $\mathbb{A}$.

The magic square of 2×2 matrices

The Tits construction can also be adapted for 2×2 matrix algebras. In this case the underlying vector space is

$$L_2(\mathbb{A},\mathbb{B}) = Der\, H_2(\mathbb{A}) \oplus H_2'(\mathbb{A}) \otimes Im\mathbb{B} \oplus \mathfrak{so}(Im\mathbb{B})$$

which admits a Lie algebra structure with the Lie brackets given as follows,

$$\begin{aligned}
([D, A\otimes x]) &= D(A)\otimes x\\
([E, A\otimes x]) &= A\otimes E(x)\\
([D, E]) &= 0\\
([A\otimes x, B\otimes y]) &= \frac{1}{4}\langle A, B\rangle D_{x,y} - \langle x, y\rangle [R_A, R_B]
\end{aligned}$$

where the symbols used in this set of brackets are defined in the same way as the ones used in the 3×3 case. This gives the compact magic square for 2×2 matrix algebras

$$L_2(\mathbb{A},\mathbb{B}) = \mathfrak{so}(\nu_1+\nu_2),$$

where ν_1, ν_2 are the dimension of $\mathbb{A}, \mathbb{B}$ over $\mathbb{R}$. The magic square is as follows

$\mathbb{A}\backslash\mathbb{B}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathfrak{so}(2)$	$\mathfrak{so}(3)$	$\mathfrak{so}(5)$	$\mathfrak{so}(9)$
$\mathbb{C}$	$\mathfrak{so}(3)$	$\mathfrak{so}(4)$	$\mathfrak{so}(6)$	$\mathfrak{so}(10)$
$\mathbb{H}$	$\mathfrak{so}(5)$	$\mathfrak{so}(6)$	$\mathfrak{so}(8)$	$\mathfrak{so}(12)$
$\mathbb{O}$	$\mathfrak{so}(9)$	$\mathfrak{so}(10)$	$\mathfrak{so}(12)$	$\mathfrak{so}(16)$

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