

Conformal Geometry and Special Holonomy

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1 Introduction

Riemannian holonomy group $hol(M, g)$ measures the richness of algebraic structure on a Riemannian manifold¹. For a generic metric, the holonomy group equals $SO(m)$ with $m = \dim_{\mathbb{R}} M$. Manifolds with special holonomy include Kähler manifolds with $hol(M, g) = U(n)$ and Calabi-Yau manifolds with $hol(M, g) = SU(n)$ where $m = 2n$. They play very important roles in geometry and mathematical physics such as string theory and M-theory. Riemannian holonomy groups were completely classified by Berger [Ber53] and all these geometries have been given a unified description in terms of real, complex, quaternionic and octonionic structures (that is, normed division algebras) and orientability in [HL] for symmetric spaces and [Leu02] for non-symmetric ones.

Another important branch in Riemannian geometry is the conformal geometry where one allows the Riemannian metric to be scaled by a conformal factor, i.e. $g \sim e^u g$ for any function u . In this article, we explain how one integrates conformal geometry with real, complex, quaternionic and octonionic geometries. In particular we give a uniform proof to the following theorem on rigidity of calibrated cycles in projective spaces, which is a generalization of the results of Lawson and Simons from conformal and complex geometries to quaternionic and octonionic geometries. After we have discovered this, we were informed that this result has been proved earlier by [Ohn86]. We hope that our approach from Jordan algebra provides a unified viewpoint on all these seemingly different kinds of geometries.

Main Theorem: In $\mathbb{A}\mathbb{P}^n$, where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{R}^m\}$, any stable minimal submanifold S (or more generally rectifiable current) must be complex, by which we means $T_x S$ is invariant under all the linear complex structures at x for almost every $x \in S$.

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¹All manifolds are connected compact oriented smooth manifolds.

Remark 1. *There is an S^2 -family of linear complex structures at every point of $\mathbb{H}\mathbb{P}^n$, and also an S^6 -family of linear complex structures at each point of $\mathbb{O}\mathbb{P}^2$.*

Remark 2. *When $\mathbb{A} = \mathbb{O}$, we only allow $n \leq 2$; When $\mathbb{A} = \mathbb{R}^m$, only $n=1$ is admitted, and $\mathbb{R}^m\mathbb{P}^1 = \mathbb{S}^m$. We will explain this notation in the next section.*

2 $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ and conformal geometry

In [Leu02] the second author gave a unified description of geometries of each holonomy group by first defining the group $G_{\mathbb{A}}(n)$ of twisted automorphisms of \mathbb{A}^n and its subgroup $H_{\mathbb{A}}(n)$ of special twisted automorphisms, where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$ is a normed division algebra and n equals one when $\mathbb{A} = \mathbb{O}$. They are given explicitly in the following table:

\mathbb{A}	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}
$G_{\mathbb{A}}(n)$	$O(n)$	$U(n)$	$Sp(n)Sp(1)$	$Spin(7)$
$H_{\mathbb{A}}(n)$	$SO(n)$	$SU(n)$	$Sp(n)$	G_2

Their corresponding geometries are as follows.

\mathbb{A}	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}
$G_{\mathbb{A}}(n)$	Riemannian	Kähler	Quaternionic-Kähler	$Spin(7)$
$H_{\mathbb{A}}(n)$	Volume	Calabi-Yau	Hyperkähler	G_2

Due to the nonassociativity of the octonion, there are obvious difficulties to define its modules \mathbb{O}^n and their automorphism groups $H_{\mathbb{O}}(n)$. Nonetheless, for $n \leq 3$, this problem can be resolved by considering the space of self-adjoint operators, leading to the notion of Jordan algebra which we shall describe below.

On \mathbb{R}^n , the space of self-adjoint operators is simply the space of symmetric $n \times n$ matrices, denoted by $S_n(\mathbb{R})$. The symmetrization of ordinary matrix multiplication

$$A \circ B = (AB + BA) / 2$$

makes $S_n(\mathbb{R})$ into a formally real Jordan algebra. Namely it is an algebra over \mathbb{R} whose multiplication \circ is commutative and power associative (that is, $(a \circ a) \circ a = a \circ (a \circ a)$), together with

$$a_1 \circ a_1 + \dots + a_n \circ a_n = 0 \Rightarrow a_1 = \dots = a_n = 0.$$

The same product also makes the space $S_n(\mathbb{A})$ of Hermitian symmetric matrices with entries in $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ into a Jordan algebra. When $n = 3$, an analog of the product can still be defined for $\mathbb{A} = \mathbb{O}$, making $S_3(\mathbb{O})$ into an *exceptional Jordan algebra* (see e.g. [Bae02]) even though \mathbb{O} lacks of associativity.

Inside $S_n(\mathbb{A})$ we may collect all rank one projections, which are matrices p with $p \circ p = p$ and $\text{tr } p = 1$, to form the projective space $\mathbb{A}\mathbb{P}^{n-1}$. For instance, while the module \mathbb{O}^3 does not exist, the concept of octonion lines in \mathbb{O}^3 can be replaced by rank one projection operators in $S_3(\mathbb{O})$, and the space of them forms

the octonion projective plane $\mathbb{O}\mathbb{P}^2$, which can be identified as the symmetric space $F_4/Spin(9)$.

Since $S_n(\mathbb{A})$ and $\mathbb{A}\mathbb{P}^{n-1}$ are spaces of self-adjoint operators on \mathbb{A}^n , they should share the same automorphism group $H_{\mathbb{A}}(n)$ as \mathbb{A}^n . This is indeed true in the classical cases when $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ and continues to have such an interpretation in the exceptional case $\mathbb{A} = \mathbb{O}$. The following gives a complete list of simple formally real Jordan algebras [JvNW34] and their automorphism groups (The center has removed for simplicity):

\mathbb{A}	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}	\mathbb{R}^m
$S_n(\mathbb{A})$	$S_n(\mathbb{R})$	$S_n(\mathbb{C})$	$S_n(\mathbb{H})$	$S_3(\mathbb{O})$	$S_2(\mathbb{R}^m) \simeq \mathbb{R}^m \oplus \mathbb{R}^{1,1}$
$\mathbb{A}\mathbb{P}^{n-1}$	$\mathbb{R}\mathbb{P}^{n-1}$	$\mathbb{C}\mathbb{P}^{n-1}$	$\mathbb{H}\mathbb{P}^{n-1}$	$\mathbb{O}\mathbb{P}^2$	$\mathbb{A}\mathbb{P}^1 = \mathbb{S}^m$
$H_{\mathbb{A}}(n)$	$SO(n)$	$SU(n)$	$Sp(n)$	F_4	$SO(m+1)$

Amazingly there is one more item in the list of Jordan algebras besides those coming from normed division algebras, namely the spin factor $S_2(\mathbb{R}^m) \simeq \mathbb{R}^m \oplus \mathbb{R}^{1,1}$. It consists of 2×2 matrices of the form

$$\begin{pmatrix} a-b & v \\ v & a+b \end{pmatrix} \leftrightarrow \begin{pmatrix} v \\ b \\ a \end{pmatrix}$$

where $v \in \mathbb{R}^m$ and $a, b \in \mathbb{R}$, and we set $v \cdot w = v^t w$ for $v, w \in \mathbb{R}^m$ to carry out matrix multiplication. The embedded projective space is

$$\left\{ \begin{pmatrix} v \\ b \\ \frac{1}{2} \end{pmatrix} : \|v\|^2 + b^2 = \frac{1}{4} \right\} \cong \mathbb{S}^m.$$

Notice that the automorphism group $SO(m+1)$ of $S_2(\mathbb{R}^m)$ is also the isometry group of \mathbb{S}^m , and it is contained as a maximal compact subgroup in the non-compact group $\text{Conf}(\mathbb{S}^m) = SO(m+1, 1)$. A natural question arises: For $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$, is there a symmetry group of $\mathbb{A}\mathbb{P}^{n-1}$ which gives an analog to the conformal symmetry $SO(m+1, 1)$ of \mathbb{S}^m ?

To answer this question, one identifies \mathbb{S}^m as the conformal boundary of the hyperbolic ball

$$B^{m+1} := \{M \in S_2(\mathbb{R}^m) : \det M = 1\} \cong SO(m+1, 1)/SO(m+1)$$

on which $SO(m+1, 1)$ acts as isometries. Under this identification, one has $\text{Conf}(\mathbb{S}^m) \cong \text{Isom}(B^{m+1}) = SO(m+1, 1)$ which preserves collinearity in the sense that $\text{Conf}(\mathbb{S}^m)$ maps circles to circles in \mathbb{S}^m .

Now for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, if we collect the symmetries of $\mathbb{A}\mathbb{P}^{n-1}$ which is linear but not necessarily isometries, we obtain the group $SL(n, \mathbb{A})$ [SBG⁺95]. Analogously $\mathbb{A}\mathbb{P}^{n-1}$ can be identified as a part of the conformal boundary of $\{M \in S_n(\mathbb{A}) : \det M = 1\} \cong SL(n, \mathbb{A})/SU(n, \mathbb{A})$ on which $SL(n, \mathbb{A})$ acts as isometries. We get the answer for $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$: $SL(n, \mathbb{A})$ can be regarded as the conformal symmetry of $\mathbb{A}\mathbb{P}^{n-1}$, which plays the same role as $SO(m+1, 1)$ acting on \mathbb{S}^m .

In general, let's denote these non-compact symmetry groups as $N_{\mathbb{A}}(n)$ which are listed below. Notice that $H_{\mathbb{A}}(n)$ sits inside $N_{\mathbb{A}}(n)$ as a maximal compact subgroup, and $N_{\mathbb{A}}(n)/H_{\mathbb{A}}(n)$ can be identified with the space of symmetric matrices with determinant one.

\mathbb{A}	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}	\mathbb{R}^m
$H_{\mathbb{A}}(n)$	$SO(n)$	$SU(n)$	$Sp(n)$	F_4	$SO(m+1)$
$N_{\mathbb{A}}(n)$	$SL(n, \mathbb{R})$	$SL(n, \mathbb{C})$	$SL(n, \mathbb{H})$	E_6^{-26}	$SO(m+1, 1)$

We may observe that when $m = 1, 2, 4$ and 8 , $N_{\mathbb{R}^m}(2) = SL(2, \mathbb{A})$ with \mathbb{A} being real, complex, quaternion and octonion respectively. Hence, $sl(2, \mathbb{R}) = so(2, 1)$, $sl(2, \mathbb{C}) = so(3, 1)$, $sl(2, \mathbb{H}) = so(5, 1)$, $sl(2, \mathbb{O}) = so(9, 1)$. In general we have $sl(2, \mathbb{A}) = so(\mathbb{A} \oplus \mathbb{R}^{1,1})$ [Bae02].

The above point of view integrates conformal geometry with real, complex, quaternionic and octonionic geometries. In the next section we will illustrate this viewpoint by studying the variation of volume of cycles under the conformal symmetry $N_{\mathbb{A}}(n)$ of $\mathbb{A}\mathbb{P}^{n-1}$ in a unified manner.

Remark 3. In [AB03], Atiyah and Berndt studied the complexified version of $\mathbb{A}\mathbb{P}^{n-1}$ with $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}$. We can extend these descriptions to $\mathbb{A} = \mathbb{R}^m$ as in the following table:

\mathbb{A}	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}	\mathbb{R}^m
$(\mathbb{A} \otimes \mathbb{C})\mathbb{P}^{n-1}$	$\mathbb{C}\mathbb{P}^{n-1}$	$(\mathbb{C}\mathbb{P}^{n-1})^2$	$Gr_{\mathbb{C}}(2, 2n-2)$	$\frac{E_6}{Spin(10)U(1)}$	$\frac{O(m+2)}{O(m)O(2)}$
$H_{\mathbb{A} \otimes \mathbb{C}}(n)$	$SU(n)$	$SU(n)^2$	$SU(2n)$	E_6	$SO(m+2)$
$N_{\mathbb{A} \otimes \mathbb{C}}(n)$	$Sp(2n, \mathbb{R})$	$SU(n, n)$	$O^*(4n)$	E_7^{-25}	$SO(m+2, 2)$

Notice that the maximal compact subgroup of $N_{\mathbb{A} \otimes \mathbb{C}}(n)$ is the product of $H_{\mathbb{A} \otimes \mathbb{C}}(n)$ with $U(1)$. Furthermore,

$$\frac{N_{\mathbb{A} \otimes \mathbb{C}}(n)}{H_{\mathbb{A} \otimes \mathbb{C}}(n)U(1)} = S_n^+(\mathbb{A}) + iS_n(\mathbb{A})$$

is a tube domain (see for example [Gro94]). This gives a complete list of tube domains.

They also have a quaternionic analog:

\mathbb{A}	\mathbb{R}	\mathbb{C}	\mathbb{H}	\mathbb{O}	\mathbb{R}^m
$(\mathbb{A} \otimes \mathbb{H})\mathbb{P}^{n-1}$	$\mathbb{H}\mathbb{P}^{n-1}$	$Gr_{\mathbb{C}}(2, 2n-2)$	$Gr_{\mathbb{R}}(4, 4n-4)$	$\frac{E_7}{Spin(12)O(4)}$	$\frac{O(m+4)}{O(m)O(4)}$
$H_{\mathbb{A} \otimes \mathbb{H}}(n)$	$Sp(n)$	$SU(2n)$	$SO(4n)$	E_7	$SO(m+4)$
$N_{\mathbb{A} \otimes \mathbb{H}}(n)$	$Sp(n, 1)$	$SU(2n, 1)$	$SO(4n, 4)$	E_8^{-24}	$SO(m+4, 4)$

3 Cycles under conformal symmetries

In the last section, we regard $N_{\mathbb{A}}(n+1)$ as the conformal symmetry group of $\mathbb{A}\mathbb{P}^n$. Its Lie algebra

$$\mathfrak{n}_{\mathbb{A}}(n+1) = \mathfrak{h}_{\mathbb{A}}(n+1) \oplus S'_{n+1}(\mathbb{A})$$

induces vector fields which acts infinitesimally on $\mathbb{A}\mathbb{P}^n$. Here the Lie algebra $\mathfrak{h}_{\mathbb{A}}(n+1)$ of $H_{\mathbb{A}}(n+1)$ induces Killing vector fields, and $S'_{n+1}(\mathbb{A})$ consists of trace-free symmetric matrices, which can be regarded as constant vector fields in $S'_{n+1}(\mathbb{A})$, projecting to conformal vector fields on $\mathbb{A}\mathbb{P}^n \subset S'_{n+1}(\mathbb{A})$. We are adopting the metric

$$\langle A, B \rangle := 2 \operatorname{Re}(\operatorname{tr} AB) = 2 \operatorname{Re}(\operatorname{tr} A \circ B)$$

on $S'_{n+1}(\mathbb{A})$ which induces the standard metric on $\mathbb{A}\mathbb{P}^n$.

We would like to compute the average second variation of the volume of a cycle in $\mathbb{A}\mathbb{P}^n$ under the action of $\mathfrak{n}_{\mathbb{A}}(n+1)$. First, Let us quickly review the terminology and set up some notations.

3.1 Terminology and notations

For a global vector field V on a Riemannian manifold M , the second variation $\mathcal{Q}_S(V)$ of the volume \mathbf{M} of a rectifiable current S under V is defined as

$$\mathcal{Q}_S(V) := \left. \frac{d^2}{dt^2} \right|_{t=0} \mathbf{M}((\phi_t)_* S) = \int_M \left. \frac{d^2}{dt^2} \right|_{t=0} \|(\phi_t)_* S_x\| d\nu_S(x),$$

where ϕ_t is the flow induced by V , S_x denotes the unit simple vector representing the oriented tangent space of S at x , and ν_S denotes the Borel measure associated with S . S is said to be stable if $\mathcal{Q}_S(V) \leq 0$ for all vector fields V on M . We will denote the integrand $\left. \frac{d^2}{dt^2} \right|_{t=0} \|(\phi_t)_* \xi\|$ by $\mathcal{Q}_\xi(V)$, the second variation of an oriented orthonormal p -frame ξ under V . One has the following second variation formula for a gradient vector field V [LS73]:

$$\begin{aligned} \mathcal{Q}_\xi(V) &= \langle \mathcal{A}_{V,V} \xi, \xi \rangle + 2\|\mathcal{A}_V \xi\|^2 - (\langle \mathcal{A}_V \xi, \xi \rangle)^2 \\ &= \left(\sum_{j=1}^p \langle \mathcal{A}_V e_j, e_j \rangle \right)^2 + 2 \sum_{j=1}^p \sum_{k=1}^q (\langle \mathcal{A}_V e_j, n_k \rangle)^2 + \sum_{j=1}^p \langle \mathcal{A}_{V,V} e_j, e_j \rangle, \end{aligned} \tag{3.1}$$

where $\xi = e_1 \wedge \dots \wedge e_p$, which is extended to an orthonormal basis $\{e_1, \dots, e_p, n_1, \dots, n_q\}$ of TM . Here for any smooth vector fields V and W , $\mathcal{A}_V(u)$, $\mathcal{A}_{V,W}$ are endomorphisms of TM defined by

$$\begin{aligned} \mathcal{A}_V X &:= \nabla_X V; \\ \mathcal{A}_{V,W} X &:= (\nabla_V \mathcal{A}_W) X = \nabla_V \nabla_{\tilde{X}} W - \nabla_{\nabla_V \tilde{X}} W, \end{aligned} \tag{3.2}$$

where ∇ is the Levi-Civita connection, \tilde{X} is a smooth local extension of $X \in TM$. An endomorphism L of TM is extended to operate on $\bigwedge^p TM$ by Leibniz rule:

$$L(e_1 \wedge \dots \wedge e_p) = \sum_{j=1}^p e_1 \wedge \dots \wedge L e_j \wedge \dots \wedge e_p.$$

From the above second variation formula, we see that \mathcal{Q}_ξ , and hence \mathcal{Q}_S , is a quadratic form on the space of smooth vector fields on M , and we may restrict it to a finite-dimensional subspace F of vector fields and take the trace $(\operatorname{tr} \mathcal{Q}_\xi|_F) = \sum \mathcal{Q}_\xi(V)$, where V runs through an orthonormal basis of F .

3.2 Main theorem

Coming back to our situation $M = \mathbb{A}\mathbb{P}^n$, since vector fields induced by $\mathfrak{h}_{\mathbb{A}}(n+1)$ preserve metric and does not contribute to the second variation, we have

$$\text{tr } \mathcal{Q}_\xi|_{\mathfrak{h}_{\mathbb{A}}(n+1)} = \text{tr } \mathcal{Q}_\xi|_{S'_{n+1}(\mathbb{A})}$$

and so we may concentrate on $F = S'_{n+1}(\mathbb{A})$.

Moreover, notice that $\mathbb{A}\mathbb{P}^n$ is an orbit of the group $H_{\mathbb{A}}(n+1)$ acting on $S'_{n+1}(\mathbb{A})$. This symmetry helps to reduce a lot of calculations, as illustrated by the following lemma:

Lemma 4. *Let G act isometrically on an inner product space \mathbb{V} , and $M \subset \mathbb{V}$ be a G -invariant submanifold. The projection of each $u \in \mathbb{V}$ gives a vector field V_u on M , and the space of all these vector fields is denoted by F . Then*

$$\text{tr } \mathcal{Q}_\xi|_F = \text{tr } \mathcal{Q}_{g \cdot \xi}|_F$$

for all $g \in G$.

Proof. Since the metric on M is G -invariant, the Levi-Civita connection ∇ is G -equivariant, that is,

$$\nabla_{g_* \cdot X}(g_* \cdot V) = g_* \cdot (\nabla_X V).$$

Hence one has

$$\mathcal{A}_V(g \cdot \xi) = g \cdot (\mathcal{A}_{g_*^{-1}V} \cdot \xi); \mathcal{A}_{V,W}(g \cdot \xi) = g \cdot (\mathcal{A}_{g_*^{-1}V, g_*^{-1}W} \cdot \xi).$$

Applying to the second variation formula, we get

$$\mathcal{Q}_{g \cdot \xi}(V_u) = \mathcal{Q}_\xi(g_*^{-1}V_u) = \mathcal{Q}_\xi(V_{g_*^{-1}u}),$$

where the last equality is due to G -invariance of metric. And so

$$\text{tr } \mathcal{Q}_\eta = \sum_u \mathcal{Q}_\eta(u) = \sum_u \mathcal{Q}_\xi(g_*^{-1}u) = \text{tr } \mathcal{Q}_\xi,$$

where u , and hence $g_*^{-1}u$, runs through an orthonormal basis of \mathbb{V} . The last equality follows from the fact that trace is independent of choice of orthonormal basis. □

By the above lemma, where we take $M = \mathbb{A}\mathbb{P}^n, \mathbb{V} = S'_{n+1}(\mathbb{A})$ and $G = H_{\mathbb{A}}(n+1)$, it suffices to consider average second variation of a p -frame $\xi = e_1 \wedge \cdots \wedge e_p$ at a particular point $x \in \mathbb{A}\mathbb{P}^n$, because p -frames at another point can be moved to x by some $g \in H_{\mathbb{A}}(n+1)$. Let's fix $x = \mathbb{E}_{n+1, n+1} \in \mathbb{A}\mathbb{P}^n$, which is the matrix with value 1 at the $(n+1, n+1)$ position and all other entries zero.

We shall need the following formula, whose proof is given in the appendix:

Theorem 5. *Assume that $M = G/K \subset \mathbb{V}$ is a compact symmetric space which is a G -orbit of an orthogonal representation \mathbb{V} of G . The projection of each $u \in \mathbb{V}$ gives a vector field V_u on M . The average second variation of an oriented orthonormal p -frame $\xi = e_1 \wedge \dots \wedge e_p$ at $x \in M$ under all such vector fields is given by*

$$\text{tr } Q_\xi = \sum_{j,k=1}^{p,q} (2 \|\text{II}(e_j, n_k)\|^2 - \langle \text{II}(e_j, e_j), \text{II}(n_k, n_k) \rangle),$$

where II is the second fundamental form of $M \subset \mathbb{V}$ at x , and $\{e_j\}_{j=1}^p \cup \{n_k\}_{k=1}^q$ is an orthonormal basis of TM .

With the above formula, it remains to compute the second fundamental form of $\mathbb{A}\mathbb{P}^n$. Let's take the following coordinates around x :

$$\begin{aligned} \mathbb{A}^n &\rightarrow \mathbb{A}\mathbb{P}^n \subset S'_{n+1}(\mathbb{A}), \\ Q &\mapsto \frac{1}{1 + \|Q\|^2} \begin{pmatrix} Q \\ 1 \end{pmatrix} (Q^* \ 1), \end{aligned}$$

Here we adopt the following notations:

$$Q = \sum_{l=0}^{\Lambda} \mathbf{i}_l X_l,$$

where X_l are column n -vectors, $\mathbf{i}_0 := 1$, and for $1 \leq l \leq \Lambda$, \mathbf{i}_l are the linearly independent imaginary square roots of unity in \mathbb{A} . Recall that for the case $\mathbb{A} = \mathbb{R}^m$, $n = 1$, $\Lambda = 0$, $Q = X_0$ is an element in \mathbb{R}^m with $Q^* = Q$ and $Q \cdot Q := \langle Q, Q \rangle$. For the other four cases, the entries of X_l are real numbers.

The basis of coordinate tangent vector fields is $\{\frac{\partial}{\partial x_l^j} : 0 \leq l \leq \Lambda, 1 \leq j \leq N\}$, where $\frac{\partial}{\partial x_l^j}$ denote the \mathbf{i}_l -directions. $N = m$ in the case of $\mathbb{A} = \mathbb{R}^m$, and $N = n$ for all the other four cases. Using product rule (which is valid for multiplication in \mathbb{A}),

$$\begin{aligned} \left. \frac{\partial}{\partial x_l^j} \right|_Q &= \frac{1}{1 + \|Q\|^2} \begin{pmatrix} \mathbf{i}_l w_j \\ 0 \end{pmatrix} (Q^* \ 1) \\ &+ \frac{1}{1 + \|Q\|^2} \begin{pmatrix} Q \\ 1 \end{pmatrix} (\bar{\mathbf{i}}_l w_j^T \ 0) \\ &- \frac{2X_l^T w_j}{(1 + \|Q\|^2)^2} \begin{pmatrix} Q \\ 1 \end{pmatrix} (Q^* \ 1), \end{aligned}$$

where w_j stands for the column n -vector with j -th coordinate 1 and other coordinates zero, and T stands for transpose. Recall that when $\mathbb{A} = \mathbb{R}^m$, n equals 1, and so transpose of an element is just itself. Differentiating both sides along $\frac{\partial}{\partial x_r^k}$

at $0 \in \mathbb{A}^n$,

$$\begin{aligned} & \left. \frac{\partial}{\partial x_r^k} \right|_0 \left(\frac{\partial}{\partial x_l^j} \right) \\ &= \begin{cases} \begin{pmatrix} 2\delta_{jk} & 0 \\ 0 & -2\delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} = \mathbb{R}^m, \\ \begin{pmatrix} \mathbf{i}_r \bar{\mathbf{i}}_l \mathbb{E}_{kj} + \mathbf{i}_l \bar{\mathbf{i}}_r \mathbb{E}_{jk} & 0 \\ 0 & -(\mathbf{i}_r \bar{\mathbf{i}}_l + \mathbf{i}_l \bar{\mathbf{i}}_r) \delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \end{cases} \end{aligned}$$

which is already perpendicular to $T_x \mathbb{A} \mathbb{P}^n$, because

$$\left. \frac{\partial}{\partial x_l^j} \right|_0 = \begin{pmatrix} 0 & \mathbf{i}_l w_j \\ \bar{\mathbf{i}}_l w_j^T & 0 \end{pmatrix}.$$

Under the metric $\langle A, B \rangle = 2 \operatorname{Re} \operatorname{tr}(AB)$, our coordinate vectors are pairwise orthogonal, each has length 2. We scale them to get an orthonormal basis $\{\frac{1}{2} \frac{\partial}{\partial x_l^j} : 1 \leq j \leq n, 0 \leq l \leq \Lambda\}$.

We conclude that

Lemma 6. *The second fundamental form $\mathbb{II}(\frac{1}{2} \frac{\partial}{\partial x_l^j}, \frac{1}{2} \frac{\partial}{\partial x_r^k})$ of $\mathbb{A} \mathbb{P}^n \subset S'_{n+1}(\mathbb{A})$ at x is given by*

$$\begin{cases} \frac{1}{2} \begin{pmatrix} \delta_{jk} & 0 \\ 0 & -\delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} = \mathbb{R}^m, \\ \frac{1}{4} \begin{pmatrix} \mathbf{i}_r \bar{\mathbf{i}}_l \mathbb{E}_{kj} + \mathbf{i}_l \bar{\mathbf{i}}_r \mathbb{E}_{jk} & 0 \\ 0 & -(\mathbf{i}_r \bar{\mathbf{i}}_l + \mathbf{i}_l \bar{\mathbf{i}}_r) \delta_{jk} \end{pmatrix} & \text{for } \mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}\}. \end{cases}$$

Now we are ready to compute $\operatorname{tr} \mathcal{Q}_\xi$ for an orthonormal p -frame $\xi = e_1 \wedge \cdots \wedge e_p$ at $x \in \mathbb{A} \mathbb{P}^n$. Complete $B = \{e_j\}_{j=1}^p$ to an orthonormal basis $\{e_j, n_k\}$ in the form

$$\begin{Bmatrix} v_1, \mathbb{J}_1 v_1, \dots, \mathbb{J}_\Lambda v_1 \\ \vdots \\ v_N, \mathbb{J}_1 v_N, \dots, \mathbb{J}_\Lambda v_N \end{Bmatrix}$$

where $\mathbb{J}_l : T_x \mathbb{A} \mathbb{P}^n \rightarrow T_x \mathbb{A} \mathbb{P}^n$ is the differential of left multiplication of \mathbf{i}_l on $\mathbb{A}^n \subset \mathbb{A} \mathbb{P}^n$.

Such an orthonormal basis can be brought to the basis of normalized coordinate vectors by the action of the isotropy group $K < G$. This is easy for $\mathbb{R} \mathbb{P}^n$, $\mathbb{C} \mathbb{P}^n$ and $\mathbb{H} \mathbb{P}^n$: $SO(n)$, $SU(n)$ and $Sp(n)$ acts transitively on orthonormal frames, unitary frames and quaternionic unitary frames respectively. For $\mathbb{O} \mathbb{P}^2$, $K = \operatorname{Spin}(9) < F_4$, we argue as follows: $T_x \mathbb{O} \mathbb{P}^2$ is the spinor representation of $\operatorname{Spin}(9)$. Under this action

$$T_x \mathbb{O} \mathbb{P}^2 \supset \mathbb{S}^{15} \cong \operatorname{Spin}(9)/\operatorname{Spin}(7)$$

(see P.283 of [DH93]). Hence we can use $\sigma \in \text{Spin}(9)$ to bring $\frac{1}{2} \frac{\partial}{\partial x_0^1}$ to v_1 . $\text{Spin}(7)$ fixes v_1 and hence acts on $T_{v_1} \mathbb{S}^{15}$, which splits into the vector representation V_7 and the spinor representation of $\text{Spin}(7)$. $\left\{ \sigma \left(\frac{1}{2} \frac{\partial}{\partial x_l^1} \right) \right\}_{l=1}^7$ and $\{\mathbb{J}_l v_1\}_{l=1}^7$ form two bases of V_7 having the same orientation. Then we can bring $\left\{ \sigma \left(\frac{1}{2} \frac{\partial}{\partial x_l^1} \right) \right\}_{l=1}^7$ to $\{\mathbb{J}_l v_1\}_{l=1}^7$ by an element in $\text{Spin}(7)$. $\left\{ \sigma \left(\frac{1}{2} \frac{\partial}{\partial x_l^2} \right) \right\}_{l=1}^7$ can be brought to $\{\mathbb{J}_l v_2\}_{l=1}^7$ by $\text{Spin}(7)$ using similar reasoning, because

$$\text{Spin}(7)/G_2 \cong \mathbb{S}^7 \text{ and } G_2/\text{SU}(3) \cong \mathbb{S}^6$$

and $\text{SU}(3)$ acts transitively on the collection of unitary bases of \mathbb{C}^3 .

By Lemma 4, average second variations of ξ and $g \cdot \xi$ are the same for all $g \in G$, and hence we may assume

$$\mathbb{J}_l v_j = \frac{1}{2} \frac{\partial}{\partial x_l^j},$$

so that we can apply Lemma 6 directly.

For the case $\mathbb{A} = \mathbb{R}^m$ in which $\mathbb{A}\mathbb{P}^1 = \mathbb{S}^m$, Lemma 6 gives

$$\left\| \text{II} \left(\frac{1}{2} \frac{\partial}{\partial x^j}, \frac{1}{2} \frac{\partial}{\partial x^k} \right) \right\|^2 = \delta_{jk},$$

which is the usual formula for the second fundamental form of $\mathbb{S}^m \subset \mathbb{R}^{m+1}$. Together with Theorem 5, the result of Lawson and Simons [LS73] is reproduced:

$$\text{tr } \mathcal{Q}_\xi = \sum_{j,k=1}^{p,q} (-1) = -pq \leq 0,$$

where $p+q = m$, implying that the average second variation of a rectifiable current of non-zero volume in \mathbb{S}^n is negative for $0 < p < m$, and hence cannot be stable.

Now let's turn to the other four cases. Lemma 6 gives

$$\| \text{II}(e_j, n_k) \|^2 = \begin{cases} 0 & \text{for } n_k = \pm \mathbb{J}_l e_j \text{ for some } 1 \leq l \leq \Lambda, \\ \frac{1}{4} & \text{otherwise,} \end{cases}$$

and

$$\langle \text{II}(e_j, e_j), \text{II}(n_k, n_k) \rangle = \begin{cases} 1 & \text{for } n_k = \pm \mathbb{J}_l e_j \text{ for some } 1 \leq l \leq \Lambda, \\ \frac{1}{2} & \text{otherwise,} \end{cases}$$

so the summand appeared in Theorem 5 is

$$\begin{aligned} & 2 \| \text{II}(e_j, n_k) \|^2 - \langle \text{II}(e_j, e_j), \text{II}(n_k, n_k) \rangle \\ &= \begin{cases} -1 & \text{for } n_k = \pm \mathbb{J}_l e_j \text{ for some } 1 \leq l \leq \Lambda, \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

meaning that for each e_j , every $\mathbb{J}_l e_j$ -direction normal to ξ contributes -1 to $\text{tr } \mathcal{Q}_\xi$, and all other normal directions have no effect. Hence

$$\begin{aligned} \text{tr } \mathcal{Q}_\xi &= - \sum_{j=1}^p (\text{number of } l \text{ such that } \pm \mathbb{J}_l e_j \notin B) \\ &= - \sum_{j=1}^p \sum_{l=1}^\Lambda \|e_1 \wedge \cdots \wedge \mathbb{J}_l e_j \wedge \cdots \wedge e_p\|^2 \\ &= - \sum_{l=1}^\Lambda \|\mathbb{J}_l \cdot \xi\|^2 \leq 0. \end{aligned}$$

(Here \mathbb{J} acts on ξ by Leibniz rule.) Equality holds if and only if $\|\mathbb{J}_l \cdot \xi\|^2 = 0$ for all $1 \leq l \leq \Lambda$, meaning that ξ is invariant under each \mathbb{J}_l , and hence invariant under the $\mathbb{S}^{\Lambda-1}$ -family of complex structures. Hence we obtain the following theorem:

Theorem 7. *In $\mathbb{A}\mathbb{P}^n$, where $\mathbb{A} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}, \mathbb{R}^m\}$, any stable minimal submanifold S (or more generally rectifiable current) must be complex, by which we means $T_x S$ is invariant under all the linear complex structures at x for almost every $x \in S$.*

We remark that in $\mathbb{H}\mathbb{P}^n$, a quaternionic submanifold must be totally geodesic.

4 Appendix: Average second variation in symmetric orbits

Our aim is to prove the following theorem, which we have used in the last section to compute the average second variation of the volume of a cycle in $\mathbb{A}\mathbb{P}^n$ along directions in $\mathfrak{h}_\mathbb{A}(n+1)$:

Theorem: *Assume that $M = G/K$ is a compact symmetric space which is a G -orbit of an orthogonal representation \mathbb{V} of G . The projection of each $u \in \mathbb{V}$ determines a vector field V_u , or simply V , on M . The average second variation of an oriented orthonormal p -frame $\xi = e_1 \wedge \cdots \wedge e_p$ at $x \in M$ under all such vector fields is given by*

$$\text{tr } \mathcal{Q}_\xi = \sum_{j,k=1}^{p,q} (2 \|\text{II}(e_j, n_k)\|^2 - \langle \text{II}(e_j, e_j), \text{II}(n_k, n_k) \rangle)$$

where II is the second fundamental form of $M \subset \mathbb{V}$ at x , and $\{e_j\}_{j=1}^p \cup \{n_k\}_{k=1}^q$ is an orthonormal basis of $T_x M$.

The method of proof is similar to [LS73]. The Lie algebra \mathfrak{g} of G decomposes:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$$

where $\mathfrak{m} := \mathfrak{k}^\perp$. On G we have a natural G -invariant metric given by negative of the Killing form, which can be scaled such that \mathfrak{m} is isometric to $T_x M$. We shall

use the same symbol to denote an element of \mathfrak{g} , its induced vector field on \mathbb{V} , and the restricted Killing vector field on M . Recall that

$$[g_1, g_2]_M = -[g_1, g_2] \tag{4.3}$$

where $[\cdot, \cdot]_M$ is the Lie bracket for vector fields on M , and $[\cdot, \cdot]$ is the Lie bracket on \mathfrak{g} . On the right hand side g_1, g_2 denote elements in \mathfrak{g} , while on the left hand side they denote their induced Killing vector fields on M .

Let's complete $\xi = e_1 \wedge \dots \wedge e_p$ to an orthonormal basis $\{e_1, \dots, e_p, n_1, \dots, n_q\}$ of $T_x M \cong \mathfrak{m}$, and further take an orthonormal basis $\{\beta_1, \dots, \beta_r\}$ of \mathfrak{k} , so that $\{\beta_1, \dots, \beta_r, e_1, \dots, e_p, n_1, \dots, n_q\}$ forms an orthonormal basis of \mathfrak{g} .

We now express the projection $V = V_u$ of $u \in \mathbb{V}$ in terms of Killing vector fields induced by \mathfrak{g} on M .

Lemma 8.

$$V = \sum_{\mu=1}^r \langle u, \beta_\mu \rangle \beta_\mu + \sum_{\nu=1}^p \langle u, e_\nu \rangle e_\nu + \sum_{\gamma=1}^q \langle u, n_\gamma \rangle n_\gamma.$$

Proof. Denote the basis $\{\beta_1, \dots, \beta_r, e_1, \dots, e_p, n_1, \dots, n_q\}$ of \mathfrak{g} by A .

At $x \in M$ the above equation is obvious, because $\beta_\mu(x) = 0$, and $\{e_1, \dots, e_p, n_1, \dots, n_q\}$ forms an orthonormal basis of $T_x M$.

At another point $y \in M$, let $\{\tilde{e}_1, \dots, \tilde{e}_p, \tilde{n}_1, \dots, \tilde{n}_q\}$ be an orthonormal basis of $T_y M \cong \mathfrak{m}$, and we complete it to an orthonormal basis

$$B = \{\tilde{\beta}_1, \dots, \tilde{\beta}_r, \tilde{e}_1, \dots, \tilde{e}_p, \tilde{n}_1, \dots, \tilde{n}_q\}$$

of \mathfrak{g} . Both A, B are orthonormal basis of \mathfrak{g} , so $B = AT$, where T is an orthogonal matrix.

$$V(x) = \sum_j \langle u, B_j \rangle B_j = \sum_j \langle u, A_k T_j^k \rangle A_i T_j^i = \sum_j \langle u, A_j \rangle A_j$$

since $\sum_j T_j^k T_j^i = \delta^{ki}$. □

Proof to Theorem 5: From the second variation formula (3.1), the average second variation is given by

$$\text{tr } \mathcal{Q}_\xi = \sum_u \left(\sum_{j=1}^p \langle \mathcal{A}_V e_j, e_j \rangle \right)^2 + 2 \sum_u \sum_{j=1, k=1}^{p, q} (\langle \mathcal{A}_V e_j, n_k \rangle)^2 + \sum_u \sum_{j=1}^p \langle \mathcal{A}_{V, V} e_j, e_j \rangle$$

where u runs through an orthonormal basis of \mathbb{V} , each gives a vector field $V = V_u$ on M by projection. We compute term by term for the three terms appeared in the above expression.

Recall [Hel01] that for a symmetric space,

$$\nabla_{K_1} K_2 = \frac{1}{2} [K_1, K_2]_M$$

for Killing vector fields K_1 and K_2 on M . Applying this to the expression of V given in Lemma 8,

$$\begin{aligned} \nabla_{e_j} V &= \langle u, \partial_{e_j} \beta_\mu \rangle \beta_\mu + \frac{1}{2} \langle u, \beta_\mu \rangle [e_j, \beta_\mu]_M + \langle u, \partial_{e_j} e_\nu \rangle e_\nu + \frac{1}{2} \langle u, e_\nu \rangle [e_j, e_\nu]_M \\ &\quad + \langle u, \partial_{e_j} n_\gamma \rangle n_\gamma + \frac{1}{2} \langle u, n_\gamma \rangle [e_j, n_\gamma]_M, \end{aligned} \tag{4.4}$$

where ∂ is the trivial connection of \mathbb{V} , and so ∂_v is the usual directional derivative along $v \in T_x \mathbb{V} \cong \mathbb{V}$. (Recall that $\beta_\mu, e_\nu, n_\gamma$ can be regarded as vector fields on \mathbb{V} , and so the above directional derivatives make sense.)

To simplify the above expression at x , notice that \mathfrak{k} induces zero vectors at x , and hence $\beta_\mu \in \mathfrak{k}$ vanishes at x . Together with equation (4.3) and the fact that

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, [\mathfrak{k}, \mathfrak{m}] \subset \mathfrak{m}, [\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}. \tag{4.5}$$

we have

$$\nabla_{e_j} V(x) = \langle u, \partial_{e_j} e_\nu \rangle e_\nu + \langle u, \partial_{e_j} n_\gamma \rangle n_\gamma$$

and hence

$$\begin{aligned} \langle \mathcal{A}_V e_j, e_j \rangle &= \langle \nabla_{e_j} V(x), e_j \rangle = \langle u, \partial_{e_j} e_j \rangle; \\ \langle \mathcal{A}_V e_j, n_k \rangle &= \langle \nabla_{e_j} V(x), n_k \rangle = \langle u, \partial_{e_j} n_k \rangle. \end{aligned}$$

The first term $\sum_u \left(\sum_{j=1}^p \langle \mathcal{A}_V e_j, e_j \rangle \right)^2$ is

$$\begin{aligned} \sum_u \left(\sum_{j=1}^p \langle \mathcal{A}_V e_j, e_j \rangle \right)^2 &= \sum_u \sum_{j,k=1}^p \langle u, \partial_{e_j} e_j \rangle \langle u, \partial_{e_k} e_k \rangle \\ &= \sum_{j,k=1}^p \langle \partial_{e_j} e_j, \partial_{e_k} e_k \rangle \\ &= \left\| \sum_{j=1}^p \Pi(e_j, e_j) \right\|^2, \end{aligned}$$

where $\partial_{e_j} e_j = \Pi(e_j, e_j)$ because $\nabla_{e_j} e_j = [e_j, e_j]_M/2 = 0$.

The second term $2 \sum_u \sum_{j=1, k=1}^{p,q} (\langle \mathcal{A}_V e_j, n_k \rangle)^2$ is

$$\begin{aligned} 2 \sum_u \sum_{j=1, k=1}^{p,q} (\langle \mathcal{A}_V e_j, n_k \rangle)^2 &= 2 \sum_u \sum_{j=1, k=1}^{p,q} (\langle u, \partial_{e_j} n_k \rangle)^2 \\ &= 2 \sum_{j,k=1}^{p,q} \|\partial_{e_j} n_k\|^2 \\ &= 2 \sum_{j,k=1}^{p,q} \|\Pi(e_j, n_k)\|^2, \end{aligned}$$

where $\partial_{e_j} n_k = \Pi(e_j, n_k)$ at x because $\nabla_{e_j} n_k(x) = [e_j, n_k]_M/2 = 0$.

Now we turn to compute the third term $\sum_u \sum_{j=1}^p \langle \mathcal{A}_{V,V} e_j, e_j \rangle$, which is more complicated. At x ,

$$\begin{aligned} \langle \mathcal{A}_{V,V} e_j, e_j \rangle &= \langle \nabla_V \nabla_{e_j} V - \nabla_{\nabla_V e_j} V, e_j \rangle \\ &= \langle \nabla_V \nabla_{e_j} V, e_j \rangle \\ &= \sum_{\nu=1}^p \langle u, e_\nu \rangle \langle \nabla_{e_\nu} \nabla_{e_j} V, e_j \rangle + \sum_{\gamma=1}^q \langle u, n_\gamma \rangle \langle \nabla_{n_\gamma} \nabla_{e_j} V, e_j \rangle, \end{aligned}$$

where $\nabla_{\nabla_V e_j} V(x) = 0$ because

$$\nabla_V e_j(x) = \sum_{\nu=1}^p \langle u, e_\nu \rangle \frac{[e_\nu, e_j]_M}{2} + \sum_{\gamma=1}^q \langle u, n_\gamma \rangle \frac{[n_\gamma, e_j]_M}{2} = 0.$$

We now compute the first part $\sum \langle u, e_\nu \rangle \langle \nabla_{e_\nu} \nabla_{e_j} V, e_j \rangle$ of the third term. Differentiating equation (4.4) along e_ν , we get

$$\begin{aligned} \nabla_{e_\nu} \nabla_{e_j} V(x) &= \frac{1}{2} \langle u, \partial_{e_j} \beta_\mu \rangle [e_\nu, \beta_\mu]_M + \frac{1}{2} \langle u, \partial_{e_\nu} \beta_\mu \rangle [e_j, \beta_\mu]_M \\ &\quad + \langle u, \partial_{e_\nu} \partial_{e_j} e_\alpha \rangle e_\alpha + \frac{1}{4} \langle u, e_\alpha \rangle [e_\nu, [e_j, e_\alpha]_M]_M \\ &\quad + \langle u, \partial_{e_\nu} \partial_{e_j} n_\gamma \rangle n_\gamma + \frac{1}{4} \langle u, n_\gamma \rangle [e_\nu, [e_j, n_\gamma]_M]_M. \end{aligned}$$

Using the identity $\langle [X, Y]_M, Z \rangle = -\langle Y, [X, Z]_M \rangle$ for Killing vector fields X, Y, Z , together with the relation (4.5) repeatedly, we get

$$\langle \nabla_{e_\nu} \nabla_{e_j} V(x), e_j \rangle = \langle u, \partial_{e_\nu} \partial_{e_j} e_j \rangle$$

and so

$$\begin{aligned} &\sum_u \sum_{j=1}^p \sum_{\nu=1}^p \langle u, e_\nu \rangle \langle \nabla_{e_\nu} \nabla_{e_j} V, e_j \rangle \\ &= \sum_u \sum_{j=1}^p \sum_{\nu=1}^p \langle u, e_\nu \rangle \langle u, \partial_{e_\nu} \partial_{e_j} e_j \rangle \\ &= \sum_{j, \nu=1}^p \langle \partial_{e_\nu} \partial_{e_j} e_j, e_\nu \rangle \\ &= - \left\| \sum_{j=1}^p \Pi(e_j, e_j) \right\|^2. \end{aligned} \tag{4.6}$$

Now proceed to compute the second part $\sum \langle u, n_\gamma \rangle \langle \nabla_{n_\gamma} \nabla_{e_j} V, e_j \rangle$ of the

third term. Differentiating the equation (4.4) along n_γ , we get

$$\begin{aligned} \nabla_{n_\gamma} \nabla_{e_j} V(x) &= \frac{1}{2} \langle u, \partial_{e_j} \beta_\mu \rangle [n_\gamma, \beta_\mu]_M + \frac{1}{2} \langle u, \partial_{n_\gamma} \beta_\mu \rangle [e_j, \beta_\mu]_M \\ &\quad + \langle u, \partial_{n_\gamma} \partial_{e_j} e_\nu \rangle e_\nu + \frac{1}{4} \langle u, e_\nu \rangle [n_\gamma, [e_j, e_\nu]_M]_M \\ &\quad + \langle u, \partial_{n_\gamma} \partial_{e_j} n_\alpha \rangle n_\alpha + \frac{1}{4} \langle u, n_\alpha \rangle [n_\gamma, [e_j, n_\alpha]_M]_M, \end{aligned}$$

and so

$$\langle \nabla_{n_\gamma} \nabla_{e_j} V(x), e_j \rangle = \langle u, \partial_{n_\gamma} \partial_{e_j} e_j \rangle.$$

$$\begin{aligned} &\sum_u \sum_{j=1}^p \sum_{\gamma=1}^q \langle u, n_\gamma \rangle \langle \nabla_{n_\gamma} \nabla_{e_j} V, e_j \rangle \\ &= \sum_{j,\gamma=1}^{p,q} \langle \partial_{n_\gamma} \partial_{e_j} e_j, n_\gamma \rangle \\ &= - \sum_{j,\gamma=1}^{p,q} \langle \Pi(e_j, e_j), \Pi(n_\gamma, n_\gamma) \rangle. \end{aligned} \tag{4.7}$$

Adding up equations (4.6) and (4.7), we get the third term

$$- \left\| \sum_{j=1}^p \Pi(e_j, e_j) \right\|^2 - \sum_{j,\gamma=1}^{p,q} \langle \Pi(e_j, e_j), \Pi(n_\gamma, n_\gamma) \rangle.$$

Adding up all the three terms, the average second variation is

$$\sum_{j,k=1}^{p,q} (2 \|\Pi(e_j, n_k)\|^2 - \langle \Pi(e_j, e_j), \Pi(n_k, n_k) \rangle). \blacksquare$$

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