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Moduli of Bundles over Rational Surfaces and Elliptic Curves II: Nonsimply Laced Cases

Naichung Conan Leung[1] and Jiajin Zhang[2]

[1]Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Hong Kong and [2]Department of Mathematics, Sichuan University, Chengdu, 610000, People's Republic of China

Correspondence to be sent to: jjzhang@scu.edu.cn

For any nonsimply laced Lie group G and elliptic curve Σ, we show that the moduli space of flat G bundles over Σ can be identified with the moduli space of rational surfaces with G-configurations which contain Σ as an anticanonical curve. We also construct $Lie(G)$-bundles over these surfaces. The corresponding results for simply laced groups were obtained by the authors in another paper. Thus, we have established a natural identification for these two kinds of moduli spaces for any Lie group G.

1 Introduction

There is a classical identification for the moduli space of flat E_n bundles over a fixed elliptic curve Σ and the moduli space of del Pezzo surfaces of degree $9-n$ containing Σ as an anticanonical curve (see [3–6]). For other simply laced Lie group G, that is, Lie group of ADE type, we also obtained in [13] an identification for the moduli space of flat G bundles over a fixed elliptic curve Σ and the moduli space of the pairs (S, Σ), where S is a rational surface (called ADE-surface in [13]) containing Σ as an anticanonical curve. In this paper, we extend the above identification to nonsimply laced cases. Therefore, we establish

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a one-to-one correspondence between flat G bundles over a fixed elliptic curve Σ and rational surfaces with Σ as an anticanonical curve for Lie groups of *all* types.

A nonsimply laced Lie group G is uniquely determined by a simply laced Lie group G' and its outer automorphism group. Hence, it is natural to apply the previous results for the simply laced cases in [13] to the current situation. Similar to simply laced cases, we can define *G-surfaces* and *rational surfaces with G-configurations* (see Definitions 3.3, 3.8, 3.13, 3.19). Our main result is the following.

Theorem 1.1. Let Σ be a fixed elliptic curve with identity $0 \in \Sigma$, G be any simple, compact, simply connected and nonsimply laced Lie group. Denote $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is a G-surface such that $\Sigma \in |-K_S|$. Denote $\mathcal{M}_\Sigma^G$ the moduli space of flat G-bundles over Σ. Then we have

(i) $\mathcal{S}(\Sigma, G)$ can be embedded into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(ii) This embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$ by including all rational surfaces with G-configurations, and this gives us a natural and explicit compactification $\overline{\mathcal{S}(\Sigma, G)}$ of $\mathcal{S}(\Sigma, G)$.

This paper is motivated by some duality in physics. When $G = E_n$ is a simple subgroup of $E_8 \times E_8$, these G bundles are related to the duality between F-theory and string theory. Among other things, this duality predicts that the moduli of flat E_n bundles over a fixed elliptic curve Σ can be identified with the moduli of del Pezzo surfaces with fixed anticanonical curve Σ. For details, one can consult [3, 4, 6]. Our result can be considered as a test of the above duality for other Lie groups.

As an application, this identification provides us with an intuitive explanation for $\mathcal{M}_\Sigma^G$. We also provide an interesting geometric realization of root system theory, and we can see very clearly how the Weyl group acts on the (marked) moduli space of flat G-bundles over Σ.

In the following, we illustrate briefly via pictures what G-configurations and G-surfaces are in each case and compare it with the corresponding case where G' is simply laced.

1.1 B_n-configurations as special D_{n+1}-configurations

In these cases we consider rational surfaces with fibration structure and a fixed smooth anticanonical curve Σ. A B_n-configuration comes from a D_{n+1}-configuration. Roughly speaking, by saying a rational surface S has a D_{n+1}-configuration $(l_1, \ldots, l_{n+1})$, we mean that S can be considered as a blowup of $\mathbb{F}_1$ (a *Hirzebruch surface*) at $n+1$ points on

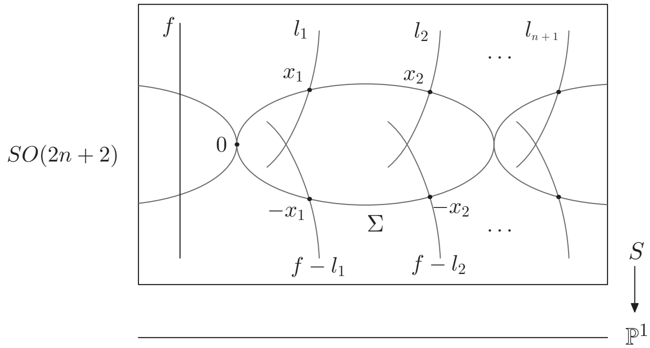

Fig. 1. A surface with a D_{n+1}-configuration $(l_1, \ldots, l_{n+1})$.

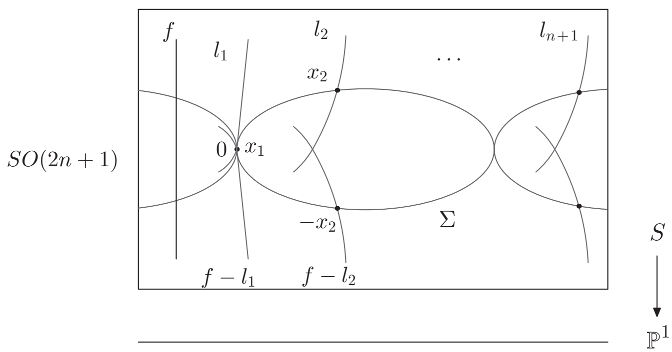

Fig. 2. A surface with a B_n-configuration $(l_1, l_2, \ldots, l_{n+1})$, where $x_1 = l_1 \cap \Sigma = 0$.

$\Sigma \in |-K_{\mathbb{F}_1}|$, such that $l_1, \ldots, l_{n+1}$ are the corresponding exceptional classes [13]. When these blown-up points are *in general position*, S is called a $G = D_{n+1}$-surface. See Figure 1 for a surface with a D_{n+1}-configuration.

Given a surface S with a D_{n+1}-configuration $\zeta = (l_1, \ldots, l_{n+1})$, if it satisfies the condition $x_1 = l_1 \cap \Sigma$, which is the identity element zero of the elliptic curve Σ, then ζ is a B_n-configuration on S (Definition 3.3). If all blown-up points but x_1 are *in general position*, S is called a B_n-surface. See Figure 2 for a surface with a B_n-configuration.

1.2 C_n-configurations as special A_{2n-1}-configurations

In these cases, we consider rational surfaces with fibration and section structure and a fixed smooth anticanonical curve Σ.

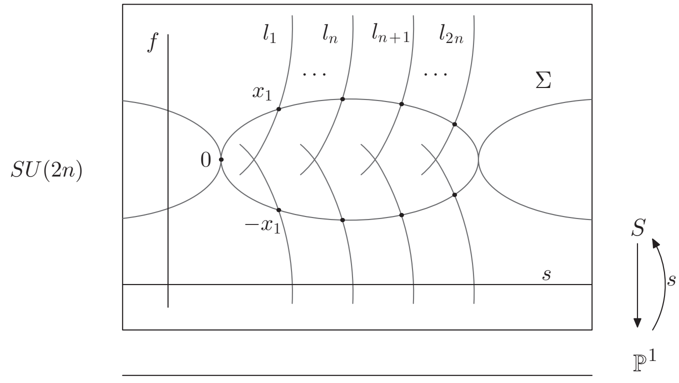

Fig. 3. A surface with an A_{2n-1}-configuration $(l_1, \ldots, l_{2n})$.

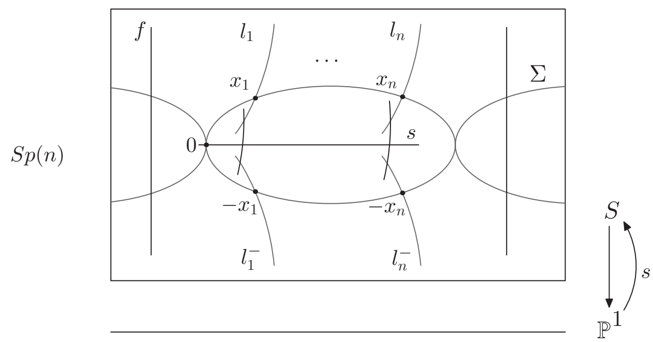

Fig. 4. A surface with a C_n-configuration $(l_1, \ldots, l_n, l_n^-, \ldots, l_1^-)$.

A C_n-configuration comes from an A_{2n-1}-configuration. By saying that a rational surface S has an A_{2n-1}-configuration $(l_1, \ldots, l_{2n})$, we mean that S can be considered as a blowup of $\mathbb{F}_1$ at $2n$ points on $\Sigma \in |-K_{\mathbb{F}_1}|$ which sum to zero, such that $l_1, \ldots, l_{2n}$ are the corresponding exceptional classes [13]. When these blown-up points are *in general position*, S is called an A_{2n-1}-surface. See Figure 3 for a surface with an A_{2n-1}-configuration.

Given a surface S with an A_{2n-1}-configuration $\zeta = (l_1, \ldots, l_{2n})$, if it satisfies the condition $x_i = -x_{2n+1-i}$ with $x_i = l_i \cap \Sigma$, for $i = 1, \ldots, n$, then ζ is called a C_n-configuration on S (Definition 3.8). If all blown-up points are *in general position*, S is called a C_n-surface. See Figure 4 for a surface with a C_n-configuration.

1.3 G_2-configurations as special D_4-configurations

In these cases we still consider rational surfaces with fibration structure and a fixed smooth anticanonical curve Σ.

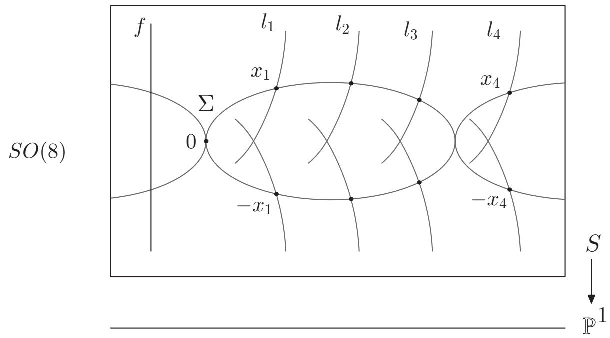

Fig. 5. A surface with a D_4-configuration $(l_1, \ldots, l_4)$.

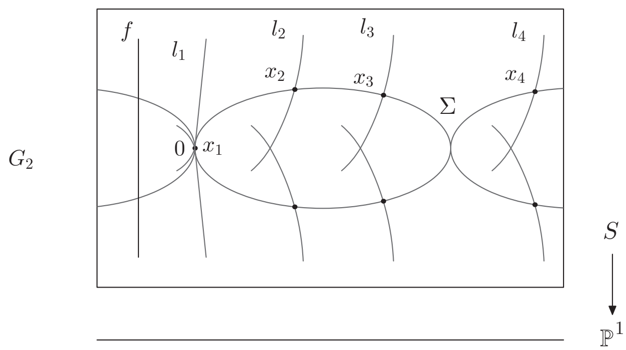

Fig. 6. A surface with a G_2-configuration (l_1, l_2, l_3, l_4), where $x_1 = 0$ and $x_4 = x_2 + x_3$ with $x_i = l_i \cap \Sigma$.

A G_2-configuration comes from a D_4-configuration. We have seen what a D_4-configuration is from Section 1.1 of the Introduction. Roughly speaking, by saying that a rational surface S has a D_4-configuration $(l_1, \ldots, l_4)$, we mean that S can be considered as a blowup of $\mathbb{F}_1$ at four points on $\Sigma \in |-K_{\mathbb{F}_1}|$, such that $l_1, \ldots, l_4$ are the corresponding exceptional classes [13]. When these blown-up points are *in general position*, S is called a $G = D_4$-surface. See Figure 5 for a surface with a D_4-configuration.

Given a surface S with a D_4-configuration $\zeta = (l_1, \ldots, l_4)$, if it satisfies two conditions $x_1 = 0$ and $x_4 = x_2 + x_3$, where $x_i = l_i \cap \Sigma$, then ζ is called a G_2-configuration on S (Definition 3.13). If all blown-up points but x_1 are *in general position*, S is called a G_2-surface. See Figure 6 for a surface with a G_2-configuration.

1.4 F_4-configurations as special E_6-configurations

In these cases we consider rational surfaces which are blowups of the projective plane $\mathbb{P}^2$ at six points in almost general position and which contain a fixed smooth anticanonical curve Σ [13].

E_6

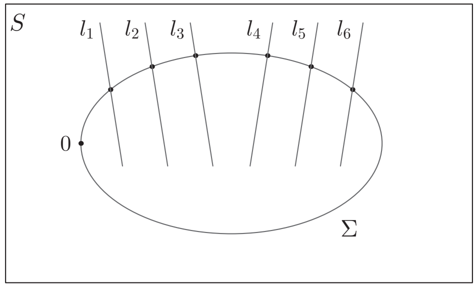

Fig. 7. A surface with an E_6-configuration $(l_1, \ldots, l_6)$.

F_4

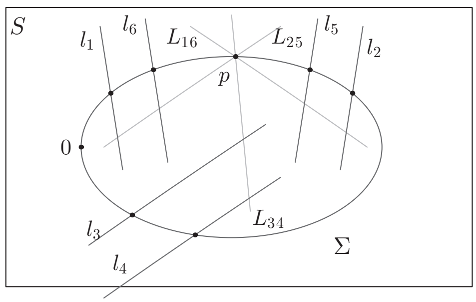

Fig. 8. A surface with an F_4-configuration $(l_1, \ldots, l_6)$, where three lines L_{16}, L_{25}, L_{34} meet at $p \in \Sigma$, or equivalently, $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$ with $x_i = l_i \cap \Sigma$.

An F_4-configuration comes from an E_6-configuration. Recall that by saying a rational surface S has an E_6-configuration $(l_1, \ldots, l_6)$, we mean that S can be considered as a blowup of $\mathbb{P}^2$ at six points on $\Sigma \in |-K_{\mathbb{P}^2}|$, such that $l_1, \ldots, l_6$ are the corresponding exceptional classes. When these blown-up points are *in general position*, S is called an E_6-surface, which is in fact a cubic surface. See Figure 7 for a surface with an E_6-configuration.

Given a surface S with an E_6-configuration $\zeta = (l_1, \ldots, l_6)$, if it satisfies the condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$, where $x_i = l_i \cap \Sigma$, then ζ is called an F_4-configuration on S (Definition 3.19). If all blown-up points are *in general position*, S is called an F_4-surface. See Figure 8 for a surface with an F_4-configuration.

Moreover, we can construct $\mathcal{G}=$ Lie(G) bundles over S with a G-configuration. By restriction, we obtain Lie(G) bundles over Σ. And we can also construct some natural fundamental representation bundles over Σ, which have interesting geometric meanings, such that the Lie algebra bundles are the automorphism bundles of these representation bundles preserving certain algebraic structures.

In this paper, the notations are the same as those in [13]. Let G be a compact, simple, and simply connected Lie group. We denote

$r(G)$: the rank of G;

$R(G)$: the root system;

$R_c(G)$: the coroot system;

$W(G)$: the Weyl group;

$\Lambda(G)$: the root lattice;

$\Lambda_c(G)$: the coroot lattice;

$\Lambda_w(G)$: the weight lattice;

$T(G)$: a maximal torus;

$ad(G)$: the adjoint group of G, i.e. $G/C(G)$, where $C(G)$ is the center of G;

$\Delta(G)$: a simple root system of G; and

$Out(G)$: the outer automorphism group of G, which is defined as the quotient of the automorphism group of G by its inner automorphism group. It is well known that $Out(G)$ is isomorphic to the diagram automorphism group of the Dynkin diagram of G.

When there is no confusion, we just ignore the letter G.

2 Reductions to Simply Laced Cases

Let G be any simple, compact, and simply connected Lie group. Then G is classified into the following seven types according to its Lie algebra:

(1) A_n-type, $G = SU(n+1)$;

(2) B_n-type, $G = Spin(2n+1)$;

(3) C_n-type, $G = Sp(n)$;

(4) D_n-type, $G = Spin(2n)$;

(5) E_n-type, $n = 6, 7, 8$;

(6) F_4-type; and

(7) G_2-type.

Among these, A_n, D_n, and E_n are of simply laced type, while B_n, C_n, F_4, and G_2 are of nonsimply laced type. And A_n, B_n, C_n, D_n are called classical Lie groups, while E_n, F_4, and G_2 are called exceptional Lie groups.

From now on, we always assume that G is a compact, simple, simply connected Lie group of nonsimply laced type, that is, of type B_n, C_n, F_4, G_2. There are two natural approaches to reduce situations to simply laced cases. One is embedding G into a simply laced Lie group G' such that G is the subgroup fixed by the outer automorphism group of G'. Another is taking the simply laced subgroup G'' of maximal rank.

In the following, we explain the first reduction. The following result is well known.

Proposition 2.1. Let G be a compact, nonsimply laced, simple, and simply connected Lie group. There exists a simple, simply connected, and simply laced Lie group G', s.t. $G \subset G'$ and $G = (G')^{\rho}$, where ρ is an outer automorphism of G' of order three for $G' = D_4$, and of order two otherwise.

Proof. By the functorial property, we just need to prove it in the Lie algebra level. For the construction of $\mathcal{G} = Lie(G)$ and $\mathcal{G}' = Lie(G')$, one can see [10] for details, where the construction of Lie algebras is determined by the construction of root systems.

Remark. For later use, we list the construction of nonsimply laced root systems via simply laced root systems.

1. $G = C_n = Sp(n), G' = A_{2n-1} = SU(2n)$.
   $\Delta(G') = \{\alpha_i, i = 1, \ldots, 2n-1\}$.
   $Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, where $\rho(\alpha_i) = \alpha_{2n-i}, i = 1, \ldots, n-1, \rho(\alpha_n) = \alpha_n$.
   $\Delta(G) = \{\beta_i = \frac{1}{2}(\alpha_i + \alpha_{2n-i}), i = 1, \ldots, n-1, \beta_n = \alpha_n\}$.

2. $G = B_n = Spin(2n+1), G' = D_{n+1} = Spin(2n+2)$.
   $\Delta(G') = \{\alpha_i, i = 1, \ldots, n+1\}$.
   $Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, where $\rho(\alpha_i) = \alpha_i, i = 3, \ldots, n+1, \rho(\alpha_1) = \alpha_2, \rho(\alpha_2) = \alpha_1$.
   $\Delta(G) = \{\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_2), \beta_i = \alpha_{i+1}, i = 2, \ldots, n\}$.

3. $G = F_4, G' = E_6$.
   $\Delta(G') = \{\alpha_i, i = 1, \ldots, 6\}$.
   $Out(G') = \{1, \rho\} \cong \mathbb{Z}_2$, where $\rho(\alpha_i) = \alpha_{6-i}, i = 1, \ldots, 5$, and $\rho(\alpha_6) = \alpha_6$.
   $\Delta(G) = \{\beta_1 = \frac{1}{2}(\alpha_1 + \alpha_5), \beta_2 = \frac{1}{2}(\alpha_2 + \alpha_4), \beta_3 = \alpha_3, \beta_4 = \alpha_6\}$.

4. $G = G_2, G' = D_4 = Spin(8)$.
   $\Delta(G') = \{\alpha_i, i = 1, \ldots, 4\}$.
   $Out(G') = \langle \rho_1, \rho_2 \rangle \cong S_3$, where ρ_1 interchanges α_1 and α_2, and ρ_2 interchanges α_1 and α_4.
   $\Delta(G) = \{\beta_1 = \frac{1}{3}(\alpha_1 + \alpha_2 + \alpha_4), \beta_2 = \alpha_3\}$.

The Dynkin diagrams of G and G' are given in Figure 9.

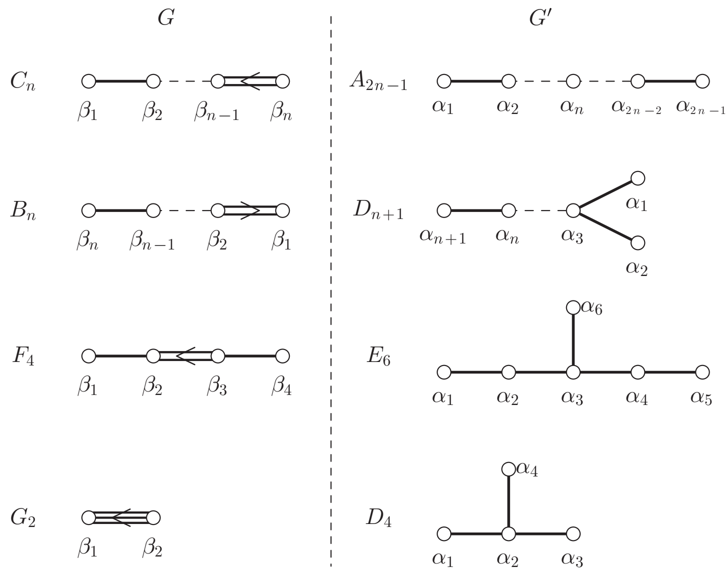

Fig. 9. Nonsimply laced G reduced to simply laced G'.

Remark. Note that $W(G)$ is the subgroup of $W(G')$ fixing the root system $R(G)$ and also the subgroup pointwise fixed by $Out(G')$. For a root α, let $S_\alpha \in W(G)$ be the reflection with respect to α, that is, $S_\alpha(x) = x + (x, \alpha)\alpha$. Thus, as a subgroup of $W(A_{2n-1})$, $W(C_n)$ is generated by $S_{\alpha_i} \circ S_{\alpha_{2n-i}}$ for $i = 1, \ldots, n-1$ and S_{α_n}. As a subgroup of $W(D_{n+1})$, $W(B_n)$ is generated by $S_{\alpha_1} \circ S_{\alpha_2}$ and S_{α_i} for $i = 3, \ldots, n+1$. As a subgroup of $W(E_6)$, $W(F_4)$ is generated by $S_{\alpha_1} \circ S_{\alpha_5}$, $S_{\alpha_2} \circ S_{\alpha_4}$, S_{α_3} and S_{α_6}. As a subgroup of $W(D_4)$, $W(G_2)$ is generated by $S_{\alpha_1} \circ S_{\alpha_2} \circ S_{\alpha_4}$ and S_{α_3}.

In the following, let Σ be a fixed elliptic curve with identity element 0, and fix a primitive d^{th} root of $Jac(\Sigma) \cong \Sigma$, where $d = 2$ for D_n case, $d = 9 - n$ for E_n case, and $d = n + 1$ for A_n case, respectively (see [13]). Recall that for any compact, simple, and simply connected Lie group H, the moduli space of flat H bundles over Σ is

$$\mathcal{M}_\Sigma^H \cong (\Lambda_c(H) \otimes \Sigma)/W(H).$$

For G', the group $Out(G')$ acts on

$$(\Lambda_c(G') \otimes \Sigma)/W(G')$$

naturally. Let $((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$ be the fixed part.

Then, we have a natural map

$$\chi : (\Lambda_c(G) \otimes \Sigma)/W(G) \to ((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}.$$

The image of χ is contained in a connected component of the fixed part.

Lemma 2.2. The map

$$\chi : (\Lambda_c(G) \otimes \Sigma)/W(G) \to ((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$$

is injective.

Proof. It suffices to prove that for any $x, y \in \Lambda(G) \otimes \Sigma$, if $\exists\, w' \in W(G')$, such that $w'(x) = y$, then $\exists\, w \in W(G)$, such that $w(x) = y$. For A_n and D_n cases, this is obvious if we check the root lattices. For E_6 case, we can also check it directly with the help of computer. Of course, we can also check this case by hand following the discussion in Section 2.4.1.

Corollary 2.3. (i) The fixed part $((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$ is determined by the condition $\rho(x) = x$, up to $W(G')$-action, where $x \in \Lambda_c(G') \otimes \Sigma$, and ρ is a generator of $Out(G')$, of order three for $G' = D_4$ and of order two for $G' = A_n,\ E_n$.

(ii) The moduli space $\mathcal{M}^G_\Sigma \cong (\Lambda_c(G) \otimes \Sigma)/W(G)$ is a connected component of the fixed part

$$\left(\mathcal{M}^{G'}_\Sigma\right)^{Out(G')} \cong ((\Lambda_c(G') \otimes \Sigma)/W(G'))^{Out(G')}$$

containing the trivial G' bundle.

Proof. (i) For any $x \in \Lambda_c(G') \otimes \Sigma$, denote by $\bar{x}$ the class in $(\Lambda_c(G') \otimes \Sigma)/W(G')$. Then $\rho(\bar{x}) = \bar{x}$ if and only if there exists $w \in W(G')$, such that $\rho(x) = w(x)$. Thus, $w^{-1}\rho(x) = x$. But $w^{-1}\rho \in Out(G')$ since $Out(G') = Aut(G')/W(G')$. Thus, we can take a new simple root system such that $w^{-1}\rho$ is the generator of the diagram automorphism (the automorphism of order three for D_4).

(ii) By (i), $(\Lambda_c(G) \otimes \Sigma)/W(G)$ and $(\mathcal{M}^{G'}_\Sigma)^{Out(G')}$ are both orbifolds with the same dimension. Thus, the result follows from Lemma 2.2.

If we express the moduli space of flat G bundles over Σ as $(T \times T)/W(G)$, where T is a maximal torus of G, then we have the following corollary.

Corollary 2.4. If two elements of $T \times T$ are conjugate under $W(G')$, then they are also conjugate under $W(G)$.

Another method is to reduce G to its simply laced subgroup G'' of maximal rank, and apply the results for simply laced cases to current situation. In another occasion, we will discuss our moduli space of G-bundles from this aspect in detail. Here, we just mention the following well-known fact from Lie theory.

Proposition 2.5. There exists canonically a simply laced Lie subgroup G'' of G, which is of maximal rank, that is, G'' and G share a common maximal torus. And there is a short exact sequence

$$1 \to W(G'') \to W(G) \to Out(G'') \to 1,$$

where $Out(G'')$ is the outer automorphism group of G''. Thus, if we write the moduli space as $\mathcal{M}_\Sigma^G = (T \times T)/W$, then

$$\mathcal{M}_\Sigma^G = \mathcal{M}_\Sigma^{G''}/Out(G''). \qquad \square$$

Remark. We give this construction of G'' in each case.

(1) For $G = Sp(n),\ G'' = SU(2)^n$. $Out(G'')$ is the group S_n of permutations of the n copies of $SU(2)$ in G''.

(2) For $G = G_2,\ G'' = SU(3)$. $Out(G'')$ is the group $\mathbb{Z}_2$ that exchanges the three-dimensional representation of $SU(3)$ with its dual.

(3) For $G = Spin(2n+1),\ G'' = Spin(2n)$. $Out(G'')$ is the group $\mathbb{Z}_2$ that exchanges the two spin representations of $Spin(2n)$.

(4) For $G = F_4,\ G'' = Spin(8)$. $Out(G'')$ is the triality group S_3 that permutes the three eight-dimensional representations of $Spin(8)$.

3 Flat G Bundles over Elliptic Curves and Rational Surfaces: Nonsimply Laced Cases

In this section, we study case by case the G bundles over elliptic curves and rational surfaces for a nonsimply laced Lie group G.

3.1 The $B_n (n \geq 2)$ bundles

According to the arguments of the previous section, for $G = Spin(2n+1)$ we can take $G' = Spin(2n+2)$, such that $G = (G')^{Out(G')}$.

Let $S = Y_{n+1}$ be a rational surface with a D_{n+1}-configuration [13] which contains Σ as a smooth anticanonical curve. Recall [13] that Y_{n+1} is a blowup of $\mathbb{F}_1$ at $n+1$ points $x_1, \ldots, x_{n+1}$ on Σ, with corresponding exceptional classes $l_1, \ldots, l_{n+1}$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Y_{n+1} is $H^2(Y_{n+1}, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \ldots, l_{n+1}$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^{n+1} l_i)$.

We know from [13] that

$$P_{n+1} := \{x \in H^2(Y_{n+1}, \mathbb{Z}) \mid x \cdot K = x \cdot f = 0\}$$

is a root lattice of D_{n+1} type. We take a simple root system of G' as

$$\Delta(D_{n+1}) = \{\alpha_1 = l_1 - l_2, \alpha_2 = f - l_1 - l_2, \alpha_3 = l_2 - l_3, \ldots, \alpha_{n+1} = l_n - l_{n+1}\}.$$

Let ρ be the generator of $Out(G') \cong \mathbb{Z}_2$, such that $\rho(\alpha_1) = \alpha_2, \rho(\alpha_2) = \alpha_1$ and $\rho(\alpha_i) = \alpha_i$ for $i = 3, \ldots, n+1$.

From [13], we know that the pair (S, Σ) determines a homomorphism

$$u \in Hom(\Lambda(G'), \Sigma)$$

which is given by the restriction map

$$u(\alpha) = \mathcal{O}(\alpha)|_{\Sigma}.$$

Lemma 3.1. Let $u \in Hom(\Lambda(G'), \Sigma)$ correspond to a pair (S, Σ), where S is a surface with a D_{n+1}-configuration. Then $\rho \cdot u = u$ if and only if $2x_1 = 0$.

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_{\Sigma}, u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_{\Sigma} = x_1 - x_2$, and $u(\alpha_2) = \mathcal{O}(f - l_1 - l_2)|_{\Sigma} = -x_1 - x_2$. Hence, $\rho \cdot u = u \Leftrightarrow u(\alpha_1) = u(\alpha_2) \Leftrightarrow x_1 - x_2 = -x_1 - x_2 \Leftrightarrow 2x_1 = 0 \Leftrightarrow x_1$ is one of the four points of order two on the elliptic curve Σ.

As in [13], we denote by $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = D_{n+1}$-surfaces with a fixed anticanonical curve Σ, and by $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with D_{n+1}-configurations (Figure 1). From [13], we know that $\phi : \overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}_{\Sigma}^{G'}$ is an isomorphism.

Corollary 3.2. For $u \in \mathcal{M}_{\Sigma}^{G} \hookrightarrow (\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Y_{n+1}(x_1, \ldots, x_{n+1})$ with $x_1 = 0$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.

Proof. By Lemma 3.1, $u \in (\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$ if and only if $2x_1 = 0$. There are four connected components corresponding to four points of order two on Σ. Since $\mathcal{M}_{\Sigma}^{G}$ is the component containing the trivial G' bundle, we have $x_1 = 0$. Recall (Section 4 of [13]) that $Y_{n+1}(x_1, \ldots, x_{n+1}) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \ldots, x_{n+1}$ are in general position, which implies in particular $x_1 \neq 0$. Hence, $\phi^{-1}(u)$ corresponds to a boundary point.

Denote $S = Y'_{n+1}(x_1 = 0, x_2, \ldots, x_{n+1})$ (or Y'_{n+1} for brevity) the blowup of $\mathbb{F}_1$ at $n+1$ points $x_1 = 0, x_2, \ldots, x_{n+1}$ on Σ, with exceptional divisors $l_1, l_2, \ldots, l_{n+1}$, where $\Sigma \in |-K_S|$. Similar to the simply laced cases, we give the following definition.

Definition 3.3. A B_n-*exceptional system* on S is an n-tuple $(e_1, e_2, \ldots, e_{n+1})$, where e_i's are exceptional divisors such that $e_i \cdot e_j = 0 = e_i \cdot f, i \neq j$ and $y_1 = e_1 \cap \Sigma = 0$ is the identity of Σ. A B_n-*configuration* on S is a B_n-exceptional system $\zeta_{B_n} = (e_1, e_2, \ldots, e_{n+1})$ such that we can consider S as a blowup of $\mathbb{F}_1$ at $n+1$ points $y_1 = 0, y_2, \ldots, y_{n+1}$ on Σ, that is $S = Y'_{n+1}(y_1 = 0, y_2, \ldots, y_{n+1})$, with corresponding exceptional divisors $e_1, e_2, \ldots, e_{n+1}$. When S has a B_n-configuration, we call S a *(rational) surface with a* B_n-*configuration* (see Figure 2).

When $x_2, \ldots, x_{n+1} \in \Sigma$ with $x_i \neq 0$, for all i are in general position (refer to Section 4 of [13] for definition), any B_n-exceptional system on S consists of exceptional curves. Such a surface is called a B_n-*surface*. So a B_n-surface must have a B_n-configuration.

Lemma 3.4. (i) Let S be a rational surface with a B_n-configuration. Then the Weyl group $W(B_n)$ acts on all B_n-exceptional systems on S simply transitively.

(ii) Let S be a B_n-surface. Then the Weyl group $W(B_n)$ acts on all B_n-configurations simply transitively.

Proof. It suffices to prove (i). Let $(e_1, e_2, \ldots, e_{n+1})$ be a B_n-exceptional system on S. By Definition 3.3, $e_i = l_{\sigma(i)}$ or $f - l_{\sigma(i)}$ for $i \neq 1$, where σ is a permutation of $\{2, \ldots, n+1\}$. Note that the Weyl group $W(B_n)$ acts as the group generated by permutations of the n pairs $\{(l_i, f - l_i) \mid i = 2, \ldots, n+1\}$ and interchanging of l_i and $f - l_i$ in each pair $(l_i, f - l_i)_{i \geq 2}$. Then the result follows.

Let $\mathcal{S}(\Sigma, B_n)$ be the moduli space of pairs (S, Σ) where S is a B_n-surface (so the blown-up points $x_1 = 0, x_2, \ldots, x_{n+1}$ are in general position), and $\Sigma \in |-K_S|$. Denote $\mathcal{M}_\Sigma^{B_n}$ the moduli space of flat B_n bundles over Σ. Then applying Corollary 3.2, we have the following identification.

Proposition 3.5. (i) $\mathcal{S}(\Sigma, B_n)$ is embedded into $\mathcal{M}_\Sigma^{B_n}$ as an open dense subset.

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, B_n)} \cong \mathcal{M}_{\Sigma}^{B_n},$$

by including all rational surfaces with B_n-configurations.

Proof. The proof is similar to that in ADE cases [13]. Firstly, we have $\mathcal{M}_{\Sigma}^{B_n} \cong \Lambda_c(B_n) \otimes_{\mathbb{Z}} \Sigma / W(B_n)$, and $\Lambda_c(B_n) \otimes_{\mathbb{Z}} \Sigma / W(B_n) \cong Hom(\Lambda(B_n), \Sigma)/ W(B_n)$ when we fixed the square root of unity of $Jac(\Sigma) \cong \Sigma$. Refer to Section 3 of [13] for the detail.

Secondly, the restriction from S to Σ induces a map (again denoted by ϕ)

$$\phi : \mathcal{S}(\Sigma, B_n) \to Hom(\Lambda(B_n), \Sigma)/ W(B_n).$$

This map is well defined; since by Lemma 3.4, choosing and fixing a B_n-configuration on S is equivalent to choosing and fixing a system of simple roots $\Delta(B_n)$.

Thirdly, the map ϕ is injective. For this, we take a simple root system of B_n as

$$\beta_1 = f - 2\, l_2 \quad \text{and} \quad \beta_k = 2\, \alpha_{k+1} \quad \text{for} \quad 2 \le k \le n.$$

Then the restriction induces an element $u \in Hom(\Lambda(B_n), \Sigma)$, which satisfies the following system of linear equations:

$$\begin{cases} -2\, x_2 = p_1, \\ 2(x_k - x_{k+1}) = p_k, \quad k = 2, \dots, n, \end{cases}$$

where $p_i = u(\beta_i)$. Obviously, the solution of this system of linear equations exists uniquely for given p_i with $1 \le i \le n$.

Finally, statement (ii) comes from Corollary 3.2 and the existence of the solutions to the above system of linear equations.

Remark. The situation here is very similar to that in the compactification theory of the moduli space of (projective) $K3$ surfaces. A natural question is how to extend the global Torelli theorem to the boundary components of a compactification (see, e.g. [11, 16]). If we consider the map $\phi : \mathcal{S}(\Sigma, G) \to \mathcal{M}_{\Sigma}^{G}$ [13] for $G = A_n, D_n$, or E_n as a type of period map, then the main result of [13] is a type of global Torelli theorem. And Proposition 3.5 implies that we can extend the theorem of Torelli type in D_{n+1} case to a boundary component of the natural compactification.

In the following, we let $S = Y_{n+1}(x_1, \dots, x_{n+1})$ be the blowup of $\mathbb{F}_1$ at $n+1$ points. We can construct a Lie algebra bundle on S. Here we do not need the existence of the anticanonical curve Σ. According to Section 2, we have a root system of B_n type, consisting of divisors on S:

$$R(B_n) \triangleq \{\pm(f-2\,l_i), 2(l_i-l_j), \pm 2(f-l_i-l_j) \ \mid \ i \neq j, 2 \leq i, j \leq n+1\}.$$

Thus, we can construct a Lie algebra bundle of B_n-type over S:

$$\mathscr{B}_n \triangleq \mathcal{O}^{\oplus n} \bigoplus_{D \in R(B_n)} \mathcal{O}(D).$$

The fiberwise Lie algebra structure of $\mathscr{B}_n$ is defined as follows (the argument here is the same as that in [13]).

Fix the system of simple roots of R_n as

$$\Delta(B_n) = \{\alpha_1 = f - 2l_2, \alpha_2 = 2(l_2 - l_3), \ldots, \alpha_n = 2(l_n - l_{n+1})\},$$

and take a trivialization of $\mathscr{B}_n$. Then over a trivializing open subset $U, \mathscr{B}_n|_U \cong U \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha \in R_n} \mathbb{C}_\alpha)$. Take a Chevalley basis $\{x^U_\alpha, \alpha \in R_n; h_i, 1 \leq i \leq n\}$ for $\mathscr{B}_n|_U$ and define the Lie algebra structure by the following four relations, namely, Serre's relations on Chevalley basis (see [8], p. 147):

(a) $[h_i h_j] = 0, 1 \leq i, j \leq n$.
(b) $[h_i x^U_\alpha] = \langle \alpha, \alpha_i \rangle x^U_\alpha, 1 \leq i \leq n, \alpha \in R_n$.
(c) $[x^U_\alpha x^U_{-\alpha}] = h_\alpha$ is a $\mathbb{Z}$-linearly combination of $h_1, \ldots, h_n$.
(d) If α, β are independent roots, and $\beta - r\alpha, \ldots, \beta + q\alpha$ are the α-string through β, then $[x^U_\alpha x^U_\beta] = 0$ if $q = 0$, while $[x^U_\alpha x^U_\beta] = \pm(r+1)x^U_{\alpha+\beta}$ if $\alpha + \beta \in R_n$.

Note that $h_i, 1 \leq i \leq n$ are independent of any trivialization, so the relation (a) is always invariant under different trivializations. If $\mathscr{B}_n|_V \cong V \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha \in R_n})$ is another trivialization and f^{UV}_α is the transition function for the line bundle $\mathcal{O}(\alpha)(\alpha \in R_n)$, that is, $x^U_\alpha = f^{UV}_\alpha x^V_\alpha$, then the relation (b) is

$$\left[h_i\left(f^{UV}_\alpha x^V_\alpha\right)\right] = \langle \alpha, \alpha_i \rangle\, f^{UV}_\alpha x^V_\alpha,$$

that is,

$$\left[h_i x^V_\alpha\right] = \langle \alpha, \alpha_i \rangle x^V_\alpha.$$

So (b) is also invariant. Relation (c) is also invariant since $(f^{UV}_\alpha)^{-1}$ is the transition function for $\mathcal{O}(-\alpha)(\alpha \in R_n)$. Finally, (d) is invariant since $f^{UV}_\alpha f^{UV}_\beta$ is the transition function for $\mathcal{O}(\alpha+\beta)(\alpha, \beta \in R_n)$.

Therefore, the Lie algebra structure is compatible with the trivialization. Hence, it is well defined.

When the surface S contains Σ as an anticanonical curve, restricting the above bundle to this anticanonical curve Σ, we obtain a Lie algebra bundle of B_n-type over Σ, which determines uniquely a flat B_n bundle over Σ. On the other hand, when $x_1 = 0$, we can identify these two line bundles $\mathcal{O}_\Sigma(l_1)$ and $\mathcal{O}_\Sigma(f - l_1)$ when restricting them to Σ. Recall that the spinor bundles S^+_{n+1} and S^-_{n+1} of D_{n+1} are defined as follows [12, 13] (here we omit the subscription $n+1$ for brevity):

$$\mathcal{S}^+ = \bigoplus_{D^2=D\cdot K=-1, D\cdot f=1} \mathcal{O}(D) \text{ and}$$
$$\mathcal{S}^- = \bigoplus_{T^2=-2, T\cdot K=0, T\cdot f=1} \mathcal{O}(T).$$

The identification of $\mathcal{O}_\Sigma(l_1) \cong \mathcal{O}_\Sigma(f - l_1)$ induces an identification of these two spinor bundles S^+ and S^-, which is given by (of course, when restricted to Σ)

$$S^+ \otimes \mathcal{O}(-l_1) \cong S^-.$$

From the representation theory, we know that this determines a flat B_n bundle over Σ.

Conversely, if $\mathcal{S}^+|_\Sigma \cong \mathcal{S}^-|_\Sigma$, then we must have $x_1 = 0$ (up to renumbering). For example, we consider the $n = 2$ case. Note that

$$\mathcal{S}^+|_\Sigma \otimes \mathcal{O}(-(0)) = \mathcal{O} \oplus \mathcal{O}((-x_1 - x_2) - (0)) \oplus \mathcal{O}((-x_1 - x_3) - (0)) \oplus \mathcal{O}((-x_2 - x_3) - (0)),$$
$$\mathcal{S}^-|_\Sigma = \mathcal{O}((0) - (x_1)) \oplus \mathcal{O}((0) - (x_2)) \oplus \mathcal{O}((0) - (x_3)) \oplus \mathcal{O}(3(0) - (x_1) - (x_2) - (x_3)).$$

Where for a point $x \in \Sigma, (x)$ means the divisor of degree one, and $\mathcal{O}((x))$ means the line bundle determined by this divisor. Thus, $\mathcal{S}^+_\Sigma \otimes \mathcal{O}(-(0)) = \mathcal{S}^-_\Sigma$ implies that $x_1 = 0$ (up to renumbering). The general case follows from similar arguments.

3.2 The C_n bundles

We take $G = C_n \subset G' = A_{2n-1}$, where $C_n = Sp(n)$ and $A_{2n-1} = SU(2n)$. They satisfy the relation $G = (G')^{Out(G')}$.

Let $S = Z_{2n}$ be a rational surface with an A_{2n-1}-configuration (see [13] or Figure 3) which contains Σ as a smooth anticanonical curve. Recall [13] that Z_{2n} is a (successive) blowup of $\mathbb{F}_1$ at $2n$ points $x_1, \ldots, x_{2n}$ on Σ, with corresponding exceptional classes $l_1, \ldots, l_{2n}$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Z_{2n} is $H^2(Z_{2n},\mathbb{Z})$, which is a lattice with basis $s, f, l_1, \ldots, l_{2n}$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^{2n} l_i)$.

Recall that

$$P_{2n-1} := \{x \in H^2(Z_{2n},\mathbb{Z}) \mid x \cdot K = x \cdot f = x \cdot s = 0\}$$

is a root lattice of A_{2n-1} type. And we can take a simple root system of A_{2n-1} as

$$\Delta(A_{2n-1}) = \{\alpha_i = l_i - l_{i+1} \mid 1 \leq i \leq 2n-1\}.$$

Note that [13] we have used the convention that $\sum_{i=1}^{2n} x_i = 0$.

Let ρ be the generator of $Out(G') \cong \mathbb{Z}_2$, such that $\rho(\alpha_i) = \alpha_{2n-i}$ for $i = 1, \ldots, 2n-1$.

When the above simple root system is chosen, the pair (S, Σ) determines a homomorphism $u \in Hom(\Lambda(G'), \Sigma)$ which is given by the restriction map

$$u(\alpha) = \mathcal{O}(\alpha)|_{\Sigma}.$$

Lemma 3.6. Let $u \in Hom(\Lambda(G'), \Sigma)$ be an element corresponding to a pair (S, Σ), where S is a surface with an A_{2n-1}-configuration. Then $\rho \cdot u = u$ if and only if $n(x_i + x_{2n+1-i}) = 0$ for $i = 1, \ldots, n$.

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_{\Sigma}, u(\alpha_i) = \mathcal{O}(l_i - l_{i+1})|_{\Sigma} = x_i - x_{i+1}$ for $i = 1, \ldots, 2n-1$. Hence, $\rho \cdot u = u \Leftrightarrow u(\alpha_i) = u(\alpha_{2n-i}) \Leftrightarrow x_i - x_{i+1} = x_{2n-i} - x_{2n-i+1} \Leftrightarrow n(x_i + x_{2n-i+1}) = 0$ since $\sum_{i=1}^{2n} x_i = 0$.

As in [13], we denote by $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = A_{2n-1}$-surfaces with a fixed anticanonical curve Σ, and by $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with A_{2n-1}-configurations. From [13], we know that there is an isomorphism $\phi : \overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}_{\Sigma}^{G'}$.

Corollary 3.7. For $u \in \mathcal{M}_{\Sigma}^{G} \hookrightarrow (\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Z_{2n}(x_1, \ldots, x_{2n})$ with $x_i + x_{2n+1-i} = 0$ for $i = 1, \ldots, n$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.

Proof. By Lemma 3.6, $u \in (\mathcal{M}_{\Sigma}^{G'})^{Out(G')}$ if and only if $n(x_i + x_{2n+1-i}) = 0$ for $i = 1, \ldots, n$. There are n^2 connected components corresponding to n^2 points of order n on Σ. Since $\mathcal{M}_{\Sigma}^{G}$ is the component containing the trivial G' bundle, we have $x_i + x_{2n+1-i} = 0$ for $i = 1, \ldots, n$. Recall (Section 4 in [13]) that $Z_{2n}(x_1, \ldots, x_{2n}) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \ldots, x_{2n}$ are in general position, which implies in particular $x_i \neq -x_{2n+1-i}$. Hence, $\phi^{-1}(u)$ corresponds to a boundary point.

Denote $S = Z'_{2n}(\pm x_1, \dots, \pm x_n)$ the blowup of $\mathbb{F}_1$ at n pairs of points $(x_1, -x_1), \dots, (x_n, -x_n)$ on Σ, with n pairs of corresponding exceptional divisors $(l_1, l_1^-), \dots, (l_n, l_n^-)$, where l_i (resp. l_i^-) is the exceptional divisor corresponding to the blowing up at x_i (resp. $-x_i$). Similar to the other cases, we give the following definitions.

Definition 3.8. A C_n-*exceptional system* on S is an n-tuple of pairs

$$((e_1, e_1^-), \dots, (e_n, e_n^-)),$$

where $(e_i, e_i^-) = (l_{\sigma(i)}, l_{\sigma(i)}^-)$ or $(l_{\sigma(i)}^-, l_{\sigma(i)}), i = 1, \dots, n$, and σ is a permutation of $1, \dots, n$. A C_n-*configuration* on S is a C_n-exceptional system $\zeta_{C_n} = ((e_1, e_1^-), \dots, (e_n, e_n^-))$ such that we can blow down successively $e_1^-, \dots, e_n^-, e_n, \dots, e_1$ such that the resulting surface is $\mathbb{F}_1$ (see Figure 4).

We say that $x_1, x_2, \dots, x_n \in \Sigma \subset \mathbb{F}_1$ are n points *in general position*, if they

(i) are distinct points, and

(ii) for any $i, j, x_i + x_j \neq 0$.

Equivalently, $x_1, x_2, \dots, x_n \in \Sigma \subset \mathbb{F}_1$ are in general position if and only if any C_n-exceptional system on $S = Z'_{2n}(\pm x_1, \dots, \pm x_n)$ consists of smooth exceptional curves. Such a surface is called a C_n-*surface*. Thus, a C_n-surface must have a C_n-configuration.

Lemma 3.9. (i) Let S be a surface with a C_n-configuration. Then the Weyl group $W(C_n)$ acts on all C_n-exceptional systems on S simply transitively.

(ii) Let S be a C_n-surface. Then the Weyl group $W(C_n)$ acts on all C_n- configurations on S simply transitively.

Proof. It suffices to prove (i). The Weyl group $W(C_n)$ acts as the group generated by permutations of the n pairs $\{(l_i, l_i^-) \mid i = 1, \dots, n\}$ and interchanging of l_i and l_i^- for each i. From this, we see that $W(C_n)$ acts on all G-configurations simply transitively.

Denote by $\mathcal{S}(\Sigma, C_n)$ the moduli space of pairs (Z'_{2n}, Σ), where Z'_{2n} is a C_n-surface, that is, the blowup of $\mathbb{F}_1$ at $2n$ points $\pm x_1, \dots, \pm x_n$ such that $x_1, \dots, x_n$ are in general position. Denote by $\mathcal{M}_{\Sigma}^{C_n}$ the moduli space of flat C_n bundles over Σ. By Corollary 3.7, we have the following identification.

Proposition 3.10. (i) $\mathcal{S}(\Sigma, C_n)$ is embedded into $\mathcal{M}_{\Sigma}^{C_n}$ as an open dense subset.

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, C_n)} \cong \mathcal{M}_{\Sigma}^{C_n},$$

by including all rational surfaces with C_n-configurations.

Proof. The proof is basically the same as the one in B_n case. We only need to replace the corresponding parts by the following two things. Firstly, according to Section 2, we can take a simple root system as

$$\Delta(C_n) = \{\beta_k = \varepsilon_k - \varepsilon_{k+1},\ 1 \le k \le n-1,\ \beta_n = 2\varepsilon_n\},$$

where $\varepsilon_k = l_k - l_k^-$, $1 \le k \le n$.

Secondly, the restriction map gives us the following system of linear equations:

$$\begin{cases} 4x_n = p_n, \\ 2(x_k - x_{k+1}) = p_k,\ k = 1, \dots, n-1. \end{cases}$$

The solution of this system exists uniquely.

Remark. As in B_n case, the above proposition is also similar to extending the Torelli theorem to a certain boundary component.

Remark. Obviously, this description in Proposition 3.10 coincides with the well-known description of flat C_n bundles over elliptic curves [6]. A flat $C_n = Sp(n)$ bundle over Σ corresponds to n pairs (unordered) of points $(x_i, -x_i), i = 1, \dots, n$ on Σ, uniquely up to isomorphism. And one pair $(x_i, -x_i)$ will determine exactly one point on $\mathbb{CP}^1$, since the rational map determined by the linear system $|2(0)|$ induces a double covering from Σ onto $\mathbb{CP}^1$. So the moduli space of flat C_n bundles over Σ is just isomorphic to $S^n(\mathbb{CP}^1) = \mathbb{CP}^n$, the ordinary projective n space.

As in B_n case, we construct a Lie algebra bundle of C_n type over Z'_{2n}:

$$\mathscr{C}_n = \mathcal{O}^{\oplus n} \bigoplus_{D \in R(C_n)} \mathcal{O}(D),$$

where $R(C_n)$ is the root system of C_n according to Section 2:

$$R(C_n) = \{\pm 2(l_i - l_i^-), \pm((l_i - l_i^-) \pm (l_j - l_j^-)) \mid i \neq j, 1 \leq i, j \leq n\}.$$

Recall [13] that the first fundamental representation bundle of $\mathscr{A}_{2n-1}$ is

$$\mathcal{V}_{2n-1} = \bigoplus_{i=1}^{2n} \mathcal{O}(l_i).$$

The condition that $x_i + x_{2n+1-i} = 0, 1 \leq i \leq n$ is equivalent to an identification of the following two fundamental representation bundles $\wedge^i(\mathcal{V}_{2n-1})$ and $\wedge^{2n-i}(\mathcal{V}_{2n-1})$ with $i = 1, \ldots, n-1$, which is given by (of course, when restricted to Σ)

$$(\wedge^i(\mathcal{V}_{2n-1}))^* \otimes \det(\mathcal{V}_{2n-1}) \cong \wedge^{2n-i}(\mathcal{V}_{2n-1}).$$

Note that when restricted to Σ, the line bundle $det(\mathcal{V}_{2n-1}) = \mathcal{O}(l_1 + \cdots + l_{2n})$ is isomorphic to $\mathcal{O}(nf)|_\Sigma = \mathcal{O}_\Sigma(2n(0))$, by our assumption that $\sum x_i = 0$. This identification determines uniquely a flat C_n bundle over Σ.

3.3 The G_2 bundles

For $G = G_2$, we take $G' = D_4 = Spin(8)$ such that $G = (G')^{Out(G')}$.

Let $S = Y_4$ be a rational surface with a D_4-configuration [13] which contains Σ as a smooth anticanonical curve. Recall ([13] or Figure 5) that Y_4 is a (successive) blowup of $\mathbb{F}_1$ at four points $x_1, \ldots, x_4$ on Σ, with corresponding exceptional classes $l_1, \ldots, l_4$. Let f be the class of fibers in $\mathbb{F}_1$, and s be the section such that $0 = s \cap \Sigma$ is the identity element of Σ. The Picard group of Y_4 is $H^2(Y_4, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \ldots, l_4$. The canonical line bundle $K = -(2s + 3f - \sum_{i=1}^4 l_i)$.

Recall that

$$P_4 := \{x \in H^2(Y_4, \mathbb{Z}) \mid x \cdot K = x \cdot f = 0\}$$

is a root lattice of D_4-type. And we can take a simple root system of D_4 as

$$\Delta(D_4) = \{\alpha_1 = l_1 - l_2, \alpha_2 = f - l_1 - l_2, \alpha_3 = l_2 - l_3, \alpha_4 = l_3 - l_4\}.$$

Let $\rho \in Out(G') \cong S_3$ (the permutation group of three letters) be the triality automorphism of order three, such that $\rho(\alpha_1) = \alpha_2, \rho(\alpha_2) = \alpha_4, \rho(\alpha_4) = \alpha_1$, and $\rho(\alpha_3) = \alpha_3$.

When the above simple root system is chosen, the pair (S, Σ) determines a homomorphism $u \in Hom(\Lambda(G'), \Sigma)$ which is given by the restriction map

$$u(\alpha) = \mathcal{O}(\alpha)|_\Sigma.$$

Lemma 3.11. Let $u \in Hom(\Lambda(G'), \Sigma)$ correspond to the pair (S, Σ), where S is a surface with a D_4-configuration. Then $\rho \cdot u = u$ if and only if $2x_1 = 0$ and $x_1 + x_4 = x_2 + x_3$.

Proof. Since u is the restriction map: $\alpha_i \mapsto \mathcal{O}(\alpha_i)|_\Sigma, u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_\Sigma = x_1 - x_2$, $u(\alpha_2) = -x_1 - x_2, u(\alpha_4) = x_3 - x_4$, and $u(\alpha_3) = x_2 - x_3$. Hence, $\rho \cdot u = u \Leftrightarrow u(\alpha_1) = u(\alpha_2) = u(\alpha_4) \Leftrightarrow x_1 - x_2 = -x_1 - x_2 = x_3 - x_4 \Leftrightarrow 2x_1 = 0$ and $x_1 + x_4 = x_2 + x_3$.

Denote by $\mathcal{S}(\Sigma, G')$ the moduli space of $G' = D_4$-surfaces with a fixed anticanonical curve Σ, and by $\overline{\mathcal{S}(\Sigma, G')}$ the natural compactification by including all rational surfaces with D_4-configurations. From [13], we know that $\overline{\mathcal{S}(\Sigma, G')} \xrightarrow{\sim} \mathcal{M}_\Sigma^{G'}$. Let ϕ be the isomorphism.

Corollary 3.12. For $u \in \mathcal{M}_\Sigma^G \hookrightarrow (\mathcal{M}_\Sigma^{G'})^{Out(G')}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')}$ represents a class of surfaces $Y_4(x_1, \dots, x_4)$ with $x_1 = 0$ and $x_4 = x_2 + x_3$, and such a surface corresponds to a boundary point in the moduli space, that is, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, G')} \backslash \mathcal{S}(\Sigma, G')$.

Proof. By Lemma 3.11, $u \in (\mathcal{M}_\Sigma^{G'})^{Out(G')}$ if and only if $2x_1 = 0$ and $x_1 + x_4 = x_2 + x_3$. There are four connected components corresponding to four points of order two on Σ. Since $\mathcal{M}_\Sigma^G$ is the component containing the trivial G' bundle, we have $x_1 = 0$ and $x_4 = x_2 + x_3$. Recall that $Y_4(x_1, \dots, x_4) \in \mathcal{S}(\Sigma, G')$ if and only if $0, x_1, \dots, x_4$ are in general position, which implies in particular $x_1 \neq 0$. Hence $\phi^{-1}(u)$ corresponds to a boundary point.

Denote $S = Y_4'(x_1, \dots, x_4)$ the blowup of $\mathbb{F}_1$ at four points $x_1, \dots, x_4$ on Σ, with $x_1 = 0$ and $x_4 = x_2 + x_3$. Let $l_1, \dots, l_4$ be the corresponding exceptional classes. We give the following definition.

Definition 3.13. A G_2-*exceptional system* on S is an ordered triple (e_1, e_2, e_3, e_4) of exceptional divisors such that $e_i \cdot e_j = 0 = e_i \cdot f, i \neq j$ and $y_1 = 0,\ y_4 = y_2 + y_3$, where $y_i = e_i \cdot \Sigma$. A G_2-*configuration* on S is a G_2-exceptional system $\zeta_{G_2} = (e_1, e_2, e_3, e_4)$ such that we can consider S as a blowup of $\mathbb{F}_1$ at these four points $y_1 = 0, y_2, y_3, y_4$ on Σ, that is $S = Y_4'(y_1 = 0, y_2, y_3, y_4)$, with corresponding exceptional divisors e_1, e_2, e_3, e_4. When S has a G_2-configuration (of course $\Sigma \in |-K_S|$), we call S a *(rational) surface with a* G_2-*configuration*. For $S = Y_4'(x_1, \dots, x_4)$ with $x_1 = 0$ and $x_4 = x_2 + x_3$, when $x_1, \pm x_2, \pm x_3, \pm x_4$ are distinct points on Σ, any G_2-exceptional system on S consists of exceptional curves. Such a surface is called a G_2-*surface*. So a G_2-surface must have a G_2-configuration. These four points $x_1, x_2, x_3, x_4 \in \Sigma$ are said to be *in general position*.

A G_2-configuration is illustrated in Figure 6.

Lemma 3.14. (i) Let $S = Y_4'(x_1, \ldots, x_4)$ with $x_1 = 0$ and $x_4 = x_2 + x_3$ be a surface with a G_2-configuration. Then the Weyl group $W(G_2)$ acts on all G_2-exceptional systems on S simply transitively.

(ii) Let S be a G_2-surface. Then the Weyl group $W(G_2)$ acts on all G_2-configurations on S simply transitively.

Proof. It suffices to prove (i). By an explicit computation, there are $12 G_2$-configurations $(l_1, l_2, l_3, l_4), (f - l_1, f - l_2, f - l_3, f - l_4), (f - l_1, f - l_2, l_4, l_3), (f - l_1, l_4, f - l_2, l_3)$, and so on. The rule is keeping the relation $x_2 + x_3 = x_4$ fixed. The Weyl group $W(G_2)$ is the automorphism group of the subroot system A_2 with simple roots $\{3(l_2 - l_3), 3(l_3 - (f - l_4))\}$, so $W(G_2) \cong \mathbb{Z}_2 \rtimes W(A_2) = \mathbb{Z}_2 \rtimes S_3$. We can also consider $W(G_2)$ as the subgroup of $W(D_4)$ generated by two elements $S_{\alpha_1} S_{\alpha_2} S_{\alpha_4}$ and S_{α_3}, where S_α means the reflection with respect to a root α of D_4. Thus, we can directly check that $W(G_2)$ acts on all G_2-exceptional systems simply transitively.

Proposition 3.15. Let $\mathcal{S}(\Sigma, G_2)$ be the moduli space of pairs (Y_4', Σ), where Y_4' is a G_2-surface, and $\mathcal{M}_\Sigma^{G_2}$ be the moduli space of flat G_2 bundles over Σ. Then

(i) $\mathcal{S}(\Sigma, G_2)$ is embedded into $\mathcal{M}_\Sigma^{G_2}$ as an open dense subset;

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, G_2)} \cong \mathcal{M}_\Sigma^{G_2},$$

by including all rational surfaces with G_2-configurations.

Proof. We just noted that only the following two things are different from their counterparts of the proofs in B_n, C_n cases.

(i) Take a simple root system of G_2 as

$$\Delta(G_2) = \{\beta_1 = f - 2l_2 + l_3 - l_4,\ \beta_2 = 3(l_2 - l_3)\}.$$

(ii) Then the restriction to Σ gives us the following system of linear equations:

$$\begin{cases} 3x_2 = -p_1, \\ 3(x_2 - x_3) = p_2. \end{cases}$$

As before, we construct a Lie algebra bundle of G_2-type over $S = Y_4'$. For brevity, denote $\varepsilon_1 = l_2,\ \varepsilon_2 = l_3$, and$\varepsilon_3 = f - l_4$. Then

$$\mathscr{G}_2 = \mathcal{O}^{\bigoplus 2} \bigoplus_{D \in R(G_2)} \mathcal{O}(D),$$

where $R(G_2)$ is the root system of G_2:

$$R(G_2) = \{\pm 3(\varepsilon_i - \varepsilon_j), \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k) \mid i \neq j \neq k, 1 \leq i, j, k \leq 3\}.$$

Recall [12] the three fundamental representation bundles of rank 8 of D_4 are defined as:

$$\begin{cases} \mathscr{W}_4 = \bigoplus_{C^2 = C\cdot K = -1, C\cdot f = 0} \mathcal{O}(C), \\ \mathcal{S}_4^+ = \bigoplus_{D^2 = D\cdot K = -1, D\cdot f = 1} \mathcal{O}(D), \\ \mathcal{S}_4^- = \bigoplus_{T^2 = -2, T\cdot K = 0, T\cdot f = 1} \mathcal{O}(T). \end{cases}$$

The conditions $x_1 = 0$, $x_4 = x_2 + x_3$ enable us to identify S_4^+, S_4^-, and $\mathscr{W}_4$ when restricted to Σ, by

$$S_4^+ \otimes \mathcal{O}(-l_1) \cong S_4^- \text{ and } S_4^+ \cong \mathscr{W}_4 \otimes \mathcal{O}(s).$$

Likewise, these identifications determine uniquely a flat G_2 bundle over Σ. Conversely, the identification of these three bundles restricted to Σ implies the conditions $x_1 = 0$ and $x_4 = x_2 + x_3$ (up to renumbering). Note that

$$\begin{aligned} \mathscr{W}_4|_\Sigma &= \bigoplus \mathcal{O}_\Sigma(l_i) \bigoplus \mathcal{O}_\Sigma(f - l_i) = \bigoplus \mathcal{O}((x_i)) \bigoplus \mathcal{O}((-x_i)), \\ \mathcal{S}_4^-|_\Sigma &= \bigoplus_i \mathcal{O}((0) - (x_i)) \bigoplus_j \mathcal{O}(3(0) - \sum_{i\neq j}(x_i)), \text{ and} \\ \mathcal{S}_4^+|_\Sigma &= \mathcal{O}((0)) \bigoplus_{i \neq j} \mathcal{O}((-x_i - x_j)) \bigoplus \mathcal{O}\left(\left(-\sum x_i\right)\right). \end{aligned}$$

So $\mathscr{W}_4|_\Sigma = \mathcal{S}_4^-$ implies $x_1 = 0$, and $\mathscr{W}_4|_\Sigma = \mathcal{S}_4^+$ implies $x_4 = x_2 + x_3$.

3.4 The F_4 bundles

First we recall some fundamental facts on E_6 root systems and cubic surfaces, which are of independent interest.

3.4.1 *The root system of E_6 revisited*

The relation between the root system of E_6-type and smooth cubic surfaces in $\mathbb{CP}^3$ has been studied for a very long time [2, 7, 15]. There are 27 *lines* on such a cubic surface S (a curve on S is a line if and only if it is an exceptional curve). And every E_6-*exceptional system* on S is an ordered 6-tuples of lines $(e_1, \ldots, e_6)$ which are pairwise disjoint. The

Weyl group $W(E_6)$ is the symmetry group of all E_6-exceptional systems, that is, $W(E_6)$ acts simply transitively on the set of all E_6-exceptional systems. Now we consider the unordered 6-tuple $L = \{e_1, \ldots, e_6\}$. There are 72 such 6-tuples. This corresponds to 36 *Schläfli's double-sixes* $\{L; L'\}$ [7]. In the following, we consider a cubic surface S as the blowup of $\mathbb{P}^2$ at six points $x_1, \ldots, x_6$ in general position, that is $S = X_6(x_1, \ldots, x_6)$, with corresponding exceptional curves $l_1, \ldots, l_6$. Fix a simple root system of E_6 as

$$\Delta(E_6) = \{\alpha_1, \ldots, \alpha_6\},$$

where $\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = h - l_1 - l_2 - l_3$, and $\alpha_i = l_{i-1} - l_i$, for $i = 4, 5, 6$ [13].

Lemma 3.16. One double-six $\{L; L'\}$ corresponds to exactly one positive root of E_6.

Proof. First take $L_0 = \{l_1, \ldots, l_6\}$, then $L_0' = \{l_1', \ldots, l_6'\} = s_{\alpha_0}(L_0)$, where $\alpha_0 = 2h - \sum l_i$ is a positive root and $l_i' = s_{\alpha_0}(l_i) = 2h - \sum_{j \neq i} l_j$. $\{L_0; L_0'\}$ forms a double-six and $\alpha_0 (\succ 0)$ is uniquely determined by $\{L_0; L_0'\}$, since $W(E_6)$ acts simply and transitively. If $L = g(L_0)$ with $g \in W(E_6)$, then $\{g(L_0); g(L_0')\}$ is also a double-six. Let $g(L_0') = S_\alpha(g(L_0))$, then $L_0' = (g^{-1} S_\alpha g)(L_0)$. So $g^{-1} S_\alpha g = S_{\alpha_0}$. Then $S_\alpha = g S_{\alpha_0} g^{-1} = S_{g(\alpha_0)}$. This implies $\alpha = \pm g(\alpha_0)$. Take $\alpha \succ 0$. Now, if $\alpha = \alpha_0$, then by a result on page 44 of [9], $g \in S_6$, that is, g is a permutation of the six lines l_i's. Thus $\{L; L'\}$ and $\{L_0; L_0'\}$ are the same one.

Remark. Let ρ be an outer automorphism of E_6 of order two, such that $\rho(\alpha_1) = \alpha_6, \rho(\alpha_2) = \alpha_5$ and ρ fixes other simple roots. Consider F_4 as the fixed part of E_6 by ρ. Then the coroot lattice $\Lambda_c(F_4)$ of F_4 is

$$\begin{aligned}
\Lambda_c(F_4) &= \Lambda_c(E_6)^\rho \\
&= \Lambda(E_6)^\rho \\
&= \left\{ah + \sum a_i l_i \mid a_1 + a_6 = a_2 + a_5 = a_3 + a_4 = -a\right\} \\
&= \mathbb{Z}\langle h - l_1 - l_2 - l_3, l_1 - l_6, l_2 - l_5, l_3 - l_4\rangle \\
&= \Lambda(D_4).
\end{aligned}$$

And the Weyl group of F_4 is

$$\begin{aligned}
W(F_4) &= \{w \in W(E_6) \mid w \text{ preserves } \Lambda_c(F_4) = \Lambda(D_4)\} \\
&= Aut(\Lambda(D_4)) \\
&= S_3 \rtimes W(D_4).
\end{aligned}$$

Remark. If three lines e_1, e_2, e_3 pairwise intersect, we say that they form a *triangle*. Denote by $\Delta = \{e_1, e_2, e_3\}$ a (unordered) triangle, and by $\overrightarrow{\Delta} = (e_1, e_2, e_3)$ an ordered triangle. Every line belongs to five triangles, so there are $27 \times 5/3 = 45$ triangles. And if $\{e_1, e_2, e_3\}$ is a triangle, then $-K = e_1 + e_2 + e_3$. $W(E_6)$ acts on all these 45 triangles transitively, and $W(F_4)$ is the isotropy subgroup of the triangle $\Delta_0 = \{h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4\}$. Moreover, $W(D_4)$ is the isotropy subgroup of the ordered triangle $\overrightarrow{\Delta}_0 = (h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4)$. The reason is the following.

Let $\Delta = \{e_1, e_2, e_3\}$ and $\Delta' = \{f_1, f_2, f_3\}$ be any two triangles. Since $K^2 = 3$, the position of these two triangles must be one of the following two cases: (1) They have a common edge and other edges do not intersect. (2) Each edge of Δ intersects with exactly one edge of Δ'. So we just check two special triangles in the above cases. What remains to do is a direct checking.

From the above, we can easily write down the 45 (left or right) cosets of $W(F_4)$ in $W(E_6)$.

3.4.2 *F_4 bundles and rational surfaces*

For $G = F_4$, we take $G' = E_6$, such that $F_4 = (E_6)^{Out(E_6)}$.

Let $S = X_6(x_1, \dots, x_6)$ be a surface with an E_6-configuration (Figure 7), that is, S is a blowup of $\mathbb{P}^2$ at six points $x_1, \dots, x_6 \in \Sigma$, where $\Sigma \in |-K_S|$. Take the simple root system $\Delta(E_6)$ and $\rho \in Out(E_6)$ just as in Section 2.4.1.

Once a simple root system is fixed, the restriction from S to Σ induces a homomorphism $u \in Hom(\Lambda(E_6), \Sigma)$.

Lemma 3.17. Let $u \in Hom(\Lambda(E_6), \Sigma)$ be an element corresponding to a pair (S, Σ), where S is a surface with an E_6-configuration. Then $\rho \cdot u = u$ if and only if $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$.

Proof. Since u is induced by the restriction to Σ, $u(\alpha_1) = \mathcal{O}(l_1 - l_2)|_\Sigma = x_1 - x_2$, $u(\alpha_2) = x_2 - x_3$, $u(\alpha_5) = x_4 - x_5$, $u(\alpha_6) = x_5 - x_6$. Therefore, $\rho \cdot u = u \Leftrightarrow u(\alpha_1) = u(\alpha_6), u(\alpha_2) = u(\alpha_5) \Leftrightarrow x_1 + x_6 = x_2 + x_5 = x_3 + x_4$.

Denote by $\mathcal{S}(\Sigma, E_6)$ the moduli space of $G' = E_6$-surfaces [13] with a fixed anticanonical curve Σ, and by $\overline{\mathcal{S}(\Sigma, E_6)}$ the natural compactification by including all rational surfaces with E_6-configurations. From [13], we know that there is an isomorphism $\phi : \overline{\mathcal{S}(\Sigma, E_6)} \xrightarrow{\sim} \mathcal{M}_\Sigma^{E_6}$. Thus, we have the following corollary.

Corollary 3.18. For $u \in \mathcal{M}_\Sigma^{F_4} \subset (\mathcal{M}_\Sigma^{E_6})^{Out(E_6)}$, $\phi^{-1}(u) \in \overline{\mathcal{S}(\Sigma, E_6)}$ represents a class of surfaces $X_6(x_1, \ldots, x_6)$ with $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$.

Denote $S = X_6'(x_1, \ldots, x_6)$ the blowup of $\mathbb{P}^2$ at six points $x_1, \ldots, x_6$ on Σ which satisfies the condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4$, with corresponding exceptional classes $l_1, \ldots, l_6$. The condition $x_1 + x_6 = x_2 + x_5 = x_3 + x_4 := p$ implies that the three lines L_{16}, L_{25}, and L_{34} in $\mathbb{P}^2$ intersect at one point $-p \in \Sigma$, where L_{ij} means the line in $\mathbb{P}^2$ passing through two points x_i and x_j. So after blowing up $\mathbb{P}^2$ at $x_i \in \Sigma, 1 \leq i \leq 6$, the three (-1) curves $h - l_1 - l_6, h - l_2 - l_5$, and $h - l_3 - l_4$ intersect at one point $-p \in \Sigma$. So they form a special triangle (see Section 2.4.1). As before, we give the following definition.

Definition 3.19. An F_4-*exceptional system* on $S = X_6'$ is a 6-tuple $(e_1, \ldots, e_6)$ consisting of six exceptional divisors which are pairwise disjoint, such that $y_1 + y_6 = y_2 + y_5 = y_3 + y_4$, where $\mathcal{O}_\Sigma(y_i) = \mathcal{O}(e_i)|_\Sigma$. And an F_4-*configuration* $\zeta_{F_4} = (e_1, \ldots, e_6)$ just means an F_4-exceptional system on S such that we can consider S as a blowup of $\mathbb{P}^2$ at six points $y_1, \ldots, y_6$ with corresponding exceptional divisors $e_1, \ldots, e_6$. For $S = X_6'(x_1, \ldots, x_6)$, when $x_1, \ldots, x_6$ are in general position, any F_4-exceptional system on S consists of exceptional curves. Such a surface is called an F_4-*surface*.

So an F_4-surface is automatically an E_6-surface (namely, a del Pezzo surface of degree three). And any F_4-exceptional system on an F_4-surface is always an F_4-configuration. See Figure 8 for an F_4-configuration.

According to the discussions in Section 2.4.1, the Weyl group $W(F_4)$ is the automorphism group of the subroot system of type D_4 with simple roots $\{l_1 - l_6, l_2 - l_5, l_3 - l_4, h - l_1 - l_2 - l_3\}$, and $W(F_4) \cong S_3 \ltimes W(D_4)$. Therefore, we have the following lemma.

Lemma 3.20. (i) Let $S = X_6'$ be a surface with an F_4-configuration. Then the Weyl group $W(F_4)$ acts on all F_4-exceptional systems on S simply transitively.

(ii) Moreover, if S is an F_4-surface, then the Weyl group $W(F_4)$ acts on all F_4-configurations on S simply transitively.

Proposition 3.21. Let $\mathcal{S}(\Sigma, F_4)$ be the moduli space of pairs (X_6', Σ), where X_6' is an F_4-surface containing Σ as an anticanonical curve, and $\mathcal{M}_\Sigma^{F_4}$ be the moduli space of flat F_4 bundles over Σ. Then

(i) $\mathcal{S}(\Sigma, F_4)$ is embedded into $\mathcal{M}_{\Sigma}^{F_4}$ as an open dense subset.

(ii) Moreover, this embedding can be extended naturally to an isomorphism

$$\overline{\mathcal{S}(\Sigma, F_4)} \cong \mathcal{M}_{\Sigma}^{F_4},$$

by including all rational surfaces with F_4-configurations.

Proof. Firstly, we can take the simple root system of F_4 as

$$\Delta(F_4) = \{\beta_1, \beta_2, \beta_3, \beta_4\},$$

where $\beta_1 = l_1 - l_2 + l_5 - l_6, \beta_2 = l_2 - l_3 + l_4 - l_5, \beta_3 = 2(h - l_1 - l_2 - l_3)$, and $\beta_4 = 2(l_3 - l_4)$.

Secondly, the restriction to Σ induces the following system of linear equations:

$$\begin{cases} x_1 - x_2 + x_5 - x_6 = p_1, \\ x_2 - x_3 + x_4 - x_5 = p_2, \\ 2(-x_1 - x_2 - x_3) = p_3, \\ 2(x_3 - x_4) = p_4, \\ x_1 + x_6 = x_2 + x_5 = x_3 + x_4. \end{cases}$$

Since the determinant is nonzero, the result follows by the same argument as in B_n case.

The Lie algebra bundle of type F_4 over X_6' can be constructed as (for brevity, we denote $\varepsilon_1 = l_2 - l_3 + l_4 - l_5, \varepsilon_2 = l_2 + l_3 - l_4 - l_5, \varepsilon_3 = 2h - 2l_1 - l_2 - l_3 - l_4 - l_5$, and $\varepsilon_4 = 2h - 2l_6 - l_2 - l_3 - l_4 - l_5$)

$$\mathscr{F}_4 = \mathcal{O}^{\bigoplus 4} \bigoplus_{D \in R(F_4)} \mathcal{O}(D),$$

where $R(F_4)$ is the root system of F_4:

$$R(F_4) = \left\{ \pm\varepsilon_i,\ \pm(\varepsilon_i \pm \varepsilon_j),\ \pm\frac{1}{2}(\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4) \mid i \neq j \right\}.$$

Remark. The 27 lines determine the 27-dimensional fundamental representation of E_6. Restricted to Σ, they give us a representation bundle of rank 27 (of $\mathscr{F}_4$) over Σ. The weights associated to the three special lines $h - l_1 - l_6, h - l_2 - l_5, h - l_3 - l_4$ restrict to zero and these three weights add to zero before restriction (since $(h - l_1 - l_6) + (h - l_2 - l_5) + (h - l_3 - l_4) = -K$). The remaining 24 weights associated to other 24 lines restrict to the 24 short roots of $\mathscr{F}_4$. The 24 lines and a rank 2 bundle V determine the 26-dimensional irreducible fundamental representation U of $\mathscr{F}_4$. Here V is determined as follows. Since

$\mathcal{O}_\Sigma(h - l_1 - l_6) = \mathcal{O}_\Sigma(h - l_2 - l_5) = \mathcal{O}_\Sigma(h - l_3 - l_4) = \mathcal{O}_\Sigma((-p))$, taking the trace, we have the following exact sequence:

$$0 \to \ker(\mathrm{tr}) \to \mathcal{O}_\Sigma((-p))^{\bigoplus 3} \to \mathcal{O}_\Sigma((-p)) \to 0.$$

Then we take $V = \ker(\mathrm{tr})$.

For more details on the 26-dimensional fundamental representation of F_4, refer to [1].

4 Conclusion

We summarize our results in [13] and this paper as follows. Let Σ be a fixed elliptic curve with identity $0 \in \Sigma$. Let G be any compact, simple, and simply connected Lie groups, simply laced or not. Denote by $\mathcal{S}(\Sigma, G)$ the moduli space of G-surfaces containing a fixed anticanonical curve Σ. Denote by $\mathcal{M}_\Sigma^G$ the moduli space of flat G bundles over Σ. Then we have the following theorem.

Theorem 4.1. (i) We can construct Lie algebra $Lie(G)$-bundles over each G-surface.

(ii) The restriction of these Lie algebra bundles to the anticanonical curve Σ induces an embedding of $\mathcal{S}(\Sigma, G)$ into $\mathcal{M}_\Sigma^G$ as an open dense subset.

(iii) This embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$, where $\overline{\mathcal{S}(\Sigma, G)}$ is a natural and explicit compactification of $\mathcal{S}(\Sigma, G)$, by including all rational surfaces with G-configurations.

Remark. (i) The result is known for $G = E_n (n = 6, 7, 8)$ case (see [3, 4, 6]).

(ii) We have mentioned in the beginning of Section 1 that there is another reduction of the nonsimply laced cases to simply laced cases. In fact, using this reduction, we will obtain the same result, just following the steps as above.

According to a theorem of [14], the moduli space $\mathcal{S}(\Sigma, G)$ is a weighted projective space. Thus, the compactification $\overline{\mathcal{S}(\Sigma, G)}$ is a weighted projective space. Conversely, we believe that the above identification between $\mathcal{S}(\Sigma, G)$ and $\overline{\mathcal{S}(\Sigma, G)}$ will give us another proof for this theorem. This is already done in E_n case by [3, 4, 6] and so on.

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