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Moduli of bundles over rational surfaces and elliptic curves I: Simply laced cases

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Abstract

It is well known that del Pezzo surfaces of degree $9 - n$ one-to-one correspond to flat E_n bundles over an elliptic curve. In this paper, we construct ADE-bundles over a broader class of rational surfaces that we call ADE-surfaces, and extend the above correspondence to all flat G-bundles over an elliptic curve, where G is any simply laced, simple, compact and simply connected Lie group. In what follows, we will construct G-bundles for a non-simply laced Lie group G over these rational surfaces, and extend the above correspondence to non-simply laced cases.

1. *Introduction*

Let S be a smooth rational surface. If the anti-canonical line bundle $-K_S$ is ample, then S is called a *del Pezzo surface*. It is well known that a del Pezzo surface can be classified as a blow-up of $\mathbb{CP}^2$ at $n(n \leqslant 8)$ points in general position or $\mathbb{CP}^1 \times \mathbb{CP}^1$. When these blown-up points are in *almost general position*, such a surface is called a *generalized del Pezzo surface*, according to Demazure [**7**]. It is also well known that the sub-lattice $K_S^{\perp}$ of $\mathrm{Pic}(S)$ is a root lattice of type E_n. For more results on (generalized) del Pezzo surfaces one can see [**7**, **22**]. Thus there is a natural Lie algebra bundle of type E_n over S. By restriction to a fixed smooth anti-canonical curve Σ, one obtains a flat E_n bundle over Σ. Moreover, Donagi [**8**, **9**] and Friedman, Morgan and Witten [**11**, **12**] prove that the moduli space of del Pezzo surfaces with a fixed anti-canonical curve Σ can be identified with the moduli space of flat E_n bundles over this elliptic curve Σ.

In this paper, we extend this correspondence to all compact, simple, simply laced and simply connected Lie groups and to a broader class of rational surfaces, that are called ADE-surfaces. This paper contains parts of the preprint [**17**], especially the construction of Lie algebra bundles and their (fundamental) representation bundles, and we shall refer to [**17**] for some of the proofs. In the following we sketch the contents briefly.

In Section 2, we first analyze the structure of the Picard lattice of a rational surface which is a blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_1$ at some points. We shall see that there is a sub-lattice of the Picard lattice that is a root lattice of ADE-type.

Next we generalize the definition of del Pezzo surfaces to that of ADE-surfaces, where an E_n surface is just a del Pezzo surface of degree $9 - n$. Roughly speaking, an ADE-surface S is a rational surface with a smooth rational curve C on S such that the sub-lattice $\langle K_S, C\rangle^{\perp}$ of $\mathrm{Pic}(S)$ is an irreducible root lattice (see Definition 2.5). The condition in Definition 2.5 implies that $C^2 = -1, 0$ or 1 and that the sub-lattice $\langle K_S, C\rangle^{\perp}$ is a root lattice of type E_n, D_n or A_n, respectively (Proposition 2.6). Therefore such a surface is called a rational surface of E_n-type, D_n-type or A_n-type accordingly.

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Note that the definition of an E_n surface implies that, after blowing down the (-1) curve C, the anti-canonical line bundle $-K$ will be ample. Thus the resulting surface is just a del Pezzo surface. Thus the definition of ADE-surfaces naturally generalizes that of del Pezzo surfaces.

After this, we prove that an ADE-surface is nothing but a blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at some points in general position. This gives us an explicit construction for any ADE-surface.

In Section 3, we construct Lie algebra bundles of ADE-type, and their natural representation bundles over those surfaces discussed in Section 2. By a Lie algebra bundle over a surface S, we mean a vector bundle that has a fiberwise Lie algebra structure, and this structure is compatible with any trivialization. Similarly, by a representation bundle, we mean a vector bundle that is a fiberwise representation of a Lie algebra bundle, and this fiberwise representation is compatible with any trivialization.

More precisely, let S be an ADE-surface. Since the sub-lattice $\langle K_S, C\rangle^{\perp}$ of $\mathrm{Pic}(S)$ is a root lattice, we can explicitly construct a natural Lie algebra bundle of corresponding type over S, using the root system of the root lattice $\langle K_S, C\rangle^{\perp}$. Using the lines and rulings on S, we can also construct natural fundamental representation bundles over S.

In Section 4, we shortly recall some well-known facts about flat G-bundles over elliptic curves.

In Section 5, we relate the above Lie algebra bundles of ADE-type over ADE rational surfaces to flat G-bundles over an elliptic curve Σ, where G is a compact Lie group of corresponding type. If an ADE rational surface S contains a fixed smooth elliptic curve Σ as an anti-canonical curve then, by restriction, one obtains a flat ADE-bundle over Σ. We can prove that this restriction identifies the moduli space of flat ADE-bundles over Σ and the moduli space of the pairs $(S, \Sigma \in |-K_S|)$ with extra structure ζ_G which is called a *G-configuration* (Definition 5.2). Our main result in this paper is the following theorem.

THEOREM 1.1. *Let Σ be a fixed elliptic curve and let G be a simple, compact, simply laced and simply connected Lie group. Denote by $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is an* ADE *rational surface with $\Sigma \in |-K_S|$. Denote by $\mathcal{M}_\Sigma^G$ the moduli space of flat G-bundles over Σ. Then, by restriction, we have the following properties.*

(i) *The moduli space $\mathcal{S}(\Sigma, G)$ can be embedded into $\mathcal{M}_\Sigma^G$ as an open dense subset.*

(ii) *There exists a natural and explicit compactification for $\mathcal{S}(\Sigma, G)$, denoted by $\overline{\mathcal{S}(\Sigma, G)}$, such that this embedding can be extended to an isomorphism from $\overline{\mathcal{S}(\Sigma, G)}$ onto $\mathcal{M}_\Sigma^G$.*

(iii) *Any surface corresponding to a boundary point in $\overline{\mathcal{S}(\Sigma, G)} \setminus \mathcal{S}(\Sigma, G)$ is equipped with a G-configuration, and, on such a surface, any smooth rational curve has a self-intersection number at least -2. Furthermore, in the E_n case, all (-2) curves form chains of ADE-type, and the anti-canonical model of such a surface admits at worst ADE-singularities.*

Physically, if $G = E_n$ is a simple subgroup of $E_8 \times E_8$, then these G-bundles are related to the duality between F-theory and string theory. Among other things, this duality predicts that the moduli of flat E_n bundles over a fixed elliptic curve Σ can be identified with the moduli of del Pezzo surfaces with a fixed anti-canonical curve Σ. For details, one can consult [**8**, **9**, **11**, **12**]. Our result can be considered as a test of the above duality for other Lie groups.

As an application, we have a more intuitive explanation for the well-known moduli space $\mathcal{M}_\Sigma^G$ of flat G-bundles over a fixed elliptic curve Σ. Furthermore, we can see very clearly how the Weyl group of G acts on the marked moduli space of flat G-bundles over Σ.

NOTATION. In this paper, we fix some notation from Lie theory. Let G be a compact, simple and simply connected Lie group. We denote by:

$r(G)$ the rank of G;

$R(G)$ the root system;

$R_c(G)$ the coroot system;
$W(G)$ the Weyl group;
$\Lambda(G)$ the root lattice;
$\Lambda_c(G)$ the coroot lattice;
$\Lambda_w(G)$ the weight lattice;
$T(G)$ a maximal torus;
$\mathrm{ad}(G)$ the adjoint group of G, that is, $G/C(G)$, where $C(G)$ is the center of G; and
$\Delta(G)$ a simple root system of G.

When there is no confusion, we just ignore the letter G.

2. *Rational surfaces of ADE-type*

Before defining what ADE-surfaces are, we first give their explicit constructions.

2.1. E_n *case*

First we consider the E_n case, that is, the case of del Pezzo surfaces. We start with a complex projective plane $\mathbb{P}^2$ and n points $x_1, \ldots, x_n$ on $\mathbb{P}^2$ with $n \leqslant 8$. Note that $x_2, \ldots, x_n$ may be *infinitely near* points. For example, we say that x_2 is *infinitely near* x_1 if x_2 lies on the exceptional curve obtained by blowing up x_1. Blowing up $\mathbb{P}^2$ at these points in turn, we obtain a rational surface, denoted by $X_n(x_1, \ldots, x_n)$ or by X_n for brevity.

These points are said to be *in general position* if they satisfy the following conditions.

(i) They are distinct points.
(ii) No three of them are collinear.
(iii) No six of them lie on a common conic curve.
(iv) No cubics pass through eight of them with one of the eight points a double point.

The following result is well known (see [**7**, **22**]).

LEMMA 2.1. *Let $x_i \in \mathbb{P}^2$, with $i = 1, \ldots, n$ and $n \leqslant 8$. Then the following conditions are equivalent to each other.*

(i) *These points are in general position.*
(ii) *The self-intersection number of any rational curve on X_n is greater than or equal to -1.*
(iii) *The anti-canonical class $-K_{X_n}$ is ample.*

A surface X_n is called a *del Pezzo surface* if it satisfies one of the above equivalent conditions.

We say that $x_i \in \mathbb{P}^2$, where $i = 1, \ldots, n$ with $n \leqslant 8$ are in *almost general position* if any smooth rational curve on X_n has a self-intersection number at least -2, and such a surface is called a *generalized del Pezzo surface* (see [**7**]).

Let h be the class of lines in $\mathbb{P}^2$ and let l_i be the exceptional divisor corresponding to the blow-up at $x_i \in \mathbb{P}^2$, where $i = 1, \ldots, n$. Denote by $\mathrm{Pic}(X_n)$ the Picard group of X_n, which is isomorphic to $H^2(X_n, \mathbb{Z})$. Then $\mathrm{Pic}(X_n)$ is a lattice with basis $h, l_1, \ldots, l_n$, of signature $(1, n)$. Let $K = -3h + l_1 + \ldots + l_n$ be the canonical class. We extend the definition of the Lie algebras E_n, where $n = 6, 7, 8$, to all n with $0 \leqslant n \leqslant 8$ by setting $E_0 = 0, E_1 = \mathbb{C}, E_2 = A_1 \times \mathbb{C}, E_3 = A_1 \times A_2, E_4 = A_4$ and $E_5 = D_5$.

We define the following:

$$P_n = \{x \in \mathrm{Pic}(X_n) \mid x \cdot K = 0\};$$
$$R_n = \{x \in \mathrm{Pic}(X_n) \mid x \cdot K = 0,\ x^2 = -2\} \subset P_n;$$
$$I_n = \{x \in \mathrm{Pic}(X_n) \mid x^2 = -1 = x \cdot K\};$$
$$C_n = \{\zeta_n = (e_1, \ldots, e_n) \mid e_i \in I_n,\ e_i \cdot e_j = 0,\ i \neq j\}.$$

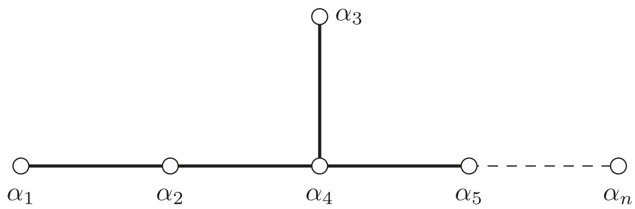

FIGURE 1. *The root system E_n.*

An element of I_n is called an *exceptional divisor*, and an element $\zeta_n \in C_n$ is called an *exceptional system* (of divisors) (see [**7**, **22**]).

LEMMA 2.2. (i) *R_n is a root system of type E_n with a system of simple roots: $\alpha_1 = l_1 - l_2$, $\alpha_2 = l_2 - l_3$, $\alpha_3 = h - l_1 - l_2 - l_3$, $\alpha_4 = l_3 - l_4, \ldots, \alpha_n = l_{n-1} - l_n$. Its root lattice is just P_n, and its weight lattice is $Q_n = \mathrm{Pic}(X_n)/\mathbb{Z}K$. Let $l \in I_n$; then $R_n \cap l^\perp$ is a root system of type E_{n-1} and $P_n \cap l^\perp$ is its root lattice.*
(ii) *The Weyl group $W(E_n)$ acts on C_n simply transitively.*

Proof. (i) For the proof that R_n is a root system of type E_n with given simple roots; see Manin's book [**22**]. We denote by $\mathrm{Pic}(X_n)$ a lattice with $\mathbb{Z}$-basis $h, l_1, \ldots, l_n$. Obviously, $\{e_0 = l_1, e_1 = \alpha_1, \ldots, e_n = \alpha_n\}$ forms another $\mathbb{Z}$-basis. We take any $x \in P_n \subset \mathrm{Pic}(X_n)$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = 0$ implies that $a_0 = 0$. Thus P_n is the root lattice of R_n.
The natural pairing $P_n \otimes \mathrm{Pic}(X_n) \to \mathbb{Z}$ induces a perfect pairing as follows:

$$P_n \otimes (\mathrm{Pic}(X_n)/\mathbb{Z}K) \longrightarrow \mathbb{Z}.$$

Thus the weight lattice is just $\mathrm{Pic}(X_n)/\mathbb{Z}K$.
For the last assertion, we can assume that $l = l_8$; then it is obviously true.
(ii) See [**22**]. □

The Dynkin diagram is shown in Figure 1.

2.2. *D_n case*

Next we consider the D_n case. Let $Y = \mathbb{F}_1$ be a *Hirzebruch surface*. We fix the ruling f and the section s, where $s^2 = -1$. In fact $\mathbb{F}_1$ is the blow-up of $\mathbb{P}^2$ at one point x_0. Thus $f = h - l_0$ and $s = l_0$, where h is the class of lines on $\mathbb{P}^2$ and l_0 is the exceptional curve. Blowing up Y at n points $x_1, \ldots, x_n$, we obtain Y_n. The Picard group $\mathrm{Pic}(Y_n)$ of Y_n is $H^2(Y_n, \mathbb{Z})$, which is a lattice with basis $s, f, l_1, \ldots, l_n$. The canonical class $K = -(2s + 3f - \sum_{i=1}^n l_i)$.
We define the following:

$$P_n = \{x \in \mathrm{Pic}(Y_n) \mid x \cdot K = 0 = x \cdot f\};$$
$$R_n = \{x \in \mathrm{Pic}(Y_n) \mid x \cdot K = 0 = x \cdot f,\ x^2 = -2\};$$
$$I_n = \{x \in \mathrm{Pic}(Y_n) \mid x^2 = -1 = x \cdot K,\ x \cdot f = 0\};$$
$$C_n = \left\{\zeta_n = (e_1, \ldots, e_n) \mid e_i \in I_n, e_i \cdot e_j = 0, i \neq j;\ \sum e_i \cdot s \equiv 0 \bmod 2\right\}.$$

In a similar way to before, an element $\zeta_n \in C_n$ is called an *exceptional system* (of divisors).

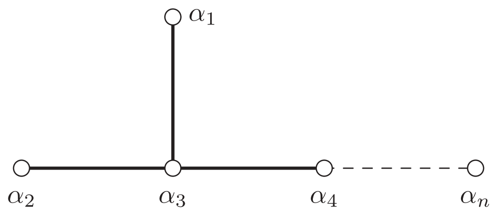

FIGURE 2. *The root system D_n.*

LEMMA 2.3. (i) *R_n is a root system of type D_n with a system of simple roots: $\alpha_1 = f - l_1 - l_2, \alpha_2 = l_1 - l_2, \dots, \alpha_n = l_{n-1} - l_n$. Its root lattice is just P_n and its weight lattice is $Q_n = \mathrm{Pic}(Y_n)/\mathbb{Z}\langle f, K\rangle$.*

(ii) *The Weyl group $W(D_n)$ acts on C_n simply transitively.*

Proof. (i) $\mathrm{Pic}(Y_n)$ is a lattice with $\mathbb{Z}$-basis $s, f, l_i, i = 1, \dots, n$. Let $x = as + bf + \sum c_i l_i \in R_n$, where $a, b, c_i \in \mathbb{Z}$. Then we have a system of linear equations as follows:

$$x^2 = -2,$$
$$x \cdot K = 0 = x \cdot f.$$

Solving this, we obtain

$$a = 0,$$
$$\sum c_i^2 = 2,$$
$$2b = -\sum c_i.$$

Thus, $x = \pm(l_i - l_j), i \neq j$, or $x = \pm(f - l_i - l_j), i \neq j$. That is $R_n = \{\pm(l_i - l_j), \pm(f - l_i - l_j) |\ i \neq j\}$. This implies that R_n is a root system of D_n-type with indicated simple roots.

Obviously, $\{e_1 = s, e_2 = l_1, e_{i+2} = \alpha_i, i = 1, \dots, n\}$ forms another $\mathbb{Z}$-basis. We take any $x \in P_n \subset \mathrm{Pic}(Y_n)$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = 0 = x \cdot f$ implies that $a_1 = a_2 = 0$. Hence P_n is the root lattice of R_n.

The natural pairing $P_n \otimes \mathrm{Pic}(Y_n) \to \mathbb{Z}$ has kernel $\mathbb{Z}\langle f, -2s + \sum l_i\rangle = \mathbb{Z}\langle f, K\rangle$. Thus the pairing induces a perfect pairing $P_n \otimes (\mathrm{Pic}(Y_n)/\mathbb{Z}\langle f, K\rangle) \to \mathbb{Z}$. Hence the weight lattice is just $\mathrm{Pic}(Y_n)/\mathbb{Z}\langle f, K\rangle$.

(ii) A simple computation shows that

$$I_n = \{l_i, f - l_i | i = 1, \dots, n\}.$$

Thus all the elements of C_n are of the form $\zeta_n = (u_1, \dots, u_n)$, where the number of the u_i, such that $u_i = f - l_k$ for some k, is even. Then, by the structure of $W(D_n)$, the result is clear. □

The Dynkin diagram is shown in Figure 2.

2.3. A_{n-1} *case*

In the following we consider the A_{n-1} case. For this, let Z_n be the same surface as Y_n.

We define the following:

$$P_{n-1} = \{x \in \mathrm{Pic}(Z_n) \mid x \cdot K = x \cdot f = x \cdot s = 0\};$$
$$R_{n-1} = \{x \in \mathrm{Pic}(Z_n) \mid x \cdot K = x \cdot f = x \cdot s = 0,\ x^2 = -2\};$$

$$I_{n-1} = \{x \in \operatorname{Pic}(Z_n) \mid x^2 = -1 = x \cdot K,\ x \cdot f = 0 = x \cdot s\};$$
$$C_{n-1} = \{\zeta_n = (e_1, \dots, e_n) \mid e_i \in I_{n-1}, e_i \cdot e_j = 0, i \neq j\}.$$

As before, an element of $\zeta_n \in C_{n-1}$ is called an *exceptional system* (of divisors).

LEMMA 2.4. (i) *R_{n-1} is a root system of type A_{n-1} with a system of simple roots: $\alpha_1 = l_1 - l_2, \dots, \alpha_{n-1} = l_{n-1} - l_n$. Its root lattice is just P_{n-1} and its weight lattice is* $\operatorname{Pic}(Z_n)/\mathbb{Z}\langle f, s, K\rangle$.

(ii) *The Weyl group $W(A_{n-1})$ acts on C_{n-1} simply transitively. In fact, $W(A_{n-1})$ acts as the permutation group of $l_1, \dots, l_n$.*

(iii) *Let e be a (-1) curve that does not meet s. Then there exist i, j with $i \neq j$ such that $e = s + f - l_i - l_j$, and when $n \geqslant 4$ we have that $\langle K, s, f, e\rangle^\perp$ is a reducible root lattice of type $A_1 \times A_{n-3}$; when $n = 3$ we have that $\langle K, s, f, e\rangle^\perp$ is not a root lattice; when $n = 2$ we have that $\langle K, s, f, e\rangle^\perp$ is the same as P_1, which is of type A_1.*

(iv) *Let $e_i, 1 \leqslant i \leqslant k,\ k \geqslant 2$ be (-1) curves such that $s, e_i, 1 \leqslant i \leqslant k$ are disjoint pairwise. Then, when $k \neq 3$ we have that $\langle K, s, f, e_i, 1 \leqslant i \leqslant k\rangle^\perp$ is not a root lattice. When $k = 3$, either of the following two conditions holds:* (a) *if $e_1 = s + f - l_{i_2} - l_{i_3},\ e_2 = s + f - l_{i_1} - l_{i_3}$ and $e_3 = s + f - l_{i_1} - l_{i_2}$, then $\langle K, s, f, e_1, e_2, e_3\rangle^\perp$ is a root lattice of A-type;* (b) *$\langle K, s, f, e_1, e_2, e_3\rangle^\perp$ is not a root lattice.*

Proof. (i) $\operatorname{Pic}(Z_n)$ is a lattice with $\mathbb{Z}$-basis $s, f, l_i, i = 1, \dots, n$. A simple computation shows that

$$R_{n-1} = \{l_i - l_j \mid i \neq j\}.$$

Then it is obviously a root system of type A_{n-1} with given simple roots.

Obviously, $\{e_1 = s, e_2 = f, e_3 = l_1, e_{i+3} = \alpha_i, i = 1, \dots, n\}$ forms another $\mathbb{Z}$-basis. We take any $x \in P_{n-1} \subset \operatorname{Pic}(Z_n)$. Let $x = \sum a_i \cdot e_i$. Then $x \cdot K = x \cdot f = x \cdot s = 0$ implies that $a_1 = a_2 = a_3 = 0$. Thus P_{n-1} is the root lattice of R_{n-1}.

The natural pairing

$$P_{n-1} \otimes \operatorname{Pic}(Z_n) \longrightarrow \mathbb{Z}$$

has a kernel given by

$$\mathbb{Z}\langle f, s, \sum l_i\rangle = \mathbb{Z}\langle f, s, K\rangle.$$

Thus the pairing induces a perfect pairing as follows:

$$P_{n-1} \otimes \operatorname{Pic}(Z_n)/\mathbb{Z}\langle f, s, K\rangle \longrightarrow \mathbb{Z}.$$

Hence the weight lattice is just $\operatorname{Pic}(Z_n)/\mathbb{Z}\langle f, s, K\rangle$.

(ii) In fact $I_{n-1} = \{l_1, \dots, l_n\}$. Hence an element of C_{n-1} is just a permutation of $l_1, \dots, l_n$.

(iii) Let $e = as + bf + \sum c_i l_i$; then e is a (-1) curve and $e \cdot s = 0$ imply that e must be of the form $s + f - l_i - l_j$, with $i \neq j$. Without loss of generality, we can assume that $e = s + f - l_1 - l_2$. Then the result follows from a simple computation.

(iv) First let $k = 2$. From the proof of (iii), we know both e_1 and e_2 are of the form $s + f - l_i - l_j$, with $i \neq j$. Since $e_1 \cdot e_2 = 0$, it follows that we can assume $e_1 = s + f - l_1 - l_2$ and $e_2 = s + f - l_1 - l_3$. Then the result follows easily. For $k = 3$, if $e_1 = s + f - l_{i_2} - l_{i_3},\ e_2 = s + f - l_{i_1} - l_{i_3}$ and $e_3 = s + f - l_{i_1} - l_{i_2}$ then $\langle K, s, f, e_1, e_2, e_3\rangle^\perp = \langle K, s, f, l_{i_1}, l_{i_2}, l_{i_3}\rangle^\perp$. We can assume that $l_{i_1} = l_1, l_{i_2} = l_2, l_{i_3} = l_3$. Then $\langle K, s, f, l_1, l_2, l_3\rangle^\perp$ is a root lattice of A-type. Other cases are similar. □

The Dynkin diagram is shown in Figure 3.

FIGURE 3. *The root system A_{n-1}.*

Note that Lemma 2.3 and Lemma 2.4(i) and (ii) are still true if we replace $\mathbb{F}_1$ by any *Hirzebruch surface* $\mathbb{F}_k (k \geqslant 0)$.

2.4. ADE-*surfaces*

Now we show that, in a suitable sense, the converse of the above lemmas is also true. As promised in the introduction, we see that the following definition generalizes that of *del Pezzo surfaces*.

DEFINITION 2.5. Let (S, C) be a pair consisting of a smooth rational surface S and a smooth rational curve $C \subset S$ with $C^2 \neq 4$. The pair (S, C) is said to be of ADE-type (or an ADE-surface) if it satisfies the following two conditions:

(i) Any (smooth) rational curve on S has a self-intersection number at least -1.

(ii) The sub-lattice $\langle K_S, C\rangle^{\perp}$ of $\mathrm{Pic}(S)$ is an irreducible root lattice of rank equal to $\mathrm{rank}(\mathrm{Pic}(S)) - 2$.

The following proposition shows that such surfaces can be classified into three types.

PROPOSITION 2.6. *Let (S, C) be a rational surface of ADE-type. Let $n = \mathrm{rank}(\mathrm{Pic}(S)) - 2$. Then $C^2 \in \{-1, 0, 1\}$ and the following properties hold:*

(i) *when $C^2 = -1$, we have that $\langle K_S, C\rangle^{\perp}$ is of E_n-type, where $4 \leqslant n \leqslant 8$;*

(ii) *when $C^2 = 0$, we have that $\langle K_S, C\rangle^{\perp}$ is of D_n-type, where $n \geqslant 3$;*

(iii) *when $C^2 = 1$, we have that $\langle K_S, C\rangle^{\perp}$ is of A_n-type.*

Proof. By the first condition in Definition 2.5, it is clear that $C^2 \geqslant -1$. Therefore there are the following four cases.

First, suppose that $C^2 = -1$. Then we can contract C to obtain a smooth surface $\widetilde{S}$. Let $\pi : S \to \widetilde{S}$ be the blow-down. Then the projection

$$\mathrm{Pic}(S) = \mathrm{Pic}(\widetilde{S}) \oplus \mathbb{Z}\langle C\rangle \longrightarrow \mathrm{Pic}(\widetilde{S})$$

induces an isomorphism $\langle K_S, C\rangle^{\perp} \cong \langle K_{\widetilde{S}}\rangle^{\perp}$. However, the latter is an irreducible root system if and only if $\widetilde{S}$ is a blow-up of $\mathbb{CP}^2$ at $n(4 \leqslant n \leqslant 8)$ points. At this time $\langle K_{\widetilde{S}}\rangle^{\perp}$ is a root system of E_n-type. Thus S is a blow-up of $\mathbb{CP}^2$ at $n + 1(4 \leqslant n \leqslant 8)$ points.

Second, suppose that $C^2 = 0$. Then by the Riemann–Roch theorem, the linear system $|C|$ defines a ruling over $\mathbb{P}^1$ with fiber C. Contracting all (-1) curves in fiber, we obtain a relatively minimal model (not unique), which is $\mathbb{P}^1 \times \mathbb{P}^1$ or the Hirzebruch surface $\mathbb{F}_1$. Thus, S is a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points. Moreover, the lattice $\langle K_S, C\rangle^{\perp}$ must be of D_n-type by Lemma 2.3.

Third, suppose that $C^2 = 1$. Then, blowing up one point $p_0 \in C$, we obtain $\widetilde{S}$, which is a ruling over $\mathbb{P}^1$ with fiber $\widetilde{C} = C - E$ and section E, where E is the exceptional curve associated to this blow-up. Contracting all (-1) curves which do not intersect with E in fiber, we will obtain $\mathbb{F}_1$. Thus $\widetilde{S}$ is a blow-up of $\mathbb{F}_1$ at n points. Furthermore, we have $\langle K_S, C\rangle^{\perp} \cong \langle K_{\widetilde{S}}, \widetilde{C}, E\rangle^{\perp}$. Therefore the lattice is a root lattice of A_n-type by Lemma 2.4.

Finally, suppose that $C^2 \geqslant 2$. Note that, since we assume that $C^2 \neq 4$, the situation of Lemma 2.4(iv(a)) cannot happen. Hence we only need to discuss the case where $C^2 = 2$, because the discussion on general cases is similar. Blowing up S at two points $p, q \in C$, with $p \neq q$, we obtain $\widetilde{S}$ with exceptional curves E_p, and E_q. Let $\widetilde{C} = C - E_p - E_q$ be the strict transform of C, then $|\widetilde{C}|$ defines a ruling with fiber $\widetilde{C}$ and section $s = E_p$ (fixed). In a similar way to before, contracting all (-1) curves E in fiber that satisfy $E \cdot \widetilde{C} = 0 = E \cdot s$, we will obtain $\mathbb{F}_1$. Then $\widetilde{S}$ can be considered as a blow-up of $\mathbb{F}_1$ at n points. Note that $\langle K_S, C\rangle^{\perp} \cong \langle K_{\widetilde{S}}, \widetilde{C}, s, E_q\rangle^{\perp}$. We know that $\langle K_{\widetilde{S}}, \widetilde{C}, s\rangle^{\perp}$ is a root lattice of A_n-type from Lemma 2.4. Then the result also follows from Lemma 2.4. □

REMARK 2.7. We extend the definition of E_n surfaces to all n with $0 \leqslant n \leqslant 8$, by defining $E_n (n \leqslant 3)$ surfaces to be del Pezzo surfaces of degree $9 - n$.

COROLLARY 2.8. *On an ADE-surface, any exceptional divisor perpendicular to C is represented by an irreducible curve. Therefore, any exceptional system consists of exceptional curves.*

Proof. In the E_n case, the result follows from Proposition 2.6 and Lemma 2.1. In D_n and A_n cases, according to Proposition 2.6, the result is obvious. □

In the following we generalize the definition for $n \leqslant 8$ points being *in general position* to any $n \geqslant 0$. Define $S = \mathbb{P}^2$ (or $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$). Denote by $S_n(x_1, \ldots, x_n)$ (or S_n for brevity) the blow-up of S at n points $x_1, \ldots, x_n$. We say that $x_1, \ldots, x_n$ are *in general position* if any smooth rational curve on S_n has a self-intersection number at least -1. Furthermore, we say that $x_1, \ldots, x_n$ are *in almost general position* if any smooth rational curve on S_n has a self-intersection number at least -2.

COROLLARY 2.9. *Let (S, C) be an ADE-surface. Then the following properties hold.*

(i) *In the E_n case, blowing down the (-1) curve C, we obtain a del Pezzo surface of degree $9 - n$.*

(ii) *In the D_n case, S is just a blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ or $\mathbb{F}_1$ at n points in general position with C as the natural ruling.*

(iii) *In the A_n case, let $\widetilde{S}$ be the blow-up of S at a point on C, with the exceptional curve E; then $\widetilde{S}$ is a blow-up of $\mathbb{F}_1$ at $n+1$ points, and the strict transform $\widetilde{C}$ of C defines a ruling with E as the section of $\mathbb{F}_1$.*

3. *Lie algebra bundles over rational surfaces of ADE-type and their representation bundles*

When G is of ADE-type, for each ADE-surface S we can construct a natural $\mathcal{G} = \mathrm{Lie}(G)$ bundle and natural fundamental representation bundles over S, which are determined by the lines (or exceptional divisors in general) and rulings on S.

DEFINITION 3.1. By a *Lie algebra $\mathcal{G} = \mathrm{Lie}(G)$ bundle* we mean a vector bundle that fiberwise carries a Lie algebra structure of $\mathcal{G}$-type, and this Lie algebra structure is compatible with trivialization of this bundle. By a *representation bundle* of a $\mathcal{G}$-bundle, we mean a vector bundle $\mathcal{V}$ that fiberwise is a representation of $\mathcal{G}$, and the action of $\mathcal{G}$ on $\mathcal{V}$ is compatible with their trivialization.

We describe these bundles in the following, and give the detailed arguments only in the E_n case, since other cases are similar.

3.1. *E_n bundles over E_n surfaces*

Let (S,C) be an E_n surface. Recall that $S = X_{n+1}(x_1,\dots,x_{n+1})$ and C is the exceptional divisor associated to the blow-up at x_{n+1}. Define $\widetilde{S} = X_n(x_1,\dots,x_n)$. Since $\langle K_S, C\rangle^{\perp} \cong K_{\widetilde{S}}^{\perp}$, we can just consider the surface $\widetilde{S} = X_n(x_1,\dots,x_n)$.

Since we have a root system of E_n-type attached to X_n, inspired by the Cartan decomposition of a complex simple Lie algebra, we can construct a Lie algebra bundle over X_n as follows:

$$\mathscr{E}_n = \mathcal{O}^{\oplus n} \bigoplus_{D\in R_n} \mathcal{O}(D).$$

The fiberwise Lie algebra structure of $\mathscr{E}_n$ is defined as the following. We fix the system of simple roots of R_n as

$$\Delta(E_n) = \{\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = h - l_1 - l_2 - l_3, \dots, \alpha_n = l_{n-1} - l_n\},$$

and take a trivialization of $\mathscr{E}_n$. Then, over a trivializing open subset U, we have $\mathscr{E}_n|_U \cong U \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha\in R_n} \mathbb{C}_\alpha)$. We take a Chevalley basis $\{x_\alpha^U, \alpha \in R_n; h_i, 1 \leqslant i \leqslant n\}$ for $\mathscr{E}_n|_U$ and define the Lie algebra structure by the following four relations, namely, Serre's relations on a Chevalley basis (see [**14**, p. 147]):

(a) $[h_i h_j] = 0, 1 \leqslant i,j \leqslant n$;
(b) $[h_i x_\alpha^U] = \langle \alpha, \alpha_i\rangle x_\alpha^U, 1 \leqslant i \leqslant n, \alpha \in R_n$;
(c) $[x_\alpha^U x_{-\alpha}^U] = h_\alpha$ is a $\mathbb{Z}$-linear combination of $h_1,\dots,h_n$;
(d) if α and β are independent roots, and $\beta - r\alpha, \dots, \beta + q\alpha$ are the α-strings through β, then $[x_\alpha^U x_\beta^U] = 0$ if $q = 0$, while $[x_\alpha^U x_\beta^U] = \pm(r+1)x_{\alpha+\beta}^U$ if $\alpha+\beta \in R_n$.

Note that h_i, with $1 \leqslant i \leqslant n$, are independent of any trivialization, and so the relation (a) is always invariant under different trivializations. If $\mathscr{E}_n|_V \cong V \times (\mathbb{C}^{\oplus n} \bigoplus_{\alpha\in R_n})$ is another trivialization, and f_α^{UV} is the transition function for the line bundle $\mathcal{O}(\alpha)(\alpha \in R_n)$, that is, $x_\alpha^U = f_\alpha^{UV} x_\alpha^V$, then the relation (b) is given by

$$[h_i(f_\alpha^{UV} x_\alpha^V)] = \langle \alpha, \alpha_i\rangle f_\alpha^{UV} x_\alpha^V,$$

that is, we have

$$[h_i x_\alpha^V] = \langle \alpha, \alpha_i\rangle x_\alpha^V.$$

Thus (b) is also invariant; and (c) is also invariant since $(f_\alpha^{UV})^{-1}$ is the transition function for $\mathcal{O}(-\alpha)(\alpha \in R_n)$. Finally, (d) is invariant since $f_\alpha^{UV} f_\beta^{UV}$ is the transition function for $\mathcal{O}(\alpha + \beta)(\alpha,\beta \in R_n)$.

Therefore, the Lie algebra structure is compatible with the trivialization. Hence it is well defined. In other words, we can construct globally a Lie algebra bundle over a surface once we are given a root system consisting of divisors on this surface.

The following relations are intricate. One is the relation between I_n (the set of all exceptional divisors) and the fundamental representation associated to the highest weight λ_n that is dual to the simple root α_n (see Figure 1). Another one is the relation between the set of rulings and the fundamental representation associated to the highest weight λ_1 that is dual to the simple root α_1 (Figure 1). We explain the relations in the following.

Let $\mathbb{L}_n$ be the fundamental representation with the highest weight λ_n. Then we have the relations listed in Table 1 between the dimension of $\mathbb{L}_n$ and the number of I_n.

Denote by Ru_n the set of all rulings on X_n. Let $\mathbb{R}_n$ be the fundamental representation with the highest weight λ_1. Then we have the relations listed in Table 2 between the dimension of $\mathbb{R}_n$ and the number of Ru_n.

Inspired by these, we can construct a fundamental representation bundle $\mathscr{L}_n$ and $\mathscr{R}_n$ using the exceptional divisors and the rulings, respectively, on X_n as follows:

$$\mathscr{L}_n = \bigoplus_{l \in I_n} \mathcal{O}(l) \text{ when } n \leqslant 7,$$
$$\mathscr{L}_8 = \bigoplus_{l \in I_8} \mathcal{O}(l) \oplus \mathcal{O}(-K)^{\oplus 8};$$

respectively,

$$\mathscr{R}_n = \bigoplus_{R \in \mathrm{Ru}_n} \mathcal{O}(R) \text{ when } n \leqslant 6,$$
$$\mathscr{R}_7 = \bigoplus_{R \in \mathrm{Ru}_7} \mathcal{O}(R) \oplus \mathcal{O}(-K)^{\oplus 7}.$$

The fiberwise action is defined naturally, which is in fact compatible with any trivialization.

For example, we consider the bundle $\mathscr{L}_n$ and suppose that $n \leqslant 7$. Take U and V as before, and suppose that they also trivialize $\mathscr{L}_n$, that is $\mathscr{L}_n|_U \cong U \times (\bigoplus_{l \in I_n} \mathbb{C}_l)$ and $\mathscr{L}_n|_V \cong V \times (\bigoplus_{l \in I_n} \mathbb{C}_l)$. Take e_l^U or $e_l^V = g^{VU} e_l^U$ to be the basis of $\mathbb{C}_l$ over U or V, respectively. Then define $x_\alpha^U . e_l^U$ to be equal to $e_{l'}^U$ if $l' = \alpha + l \in I_n$ and be equal to 0 otherwise. Furthermore define $h_\alpha . e_l^U = (\alpha \cdot l) e_l^U$.

Note that the situation here is slightly different from some standard usage, for example [**6**, **14**], since the self-intersection number of an element of R_n or I_n is negative. However, this does not matter if we take the simple root system to be $\{-\alpha_1, \ldots, -\alpha_n\}$, and the pairing to be $(x, y) := -(x \cdot y)$. First since $\lambda_n(-\alpha_i) = (-\alpha_i, l_n) = \alpha_i \cdot l_n = \delta_{in}$, we have $\lambda_n \cong (\cdot, l_n)$. Second the action is irreducible since the Weyl group acts on I_n transitively. Lastly $e_{l_n}^U$ is the maximal vector of weight λ_n. Therefore this fiberwise action does define the highest weight module with the highest weight λ_n (see [**14**]).

Obviously, this fiberwise Lie algebra action is compatible with the trivialization.

For $\mathscr{L}_8$, note that the bijection $I_8 \to R_8$ given by $l \mapsto l + K$ induces an isomorphism

$$\mathscr{E}_8 \cong \mathscr{L}_8 \otimes \mathcal{O}(K).$$

This implies that $\mathscr{L}_8$ is just the adjoint representation bundle.

Similarly, $\mathscr{R}_n$ is the fundamental representation bundle with the highest weight $\lambda_1 \cong (\cdot, h - l_1)$ and the maximal vector $e_{h-l_1}^U$, where the simple root system and the pairing are defined as above. We also have that $\mathscr{R}_7 \otimes \mathcal{O}(K) \cong \mathscr{E}_7$ is the adjoint representation bundle.

Table 1. *Lines and fundamental representations.*

n	1	2	3	4	5	6	7	8
$\dim \mathbb{L}_n$	1	3	6	10	16	27	56	248
$\lvert I_n \rvert$	1	3	6	10	16	27	56	240

Table 2. *Rulings and fundamental representations.*

n	1	2	3	4	5	6	7	8
$\dim \mathbb{R}_n$	1	2	3	5	10	27	133	3875
$\lvert \mathrm{Ru}_n \rvert$	1	2	3	5	10	27	126	2160

Example 1. We look at the $sl(2)$ sub-bundle

$$\mathcal{O} \oplus \mathcal{O}(\alpha) \oplus (-\alpha),$$

where $\alpha = l_1 - l_2$. Then the bundle $\mathcal{O}(l_1) \oplus \mathcal{O}(l_2)$ is the standard representation bundle. Furthermore, the line bundle $\mathcal{O}(h - l_1 - l_2)$ is a trivial representation.

In fact, the Lie algebra bundle $\mathscr{E}_n$ is uniquely determined by its representation bundles $\mathscr{L}_n$ and $\mathscr{R}_n$, according to [**1**]. Concretely (see [**17**] for more details), the following properties hold:

(i) $\mathscr{E}_4$ is the automorphism bundle of $\mathscr{R}_4$ preserving $\wedge^5 \mathscr{R}_4 \cong \mathcal{O}(-2K)$;

(ii) $\mathscr{E}_5$ is the automorphism bundle of $\mathscr{R}_5$ preserving $q_5 : \mathscr{R}_5 \otimes \mathscr{R}_5 \to \mathcal{O}(-K)$, where q_5 is defined by $\mathcal{O}(R') \otimes \mathcal{O}(R'') \to \mathcal{O}(-K)$ if $R' + R'' = -K$, and 0 otherwise;

(iii) $\mathscr{E}_6$ is the automorphism bundle of $\mathscr{R}_6$ and $\mathscr{L}_6$ preserving

$$c_6 : \mathscr{L}_6 \otimes \mathscr{L}_6 \longrightarrow \mathscr{R}_6 \quad \text{and} \quad c_6^* : \mathscr{R}_6 \otimes \mathscr{R}_6 \longrightarrow \mathscr{L}_6 \otimes \mathcal{O}(-K),$$

where c_6 is defined by the map $(l_i,\ l_j) \mapsto 2h - \sum_{k \neq i,j} l_k$ and c_6^* is defined by the map $(h - l_i, h - l_j) \mapsto h - l_i - l_j$.

(iv) $\mathscr{E}_7$ is the automorphism bundle of $\mathscr{L}_7$ preserving

$$f_7 : \mathscr{L}_7 \otimes \mathscr{L}_7 \otimes \mathscr{L}_7 \otimes \mathscr{L}_7 \longrightarrow \mathcal{O}(-2K),$$

where f_7 is defined by the map $(C_1, C_2, C_3, C_4) \mapsto -2K$ if $C_1 + C_2 + C_3 + C_4 = -2K$, and 0 otherwise;

(v) $\mathscr{E}_8$ is the automorphism bundle of $\mathscr{L}_8$ preserving

$$\mathscr{L}_8 \wedge \mathscr{L}_8 \longrightarrow \mathscr{L}_8 \otimes \mathcal{O}(-K).$$

For X_6, the bijection $Ru_6 \to I_6$ defined by $R \mapsto -(R + K)$ induces an isomorphism $\mathscr{R}_6 \cong \mathscr{L}_6^* \otimes \mathcal{O}(-K)$, which is consistent with the duality between $\mathbb{L}_6$ and $\mathbb{R}_6$ for the Lie group E_6.

3.2. *D_n bundles over rational ruled surfaces*

Let (S, C) be a D_n surface. By Proposition 2.6, it is clear that S dominates $\mathbb{F}_1$ or $\mathbb{F}_0 (= \mathbb{P}^1 \times \mathbb{P}^1)$ with ruling C. We can suppose that S dominates $\mathbb{F}_1$ since, for the other case the argument is the same. Thus $S = Y_n(x_1, \ldots, x_n)$ is the blow-up of $\mathbb{F}_1$ at n points x_i, with $i = 1, \ldots, n$, where, for any i, we find that x_i does not lie on the section s.

Since R_n is a root system of type D_n, the Lie algebra bundle can be constructed as follows:

$$\mathscr{D}_n = \mathcal{O}^{\oplus n} \bigoplus_{D \in R_n} \mathcal{O}(D).$$

Recall that, in the D_n case, we have

$$\begin{aligned} I_n &= \{C \mid C^2 = C \cdot K = -1, C \cdot f = 0\} \\ &= \{l_i, f - l_i \mid i = 1, \ldots, n\}. \end{aligned}$$

The fundamental representation bundle of $\mathscr{D}_n$ with the highest weight λ_n, where λ_n is the fundamental weight corresponding to $\alpha_n = l_{n-1} - l_n$, is

$$\mathscr{W}_n = \bigoplus_{C \in I_n} \mathcal{O}(C).$$

In fact, $\mathscr{W}_n$ is the standard representation bundle of $\mathscr{D}_n$.

Note that there are n singular fibers, and each singular fiber is of the form $l_i + l_i'$, where $l_i' = f - l_i, i = 1, \ldots, n$. The relation

$$\mathcal{O}(l_i) \otimes \mathcal{O}(l_i') = \mathcal{O}(f)$$

implies that we can define a non-degenerated fiberwise quadratic form as follows:

$$q_n : \mathscr{W}_n \otimes \mathscr{W}_n \longrightarrow \mathcal{O}(f).$$

The two spinor bundles are defined as

$$\mathcal{S}_n^+ = \bigoplus_{S^2=S\cdot K=-1, S\cdot f=1} \mathcal{O}(S) \quad \text{and} \quad \mathcal{S}_n^- = \bigoplus_{T^2=-2, T\cdot K=0, T\cdot f=1} \mathcal{O}(T).$$

Moreover, there are all kinds of structures on these representation bundles, for example, the Clifford multiplication:

$$\mathcal{S}_n^+ \otimes \mathscr{W}_n^* \longrightarrow \mathcal{S}_n^- \quad \text{and} \quad \mathcal{S}_n^- \otimes \mathscr{W}_n \longrightarrow \mathcal{S}_n^+.$$

When $n = 2m - 1$ is odd, we have the isomorphism as follows:

$$(\mathcal{S}_n^+)^* \otimes \mathcal{O}_{Y_n}((m-4)f - K) \cong \mathcal{S}_n^-.$$

When $n = 2m$ is even, we have isomorphisms as follows:

$$(\mathcal{S}_n^+)^* \otimes \mathcal{O}_{Y_n}((m-3)f - K) \cong \mathcal{S}_n^+,$$
$$(\mathcal{S}_n^-)^* \otimes \mathcal{O}_{Y_n}((m-4)f - K) \cong \mathcal{S}_n^-.$$

For more details, see [**17**].

3.3. A_{n-1} *bundles and their representation bundles*

Let S be an A_{n-1} surface. By Proposition 2.6, we can assume that $S = Z_n(x_1, \ldots, x_n)$ be the blow-up of $\mathbb{F}_1$ at n points x_i, with $i = 1, \ldots, n$, where for any i it is clear that x_i does not lie on the section s. Recall that

$$R_{n-1} = \{l_i - l_j |\ i \neq j\}$$

and

$$I_{n-1} = \{l_1, \ldots, l_n\}.$$

Since R_{n-1} is a root system of A_{n-1}-type, the Lie algebra bundle can be constructed as

$$\mathscr{A}_{n-1} = \mathcal{O}^{\oplus n-1} \bigoplus_{D\in R_{n-1}} \mathcal{O}(D),$$

and the standard representation bundle is

$$\mathcal{V}_{n-1} = \bigoplus_{C\in I_{n-1}} \mathcal{O}(C) = \bigoplus_{i=1}^{n} \mathcal{O}(l_i).$$

The kth fundamental representation bundle is just

$$\wedge^k(\mathcal{V}_{n-1}) \cong \bigoplus_{i_1<\ldots<i_k} \mathcal{O}(l_{i_1} + \ldots + l_{i_k}).$$

We also have $\mathscr{A}_{n-1} = \mathrm{End}_0(\mathcal{V}_{n-1})$.

We summarize the content of this section as follows.

Conclusion. For every ADE-surface S, there is a natural Lie algebra bundle of corresponding ADE-type over S. Furthermore, we can construct two natural fundamental representation bundles over S, using lines and rulings on S. Moreover, the Lie algebra bundle can be considered as the automorphism (Lie algebra) bundle of these fundamental representation bundles preserving natural structures.

4. *Flat G-bundles over elliptic curves*

In this section we review some well-known results about flat G-bundles over elliptic curves.

Let Σ be an elliptic curve with identity element 0. The fundamental group $\pi_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$. Let G be a compact, simple and simply connected Lie group of rank r with root system R, coroot system R_c, Weyl group W, root lattice Λ, coroot lattice Λ_c and maximal torus T. The dual lattice $\Lambda_c^\vee$ of Λ_c is the weight lattice. We denote the moduli space of flat G-bundles over Σ by $\mathcal{M}_\Sigma^G$. It is well known that we have the following isomorphisms:

$$\begin{aligned}\mathcal{M}_\Sigma^G &\cong \mathrm{Hom}(\pi_1(\Sigma), G)/\mathrm{ad}(G)\\ &\cong \mathrm{Hom}(\pi_1(\Sigma), T)/W\\ &\cong T \times T/W\\ &\cong \Sigma \otimes_{\mathbb{Z}} \Lambda_c/W.\end{aligned}$$

The second isomorphism is by Borel's theorem [5], which says that a commuting pair of elements in G can be diagonalized simultaneously. The last isomorphism comes from

$$\mathrm{Hom}(\pi_1(\Sigma), T) = \mathrm{Hom}(\pi_1(\Sigma), U(1) \otimes_{\mathbb{Z}} \Lambda_c) \cong \mathrm{Hom}(\pi_1(\Sigma), U(1)) \otimes_{\mathbb{Z}} \Lambda_c$$

and

$$\mathrm{Hom}(\pi_1(\Sigma), U(1)) \cong \mathrm{Pic}^0(\Sigma) \cong \Sigma.$$

A theorem of Bernshtein and Shvartsman [4] and Looijenga [18] says that

$$\Sigma \otimes_{\mathbb{Z}} \Lambda_c/W \cong \mathbb{WP}^r_{s_0=1,s_1,\ldots,s_r},$$

where the latter is the weighted projective space with the weights s_i, and $s_1, \ldots, s_r$ are the coefficients of the highest coroot of R_c.

One element of $\mathrm{Hom}(\Lambda, \Sigma)/W$ can only determine a flat $\mathrm{ad}(G) = G/C(G)$ bundle in general. For the adjoint group $\mathrm{ad}(G)$, the moduli space of flat $\mathrm{ad}(G)$ bundles $\mathcal{M}_\Sigma^{\mathrm{ad}(G)}$ contains $\mathrm{Hom}(\Lambda, \Sigma)/W$ as a connected component (see [11]). On the other hand, we have the following short exact sequences:

$$0 \longrightarrow \Lambda \longrightarrow \Lambda_c^\vee \longrightarrow \Gamma \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Hom}(\Gamma, \Sigma) \longrightarrow \mathrm{Hom}(\Lambda_c^\vee, \Sigma) \longrightarrow \mathrm{Hom}(\Lambda, \Sigma) \longrightarrow 0.$$

Here Γ is a finite abelian group. The second sequence is exact since Σ is a divisible abelian group. It follows that $\mathrm{Hom}(\Lambda, \Sigma)$ and $\Sigma \otimes_{\mathbb{Z}} \Lambda_c$ are isogenous as abelian varieties. Let d be the exponent of the finite group Γ. If we fix a dth root of unity in $\mathrm{Jac}(\Sigma) \cong \Sigma$ (equivalently, if we fix a cyclic subgroup of Σ of order d), then we can uniquely extend a homomorphism $f_0 \in \mathrm{Hom}(\Lambda, \Sigma)$ to a homomorphism $f \in \mathrm{Hom}(\Lambda_c^\vee, \Sigma) \cong \Lambda_c \bigotimes_{\mathbb{Z}} \Sigma$. We have explained the following.

LEMMA 4.1. *When we fix a dth root of unity in* $\mathrm{Jac}(\Sigma)$, *we have an isomorphism*

$$\mathrm{Hom}(\Lambda_c^\vee, \Sigma)/W \cong \mathrm{Hom}(\Lambda, \Sigma)/W,$$

and therefore

$$\mathcal{M}_\Sigma^G \cong \mathrm{Hom}(\Lambda, \Sigma)/W.$$

REMARK 4.2. We have constructed ADE (Lie algebra) bundles over ADE rational surfaces. We see that the restriction of such a Lie algebra bundle to the anti-canonical curve Σ will uniquely determine a flat G-bundle over Σ. To obtain a simple Lie group $G = E_n$ or D_n, we need to assume that $4 \leqslant n \leqslant 8$ or $n \geqslant 3$.

5. *Flat G-bundles over elliptic curves and rational surfaces: simply laced cases*

From this section on, we fix our ADE-surface S to be the rational surface $X_n(x_1, \ldots, x_n)$, $Y_n(x_1, \ldots, x_n)$, or $Z_n(x_1, \ldots, x_n)$. For X_n, we assume $n \leqslant 8$.

Given any smooth elliptic curve Σ with identity $0 \in \Sigma$, we assume that our surface S contains Σ as an anti-canonical curve. For this aim, we first embed Σ into $\mathbb{P}^2$ as an anti-canonical curve, using the projective embedding ϕ determined by the linear system $|3(0)|$, where (0) is the divisor of the identity element of Σ, and assume that all these blown-up points $x_i \in \Sigma$ for $i = 1, \ldots, n$, and that $0, x_1, \ldots, x_n$ are in general position. Moreover, we blow up $\mathbb{P}^2$ at 0 to obtain the embedding of Σ into $\mathbb{F}_1$ as an anti-canonical curve, and take the exceptional curve l_0 as the section s for the ruled surface $\mathbb{F}_1$.

Convention. In the Z_n case, it is well known that in order to obtain a flat SU(n)-bundle over Σ we need one more assumption as follows:

$$\sum x_i = 0 \text{ in } \Sigma.$$

We explain how the moduli space $\mathcal{M}_\Sigma^G$ is related to the moduli space of rational surfaces of the above types. Denote by $\mathcal{S}(\Sigma, G)$ the moduli space of the pairs (S, Σ), where S is an ADE rational surface of the same type as that of G and $\Sigma \in |-K_S|$.

Proposition 5.1. *There exists a well-defined map*

$$\phi : \ \mathcal{S}(\Sigma, G) \longrightarrow \mathrm{Hom}(\Lambda, \Sigma)/W,$$

where Λ is the lattice P_n or P_{n-1} defined in Section 2.

Proof. First we consider the case where $S = X_n$ is a del Pezzo surface, that is, all blown-up points are in general position. Suppose that we are given the pair $(X_n, \Sigma \in |-K_{X_n}|)$. For each element $y \in P_n$, we find that y stands for a holomorphic line bundle over S. Restricting y to Σ, we obtain a holomorphic line bundle over Σ, denoted by $\mathcal{L}_y$. The degree of $\mathcal{L}_y$ is given by

$$\deg(\mathcal{L}_y) = y \cdot (-K) = 0.$$

Thus $\mathcal{L}_y$ is an element of the Jac3obian of Σ, which is canonically isomorphic to Σ since the identity element of Σ is given. Thus we obtain a map from P_n to $\Sigma : y \mapsto \mathcal{L}_y$, which is obviously a homomorphism of abelian groups. However, for one pair (X_n, Σ), we can have different choices of simple roots in order to identify P_n with the root lattice of E_n, and all choices only differ by the action of the Weyl group $W(E_n)$. Hence, finally we obtain a well-defined map from the moduli space $\mathcal{S}(\Sigma, E_n)$ of such pairs (X_n, Σ) to the projective variety $\mathrm{Hom}(P_n, \Sigma)/W(E_n)$.

The other two cases are similar. Roughly speaking, given a pair (Y_n, Σ) or (Z_n, Σ), we obtain an element in, respectively,

$$\mathrm{Hom}(P_n, \Sigma)/W(D_n) \quad \text{or} \quad \mathrm{Hom}(P_{n-1}, \Sigma)/W(A_{n-1}).$$

□

In fact we can prove a theorem of Torelli-type for the above correspondence. Roughly speaking, the moduli space $\mathcal{S}(\Sigma, G)$ of the pairs (S, Σ) is isomorphic to $\mathrm{Hom}(\Lambda, \Sigma)/W$, where Λ is our root lattice.

MODULI OF BUNDLES I

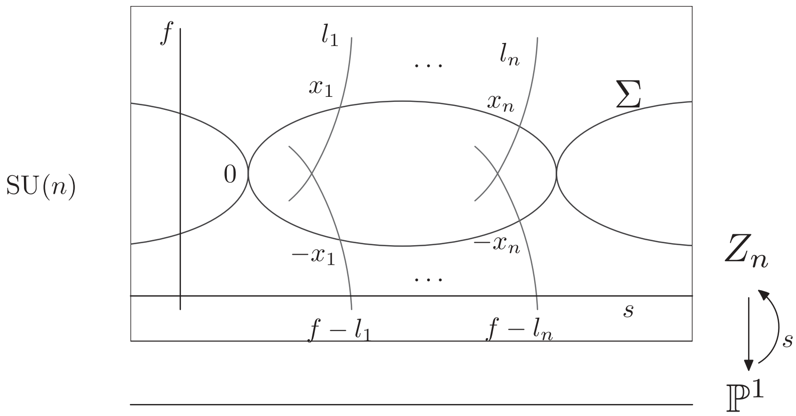

FIGURE 4. *A surface with an* A_{n-1}*-configuration* $(l_1,\dots,l_n)$. (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

DEFINITION 5.2. Let $S = X_n$, Y_n or Z_n. An exceptional system $\zeta_n = (e_1,\dots,e_n) \in C_n$ on X_n, Y_n or Z_n is called a *G-configuration* for $G = E_n$, D_n or A_{n-1}, respectively, if e_n is a (-1) curve, and if after blowing down e_n we have that e_{n-1} is a (-1) curve. Furthermore, this process can proceed successively until, after blowing down e_1, we obtain $\mathbb{P}^2$ or $\mathbb{F}_1$ for $G = E_n$ or D_n and A_{n-1}. Denote by ζ_G a *G-configuration*. When S is equipped with a G-configuration ζ_G, and S has Σ as an anti-canonical curve, we call S a *rational surface with G-configuration* and denote it by a pair (S, G).

Equivalently, a G-configuration ζ_{E_n}, ζ_{D_n} or $\zeta_{A_{n-1}}$ on $S = X_n$, Y_n or Z_n, means that S could be considered as the blow-up of $\mathbb{P}^2$, $\mathbb{F}_1$ or $\mathbb{F}_1$, respectively, at n (maybe not distinct) points $y_1,\dots,y_n \in S$ successively, such that $e_1,\dots,e_n$ are the corresponding exceptional divisors.

LEMMA 5.3. *Let S be a surface with G-configuration. Then any smooth rational curve on S has a self-intersection number at least -2. Furthermore, in the E_n case, all these (-2) curves form chains of ADE-type.*

Proof. Let L be a smooth rational curve on S. Then $L \cdot \Sigma \geqslant 0$. By the adjoint formula, we have $-2 = L^2 + L \cdot K_S$. Since Σ is linearly equivalent to $-K_S$, we have $L^2 \geqslant -2$. For the last assertion, see [7]. □

On an ADE-surface, by Corollary 2.8, any exceptional system is an ADE-configuration. Thus, we can restate the result of Lemma 2.2(ii), Lemma 2.3(ii) and Lemma 2.4(ii) as follows.

PROPOSITION 5.4. *For an ADE-surface, $W(G)$ acts on the set of all G-configurations simply transitively.*

This proposition implies that a G-configuration determines exactly an isomorphism from P_n (or P_{n-1} for A_{n-1}) to the corresponding root lattice $\Lambda(G)$.

An A_{n-1}-configuration on Z_n is illustrated in Figure 4.

A D_n-configuration on Y_n is illustrated in Figure 5.

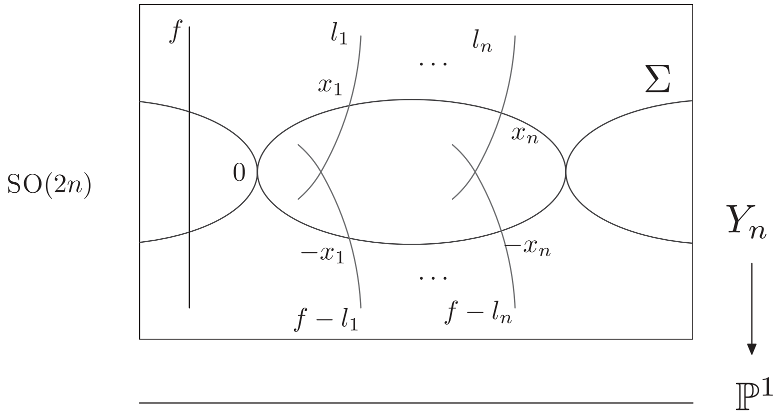

FIGURE 5. *A surface with an* D_n*-configuration* $(l_1, \ldots, l_n)$. (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

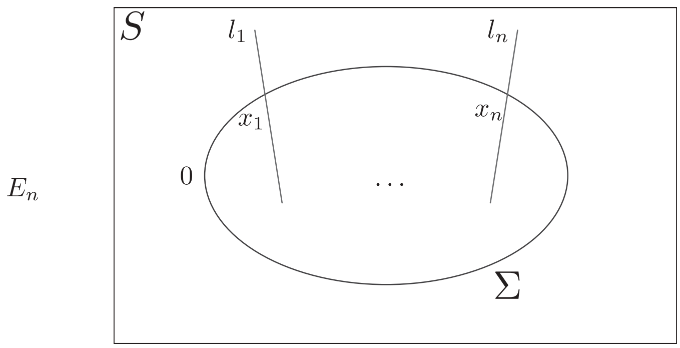

FIGURE 6. *A surface with an* E_n*-configuration* $(l_1, \ldots, l_n)$. (*A colour version of this figure is available as supplementary data at the* Journal of the London Mathematical Society *online.*)

Also an E_n-configuration on X_n is illustrated in Figure 6.

Recall the definition for ζ_{D_n}: $\zeta_{D_n} = (e_1, \ldots, e_n)$ where $e_i \cdot K_{Y_n} = -1,\ e_i \cdot f = 0,\ e_i \cdot e_j = \delta_{ij}$ and $\sum e_i \cdot s \equiv 0 \bmod 2$. Next we explain geometrically why we need to assume that $\sum e_i \cdot s \equiv 0 \bmod 2$.

DEFINITION 5.5. Let $C \subset \mathbb{P}^2$ be a curve of degree d. A point $P \in C$ is called an ordinary k-fold point of C if P is a k-fold singular point and C has k distinct tangent directions at P.

LEMMA 5.6. *Let C be a plane curve of degree d with an ordinary $(d-1)$-fold point P. Then the following properties hold.*

(i) *The point P is the only singular point of C.*

(ii) *The normalization of C is a smooth rational curve.*

(iii) *Fix a point $P \in \mathbb{P}^2$. Then the variety of all plane curves of degree d with P as an ordinary $(d-1)$-fold point is of dimension $2d$.*

(iv) *Given P and other $2d$ generic points, there exists a unique curve $C \subset \mathbb{P}^2$ of degree d, such that C has P as an ordinary $(d-1)$-fold point and passes through these $2d$ generic points.*

Proof. (i) We apply Bezout's theorem. (ii) We apply the genus formula. (iii) Let $[x, y, z]$ be the homogenous coordinates of $\mathbb{P}^2$ and let $P = [1, 0, 0]$. Then C is defined by the polynomial

$$f(x, y, z) = g(y, z) + \prod_{i=1}^{d-1} (a_i y - b_i z)x,$$

where $\deg(g) = d$. Therefore, the dimension is $2d$. □

PROPOSITION 5.7. *Let Σ be embedded into $\mathbb{F}_1$ (with section s) as a smooth anti-canonical curve and $x_1, \ldots, x_n$ are distinct points of Σ. Blowing up $\mathbb{F}_1$ at the x_i, we obtain Y_n with corresponding exceptional curves l_i, with $i = 1, \ldots, n$.*

(i) *When $n = 2k$, if $x_1, \ldots, x_n$ are in general position, then after contracting $f - l_1, \ldots, f - l_n$, we still obtain the surface $\mathbb{F}_1$. In other words, we obtain the same surface Y_{2k} by blowing up either $\{x_1, \ldots, x_n\}$, or $\{-x_1, \ldots, -x_n\}$.*

(ii) *When $n = 2k + 1$, if $x_1, \ldots, x_n$ are in general position, then after contracting $f - l_1, \ldots, f - l_n$, the resulting surface is $\mathbb{P}^1 \times \mathbb{P}^1$, but not $\mathbb{F}_1$.*

Proof. Let C be a negative rational curve in Y_n that does not intersect $f - l_i$, with $i = 1, \ldots, n$. Then C satisfies the following equations

$$C \cdot C = -m, m > 0;$$
$$C \cdot K = m - 2;$$
$$C \cdot (f - l_i) = 0, \quad i = 1, \ldots, n.$$

Since C is a rational curve and $\Sigma \in |-K|$, we have that $C \cdot (-K) \geqslant 0$. Thus $m \leqslant 2$. Then $m = 1$ or 2. Considering $\mathbb{F}_1$ as the blow-up of $\mathbb{P}^2$ at $0 \in \Sigma$ with exceptional curve s, we can assume that $C = a \cdot h - b \cdot s - \sum c_i \cdot l_i$, with $a \geqslant 0, b \geqslant 0$ and $c_i \geqslant 0$. Solving the system of equations, we obtain

$$m = 1 \text{ or } 2,$$
$$b = a - 1,$$
$$c_i = 1, \quad i = 1, \ldots, n,$$
$$a = (n - 1 + m)/2.$$

For $m = 1$, we find that $n = 2a$ is even. We have the class as follows:

$$C = ah - (a-1)s - \sum_{i=1}^{n=2a} l_i = af + s - \sum_{i=1}^{2a} l_i.$$

This means that all of the points $0, x_1, \ldots, x_n$ lie on the curve $\pi(C)$, where $\pi : Y_n \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ successively at $0, x_1, \ldots, x_n$. There exists exactly one such curve C for generic $x_1, \ldots, x_n$, and it is smooth, by Lemma 5.6. Hence, after contracting $f - l_1, \ldots, f - l_{2a}$, we still obtain $\mathbb{F}_1$.

For $m = 2$, it is clear that $n = 2a + 1$ is odd. We have the class as follows:

$$C = ah - (a-1)s - \sum_{i=1}^{n=2a+1} l_i = af + s - \sum_{i=1}^{2a+1} l_i.$$

This means that all of the points $0, x_1, \ldots, x_n$ lie on the curve $\pi(C)$, where $\pi : Y_n \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ successively at $0, x_1, \ldots, x_n$. There exists no such curve for generic $x_1, \ldots, x_n$, by Lemma 5.6. Hence, after contracting $f - l_1, \ldots, f - l_{2a+1}$, no rational curve with negative self-intersection number can survive. Therefore the resulting surface is $\mathbb{P}^1 \times \mathbb{P}^1$, but not $\mathbb{F}_1$. □

EXAMPLE 2. Blowing up $\mathbb{F}_1$ at two points x_1 and x_2 we obtain Y_2. Contracting $f - l_1$ and $f - l_2$, or contracting l_1 and l_2, we always obtain the surface $\mathbb{F}_1$. However, contracting $f - l_1$ and l_2, we just obtain the surface $\mathbb{P}^1 \times \mathbb{P}^1$, but not $\mathbb{F}_1$!

REMARK 5.8. (i) Lemma 5.6 has a corresponding version for $\mathbb{P}^1 \times \mathbb{P}^1$.

(ii) A G-configuration $\zeta_G = (e_1, \dots, e_n)$ for $S = X_n$, Y_n or Z_n just means that, after blowing down $e_n, e_{n-1}, \dots, e_1$ successively, we still obtain $\mathbb{P}^2$, $\mathbb{F}_1$ or $\mathbb{F}_1$, respectively.

Let S be an ADE-surface equipped with a G-configuration ζ_G. We denote the moduli space of the pairs (S, Σ) by $\mathcal{S}(\Sigma, G)$, where two pairs (S, Σ) and $(S^{'}, \Sigma^{'})$ are equivalent if and only if there is an isomorphism π from S to S' such that $\pi|_\Sigma$ is also an isomorphism from Σ to Σ'.

We show that $\mathcal{S}(\Sigma, G)$ is isomorphic to an open dense subset U of the variety $\mathrm{Hom}(\Lambda, \Sigma)/W$. In fact, for any element $\theta \in (\mathrm{Hom}(\Lambda, \Sigma)/W) \setminus U$, the boundary component, we can find possibly non-equivalent pairs (S, Σ) such that θ comes from the restriction. Thus, we can complete $\mathcal{S}(\Sigma, G)$ by adding these pairs and identifying them as one point. Denote the completion by $\overline{\mathcal{S}(\Sigma, G)}$. Then we can identify $\overline{\mathcal{S}(\Sigma, G)}$ with the projective variety $\mathrm{Hom}(\Lambda, \Sigma)/W$. This provides a natural compactification for the moduli space $\mathcal{S}(\Sigma, G)$.

More precisely, let $S = X_n$, Y_n or Z_n be an ADE-surface and let Λ be the root lattice of E_n, D_n or A_{n-1}, respectively, with corresponding Weyl group W. Furthermore, we fix a 3rd, 2nd or nth root of unity in $\mathrm{Jac}(\Sigma) \cong \Sigma$ in the E_n, D_n or A_{n-1} case (equivalently, we fix a cyclic subgroup of Σ of order $d = 3, 2$ or n). Then we have the following theorem.

THEOREM 5.9. (i) *There is an injective map ϕ from the moduli space $\mathcal{S}(\Sigma, G)$ onto an open dense subset of* $\mathrm{Hom}(\Lambda, \Sigma)/W$.

(ii) *The injective map ϕ can be extended to a bijective map from the completion $\overline{\mathcal{S}(\Sigma, G)}$ onto $Hom(\Lambda, \Sigma)/W$.*

(iii) *Moreover, the completion is obtained by including all rational surfaces with G-configuration to $\mathcal{S}(\Sigma, G)$. Any smooth rational curve on a surface corresponding to a boundary point has a self-intersection number at least -2, and in the E_n case these (-2) curves form chains of ADE-type.*

Proof. First we suppose $S = X_n$. We have constructed the map ϕ in Proposition 5.1. We prove the injectivity. We fix a G-configuration $\zeta_G = (l_1, \dots, l_n)$ on X_n, and a simple root system: $\alpha_1 = l_1 - l_2, \alpha_2 = l_2 - l_3, \alpha_3 = h - l_1 - l_2 - l_3, \alpha_4 = l_3 - l_4, \dots, \alpha_n = l_{n-1} - l_n$. Blowing down $l_n, l_{n-1}, \dots, l_1$ successively, we obtain $\mathbb{P}^2$ with Σ as an anti-canonical curve. For all $i = 1, \dots, n$, let $x_i \in X_n$ be the unique intersection points of l_i and Σ. Then X_n can be considered as a blow-up of $\mathbb{P}^2$ at these n points $x_i \in \Sigma$, with $i = 1, \dots, n$ with exceptional curves l_i, with $i = 1, \dots, n$.

According to previous arguments, we have a homomorphism $g \in \mathrm{Hom}(\Lambda, \Sigma)$. Let $g(\alpha_i) = p_i \in \Sigma$; then we have the following equations by the group law of Σ as an abelian group:

$$x_1 - x_2 = p_1,$$
$$x_2 - x_3 = p_2,$$
$$-x_1 - x_2 - x_3 = p_3,$$
$$x_{k-1} - x_k = p_k, k = 4, \dots, n.$$

The determinant of the coefficient matrix of this system of linear equations is ± 3. Thus it has unique solution (if we fix a third root of unity in $\mathrm{Jac}(\Sigma)$). That is, the x_i are uniquely determined by g up to Weyl group actions. The Weyl group actions just lead to choices of other G-configurations. By Proposition 5.4, this does not change the pair (X_n, Σ). Hence, ϕ is injective. These points x_i are not 'in general position' if and only if p_i will satisfy some (finitely many) equations. That means the image of ϕ must be open dense in $\mathrm{Hom}(\Lambda, \Sigma)/W$. The extendability of ϕ is also because of the existence and uniqueness of the solution of the above equations.

For the cases of Y_n and Z_n, the arguments are similar. It is easy to see that the map ϕ is well defined in both cases. For Y_n, the system of linear equations is given by

$$-x_1 - x_2 = p_1,$$
$$x_{k-1} - x_k = p_k, \quad k = 2, \ldots, n.$$

The determinant is ± 2. Thus the solution is uniquely determined (if we fix a 2nd root of unity in $\mathrm{Jac}(\Sigma)$). The remaining arguments are just the same as in the first case. Finally, for the case of Z_n, the system of equations is given by

$$\sum x_i = 0,$$
$$x_{k-1} - x_k = p_{k-1}, \quad k = 2, \ldots, n.$$

The determinant is $\pm n$. Then the solution is uniquely determined (if we fix an nth root of unity in $\mathrm{Jac}(\Sigma)$). The remaining arguments are just the same as in the E_n case. These prove (i) and (ii).

For (iii), the result follows from Lemma 5.3. □

Remark 5.10. The referee remarked that the set $\phi(\mathcal{S}(\Sigma, G))$ in Theorem 5.9 was exactly the complement of the discriminant in $\mathrm{Hom}(\Lambda, \Sigma)/W$. This is the case for E_n-type. As the referee indicated to us, this follows from the description by Looijenga [**19**, **20**] and Pinkham [**24**] of $\mathrm{Hom}(\Lambda, \Sigma)/W$ as the semi-universal deformation space of a simple elliptic singularity. The deformation space is realized as a family of affine surfaces, and the fiberwise compactification is a del Pezzo surface with an anti-canonical elliptic curve as the complement divisor. Furthermore, the -2 curves on fibers produce the vanishing cycles that determine the discriminant locus in $\mathrm{Hom}(\Lambda, \Sigma)/W$. For other cases, it is hoped to be true. However, we cannot give a proof at present. When the anti-canonical curve $C \in |-K_S|$ is a nodal rational curve, the moduli space of pairs (S, C) is considered by Looijenga in [**21**]. This is in fact a degeneration of the situation above, where the elliptic curve degenerates into a nodal curve. It is also interesting to study the configurations on such surfaces which are related to some fundamental representations.

As a conclusion of Lemma 4.1 and Theorem 5.9, we have the following.

Theorem 5.11. *When we fix a dth root of unity in* $\mathrm{Jac}(\Sigma)$*, we have a bijection*

$$\overline{\mathcal{S}(\Sigma, G)} \xrightarrow{\sim} \mathcal{M}_\Sigma^G,$$

where d is the exponent of the finite group Λ_c/Λ.

Remark 5.12 [**11**, **12**, **26**]. The moduli space of flat A_n bundles over Σ is exactly the ordinary projective space $\mathbb{CP}^n$. This can be described as follows. A flat $\mathrm{SU}(n+1)$ bundle is determined uniquely by $n+1$ points on Σ with sum equal to 0, up to isomorphism. Furthermore, $n+1$ points on Σ with sum equal to 0 are determined uniquely by a global section $H^0(\Sigma, \mathcal{O}_\Sigma(n(0)))$ up to a scalar, where (0) is the divisor of the identity element 0. Thus the moduli space of flat $\mathrm{SU}(n+1)$ bundles is isomorphic to $\mathbb{P}(H^0(\Sigma, \mathcal{O}_\Sigma((n+1)P))) = \mathbb{P}^n$.

From this we see that the moduli space of pairs $(S,\ \Sigma)$ is just the ordinary complex projective space $\mathbb{CP}^n$.

EXAMPLE 3. We look at what the pre-image of a trivial G-bundle is. For example, in the E_8 case, the trivial bundle means the element $0 \in \mathrm{Hom}(\Lambda(E_8), \Sigma)/W(G)$. By the above correspondence, all $x_i = 0$ in Σ. This means that we can blow up $\mathbb{P}^2$ at the identity element 0 (an inflection point) eight times to obtain the surface represented by this pre-image, which is a boundary point in the moduli space $\overline{\mathcal{S}(\Sigma, G)}$. Blowing it up once more, we obtain an elliptic fibration with a singular fiber of $\widetilde{E}_8$-type [**3**].

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