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Geometric structures on G_2 and Spin(7)-manifolds

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Abstract

This article studies the geometry of moduli spaces of G_2-manifolds, associative cycles, coassociative cycles and deformed Donaldson–Thomas bundles. We introduce natural symmetric cubic tensors and differential forms on these moduli spaces. They correspond to Yukawa couplings and correlation functions in M-theory.

We expect that the Yukawa coupling characterizes (co-)associative fibrations on these manifolds. We discuss the Fourier transformation along such fibrations and the analog of the Strominger–Yau–Zaslow mirror conjecture for G_2-manifolds.

We also discuss similar structures and transformations for Spin(7)-manifolds.

The *mirror symmetry conjecture* for Calabi–Yau 3-folds, which originated from the study of the string theory on $X \times \mathbb{R}^{3,1}$, has attracted much attention in both the physics and mathematics communities. Certain parts of the

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conjecture should hold true for Calabi–Yau manifolds of any dimensions. But others are designed only for 3-folds, for example, the special Kähler geometry on the moduli space, the holomorphic Chern–Simons theory and relationships to knot invariants. These features should be interpreted in the realm of the geometry of the G_2-manifold, $M = X \times S^1$. From a physical point of view, they come from the studies of M-theory on $M \times \mathbb{R}^{3,1}$.

In mathematics, manifolds with G_2-holonomy group have been studied for a while. In Berger's list of holonomy groups of Riemannian manifolds, there are a couple of exceptional holonomy groups, namely G_2 and Spin(7). Manifolds with these holonomy groups are Einstein manifolds of dimension 7 and 8, respectively. As cousins of Kähler manifolds, the analogs of complex structures, complex submanifolds and Hermitian–Yang–Mills bundles are vector cross-products [7], calibrated submanifolds [8] and Donaldson–Thomas bundles [6]. These geometries are studied by many people in the Oxford school, including Donaldson, Hitchin, Joyce, Thomas and others.

A manifold M with G_2-holonomy, or simply a G_2-manifold, can be characterized by the existence of a parallel positive 3-form Ω. Because of the natural inclusion of Lie groups $\mathrm{SU}(3) \subset G_2$, the product of a Calabi–Yau 3-fold X with a circle S^1 has a canonical G_2-structure with $\Omega = \mathrm{Re}\Omega_X - \omega_X \wedge dt$, where Ω_X and ω_X are the holomorphic volume form and the Calabi–Yau Kähler form on X.

The mirror symmetry conjecture for Calabi–Yau manifolds roughly says that there is a duality transformation from the symplectic geometry (or the A-model) to the complex geometry (or the B-model) between a pair of Calabi–Yau 3-folds. For instance, natural cubic structures, called the A- and B-Yukawa couplings, on their moduli spaces can be identified. This gives highly non-trivial predictions for the enumerative geometry of X.

From physical considerations, it is more natural to understand the *M-theory* on $M \times \mathbb{R}^{3,1}$, with M being a G_2-manifold. This is studied recently by Acharya, Atiyah, Vafa, Witten, Yau, Zaslow and others [1–4]. To better understand the M-theory and its duality transformations, or (equivalently) the geometry of G_2-manifolds, we need to study natural geometric structures on various moduli spaces attached to M. For example, we introduce the analog of the Yukawa coupling on the moduli space of G_2-metrics as follows:

$$\mathcal{Y}_M(\phi) = \int_M \Omega(\hat{\phi}, \hat{\phi}, \hat{\phi}) \wedge *\Omega,$$

where $\phi \in H^3(M, \mathbb{R})$. This cubic structure comes from the exceptional Jordan algebra structure for E_6 and it seems to be even more natural than those A- and B-Yukawa couplings for Calabi–Yau 3-folds. It will be used later to characterize different kinds of fibration structures on M.

The geometry of a G_2-manifold is reflected by its calibrated submanifolds [8] and Yang–Mills bundles [6]. Calibrated submanifolds are always volume minimizing and there are two different types in a G_2-manifold M: (i) coassociative submanifolds C of dimension 4, calibrated by $*\Omega$ and (ii) associative submanifolds A of dimension 3, calibrated by Ω. Sometimes it is important to include both points and the whole manifold in the above list, together they are calibrated by different components of $e^{\Omega+*\Omega}$.

In physics, one needs to study *supersymmetric cycles* [12] which are calibrated submanifolds together with *deformed Yang–Mills bundles* over them. In the G_2 case, there are three different types: *coassociative cycle* (C, D_E), with D_E an ASD connection on a coassociative submanifold C; (ii) *associative cycle* (A, D_E), with D_E a unitary flat connection on an associative submanifold A, it is a critical point of the following functional,

$$\int_{A\times[0,1]} \mathrm{Tr}[e^{*\Omega+\tilde{F}}].$$

(iii) *Deformed Donaldson–Thomas connection* D_E which satisfies the equation

$$F \wedge *\Omega + F^3/6 = 0,$$

and it is a critical point of the following functional,

$$\int_{M\times[0,1]} \mathrm{Tr}[e^{*\Omega+\tilde{F}}].$$

Moduli spaces of these cycles are denoted as $\mathcal{M}^{\mathrm{coa}}(M), \mathcal{M}^{\mathrm{ass}}(M)$ and $\mathcal{M}^{\mathrm{bdl}}(M)$, respectively, or simply $\mathcal{M}^{\mathrm{coa}}, \mathcal{M}^{\mathrm{ass}}$ and $\mathcal{M}^{\mathrm{bdl}}$. We will define canonical 3-forms and 4-forms on them using the Clifford multiplication on spinor bundles and the Lie algebra structure on the space of self-dual 2-forms. In physics languages, they correspond to *correlation functions* in quantum field theory. We will also introduce natural symmetric cubic tensors on these moduli spaces.

In the flat situation, such canonical 3-forms determine G_2-structures on both $\mathcal{M}^{\mathrm{coa}}$ and $\mathcal{M}^{\mathrm{ass}}$. In general, the moduli space of associative (resp. coassociative) submanifolds can be regarded as a coassociative (resp. associative) subspace of $\mathcal{M}^{\mathrm{ass}}$ (resp. $\mathcal{M}^{\mathrm{coa}}$).

We summarize these structures on various moduli spaces in following tables: structures on the moduli space of G_2-metric with unit volume.

	Moduli	Yukawa coupling	Metric tensor	Prepotential
$\mathcal{M}^{G_2}$	G_2-structures	$\mathcal{Y}$	$\mathcal{G}$	$\mathcal{F}$

Structures on the moduli space of various cycles on M.

	Moduli of	3-form	4-form	cubic tensor
M	Points	Ω	Θ	
$\mathcal{M}^{\mathrm{coa}}(M)$	Coassociative cycles	$\Omega_{\mathcal{M}^{\mathrm{coa}}(M)}$	$\Theta_{\mathcal{M}^{\mathrm{coa}}(M)}$	$\mathcal{Y}_{\mathcal{M}^{\mathrm{coa}}(M)}$
$\mathcal{M}^{\mathrm{ass}}(M)$	Associative cycles	$\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$	$\Theta_{\mathcal{M}^{\mathrm{ass}}(M)}$	$\mathcal{Y}_{\mathcal{M}^{\mathrm{ass}}(M)}$
$\mathcal{M}^{\mathrm{bdl}}(M)$	Deformed DT bundles	$\Omega_{\mathcal{M}^{\mathrm{bdl}}(M)}$	$\Theta_{\mathcal{M}^{\mathrm{bdl}}(M)}$	$\mathcal{Y}_{\mathcal{M}^{\mathrm{bdl}}(M)}$

In Section 3, we study structures of calibrated fibrations on M, they are associative T^3-fibration, coassociative T^4-fibration and coassociative K3-fibration. Then we propose a duality transformation along such fibrations on G_2-manifolds analogous to the Fourier–Mukai transformation in the geometric mirror symmetry conjecture by Strominger *et al.* [16]. Roughly speaking, it should be given by a fiberwise Fourier transformation on a coassociative T^4-fibration on M. We partially verify our proposed conjecture in the flat case, in the spirit of [10, 11].[1]

For instance the transformation of an ASD connection over a coassociative torus fiber is another ASD connection over the dual torus fiber, as studied earlier by Schenk [15] and Braam and van Baal [5].

We also study the Fourier transformation on an associative T^3-fibration on M. The Fourier transformation on a coassociative T^4-fibration takes cycles calibrated by e^Θ (resp. $*e^\Theta$) back to themselves. On an associative T^3-fibration, the Fourier transformation takes cycles calibrated by e^Θ to those calibrated by $*e^\Theta$.

In Section 4, we study the geometry of Spin(7)-manifolds in the same manner, and therefore our discussions will be brief.

1 G_2-manifolds and their moduli

1.1 Definitions

We first review some basic structures on G_2-manifolds (see [9, 14] for more details). Any G_2-manifold M admits a parallel positive 3-form Ω_M (or simply Ω) and its Ricci curvature is zero. When

$$M = \mathbb{R}^7 = \operatorname{Im}\mathbb{H} \oplus \mathbb{H} = \operatorname{Im}\mathbb{O}$$

[1]Those authors are also familiar with these transformations in the G_2 case.

with the standard metric and orientation $dx^{123}dy^{0123}$, we have

$$\Omega = dx^{123} - dx^1(dy^{23} + dy^{10}) - dx^2(dy^{31} + dy^{20}) - dx^3(dy^{12} + dy^{30}).$$

Remark. Since the Lie group G_2 preserves a vector cross-product structure on $\mathbb{R}^7$, any G_2-manifold inherits such a product structure $\times$ on its tangent spaces given by

$$\langle u \times v, w \rangle = \Omega(u, v, w).$$

The parallel 4-form $*\Omega$ will be denoted as Θ,

$$\Theta = dy^{0123} + dx^{23}(dy^{23} + dy^{10}) + dx^{31}(dy^{31} + dy^{20}) + dx^{12}(dy^{12} + dy^{30}).$$

When M is the product of a Calabi–Yau 3-fold X and S^1 with its reversed orientation, then the holonomy group equals $\mathrm{Hol}(M) = \mathrm{SU}(3) \subset G_2$. Thus $M = X \times S^1$ is a G_2-manifold. In this case we have

$$\begin{aligned} \Omega &= \operatorname{Re}\Omega_X - \omega_X \wedge d\theta, \\ \Theta &= \operatorname{Im}\Omega_X \wedge d\theta - \omega_X^2/2. \end{aligned}$$

where ω_X and Ω_X are the Ricci flat Kähler form and a holomorphic volume form on X.

Using the G_2 action, we can decompose differential forms (or cohomology classes) on M into irreducible components. For example,

$$\begin{aligned} \Lambda^1 &= \Lambda^1_7, \\ \Lambda^2 &= \Lambda^2_7 \oplus \Lambda^2_{14}, \\ \Lambda^3 &= \Lambda^3_1 \oplus \Lambda^3_7 \oplus \Lambda^3_{27}. \end{aligned}$$

When $\mathrm{Hol}(M) = G_2$, we have $H^k_7(M) = 0$ for all k and $H^3_1(M)$ (resp. $H^4_1(M)$) is generated by Ω (resp. Θ). Furthermore, the moduli space of G_2-metrics is smooth with tangent space equals $H^3(M) = H^3_1(M) \oplus H^3_{27}(M)$. If we normalize the volume to be one, the tangent space to this moduli space $\mathcal{M}^{G_2}$ becomes $H^3_{27}(M)$.

1.2 G_2-analog of Yukawa couplings

In this subsection, we introduce several symmetric tensors on the moduli space of G_2-manifolds. They are the Yukawa coupling, a metric tensor and the prepotential. First we recall a tensor $\chi \in \Omega^3(M, T_M)$ which is defined by $\iota_v \Theta = \langle \chi, v \rangle \in \Omega^3(M)$ for every tangent vector v.

Definition 1.1. *Suppose M is a compact G_2-manifold, we define*

$$\mathcal{C}_M : \bigotimes^3 H^3(M,\mathbb{R}) \to \mathbb{R},$$
$$\mathcal{C}_M(\phi_1,\phi_2,\phi_3) = \int_M \Omega(\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3) \wedge \Theta,$$

where $\hat{\phi} = *(\phi \wedge \chi) \in \Omega^1(M, T_M)$ and $\Omega(\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3) \in \Omega^3(M)$ is the evaluation of the 3-form Ω on the vector components of $\hat{\phi}_j$'s.

It is clear that $\mathcal{C}_M$ is symmetric. By using $\mathcal{C}_M$, we define $\mathcal{Y}_M$, $\mathcal{G}_M$ and $\mathcal{F}_M$ on $H^3_{27}(M)$ as follows:[2] symmetric tensors on the moduli space of G_2-manifolds. Here $\phi \in H^3_{27}(M)$.

Yukawa coupling	$\mathcal{Y}_M(\phi) = \mathcal{C}_M(\phi,\phi,\phi) = C\int_M \Omega \wedge *\Omega$
Metric tensor	$\mathcal{G}_M(\phi) = \mathcal{C}_M(\phi,\phi,\Omega) = C'\int_M \phi \wedge *\phi$
Prepotential	$\mathcal{F}_M = \mathcal{C}_M(\Omega,\Omega,\Omega)$

We will use this Yukawa coupling and $p_1(M)$ in Section 3.1 to describe structures of calibrated fibrations on M.

1.2.1 Reduction to CY 3-folds

When $M = X \times S^1$ with $\mathrm{Hol}(X) = \mathrm{SU}(3)$, i.e., X is a Calabi–Yau 3-fold, any deformation of the Einstein structure on M remains a product. There is a natural isomorphism of complex vector spaces

$$H^3_{27}(M,\mathbb{C}) \cong H^{1,1}(X) + H^{2,1}(X) + H^{1,2}(X).$$

For example, if ϕ is a primitive form in $H^{1,1}(X)$ (resp. $H^{2,1}(X)$), then $\phi \wedge dt$ (resp. ϕ) is in $H^3_{27}(M,\mathbb{C})$. Moreover, $\mathcal{Y}_M$ restricts to the Yukawa couplings on $H^{1,1}(X)$, $H^{2,1}(X)$ and $H^{1,2}(X)$ (e.g., [11]).

Remark. The cubic structure on the G_2-module Λ^3_{27} is the exceptional Jordan algebra structure on $\mathbb{R}^{27}$, whose automorphism group determines an exceptional Lie group of type E_6. When we reduce from G_2 to $\mathrm{SU}(3)$, i.e., $M = X \times S^1$, the Kähler form ω_X is naturally an element in Λ^3_{27} which splits $\Lambda^3_{27} = 27$ into $1 + 26$. In this case, the automorphism group of the algebraic structure of this 26 dimensional vector space determines an exceptional Lie group of type F_4. Roughly speaking, the Yukawa coupling on a G_2-manifold arises from a E_6 structure. When the holonomy group reduces to $\mathrm{SU}(3)$, then the E_6 structure also reduces to a F_4 structure.

[2]Recall $H^3_{27}(M) = H^3(M) \cap \mathrm{Ker}(\wedge\Omega) \cap \mathrm{Ker}(\wedge\Theta)$.

1.2.2 Including B-fields

From physical considerations, Acharya [1] argues that when M is a smooth G_2-manifold, then the low energy theory for M-theory on $M \times \mathbb{R}^{3,1}$ is an $N = 1$ supergravity theory with $b_2(M)$ Abelian vector multiplets plus $b_3(M)$ neutral chiral multiplets. We also include one scalar field and consider the enlarged moduli space $\mathcal{M}^{G_2+B} = \mathcal{M}^{G_2} \times H^2(M, U(1)) \times H^0(M, U(1))$. Note $b_0(M) = 1$.

We are going to construct the corresponding Yukawa coupling for this moduli space. We define (i) a symmetric cubic tensor $\mathcal{C}'$ on $H^2(M, \mathbb{R})$,

$$\mathcal{C}'_M : \bigotimes^3 H^2(M) \to \mathbb{R},$$
$$\mathcal{C}'_M(\beta_1, \beta_2, \beta_3) = \int_M \Omega(\hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3) \wedge \Theta,$$

where $\hat{\beta} = \iota_\beta \chi \in \Omega^1(M, T_M)$ and (ii) a symmetric bilinear tensor $\mathcal{Q}_M$ on $H^2(M, \mathbb{R})$,

$$\mathcal{Q}_M : \bigotimes^2 H^2(M) \to \mathbb{R},$$
$$\mathcal{Q}_M(\beta_1, \beta_2) = \int_M \beta_1 \wedge \beta_2 \wedge \Omega.$$

Now the Yukawa coupling on the enlarged moduli space $\mathcal{M}^{G_2} \times H^2(M, U(1)) \times H^0(M, U(1))$ of G_2-manifolds (with $\mathrm{Hol} = G_2$) is defined as a combination of $\mathcal{Y}_M$, $\mathcal{C}'_M$ and $\mathcal{Q}_M$ as follows:

$$\tilde{\mathcal{Y}}_M : \bigotimes^3 (H^3_{27}(M) + H^2_{14}(M) + H^0(M)) \to \mathbb{R}.$$

When $M = X \times S^1$ for a Calabi–Yau 3-fold X, we have $H^2_{14}(M) = H^{1,1}_{\mathrm{prim}}(X, \mathbb{R})$ which corresponds to the decomposition $\mathbf{g}_2 = \mathbf{su}(3) + \mathbb{C}^3 + \mathbb{C}^{3*}$ because of $H^{2,0}(X) = H^{0,2}(X) = 0$ for a manifold with SU(3) holonomy. Therefore,

$$H^2_{14}(M, U(1)) + H^0(M, U(1)) \cong H^2(X, U(1)),$$

and an element in it is usually called a *B-field* on the Calabi–Yau manifold X. Our Yukawa coupling $\tilde{\mathcal{Y}}$ thus reduces to the usual one on Calabi–Yau manifolds coupled with B-fields.

Remark. If we do not fix the volume of M, then the moduli space of G_2-metrics with B-fields is locally isomorphic to $H^{\leq 3}(M, \mathbb{R})$. In M-theory (see e.g., [4]), it is desirable to complexify this moduli space, and this implies that the resulting space will be locally isomorphic to the total cohomology group $H^*(M, \mathbb{R})$ of M.

2 Supersymmetric cycles

2.1 Deformed Donaldson–Thomas bundles

In [6], Donaldson and Thomas study the following Yang–Mills equation for Hermitian connections D_E on a complex vector bundle E over M,

$$F_E \wedge \Theta = 0 \in \Omega^6(M, \mathrm{ad}(E)).$$

This is the Euler–Lagrange equation for the following Chern–Simons-type functional,

$$\begin{aligned} \mathcal{A}(E) &\to \mathbb{R}, \\ D_E &\to \int_M \mathrm{CS}_3(D_0, D_E) \wedge \Theta, \end{aligned}$$

where D_0 is any fixed background connection. This functional is equivalent to the following functional

$$\int_{M\times[0,1]} \mathrm{Tr}\, e^{\tilde{F}} \wedge \Theta,$$

where $\tilde{F}$ is the curvature of a connection on $M \times [0,1]$ formed by the affine path of connections on E joining D_0 and D_E.

On a G_2-manifold, we introduce the following *deformed Donaldson–Thomas equation* which should have the corresponding effect of preserving supersymmetry in M-theory [12],

$$F_E \wedge \Theta + F_E^3/6 = 0 \in \Omega^6(M, \mathrm{ad}(E)).$$

Equivalently, this equals

$$[e^{\Theta + F_E}]^{[6]} = 0 \in \Omega^6(M, \mathrm{ad}(E)).$$

It is the Euler–Lagrangian equation for the following Chern–Simons-type functional

$$D_E \to \int_{M\times[0,1]} \mathrm{Tr}[e^{\Theta+\tilde{F}}] = \int_M \mathrm{CS}(D_0, D_E) e^{\Theta}.$$

The mirror transformation along a coassociative T^4-fibration on a flat G_2-manifold, which we will discuss in Section 3.3, takes an associative section to a deformed Donaldson–Thomas bundle on the mirror manifold.

2.1.1 Geometric structures on $\mathcal{M}^{\mathrm{bdl}}(\mathbf{M})$

We define natural differential forms and a symmetric cubic tensor on the moduli space of deformed DT bundles $\mathcal{M}^{\mathrm{bdl}}(M)$ as follows: natural geometric structures on the moduli space of deformed DT bundles on M. Here $\alpha, \beta, \gamma, \delta \in \Omega^1(M, \mathrm{ad}(E))$.

3-form	$\Omega_{\mathcal{M}^{\mathrm{bdl}}(M)} = \int_M \mathrm{Tr}[\alpha \wedge \beta \wedge \gamma]_{\mathrm{skew}} e^{\Theta + F_E}$
4-form	$\Theta_{\mathcal{M}^{\mathrm{bdl}}(M)} = \int_M \mathrm{Tr}[\alpha \wedge \beta \wedge \gamma \wedge \delta]_{\mathrm{skew}} * e^{\Theta + F_E}$
Cubic	$\mathcal{Y}_{\mathcal{M}^{\mathrm{bdl}}(M)} = \int_M \langle \alpha, [\beta, \gamma] \rangle_{\mathrm{ad}} e^{\Theta + F_E}$.

Here $\langle \cdot, \cdot \rangle_{\mathrm{ad}}$ is the Killing form on $\mathbf{g}$, the Lie algebra of the gauge group. In particular, the symmetric cubic tensor $\mathcal{Y}_{\mathcal{M}^{\mathrm{bdl}}(M)}$ on $\mathcal{M}^{\mathrm{bdl}}(M)$ is trivial when G is Abelian. Note $e^{\Theta + F_E}$ in the cubic form works as $[e^{\Theta + F_E}]^{[6]}$.

Remark. When M is a flat torus $T = \mathbb{R}^7/\Lambda$, the moduli space $\mathcal{M}^{\mathrm{bdl}}(M)$ of flat $U(1)$-bundles is canonically isomorphic to the dual flat torus $T^* = \mathbb{R}^{7*}/\Lambda^*$. Under this natural identification, we have

$$\Omega_{\mathcal{M}^{\mathrm{bdl}}(T)} = \Omega_{T^*},$$
$$\Theta_{\mathcal{M}^{\mathrm{bdl}}(T)} = \Theta_{T^*}.$$

Moreover, this transformation from T to $\mathcal{M}^{\mathrm{bdl}}(T) = T^*$ is involutive, i.e., $\mathcal{M}^{\mathrm{bdl}}(T^*) = T$.

Remark. The tangent bundle of M is a G_2-bundle, moreover, the curvature tensor of the Levi–Civita connection on M satisfies the DT-equation,

$$F \wedge \Theta = 0.$$

This equation is equivalent to $F \in \Omega^2_{14}(M, \mathrm{ad}(T_M))$ and it follows from the fact that $F \in \Omega^2(M, \mathbf{g}_2(T_M))$ and the torsion freeness of the connection.

Infinitesimal deformations of T_M as a DT-bundle are parameterized by the first cohomology group $H^1(M, \mathbf{g}_2(T_M))$ of the following elliptic complex [6]:

$$0 \to \Omega^0(M, \mathbf{g}_2(T_M)) \overset{D_E}{\to} \Omega^1(M, \mathbf{g}_2(T_M)) \overset{\Theta D_E}{\to} \Omega^6(M, \mathbf{g}_2(T_M))$$
$$\overset{D_E}{\to} \Omega^7(M, \mathbf{g}_2(T_M)) \to 0.$$

When $M = X \times S^1$ with X a Calabi–Yau 3-fold with SU(3) holonomy, then there is a natural isomorphism

$$H^1(M, \mathbf{g}_2(T_M)) \cong H^{1,1}(X) + H^{2,1}(X) + H^1(X, \mathrm{End}_0(T_X)).$$

Furthermore, the above cubic tensor $\mathcal{Y}_{\mathcal{M}^{\mathrm{bdl}}(M)}$ restricts to $\mathcal{Y}_A$ on $H^{1,1}(X)$, $\mathcal{Y}_B$ on $H^{2,1}(X)$ and a similar Yukawa coupling $\mathcal{Y}_C$ on $H^1(X, \mathrm{End}_0(T_X))$.

All three seemingly unrelated Yukawa couplings on a Calabi–Yau 3-fold combine in a natural way when we study the G_2-manifold $X \times S^1$.

2.2 Associative cycles

2.2.1 Associative submanifolds

There are two types of non-trivial calibrations on a G_2-manifold M as studied by Harvey and Lawson [8], they are (i) associative submanifolds and (ii) coassociative submanifolds. They are always absolute minimal submanifolds in M.

A three-dimensional submanifold A in M is called an *associative* submanifold if the restriction of Ω to A equals the volume form on A of the induced metric.

It has the following two equivalent characterizations:

(1) The restriction of $\chi \in \Omega^3(M, T_M)$ to A is zero;
$$\chi|_A = 0 \in \Omega^3(A, T_M|_A).$$
(2) The vector cross-product on M preserves T_A, i.e., $u, v \in T_A$ implies that $u \times v \in T_A$. As a corollary, two associative submanifolds cannot intersect along a two-dimensional subspace.

McLean studies the deformation theory of associative submanifolds in [13] and identifies their infinitesimal deformations as certain twisted harmonic spinors. Unlike coassociative submanifolds, deformations of an associative submanifold can be *obstructed.* For example, if we take any smooth isolated rational curve C with normal bundle $O \oplus O(-2)$ in a Calabi–Yau 3-fold X, then $C \times S^1$ is an associative submanifold in $M = X \times S^1$ with obstructed deformations. That is the moduli space of associative submanifolds, denoted $\mathcal{B}^{\mathrm{ass}}(M)$, can be singular.

2.2.2 Associative cycles

From a physical perspective, one would couple associative submanifolds with gauge fields. This is analogous to coupling of gauge fields with special Lagrangian submanifolds of a Calabi–Yau manifold. From a mathematical perspective, as we will see, the moduli space of associative submanifolds coupled with gauge fields has very rich geometric structures.

An *associative cycle* is defined to be any pair (A, D_E) with A an associative submanifold in M and D_E a unitary flat connection on a bundle E over

A. The flatness equation, $F_E = 0$ for connections over a 3-manifold is the Euler–Lagrange equation of the standard Chern–Simons functional. Analogously, the associativity condition for a pair (A, D_E) is the Euler–Lagrange equation of the following functional:

$$\begin{aligned} \mathrm{Map}(A, M) \times \mathcal{A}(E) &\to \mathbb{R}/\mathbb{Z}, \\ \mathrm{CS}(A, D_E) &= \int_{A\times[0,1]} \mathrm{Tr}[e^{\Theta+\tilde{F}}]. \end{aligned}$$

The first term in the above functional is a direct analog of a functional used by Thomas [17] for special Lagrangian submanifolds in a Calabi–Yau 3-fold.

Next we are going to study the moduli space of associative cycles, namely the critical set of CS, and we denote it as $\mathcal{M}^{\mathrm{ass}}(M)$. This space has richer structure than the moduli space of associative submanifolds $\mathcal{B}^{\mathrm{ass}}(M)$.

The tangent space of $\mathcal{M}^{\mathrm{ass}}(M)$ at an associative cycle (A, D_E) can be identified as [13]

$$T_{(A,D_E)}\mathcal{M}^{\mathrm{ass}} = \mathrm{Ker}\,\mathbf{D} \oplus H^1(A, \mathrm{ad}(E)).$$

We define a 3-form $\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$, 4-form $\Theta_{\mathcal{M}^{\mathrm{ass}}(M)}$ and a symmetric cubic tensor $\mathcal{Y}_{\mathcal{M}^{\mathrm{ass}}(M)}$ on $\mathcal{M}^{\mathrm{ass}}(M)$ as follows: natural geometric structures on the moduli space of associative cycles on M. Here $\alpha, \beta, \gamma, \delta \in \Omega^1(A, \mathrm{ad}(E))$ and $\phi, \eta, \xi, \zeta \in \mathrm{Ker}\, D$.

3-form	$\Omega_{\mathcal{M}^{\mathrm{ass}}(M)} = \begin{cases} \int_A \mathrm{Tr}\ \alpha\wedge\beta\wedge\gamma \\ -\int_A \langle \alpha\cdot\overline{\phi}, \overline{\eta}\rangle\Omega \end{cases}$
4-form	$\Theta_{\mathcal{M}^{\mathrm{ass}}(M)} = \begin{cases} \int_A \langle\overline{\phi}, *(\alpha\wedge\beta)\cdot\overline{\eta}\rangle\Omega \\ \int_A \det(\phi,\eta,\xi,\zeta)\Omega \end{cases}$
Cubic	$\mathcal{Y}_{\mathcal{M}^{\mathrm{ass}}(M)} = \int_A \langle[\alpha,\beta],\gamma\rangle\Omega$

In the definition of the 4-form, we use the fact that the twisted spinor bundle S is a complex rank 2 bundle and therefore, as a real vector bundle $S_{\mathbb{R}}$, there is a natural trivialization of $\Lambda^4 S_{\mathbb{R}} \cong \mathbb{R}$. The action of $\mathrm{Im}\,\mathbb{H}$ on $\mathbb{H}$ has used to define 3- and 4-form.

2.2.3 An example

In this example, the above 3-form $\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$ defines a G_2-structure on the moduli space $\mathcal{M}^{\mathrm{ass}}(M)$. We consider a flat example $M = T^3 \times T^4$ with $T^3 = \mathbb{R}^3/\Lambda_3$ and $T^4 = \mathbb{R}^4/\Lambda_4$. The projection to the second factor $M \to B = T^4$ is an associative fibration. The moduli space $\mathcal{M}^{\mathrm{ass}}(M)$ of (A, D_E) with A a fiber and D_E a flat $U(1)$-connection over A, can be identified with $T^{3*} \times T^4$, where $T^{3*} = \mathbb{R}^{3*}/\Lambda_3^*$ is the dual torus to T^3. Furthermore, $\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$ and $\Theta_{\mathcal{M}^{\mathrm{ass}}(M)}$ are precisely the natural calibration 3- and 4-forms on the flat G_2-manifold $T^{3*} \times T^4$. This is a simple but fun exercise.

In particular, $\chi_{\mathcal{M}^{\mathrm{ass}}(T^3 \times T^4)} = \chi_{\mathcal{M}^{\mathrm{ass}}(T^{3*} \times T^4)}$ and therefore, $T^{3*} \times T^4 \to T^4$ defines an associative fibration structure on $\mathcal{M}^{\mathrm{ass}}(T^3 \times T^4) \to \mathcal{B}^{\mathrm{ass}}(T^3 \times T^4)$ with a coassociative section, namely, $\Omega_{\mathcal{M}^{\mathrm{ass}}(T^3 \times T^4)}$ (resp. $\chi_{\mathcal{M}^{\mathrm{ass}}(T^3 \times T^4)}$) vanishes on the section (resp. fibers) of this fibration.

In the next paragraph, we will explain that such structures exist on every $\mathcal{M}^{\mathrm{ass}}(M)$. However, when M is not flat, $\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$ and $\Theta_{\mathcal{M}^{\mathrm{ass}}(M)}$ are not parallel forms, thus they do not define G_2-structure on $\mathcal{M}^{\mathrm{ass}}(M)$ and we can only discuss their (co-)associativity in terms of vanishing of appropriate tensors.

2.2.4 Moduli space of associative submanifolds is coassociative

Every associative submanifold A defines an associative pair (A, D_E) where D_E is simply the trivial connection d. Thus we have an embedding of $\mathcal{B}^{\mathrm{ass}}(M)$ inside $\mathcal{M}^{\mathrm{ass}}(M)$. It is not difficult to see that the restriction of $\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$ to it is zero. As we will see in the next subsection, this property characterizes a coassociative submanifold is $\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$ defines a G_2-structure on $\mathcal{M}^{\mathrm{ass}}(M)$. Furthermore, we have a natural fibration structure, $\mathcal{M}^{\mathrm{ass}}(M) \to \mathcal{B}^{\mathrm{ass}}(M)$ by forgetting the connections. Fibers are associative submanifolds in the sense that the restriction $\chi_{\mathcal{M}^{\mathrm{ass}}(M)}$ to them are zero. Here $\chi_{\mathcal{M}^{\mathrm{ass}}(M)}$ is defined the same way as χ for M. The reason is fiber directions for $\mathcal{M}^{\mathrm{ass}}(M) \to \mathcal{B}^{\mathrm{ass}}(M)$ corresponds to $H^1(A, \mathrm{ad}(E))$ and each component in $\Theta_{\mathcal{M}^{\mathrm{ass}}(M)}$ involves at least two $\mathrm{Ker}\,\mathbf{D}$ components in its variables. Therefore, $\chi_{\mathcal{M}^{\mathrm{ass}}}$ must vanish on fibers.

2.3 Coassociative cycles

2.3.1 Coassociative submanifolds

A four-dimensional submanifold C in M is called a *coassociative submanifold* if it is calibrated by Θ, i.e., $\Theta|_C = \mathrm{vol}_C$. It can be characterized by

$$\Omega|_C = 0 \in \Omega^3(C, \mathbb{R}).$$

Another characterization is the vector cross-product on M preserves $N_{C/M} \subset TM$. As a corollary of this, any two coassociative submanifolds in M cannot intersect along a three-dimension subspace.

The normal bundle of C in M can be identified with the bundle of ASD 2-forms on C,

$$N_{C/M} = \Lambda^2_+(C).$$

Furthermore, the space of infinitesimal deformations of C inside M as a coassociative submanifold is isomorphic to the space of self-dual harmonic 2-forms on C, $H^2_+(C,\mathbb{R})$. In fact, the deformation theory of coassociative submanifold is unobstructed [13] and therefore their moduli space, denoted $\mathcal{B}^{\mathrm{coa}}(M)$, is always smooth.

2.3.2 Coassociative cycles

Similar to the associative cases, we will also couple coassociative submanifolds with certain gauge fields. A *coassociative cycle* is a pair (C, D_E) with C a coassociative submanifold in M and E an ASD connection on C with respect to the induced metric. Recall that a connection is called ASD if its curvature 2-form F_E lies in $\Omega^2_-(C,\mathrm{ad}(E))$, i.e., $F_E + *F_E = 0$. The tangent space of their moduli space $\mathcal{M}^{\mathrm{coa}}(M)$ can be identified as $H^2_+(C) + H^1(C,\mathrm{ad}(E))$ [13]. We define a 3-form $\Omega_{\mathcal{M}^{\mathrm{coa}}(M)}$, a 4-form $\Theta_{\mathcal{M}^{\mathrm{coa}}(M)}$ and a symmetric cubic tensor $\mathcal{Y}_{\mathcal{M}^{\mathrm{coa}}(M)}$ on $\mathcal{M}^{\mathrm{coa}}(M)$ as follows: natural geometric structures on the moduli space of coassociative cycles on M. Here $\alpha,\beta,\gamma,\delta \in \Omega^1(C,\mathrm{ad}(E))$ and $\phi,\eta,\xi \in H^2_+(C)$.

3-form	$\Omega_{\mathcal{M}^{\mathrm{coa}}(M)} = \begin{cases} \int_C [\phi,\eta]\wedge\xi \\ -\int_C \mathrm{Tr}(\phi\wedge\alpha\wedge\beta) \end{cases}$
4-form	$\Theta_{\mathcal{M}^{\mathrm{coa}}(M)} = \begin{cases} -\int_C \mathrm{Tr}(\alpha\wedge\beta\wedge\gamma\wedge\delta)_{\mathrm{skew}} \\ \int_C \mathrm{Tr}([\phi,\eta]\wedge\alpha\wedge\beta) \end{cases}$
Cubic	$\mathcal{Y}_{\mathcal{M}^{\mathrm{coa}}(M)} = \int_A \mathrm{Tr}\,\phi[\alpha,\beta]_{\mathrm{ad}(E)}$

Here, Lie algebra structure on space of 2-forms is given by $\wedge^2\mathbb{R}^4 \simeq so(\mathbb{R}^4)$.

2.3.3 An example

In this example, the above 3-form $\Omega_{\mathcal{M}^{\mathrm{coa}}(M)}$ defines a G_2-structure on the moduli space $\mathcal{M}^{\mathrm{coa}}(M)$. We consider a flat example $M = T^3 \times T^4$ with $T^3 = \mathbb{R}^3/\Lambda_3$ and $T^4 = \mathbb{R}^4/\Lambda_4$. The projection to the first factor $M \to B = T^3$ is a coassociative fibration. The moduli space $\mathcal{M}^{\mathrm{coa}}$ of (C, D_E) with C a fiber and D_E a flat $U(1)$-connection over C, can be identified with $T^3 \times T^{4*}$, where $T^{4*} = \mathbb{R}^{4*}/\Lambda_4^*$ is the dual torus to T^4. Moreover, $\Omega_{\mathcal{M}^{\mathrm{coa}}}$ and $\Theta_{\mathcal{M}^{\mathrm{coa}}}$ are precisely the natural calibration 3- and 4-form on the flat G_2-manifold $T^3 \times T^{4*}$. This is another simple but fun exercise.

2.3.4 Moduli space of coassociative submanifolds is associative

The fibration $\mathcal{M}^{\mathrm{coa}}(M) \to \mathcal{B}^{\mathrm{coa}}(M)$ is a "coassociative fibration" and the section embeds $\mathcal{B}^{\mathrm{coa}}(M)$ inside $\mathcal{M}^{\mathrm{coa}}(M)$ as an "associative" submanifold in an appropriate sense. The arguments are parallel to the case for $\mathcal{B}^{\mathrm{ass}}(M)$ in $\mathcal{M}^{\mathrm{ass}}(M)$ as we discussed in Section 2.2 and hence omitted.

2.4 Reductions to Calabi–Yau 3-folds

In this section, we assume $M = X \times S^1$ with X a Calabi–Yau 3-fold and

$$\Omega = \operatorname{Re}\Omega_X - \omega \wedge dt,$$

where Ω_X and ω_X are the holomorphic 3-form and the Kähler form on X, respectively. We will see that every moduli space of cycles (with $U(1)$-connections) on M and its natural differential forms inherit similar decompositions. That is,

$$\begin{aligned}\mathcal{M}(M) &= \mathcal{M}(X) \times S^1,\\ \Omega_{\mathcal{M}(M)} &= \operatorname{Re}\Omega_{\mathcal{M}(X)} - \omega_{\mathcal{M}(X)} \wedge dt.\end{aligned}$$

The manifold M itself can be regarded as the moduli space of zero dimensional cycles in M.

Remark. The reduction from a G_2-manifold M to a Calabi–Yau 3-fold X has a physical significance. Namely, we reduce the studies of M-theory on an 11-dimensional space time $\mathbb{R}^{3,1}\times M$ to the string theory on 10-dimensional space time $\mathbb{R}^{3,1}\times X$.

2.4.1 On the moduli of deformed DT bundles

The *deformed Hermitian Yang–Mills* connection on a bundle E on X is introduced in [12] as a supersymmetric cycle. Its curvature tensor satisfies the following equation

$$\begin{aligned}F \wedge \omega^2 &= F^3/3,\\ F^{0,2} &= 0.\end{aligned}$$

Their moduli space $\mathcal{M}^{\mathrm{bdl}}(X)$ has a natural holomorphic 3-form $\Omega_{\mathcal{M}^{\mathrm{bdl}}(X)}$ and an almost symplectic form $\omega_{\mathcal{M}^{\mathrm{bdl}}(X)}$,

$$\begin{aligned}\Omega_{\mathcal{M}^{\mathrm{bdl}}(X)}(\alpha,\beta,\gamma) &= \int_X \alpha\wedge\beta\wedge\gamma\wedge\Omega_X,\\ \omega_{\mathcal{M}^{\mathrm{bdl}}(X)}(\alpha,\beta) &= \int_X \alpha\wedge\bar\beta\wedge e^{\omega+F_E}.\end{aligned}$$

Here $\alpha, \beta, \gamma \in H^1(X, O_X) = H^{0,1}(X)$, the tangent space of $\mathcal{M}^{\mathrm{bdl}}(M)$ at D_E.

The pullback of any deformed Hermitian Yang–Mills connection D_E on X is a deformed DT connection on M. This is because the condition $F^{0,2} = 0$ for the holomorphicity of E on a 3-fold is equivalent to $F \wedge \mathrm{Re}\,\Omega_X = 0$ in $\Omega^5(X)$. We can obtain other elements in $\mathcal{M}^{\mathrm{bdl}}(M)$ by tensoring with flat $U(1)$-connection on S^1. The moduli space of flat connections on S^1 is the dual circle, also denoted as S^1. It is not difficult to verify the above mentioned decompositions for $\mathcal{M}^{\mathrm{bdl}}(M)$ and $\Omega_{\mathcal{M}^{\mathrm{bdl}}(M)}$ in this case.

2.4.2 On the moduli of associative cycles

Suppose that an associative submanifold A in $M = X \times S^1$ is of product types, then

Case (1): $A = \Sigma \times S^1$	Σ holomorphic curve in X
Case (2): $A = L \times \{t\}$	L special Lagrangian with phase $= 0$

Case (1): $A = \Sigma \times S^1$. We denote the moduli space of holomorphic curves Σ in X as $\mathcal{B}^{cx}(X)$. Similarly we consider the moduli space of pairs (Σ, D_E) where D_E is a flat $U(1)$-connection on Σ, and denote it as $\mathcal{M}^{cx}(X)$. Gopakumar and Vafa conjecture that the cohomology group of $\mathcal{M}^{cx}(X)$ admits an $\mathbf{sl}(2) \times \mathbf{sl}(2)$ action whose multiplicities determine Gromov–Witten invariants of X of every genus for the class $[\Sigma]$. The tangent space of $\mathcal{M}^{cx}(X)$ at (Σ, D_E) equals $H^0(\Sigma, N_{\Sigma/X}) \oplus H^1(\Sigma, O_\Sigma)$. It carries natural holomorphic 3-form $\Omega_{\mathcal{M}^{cx}(X)}$ and symplectic form $\omega_{\mathcal{M}^{cx}(X)}$ as follows.

Holomorphic 3-form	$\Omega_{\mathcal{M}^{cx}(X)} = \int_\Sigma \widetilde{(\phi \wedge \eta)} \wedge \alpha$
Symplectic form	$\omega_{\mathcal{M}^{cx}(X)} = \int_\Sigma \alpha \wedge \bar{\beta} \pm \langle \phi, \eta \rangle \omega_X$

Here $\alpha, \beta \in H^1(\Sigma, O_\Sigma)$ and $\phi, \eta \in H^0(\Sigma, N_{\Sigma/X})$ and $\widetilde{(\phi \wedge \eta)} \in T^*_\Sigma$ is the image $\phi \wedge \eta \in \Lambda^2 N_{\Sigma/X}$ under the natural identification $\Lambda^2 N_{\Sigma/X} \cong T^*_\Sigma$ because of $\Lambda^3 T_X \cong O_X$.

Note that any deformation of $A = \Sigma \times S^1$ as an associative submanifold in M must still be of the same form, i.e., $\mathcal{B}^{\mathrm{ass}}(M) = \mathcal{B}^{cx}(X)$. However, there are more flat $U(1)$ bundles on A than on Σ, i.e., $H^1(A, \mathbb{R}/\mathbb{Z}) \cong H^1(\Sigma, \mathbb{R}/\mathbb{Z}) \times S^1$. We have the decompositions for $\mathcal{M}^{\mathrm{coa}}(M)$ and $\Omega_{\mathcal{M}^{\mathrm{coa}}(M)}$.

Case (2): $A = L \times \{t\}$ with L a special Lagrangian submanifold of zero phase in X, i.e., calibrated by $\mathrm{Re}\,\Omega$.

We denote the moduli space of L (resp. (L, D_E) with D_E a flat $U(1)$-connection on L) in X as $\mathcal{B}^{cx}(X)$ (resp. $\mathcal{M}^{\mathrm{SL}}(X)$). McLean [13] introduces a 3-form on $\mathcal{B}^{cx}(X)$. On $\mathcal{M}^{\mathrm{SL}}(X)$, there are a holomorphic 3-form $\Omega_{\mathcal{M}^{\mathrm{SL}}(X)}$ and a symplectic form $\omega_{\mathcal{M}^{\mathrm{SL}}(X)}$ as follows.

Holomorphic 3-form	$\Omega_{\mathcal{M}^{\mathrm{SL}}(X)} = \int_L \alpha \wedge \beta \wedge \gamma$
Symplectic form	$\omega_{\mathcal{M}^{\mathrm{SL}}(X)} = \int_L \langle \alpha, J\beta \rangle \mathrm{Re}\, \Omega_X$

Here $\alpha, \beta, \gamma \in H^1(L, \mathbb{C})$, the tangent space of $\mathcal{M}^{\mathrm{SL}}(X)$ at (L, D_E) [13].

It is not difficult to verify that $\mathcal{B}^{\mathrm{ass}}(M) = \mathcal{B}^{\mathrm{SL}}(X) \times S^1$ and the decomposition for $\mathcal{M}^{\mathrm{ass}}(M)$ and $\Omega_{\mathcal{M}^{\mathrm{ass}}(M)}$.

2.4.3 On the moduli of coassociative cycles

Suppose C is a coassociative submanifold of product type in $M = X \times S^1$, then

Case (1): $C = L \times S^1$	L special Lagrangian with phase $= \pi/2$
Case (2): $C = S \times \{t\}$	S complex surface in X

Case (1): $C = L \times S^1$. We can argue as in the previous situation to obtain decompositions for $\mathcal{M}^{\mathrm{coa}}(M)$ and $\Omega_{\mathcal{M}^{\mathrm{coa}}(M)}$.

Case (2): $C = S \times \{t\}$. We denote the moduli space of complex surfaces S (resp. (S, D_E) with D_E a flat $U(1)$-connection on S) in X as $\mathcal{B}^{cx}(X)$ (resp. $\mathcal{M}^{cx}(X)$). The tangent space of $\mathcal{M}^{cx}(X)$ can be identified as $H^{2,0}(S) \oplus H^{0,1}(S)$ because of $K_X = O_X$. We have natural differential forms on $\mathcal{M}^{cx}(X)$. Here $\phi, \eta \in H^{2,0}(S)$ and $\alpha, \beta, \in H^{0,1}(S)$.

Holomorphic 3-form	$\Omega_{\mathcal{M}^{cx}(X)} = \int_S \phi \wedge \alpha \wedge \beta$
Symplectic form	$\omega_{\mathcal{M}^{cx}(X)} = \int_S (\langle \phi, J\eta \rangle + \langle \alpha, J\beta \rangle)\omega^2/2$

Again we can verify the decompositions for $\mathcal{M}^{\mathrm{coa}}(M)$ and $\Omega_{\mathcal{M}^{\mathrm{coa}}(M)}$.

3 Fibrations and Fourier transforms

We review briefly the Strominger–Yau–Zaslow's construction [16] of the conjectural mirror manifold of a Calabi–Yau 3-fold. If X is close to a large complex and Kähler structure limit point, then it should admit a special Lagrangian fibration,

$$\pi_X : X \to B,$$

calibrated by $\mathrm{Im}\Omega_X$. Moreover there is a special Lagrangian section, calibrated by $\mathrm{Re}\,\Omega_X$. The mirror manifold is conjectured to be the dual torus fibration over B. Moreover the fiberwise Fourier transformation is expected to play an important role in the mirror transformation of the complex (resp. symplectic) geometry of X to the symplectic (resp. complex) geometry of its mirror manifold.

In this section, we study calibrated fibrations on G_2-manifolds and Fourier transformations along them.

3.1 Calibrated fibrations

3.1.1 Coassociative fibrations

Starting with a special Lagrangian fibration with a section on X, the induced fibration on the G_2-manifold $M = X \times S^1$,

$$\pi : M \to B,$$

becomes a coassociative T^4-fibration with an associative section.

Recall that a general fiber in a Lagrangian fibration on any symplectic manifold is necessarily a torus. This is not true for a general fiber in a coassociative fibration on a G_2-manifold. For example, given any holomorphic fibration on a Calabi–Yau 3-fold,

$$p_X : X \to \mathbb{CP}^1 \cong S^2,$$

its general fiber is necessarily a K3 surface. The induced fibration on the G_2-manifold $M = X \times S^1$ over $S^2 \times S^1$ is again a coassociative fibration, which is now a K3-fibration.

We expect that a general fiber in a coassociative fibration is always a torus or a K3-surface.[3] This is indeed the case provided that a general fiber admits an Einstein metric. The normal bundle of a coassociative submanifold F is always the bundle of self-dual 2-forms on F. When F is a fiber of π, then

[3]We learn this from S.-T. Yau.

the normal bundle is necessarily trivial, i.e.,

$$\Lambda^2_+(F) = F \times \mathbb{R}^3.$$

This implies that

$$3\tau(F) + 2\chi(F) = 0,$$

where $\tau(F)$ and $\chi(F)$ are the signature and the Euler characteristic of F, respectively.

From physical considerations, we expect that there should be a family of G_2-metrics together with coassociative fibrations parameterized by (t_0, ∞) such that the limit as $t \to \infty$ would imply the second fundamental form of each smooth fiber would go to zero. Combining with the fact the Ricci tensor of a G_2-metric is trivial, this suggest that every smooth fiber should admit a Einstein metric. However, characteristic numbers for F saturate the Hitchin inequality for Einstein 4-manifolds, this implies that F is either flat or covered by a K3 surface.

3.1.2 Associative fibrations

Suppose M has an associative fibration,

$$\pi : M \to B.$$

As in the previous case, we expect a general fiber to admit a metric with zero Ricci curvature in the adiabatic limit, thus it must be a 3-torus T^3.

3.1.3 Conjectural characterizations

Calibrated fibrations on a Calabi–Yau 3-fold X can be either special Lagrangian T^3-fibrations or holomorphic fibrations. In the latter case, a generic fiber can be an elliptic curve T^2, an Abelian surface T^4 or a K3 surface. Wilson [18] studies such fibrations in terms of the ring structure on $H^2(X)$ and the linear form,

$$\int_X p_1(X) \cup (\cdot) : H^2(X, \mathbb{R}) \to \mathbb{R}.$$

Notice that such linear form is trivial on $H^3(X, \mathbb{R})$.

We conjecture that similar characterizations should hold true for G_2-manifolds. To be precise, let Ω_t be a family of G_2-structures on M for $t \in (0, 1)$ such that as $t \to 0$ the volume of M goes to zero while the diameter remains constant. For small t, M should admit a calibrated fibration, the structure of this fibration should be determined by the Yukawa coupling $\mathcal{Y}$ and $p_1(M)$ as follows.

On M	Characterizations
Associative T^3-fibration	$\mathcal{Y}(\Omega_t, \Omega_t, H^3) \to 0$
Coassociative T^4-fibration	$\mathcal{Y}(\Omega_t, \Omega_t, H^3) \nrightarrow 0$ and $\int p_1 \cup \Omega_t \to 0$
Coassociative K3-fibration	$\mathcal{Y}(\Omega_t, \Omega_t, H^3) \nrightarrow 0$ and $\int p_1 \cup \Omega_t \nrightarrow 0$

It is also possible that $\mathcal{Y}(\Omega_t, \Omega_t, H^3)$ becomes unbounded, but this should only happen when $\tilde{M} = X \times \mathbb{R}$.

Remark. We also expect that in above situations, $\lim_{t\to 0} \Omega_t$ is of infinite distance from any point in the moduli space of G_2-metrics on M. On the other hand, finite distance boundary points in this moduli space (i.e., incompleteness) might be related to contractions of (i) an ADE configuration of associative S^3's or (ii) an associative family of ADE configuration of S^2's and so on. Wang further conjectures that such incompleteness of the moduli space should be equivalent to degenerating families of G_2-structures on M with both the volume and the diameter bounded away zero and infinity uniformly.

3.2 Analog of the SYZ conjecture

We are going to propose an analog of the SYZ conjecture for G_2-manifolds, namely a transformation of G_2-geometry along a calibrated fibration. Acharya [1] is the first to use these T^4- or T^3-fibrations to study the mirror symmetry for G_2-manifolds from a physics point of view.

3.2.1 The G_2 mirror conjecture

There is a *reasonably* large class of G_2-manifolds such that every such M satisfies the following:

(1) *Mirror pair*: There is a family of G_2-metric g_t together with a coassociative T^4-fibration and an associative section on M. As t goes to infinity the diameter and the curvature tensor of each fiber go to zero. The moduli space $\mathcal{M}^{\mathrm{coa}}(M)$ of coassociative cycles (C, D_E) in M when C is a fiber admits a suitable compactification W_t as a G_2-manifold with $\Omega_{\mathcal{M}^{\mathrm{coa}}(M)}$ the calibrating 3-form (after suitable instanton corrections from associative cycles). The relation between M and W is involutive.

(2) *Yukawa couplings*: The Yukawa couplings $\mathcal{Y}_M$ and $\mathcal{Y}_W$ for M and W on $H^3_{27}(M)$ and $H^3_{27}(W)$ should be naturally identified after suitable corrections coming from associative cycles in M and W.
(3) *Coassociative geometry*: There is an identification between moduli spaces of coassociative pairs or points in M and moduli spaces of corresponding objects in W.
(4) *Associative geometry*: There is an identification between moduli spaces of associative pairs or deformed Donaldson–Thomas bundles in M and moduli spaces of corresponding objects in W.

The mirror symmetry conjecture for G_2-manifolds can be summarized in the following table.

Coassociative geometry on M <=> Associative geometry on W
Associative geometry on M <=> Coassociative geometry on W

3.3 Fourier transform on coassociative T^4-fibrations

In this subsection we verify various parts of the above conjecture for *flat* G_2-manifolds, i.e., $M = B \times T$ with $B = \mathbb{R}^3$ and $T = \mathbb{R}^4/\Lambda$ a flat torus. The projection map

$$\pi : M \to B$$

is a coassociative T^4-fibration on M. The large structure limit on M can be obtained by rescaling Λ. If we denote $\mathcal{M}^{\mathrm{coa}}(M)$ the moduli space of coassociative cycles (C, D_E) on M with $[C]$ representing a fiber class and D_E is a flat $U(1)$-connection on C, then it can be naturally identified with the total space $W = B \times T^*$ of the dual torus fibration

$$\pi' : W \to B.$$

Moreover, under this identification, the canonical 3-form and 4-form on the moduli space correspond to the calibrating 3-form and 4-form for the G_2-structure on W:

$$\Omega_W = \Omega_{\mathcal{M}^{\mathrm{coa}}(M)},$$
$$\Theta_W = \Theta_{\mathcal{M}^{\mathrm{coa}}(M)}.$$

We are going to perform fiberwise Fourier transformation on M. As usual we denote the coordinates on B (resp. T and T^*) as x^1, x^2, x^3 (resp. $y^0, \ldots, y^3$ and $y_0, \ldots, y_3$). The dual torus T^* is treated as the moduli space of flat $U(1)$ connections on T. The universal line bundle over $T \times T^*$ is called the Poincaré bundle P. We normalize it so that its restriction to both

$T \times 0$ and $0 \times T^*$ are trivial. It has a universal connection whose curvature 2-form equals

$$\mathbb{F} = \sum_{j=0}^{3} dy^j \wedge dy_j \in \Omega^2(T \times T^*).$$

The Fourier transformation, or the Fourier–Mukai transformation, from M to W can be described roughly as follows:

$$\mathbf{F}(\bullet) = p'_*(p^*(\bullet) \otimes P),$$

where p and p' is defined by the following commutative diagram,

$$\begin{array}{ccc} M \times_B W & \overset{p'}{\to} & W \\ p \downarrow & & \downarrow \pi' \\ M & \overset{\pi}{\to} & B \end{array}$$

and p^* is the pull back and p'_* is the push forward operator or equivalently integration along fibers of $p' : M \times_B W \longrightarrow W$. We can discuss this transformation on different levels, including differential forms, cohomology, K-theory and so on. For differential forms, we have

$$\mathbf{F} : \Omega^*(M) \longrightarrow \Omega^*(W),$$
$$\mathbf{F}(\varphi) = \int_{M/B} p^*(\varphi) \wedge e^{(\sqrt{-1}/2\pi)\mathbb{F}},$$

where $\mathbb{F}$ is the universal curvature 2-form on the Poincaré bundle P. Moreover, φ could be a differential form supported on a submanifold in M, and the same for $\mathbf{F}(\varphi)$, too. For example, the above Fourier transformation preserves the calibrating 3-form and 4-form, more precisely we have,

$$\mathbf{F}(e^{\Theta_M}) = e^{\Theta_W}$$
$$\mathbf{F}(*e^{\Theta_M}) = *e^{\Theta_W}.$$

These follow from the following integrals with orientation y^{0123},

$$\int_{M/B} e^{\Theta_M} \wedge e^{\mathbb{F}} = e^{\Theta_W} \quad \text{and} \quad \int_{M/B} (*e^{\Theta_M}) \wedge e^{\mathbb{F}} = (*e^{\Theta_W}).$$

Similarly, on the level of cohomology classes, we have

$$\mathcal{F} : H^*(M) \longrightarrow H^*(W)$$
$$\mathcal{F}(\varphi) = \int_{M/B} p^*([\varphi]) \cup \mathrm{ch}(P).$$

We can also transform a connection D_E on a bundle E over a submanifold C in M to one in W. To do this, we consider tensor product of the pullback connection on the pullback bundle $p^*E \longrightarrow C \times_B W$ with the universal connection on the Poincaré bundle restricted to $C \times_B W$. Under good circumstances, the pushforward sheaf under $p' : C \times_B W \to W$ gives a vector bundle over a submanifold in W. For instance, a flat $U(1)$ connection over $T^4 \times \{p\}$ in $M = T^4 \times T^3$ is transformed to a point in $W = T^{4*} \times T^3$, as discussed earlier.

3.3.1 Transforming coassociative cycles

Transforming ASD connections on T^4 fibers. Suppose that we relax the flatness assumption on D_E and we assume that it is an ASD connection of a torus fiber $C = T$. The moduli space of such coassociative cycles (C, D_E) has the form

$$\mathcal{M}^{\text{coa}}(M) = B \times \mathcal{M}^{\text{ASD}}(T),$$

where $\mathcal{M}^{\text{ASD}}(T)$ denotes the moduli space of ASD connections on the flat torus T.

When D_E has no flat factor, the Fourier transformation of an ASD connection over a flat torus T is studied by Schenk [15], Braam and van Baal [5] and it produces another ASD connection over the dual torus T^*. Moreover, it is an isometry from $\mathcal{M}^{\text{ASD}}(T)$ to $\mathcal{M}^{\text{ASD}}(T^*)$. Using their results, we have

$$\mathbf{F} : \mathcal{M}^{\text{coa}}(M) \overset{\approx}{\to} \mathcal{M}^{\text{coa}}(W).$$

Moreover, it is not difficult to identify the natural 3-forms $\Omega_{\mathcal{M}^{\text{coa}}}$ on these moduli spaces under $\mathbf{F}$.

Describing other coassociative cycles in M. Suppose C is any coassociative submanifold in M which is not a fiber. The intersection of C with a general fiber cannot have dimension 3 (see Subsection 2.3.1). The intersection cannot have dimension 1 neither because otherwise the normal bundle of C in M, which is associative, lies inside the relative tangent bundle of the coassociative fibration, a contradiction. Therefore, the intersection of C with a general fiber must have dimension 2.

We assume that the coassociative submanifold C in M is *semi-flat* in the sense that it intersects each fiber along a flat subtorus. As subtori in T^4, they are all translations of each other. Using the fact that the vector crossproduct on M preserves the normal directions of C, one can show that the image of C in $B \cong \mathbb{R}^3$ must be an affine plane. Therefore, up to coordinate changes, we can assume that C is of the form

$$C:\begin{cases} x^3=0,\\ y^0=B^0(x^1,x^2),\\ y^3=B^3(x^1,x^2),\end{cases}$$

for some smooth functions B^0 and B^3 in variables x^1 and x^2 satisfying

$$-\frac{\partial}{\partial x^2}B^3+\frac{\partial}{\partial x^1}B^0=\frac{\partial}{\partial x^1}B^3+\frac{\partial}{\partial x^2}B^0=0.$$

Suppose D_E is an ASD connection on C which is *semi-flat* in the sense that D_E is invariant under translations along fiber directions, we write

$$D_E=d+a_1(x^1,x^2)dx^1+a_2(x^1,x^2)dx^2+D_1(x^1,x^2)dy^1+D_2(x^1,x^2)dy^2,$$

and the ASD condition is equivalent to

$$\begin{aligned}\frac{\partial}{\partial x^2}a_1-\frac{\partial}{\partial x^1}a_2&=0,\\ -\frac{\partial}{\partial x^2}D_1+\frac{\partial}{\partial x^1}D_2=\frac{\partial}{\partial x^1}D_1+\frac{\partial}{\partial x^2}D_2&=0.\end{aligned}$$

On each T^4 fiber of $\pi:M\to B$ over a point $(x^1,x^2,0)\in B$, we obtain a flat 2-torus defined by

$$\{(y^0,y^1,y^2,y^3)\in T^4\mid y^0=B^0(x^1,x^2),\ y^3=B^3(x^1,x^2)\}$$

together with a flat $U(1)$ connection $D_E{=}d{+}D_1(x^1,x^2)dy^1{+}D_2(x^1,x^2)dy^2$ on it, because $D_1(x^1,x^2)$ and $D_2(x^1,x^2)$ are constant on each fiber.

Transforming coassociative families of subtori. Applying the Fourier transformation from this T^4 fiber to T^{4*}, it is not difficult to check that the above flat connection (resp. flat 2-torus) in T^4 will be transformed to a flat 2-torus (resp. flat connection) in T^{4*} (see [11] for more details). Putting these together over various fibers of $\pi:M\to B$, we obtain the Fourier transformation of (C,D_E) in M, which is the pair (C',D'_E) on W with

$$C':\begin{cases} x^3=0,\\ y_1=D_1(x^1,x^2),\\ y_2=D_2(x^1,x^2).\end{cases}$$

and

$$D'_E=d+a_1(x^1,x^2)dx^1+a_2(x^1,x^2)dx^2+B^0(x^1,x^2)dy_0+B^3(x^1,x^2)dy_3.$$

(C',D'_E) is again a semi-flat coassociative cycle in W and notice that the ASD connection for D_E (resp. coassociative condition for C) implies the coassociative condition for C (resp. ASD condition for D'_E). Furthermore, the Fourier transformation induces a bijection on the semi-flat part of the moduli spaces,

$$\mathbf{F} : \mathcal{M}^{\mathrm{coa}}(M)_{\text{semi-flat}} \to \mathcal{M}^{\mathrm{coa}}(W)_{\text{semi-flat}}.$$

It is easy to check that when C is horizontal, i.e., $B^0 = B^3 = 0$, then $\mathbf{F}$ preserves the natural 3-forms $\Omega_{\mathcal{M}^{\mathrm{coa}}}$ on these moduli spaces and we expect it continues to hold in general.

3.3.2 Transforming associative cycles and DT bundles

Describing associative cycles in M. Suppose A is any associative submanifold in M, then it can only intersect a general fiber in dimension 2 or 0. In the former case, we can argue as before, the image of A in $B \cong \mathbb{R}^3$ must be an affine line provided that A is *semi-flat.* We can write, up to coordinate changes,

$$A : \begin{cases} x^2 = x^3 = 0, \\ y^2 = B^2(x^1), \\ y^3 = B^3(x^1). \end{cases}$$

The associative condition on A implies that

$$\frac{\partial}{\partial x^1} B^2 = \frac{\partial}{\partial x^1} B^3 = 0,$$

that is, both $B^2(x^1)$ and $B^3(x^1)$ are constant functions.

Suppose D_E is a unitary connection on A which is invariant under translations along fiber directions, we can write

$$D_E = a(x^1) dx^1 + D^0(x^1) dy_0 + D^1(x^1) dy_1.$$

If (A, D_E) is an associative cycle, then the flatness of D_E is equivalent to

$$\frac{\partial}{\partial x^1} a = \frac{\partial}{\partial x^1} D^0 = \frac{\partial}{\partial x^1} D^1 = 0.$$

That is a, D^0 and D^1 are all constant functions.

Transformating associative families of tori. As in the earlier situation, the Fourier transformation of a semi-flat cycle (A, D_E) on M, which is not necessarily associative, is a pair (A', D'_E) on W with

$$A' : \begin{cases} x^2 = 0, \\ x^3 = 0, \\ y_0 = D^0(x^1), \\ y_1 = D^1(x^1), \end{cases}$$

and

$$D'_E = a(x^1)dx^1 + B^2(x^1)dy_2 + B^3(x^1)dy_3.$$

It is obvious that (C, D_E) is associative if and only if (C', D'_E) is associative.

Remark. Recall that associative cycles in M are critical points of the Chern–Simons functional. Using arguments in [11], we can show that the Fourier transformation preserves the Chern–Simons functional of semi-flat three cycles on M and W which are *not* necessarily associative.

Transforming associative sections to deformed DT bundles. Any section of the (trivial) coassociative fibration π on M can be expressed as a function f,

$$f : B \to T = \frac{\mathbb{H}}{\Lambda}.$$

Harvey and Lawson [8] show that the associative condition is equivalent to the following equation:

$$-\frac{\partial f}{\partial x^1}i - \frac{\partial f}{\partial x^2}j - \frac{\partial f}{\partial x^3}k = \frac{\partial f}{\partial x^1} \times \frac{\partial f}{\partial x^2} \times \frac{\partial f}{\partial x^3}.$$

If we write $f = {}^t(f^0, f^1, f^2, f^3) \in \mathbb{R}^4/\Lambda$, $I = {}^t(1, i, j, k)$ and $f^i_{x^j} = \frac{\partial f^i}{\partial x^j}$, then the above equation can be rewritten as follows:

$$\begin{aligned}\det(I, f_{x^1}, f_{x^2}, f_{x^3})_{4\times 4} = {} & (f^1_{x^1} + f^2_{x^2} + f^3_{x^3})1 + (-f^0_{x^1} + f^3_{x^2} - f^2_{x^3})i \\ & + (-f^3_{x^1} - f^0_{x^2} + f^1_{x^3})j + (f^2_{x^1} - f^1_{x^2} - f^0_{x^3})k.\end{aligned}$$

If D_E is a unitary flat connection on A, we can write

$$D_E = d + a_1 dx^1 + a_2 dx^2 + a_3 dx^3,$$

where a_k's are matrix valued functions of (x^1, x^2, x^3).

As before, or from [11], the Fourier transformation of (A, D_E) is a unitary connection D'_E on the whole manifold W given by

$$D'_E = d + a_1 dx^1 + a_2 dx^2 + a_3 dx^3 + f^0 dy_0 + f^1 dy_1 + f^2 dy_2 + f^3 dy_3.$$

We are going to show that D'_E satisfies the deformed DT equation, i.e.,

$$F_{E'} \wedge \Theta_W + F_{E'}^3/6 = 0.$$

First we notice that the curvature of D'_E is given by

$$F_{E'} = \sum_{j=1}^{3} \sum_{k=0}^{3} f^k_{x^i} dx^i \wedge dy_k,$$

and a_j's do not occur here because of the flatness of D_E. By direct but lengthy computations, we can show that the deformed DT equation for D'_E is reduced to the flatness of D_E together with the associativity of A. Therefore, the Fourier transformation in this case gives

$$\mathbf{F} : \mathcal{M}^{\mathrm{ass}}(M) \to \mathcal{M}^{\mathrm{bdl}}(W).$$

In fact, this transformation $\mathbf{F}$ preserves the Chern–Simons functionals (see Sections 2.1 and 2.2) on M and W. Namely suppose (A, D_E) is any pair on M which is not necessarily an associative cycle, then we have

$$\int_{A\times[0,1]} \mathrm{Tr}[e^{\Theta_M + \tilde{F}}] = (\mathrm{const}) \int_{W\times I} \mathrm{Tr}[e^{\Theta_W + \tilde{F}'}].$$

We expect that $\mathbf{F}$ preserves the natural 3-forms on $\mathcal{M}^{\mathrm{ass}}(M)$ and $\mathcal{M}^{\mathrm{bdl}}(W)$.

3.4 Fourier transform on associative T^3-fibrations

Suppose M has an associative T^3-fibration,

$$\pi : M \to B.$$

The fiberwise Fourier transformation $\mathbf{F}$ in this case should interchange the two types of geometry on G_2-manifolds,

$$\text{Associative geometry on } M \overset{\mathbf{F}}{\longleftrightarrow} \text{Coassociative geometry on } W.$$

Here W is the dual 3-torus fibration to π on M, and it can be identified as the moduli space $\mathcal{M}^{\mathrm{ass}}(M)$ of associative cycles (A, D_E) on M, with D_E being a flat $U(1)$-connection over a fiber A.

In the *flat* case, we have $M = T \times \mathbb{R}^4$ and $\mathcal{M}^{\text{ass}}(M) \cong W = T^* \times B$. Under these identifications, we have

$$\begin{aligned} \Omega_W &= \Omega_{\mathcal{M}^{\text{ass}}(M)}, \\ \Theta_W &= \Theta_{\mathcal{M}^{\text{ass}}(M)}. \end{aligned}$$

We can analyze semi-flat associative and coassociative cycles in M and their Fourier transformations as before. For example, semi-flat coassociative submanifolds in M are either sections or a family of affine 2-tori over an affine plane in B. Semi-flat associative submanifolds in M are either fibers or a family of affine 2-tori over an affine line in B.

Suppose (C, D_E) is a coassociative cycle in M, with C being a section of the (trivial) associative fibration on M. The coassociative condition of C can be expressed in terms of a differential equation for a function $g = (g^1, g^2, g^3)\colon \mathbb{R}^4 \to T = \mathbb{R}^3/\Lambda$ (see [8]),

$$\nabla g^1 i + \nabla g^2 j + \nabla g^3 k = \nabla g^1 \times \nabla g^2 \times \nabla g^3.$$

If we write the ASD connection as

$$D_E = d + a_0 dy^0 + a_1 dy^1 + a_2 dy^2 + a_3 dy^3,$$

then the Fourier transformation of (C, D_E) will be the following connection D'_E on W,

$$D'_E = d + a_0 dy^0 + a_1 dy^1 + a_2 dy^2 + a_3 dy^3 + g^1 dx_1 + g^2 dx_2 + g^3 dx_3.$$

Again direct but lengthy computations show that the deformed DT equation on D'_E is reduced to the above coassociativity condition of C and the ASD equation for D_E.

4 Spin(7)-geometry

There are only two kinds of exceptional holonomy groups for Riemannian manifolds, namely G_2 for dimension 7 and Spin(7) in dimension 8. Both of them play important roles in M-theory. Almost all of our earlier discussions for G_2-geometry has a Spin(7) analog. As a result, we will only indicate what geometric structures we have in Spin(7)-geometry and leave out those derivations which are the same as in their G_2-counterparts.

An 8-dimensional Riemannian manifold Z with Spin(7)-holonomy has a closed self-dual 4-form Θ_Z which can be expressed as follows:

$$\begin{aligned}\Theta_Z = &-dy^{0123} - dx^{0123} - (dx^{10} + dx^{23})(dy^{10} + dy^{23}) \\ &- (dx^{20} + dx^{31})(dy^{20} + dy^{31}) - (dx^{30} + dx^{12})(dy^{30} + dy^{12}),\end{aligned}$$

when $Z = \mathbb{R}^8$. As before, we can decompose differential forms on Z into Spin(7) irreducible components (see, e.g., [9]), for example, $\Lambda^4 = \Lambda^4_1 + \Lambda^4_7 + \Lambda^4_{27} + \Lambda^4_{35}$.

Holonomy reduction. For example, if M is a G_2-manifold, then $Z = M \times S^1$ has a Spin(7)-metric with

$$\Theta_Z = \Omega_M \wedge dt - \Theta_M.$$

This corresponds to the natural inclusion $G_2 \subset \mathrm{Spin}(7)$. We can also consider the reduction of holonomy to the subgroup $\mathrm{Spin}(6) \subset \mathrm{Spin}(7)$. Notice that Spin(6) = SU(4), namely such a Z would be a Calabi–Yau 4-fold and

$$\Theta_Z = -\frac{\omega_Z^2}{2} + \mathrm{Re}\ \Omega_Z,$$

where ω_Z and Ω_Z are the Ricci flat Kähler form and the holomorphic volume form on Z. We can compare their decompositions of differential forms as follows:

$$\begin{aligned}\Lambda^4_{35} &= \Lambda^{3,1}_{prim} + \Lambda^{1,1}_{prim} + \Lambda^{1,3}_{prim}, \\ \Lambda^4_{27} &= \Lambda^{2,2}_{prim} + \Lambda^{0,2} + \Lambda^{0,4}.\end{aligned}$$

We can reduce the holonomy group to smaller subgroups of Spin(7) and give the following tables:

Holonomy	Manifold
Spin(7)	Spin(7)-manifold
Spin(6) = SU(4)	Calabi–Yau 4-fold
Spin(5) = Sp(2)	Hyperkähler 4-fold
Spin(4) = $\mathrm{SU}(2)^2$	K3 × K3
Spin(3) = SU(2)	K3 × T^4

4.1 Spin(7)-analog of Yukawa coupling

4.1.1 An analog of the Yukawa coupling

We consider the $(3,1)$-tensor $\chi_Z \in \Omega^3(Z, T_Z)$ constructed by raising an index of Θ_Z using the metric tensor.

We define a symmetric quartic tensor $\mathcal{Q}$ on $H^4(Z,\mathbb{R})$ as follows:

$$\mathcal{Q} : \bigotimes^4 \Omega^4(Z,\mathbb{R}) \to \mathbb{R},$$
$$\mathcal{Q}(\phi_1,\phi_2,\phi_3,\phi_4) = \int_Z \Theta_Z(\hat{\phi}_1,\hat{\phi}_2,\hat{\phi}_3,\hat{\phi}_4) \wedge \Theta_Z.$$

Here $\hat{\phi} = *(\phi \wedge \chi) \in \Omega^1(Z,T_Z)$. Note that $\hat{\phi}$ is zero if $\phi \in \Omega^4_{27}(Z,\mathbb{R})$. We define the *Yukawa coupling* $\mathcal{Y}$ to be the restriction of $\mathcal{Q}$ to $H^4_{35}(Z,\mathbb{R})$. We can also define a cubic form, a quadratic form and a linear form on $H^4_{35}(Z,\mathbb{R})$ by evaluating $\mathcal{Q}$ on one, two and three Θ_Z before restricting it to $H^4_{35}(Z,\mathbb{R})$.

4.2 Deformed DT bundles and Cayley cycles

4.2.1 Deformed DT bundles

As in the G_2 case, we define a *deformed DT connection* to be a connection D_E over Z whose curvature tensor F_E satisfies

$$*F_E + \Theta_Z \wedge F_E + F_E^3/6 = 0,$$

or equivalently,

$$[e^{\Theta_Z + F_E + *F_E}]^{[6]} = 0.$$

Its moduli space $\mathcal{M}^{\mathrm{bdl}}(Z)$ carries a natural 4-form $\Theta_{\mathcal{M}^{\mathrm{bdl}}(Z)}$ defined by

$$\Theta_{\mathcal{M}^{\mathrm{bdl}}(Z)}(\alpha,\beta,\gamma,\delta) = \int_Z \mathrm{Tr}_E[\alpha \wedge \beta \wedge \gamma \wedge \delta]_{\mathrm{skew}} \wedge e^{\Theta_Z + F_E},$$

with $\alpha,\beta,\gamma,\delta \in H^1(Z,\mathrm{ad}(E))$, the first cohomology group of the elliptic complex

$$0 \to \Omega^0(Z,\mathrm{ad}(E)) \xrightarrow{D_E} \Omega^1(Z,\mathrm{ad}(E)) \xrightarrow{\pi_7 \circ D_E} \Omega^2_7(Z,\mathrm{ad}(E)) \to 0.$$

4.2.2 Cayley cycles

A submanifold C in Z of dimension 4 is called a *Cayley* submanifold if it is calibrated by Θ_Z, i.e., the restriction on Θ_Z to C equals the volume form on C with respect to the induced metric [8]. A *Cayley cycle* is defined to be any pair (C, D_E) with C a Cayley submanifold in Z and D_E is an ASD connection on a bundle E over C. The moduli space $\mathcal{M}^{\mathrm{Cay}}(Z)$ of Cayley cycles in Z also have a natural 4-form $\Theta_{\mathcal{M}^{\mathrm{Cay}}(Z)}$ defined by

$$\Theta_{\mathcal{M}^{\mathrm{Cay}}(Z)} = \begin{cases} \int_C \mathrm{Tr}\alpha \wedge \beta \wedge \gamma \wedge \delta \\ -\int_C \langle \phi, \mathrm{Tr}(\widehat{\alpha \wedge \beta}) \cdot \eta \rangle \Theta_Z \\ -\int_C \det(\phi, \eta, \xi, \zeta) \Theta_Z \end{cases}$$

where $\alpha, \beta, \gamma, \delta \in \Omega^1(C, \mathrm{ad}(E))$ and $\phi, \eta, \xi, \zeta \in \mathrm{Ker}\ \mathbf{D}$ are harmonic spinors and

$$\widehat{\alpha \wedge \beta} = \alpha \wedge \beta + *(\alpha \wedge \beta) \in \Omega^2_+(C, \mathrm{ad}(E)) = \mathrm{Im}(\mathbb{H}) \otimes \mathrm{ad}(E).$$

We leave it to our readers to verify that the Fourier transformation along Cayley T^4-fibration on a flat Spin(7)-manifold Z will transform a Cayley section cycle (C, D_E) on Z to a deformed DT bundle over the dual torus fibration. Transformations of certain non-section Cayley cycles on Z can also be analyzed as in the G_2 case.

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