ASIAN J. MATH.
Vol. 12, No. 1, pp. 121–144, March 2008

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INSTANTONS AND BRANES IN MANIFOLDS WITH VECTOR CROSS PRODUCTS*

JAE-HYOUK LEE† AND NAICHUNG CONAN LEUNG‡

Abstract. In this paper we study the geometry of manifolds with vector cross products and its complexification.

First we develop the theory of instantons and branes and study their deformations. For example they are (i) holomorphic curves and Lagrangian submanifolds in symplectic manifolds and (ii) associative submanifolds and coassociative submanifolds in G_2-manifolds.

Second we classify Kähler manifolds with the complex analog of the vector cross product, namely they are Calabi-Yau manifolds and hyperkähler manifolds. Furthermore we study instantons, Neumann branes and Dirichlet branes on these manifolds. For example they are special Lagrangian submanifolds with phase angle zero, complex hypersurfaces and special Lagrangian submanifolds with phase angle $\pi/2$ in Calabi-Yau manifolds.

Key words. Instanton, Brane, vector cross product, complex vector cross product, Calibrated submanifold

AMS subject classifications. 37J05, 53C15, 53C25, 53C26, 53C38

1. Introduction. The vector product, or the cross product, in $\mathbb{R}^3$ was generalized by Gray ([1],[6]) to the product of any number of tangent vectors, called the *vector cross product* (abbrev. VCP). The list of Riemannian manifolds with VCP structures on their tangent bundles includes symplectic (or Kähler) manifolds, G_2-manifolds and $Spin(7)$-manifolds. They are manifolds with special holonomy and they play important roles in string theory, M-theory and F-theory respectively. In this paper we develop the geometry of the VCP in general.

We also introduce the complex analog of the VCP in definition 26, called the *complex vector cross product* (abbrev. $\mathbb{C}$-VCP). We prove that there are only two classes of manifolds with $\mathbb{C}$-VCPs.

THEOREM 1. *If M is a closed Kähler manifold with a $\mathbb{C}$-VCP, then M must be either (i) a Calabi-Yau manifold, or (ii) a hyperkähler manifold.*

They are again manifolds with special holonomy and they play important roles in Mirror Symmetry. Notice that the list in the Berger classification of holonomy groups of oriented Riemannian manifolds coincides with the list of Riemannian manifolds admitting a VCP or a $\mathbb{C}$-VCP.

We study the geometry of *instantons,*which are submanifolds in M preserved by the VCP. Instantons are always absolute minimal submanifolds in M. When an instanton is not a closed submanifold in M, we require its boundary to lie inside a *brane* in order to have a Fredholm theory for the free boundary value problem. For example, when M is a symplectic manifold, then instantons and branes are holomorphic curves and Lagrangian submanifolds in M respectively. These geometric objects play important roles in understanding the symplectic geometry of M. When M is a G_2-manifold (resp. $Spin(7)$-manifold), instantons correspond to BPS states in M-theory (resp. F-theory) and branes are supersymmetric cycles.

*Received April 10, 2007; accepted for publication August 22, 2007.

†Department of Mathematics, Washington University in St. Louis, St. Louis, MO 63130-4899, USA (jhlee@math.wustl.edu).

‡Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Hong Kong (leung@ims.cuhk.edu.hk).

Manifolds w/ VCP	Symplectic Manifolds	G_2-Manifolds	$Spin\,(7)$-Manifolds	Oriented Manifolds
Instantons	Holomorphic curves	Associative submanifolds	Cayley submanifolds	Open submanifolds
Branes	Lagrangian submanifolds	Coassociative submanifolds	N/A	Hypersurfaces

We define a $\mathbf{g}_M^{\perp}$-valued tensor τ on M and we prove that instantons A in M can be characterized by the condition

$$\tau|_A = 0.$$

Such a characterization is useful in describing the deformations of instantons (see section 2.3 for details.).

For a brane C in M, its normal bundle is naturally identified with the space $\Lambda^r_{VCP} T^*_C$ of VCP forms of degree r on C. We prove that infinitesimal deformations of branes are parameterized by such differential forms on C which are closed. Furthermore they are always unobstructed, namely the moduli spaces of branes are always smooth. (See section 2.5 for details).

For the *complex* analog, the definition of instantons depends on a parameter called the phase θ. We prove that they can be characterized by the vanishing of certain differential forms on them. Such characterization is needed to describe the deformation of these instantons.

For $\mathbb{C}$-VCPs, there are two types of branes corresponding to the Dirichlet type and Neumann type boundary value problems for instantons. We call them *D-branes* and *N-branes* respectively. We will discuss some of their basic properties and they can classified as follows.

Manifolds w/ $\mathbb{C}$-VCP	Calabi-Yau manifolds	hyperkähler manifolds
Instantons	Special Lagrangian$_{\theta=0}$	I-holomorphic curves
N-Branes	Complex Hypersurfaces	J-complex Lagrangians
D-Branes	Special Lagrangian$_{\theta=-\pi/2}$	K-complex Lagrangians

In the sequel [11] to this paper, we define higher dimensional knot spaces for the manifolds with VCPs. They admit natural symplectic structures and the geometry of branes and instantons in M can be interpreted as the geometry of Lagrangians and holomorphic curves in these knot spaces. For $\mathbb{C}$-VCPs, we construct isotropic knot spaces for Calabi-Yau manifolds M and we show that they have natural holomorphic symplectic structures. Both special Lagrangian submanifolds and complex hypersurfaces in M give complex Lagrangian submanifolds in its isotropic knot space, but with respect to different complex structures in its twistor space. Relations to Mirror Symmetry will also be discussed there.

2. Instantons and branes. In this section we introduce and study instantons and branes on manifolds with (complex) vector cross products. Traditionally instantons refer to gradient flow lines of a Morse function f on a Riemannian manifold

(M, g), as studied by Witten in [16]. Morse theory can be generalized to any closed one form ϕ, because $\phi = df$ locally. Supposing that ϕ is nonvanishing, we can choose a Riemannian metric on M such that it has unit length at every point. Then the gradient flow lines for the vector field X, defined by $\phi = \iota_X g$, can be reinterpreted as one-dimensional submanifolds in M calibrated by ϕ.

Suppose that M is a symplectic manifold with a symplectic form ω. By transgression on ω, we obtain a closed one form on the free loop space of M. The instantons in the free loop space correspond to holomorphic curves in M and they are calibrated by ω. We continue to call these instantons and they play important roles in the closed String theory (see e.g. [17]). In open String theory, we consider holomorphic curves in M with boundaries lying on a Lagrangian submanifold in M, which we call a brane.

In general instantons are submanifolds (of the smallest dimension) which are preserved by the VCP, and branes are the natural boundaries for the free boundary value problems for instantons.

2.1. Manifolds with vector cross product. In this subsection we define closed and parallel vector cross product structure on a Riemannian manifold. First we need to review the VCP as introduced by Gray [6]. A basic example of *VCPs* is the standard vector product on $\mathbb{R}^3$. The vector product of any two linearly independent vectors in a plane is a vector which is orthogonal to the plane and has the length equal to the area of the parallelogram spanned by those vectors. Actually, these two properties characterize the *VCP* as in the following definition by Brown and Gray [1].

Definition 2. *On an n-dimensional Riemannian manifold M with a metric g, an r-fold Vector Cross Product (VCP) is a smooth bundle map,*

$$\chi : \wedge^r TM \to TM$$

satisfying

$$\begin{cases} g\left(\chi\left(v_1, ..., v_r\right), v_i\right) = 0\ , \quad (1 \leq i \leq r) \\ g\left(\chi\left(v_1, ..., v_r\right), \chi\left(v_1, ..., v_r\right)\right) = \left\|v_1 \wedge ... \wedge v_r\right\|^2 \end{cases}$$

where $\|\cdot\|$ is the induced metric on $\wedge^r TM$.

We also write

$$v_1 \times ... \times v_r = \chi\left(v_1, ..., v_r\right).$$

The first condition in the above definition is equivalent to the following tensor ϕ being a skew symmetric tensor of degree $r+1$, i.e. ϕ is a differential form,

$$\phi\left(v_1, ..., v_{r+1}\right) = g\left(v_1 \times ... \times v_r, v_{r+1}\right).$$

The VCP can also be characterized in terms of ϕ as follows.

Definition 3. *A VCP form of degree r+1 is a differential form $\phi \in \Omega^{r+1}(M)$ satisfying*

$$\left|\iota_{e_1 \wedge e_2 .. \wedge e_r}(\phi)\right| = 1$$

for any orthonormal vectors $e_1, ..., e_r \in T_x M$, for any x in M.

The following proposition is immediate.

PROPOSITION 4. *Let (M, g) be a Riemannian manifold and $\phi \in \Omega^{r+1}(M)$. We write $\phi(v_1, ..., v_{r+1}) = g(\chi(v_1, ..., v_r), v_{r+1})$ with $\chi : \wedge^r TM \to TM$. Then ϕ is a VCP form if and only if χ is a VCP.*

As we will see, a Hermitian almost complex structure is equivalent to a 1-fold *VCP* and the corresponding Kähler form is the corresponding *VCP form.* In fact the complete list of VCPs is surprisingly short. The classification of the linear *VCPs* on a vector space V with a positive definite inner product g, by Brown and Gray [1], can be summarized in the following.

(i) $r = 1$: If $\chi : V \to V$ is a 1-fold VCP, then $|\chi(v)| = |v|$ implies that χ is an orthogonal transformation. Polarizing $\langle \chi(v), v \rangle = 0$, we obtain $\langle \chi(u), v \rangle + \langle u, \chi(v) \rangle = 0$, that is $\chi^* = -\chi$. Together we have $\chi^2 = -id$. Namely a 1-fold VCP is equivalent to a Hermitian complex structure on V. The symmetry group of χ is isomorphic to $U(m) = O(2m) \cap GL(m, \mathbb{C})$. On the other hand, the isomorphism $U(m) = O(2m) \cap Sp(2m, \mathbb{R})$ reflects the fact that a 1-fold VCP is determined by its corresponding VCP form, or its Kähler form ϕ. The standard example is $V = \mathbb{C}^m$ with

$$\phi = dx^1 \wedge dy^1 + \cdots + dx^m \wedge dy^m.$$

(ii) $r = n - 1$: an $(n-1)$-fold VCP on an n-dimensional inner product space V is the Hodge star operator $*$ given by g on $\Lambda^{n-1}V$ and the VCP form of degree n is the induced volume form vol_V on V. The automorphism group preserving the VCP form vol_V is the group of linear transformations preserving g and vol_V, i.e. $Aut(V, vol_V) = O(n) \cap SL(n, \mathbb{R}) = SO(n)$. The standard example is $V = \mathbb{R}^n$ and

$$\phi = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

(iii) $r = 2$: a 2-fold *VCP on* a 7-dimensional vector space $\mathbb{R}^7$ is a cross product defined as $a \times b = \operatorname{Im}(ab)$ for any a, b in $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$, the set of imaginary octonion numbers. For coordinates $(x_{1,...,}x_7)$ on $\operatorname{Im}\mathbb{O}$, the corresponding *VCP* form Ω of degree 3 can be written as

$$\Omega = dx^{123} - dx^{167} + dx^{145} + dx^{257} + dx^{246} - dx^{356} + dx^{347}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$. Bryant [3] showed that the group of real-linear transformations of $\operatorname{Im}\mathbb{O}$ preserving the *VCP* form Ω actually preserves g and VCP and it is the rank two exceptional Lie group G_2, the automorphism group of the octonion $\mathbb{O}$, i.e. $Aut(\operatorname{Im}\mathbb{O}, \Omega) = G_2 \subset SO(\operatorname{Im}\mathbb{O}) = SO(7)$.

(iv) $r = 3$: a 3-fold *VCP* on an 8-dimensional vector space $\mathbb{R}^8$ is a cross product defined as $a \times b \times c = \left(a\left(\bar{b}c\right) - c\left(\bar{b}a\right)\right)/2$ for any a, b and c in $\mathbb{R}^8 \cong \mathbb{O}$, the octonion numbers. For coordinates $(x_{1,...,}x_8)$ on $\mathbb{O}$, the corresponding *VCP* form Θ of degree 4 can be written as

$$\begin{aligned}\Theta = &-dx^{1234} - dx^{5678} - \left(dx^{21} + dx^{34}\right)\left(dx^{65} + dx^{78}\right) \\ &- \left(dx^{31} + dx^{42}\right)\left(dx^{75} + dx^{86}\right) - \left(dx^{41} + dx^{23}\right)\left(dx^{85} + dx^{67}\right).\end{aligned}$$

Bryant [3] also showed that the group of real-linear transformations of $\mathbb{O}$ preserving the *VCP* form Θ on $\mathbb{O}$ preserves g and the VCP, and it is $Spin(7)$, i.e. $Aut\left(\mathbb{R}^8, \Theta\right) = Spin(7) \subset SO(8)$.

From the above classification of linear VCPs, the existence of a VCP on a Riemannian manifold M is equivalent to the reduction of the structure group of the

r	1	n-1	2	3
$n = \dim V$	2m	n	7	8
V	$\mathbb{C}^m$	$\mathbb{R}^n$	Im$\mathbb{O}$	$\mathbb{O}$

TABLE 1
Classification of r-fold VCP on $V \simeq \mathbb{R}^n$

frame bundle from $O(n)$ to $U(m)$, $SO(n)$, G_2 and $Spin(7)$, for $r = 1$, $n-1$, 2 and 3 respectively. Using the obstruction theory in topology, the necessary and sufficient conditions for the existence of such a reduction of structure group in the $r = n-1$, 2 and 3 cases are $w_1 = 0$, $w_1 = w_2 = 0$ and $w_1 = w_2 = p_1^2 - 4p_2 + 8\chi = 0$ respectively. Here w_i, p_i and χ are the Stiefel-Whitney class, the Pontrjagin class, and the Euler class of M, respectively (see e.g. [9]).

Given a *VCP* on a Riemannian manifold M, that is a linear VCP on each tangent space of M, we would impose certain integrability conditions for its coherence. For example, a 1-fold VCP is a Hermitian almost complex structure on M. This defines a symplectic structure or Kähler structure on M if the corresponding VCP form on M is closed or parallel respectively. In general we have the following definition.

DEFINITION 5. *Suppose that M is a Riemannian manifold with a VCP χ and ϕ is its corresponding VCP form. We call χ a closed (resp. parallel) VCP if $d\phi = 0$ (resp. $\nabla\phi = 0$, where ∇ is the Levi-Civita connection on M).*

The classification of manifolds with closed/parallel VCP is presented in the following table.

r	Closed VCP	Parallel VCP
1	Almost Kähler manifolds	Kähler manifolds
$n-1$	Oriented Riem. manifolds	Oriented Riem. manifolds
2	Almost G_2-manifolds	G_2-manifolds
3	$Spin(7)$-manifolds	$Spin(7)$-manifolds

REMARK. A *VCP* form ϕ of degree $r+1$ on M induces a *VCP* form of degree r on any oriented hypersurface H in M, namely the restriction of $\iota_\nu\phi$ to H where ν is the unit normal vector field on H. However it usually does not preserve the closedness of the VCP form. For example, Calabi [5] and Gray [6] showed that such a two form on any hypersurface in $\mathbb{R}^7 = \mathrm{Im}\,\mathbb{O}$ is never closed unless it is an affine hyperplane. This result is generalized by Bryant [2] to any codimension two submanifold in $\mathbb{O}$, where he showed that closedness of such a two form is equivalent to the submanifold being a complex hypersurfaces for some suitable complex structure on $\mathbb{O}$.

REMARK. We consider any nontrivial smooth map $f : M_1 \to M_2$ between two

Riemannian manifolds with r-fold VCPs. If f preserves their VCPs, i.e.

$$f_* (v_1 \times ... \times v_r) = (f_* v_1) \times ... \times (f_* v_r),$$

for any tangent vectors $v_1, v_2, ..., v_r$ to M_1, then Gray [7] showed that f is actually an isometric immersion unless $r = 1$ and f would be a holomorphic map. In particular any diffeomorphism of M preserving an r-fold VCP with $r \geq 2$ is automatically an isometry of M, i.e. $Diff(M, \chi) \subset Isom(M, g)$.

In the next subsection we study submanifolds in M which are preserved by the VCP.

2.2. Instantons for vector cross products. In a symplectic manifold with a compatible almost complex structure J, namely an almost Kähler manifold, an instanton is a two-dimensional submanifold which is preserved by J, and a brane is a Lagrangian submanifold, i.e. a middle-dimensional submanifold such that the restriction of the symplectic form vanishes. They play essential roles in symplectic geometry. These notions have natural analogs for manifolds with VCPs.

Definition 6. *Let M be a Riemannian manifold with a closed r-fold VCP χ. An $(r+1)$-dimensional submanifold A is called an* ***instanton*** *if it is preserved by χ. That is, for any tangent vectors $u_1, ..., u_r$ in $T_x A$ we have $u_1 \times ... \times u_r \in T_x A$.*

In this subsection we establish some basic properties for instantons. The following lemma is immediate from the definition.

Lemma 7. *Suppose that A_1 and A_2 are two instantons in M with clean intersection $A_1 \cap A_2$. Then $A_1 \cap A_2$ cannot be of codimension one in A_1 or A_2.*

When A is a holomorphic curve in an almost Kähler manifold M, its normal bundle $N_{A/M}$ has a natural complex structure whose holomorphic sections describe infinitesimal deformations of A inside M. The next proposition generalizes such a structure on $N_{A/M}$ to any VCP structure.

Proposition 8. *Let M be a Riemannian manifold with an r-fold VCP χ and A is an instanton in M. Then χ restricts to give a bundle map $\Lambda^{r-1} T_A \otimes N_{A/M} \to N_{A/M}$, where $N_{A/M}$ is the normal bundle of A in M.*

Proof. Suppose that A is an instanton in M. At any given point x in A and any tangent vectors $u_1, ..., u_{r-1}$ in $T_x A$ and any normal vector ν in $N_{A/M,x}$, we want to show that

$$u_1 \times ... \times u_{r-1} \times \nu \in N_{A/M,x}.$$

This is because for any u in $T_x A$,

$$\begin{aligned} g(u_1 \times ... \times u_{r-1} \times \nu, u) &= \phi(u_1, ..., u_{r-1}, \nu, u) \\ &= -\phi(u_1, ..., u_{r-1}, u, \nu) \\ &= -g(u_1 \times ... \times u_{r-1} \times u, \nu) \\ &= 0. \end{aligned}$$

The last equality follows from the fact that A is preserved by χ and therefore $u_1 \times ... \times u_{r-1} \times u$ lies in $T_x A$. This implies that there is a bundle map,

$$\Lambda^{r-1} T_A \otimes N_{A/M} \to N_{A/M}. \quad \square$$

In fact the normal bundle to an instanton is always a twisted spinor bundle over A and the above bundle map is given by the Clifford multiplication.

An instanton in M is always a minimal submanifold with absolute minimal volume. This will follow from an equivalent characterization of instantons, namely they are those submanifolds in M calibrated by ϕ. The theory of calibration is developed by Harvey and Lawson in [8] to produce absolute minimal submanifolds. Recall that a closed differential form ψ of degree k on a Riemannian manifold M is called a *calibrating form* if it satisfies

$$\psi(x)|_V \leq vol_V,$$

for every oriented k-plane V in T_xM, at each point x in M. Here vol_V is the volume form on V for the induced metric. A *calibrated submanifold* A is a submanifold where $\psi|_A$ is equal to the induced volume form on A. An important observation is the following: any other submanifold B, homologous to A, satisfies

$$Vol(B) \geq Vol(A) = \int_A [\psi],$$

and the equality sign holds if and only if B is also a calibrated submanifold in M. We will discover that all the examples of calibrated submanifolds studied in [8] are either instantons or branes in manifolds with VCPs or complex VCPs.

The next lemma shows that instantons are calibrated submanifolds.

Lemma 9. *Let M be a Riemannian manifold with a closed r-fold VCP χ and we denote the corresponding VCP form as ϕ . Then we have (i) ϕ is a calibrating form, (ii) an $(r+1)$-dimensional submanifold A in M is calibrated by ϕ if and only if it is an instanton.*

Proof. The closed form ϕ is a calibrating form because for any x in M and for any oriented orthonormal tangent vectors $e_1, e_2, ..., e_{r+1}$ at x, it satisfies,

$$\begin{aligned}\phi(e_1, e_2, ..., e_{r+1}) &= \langle \chi(e_1, e_2, ..., e_r), e_{r+1} \rangle \\ &\leq |\chi(e_1, e_2, ..., e_r)| \cdot |e_{r+1}| \\ &= \|e_1 \wedge e_2 \wedge ... \wedge e_r\| \cdot |e_{r+1}| \\ &= 1.\end{aligned}$$

The equality signs hold if and only if $\chi(e_1, e_2, ..., e_r) = e_{r+1}$. Namely the linear span of $e_1, e_2, ..., e_{r+1}$ is preserved by χ. □

As a corollary of this lemma and basic properties of calibration, an $(r+1)$-dimensional submanifold A in M is an instanton if and only if its total volume with respect to the induced metric satisfies the following equality,

$$Vol(A) = \int_A [\phi].$$

Examples of instantons. (i) Instantons in an oriented n-dimensional manifold (i.e. VCP form of degree n) are open subsets in M. (ii) Instantons in a Kähler manifold (i.e. parallel VCP form of degree 2) are holomorphic curves. (iii) Instantons in a G_2-manifold (i.e. parallel VCP form of degree 3) are called associative submanifolds (see

[8][10]). (iv) Instantons in a *Spin* (7)-manifold (i.e. parallel VCP form of degree 4, called the Cayley form) are called Cayley submanifolds (see [8][10]).

Suppose that (M, g, χ) has a torus symmetry group, say $M = X \times T^k$. Then an r-fold VCP on M induces an $(r-k)$-fold VCP on X. Moreover a submanifold B in X is an instanton if and only if $B \times T^k$ is an instanton in M. For instance if Σ is a holomorphic curve is a Calabi-Yau threefold X, then $\Sigma \times S^1$ is an associative submanifold in the G_2-manifold $X \times S^1$ and vice versa.

The next proposition shows that instantons arise naturally as fixed sets of symmetry.

Proposition 10. *Suppose that σ is an isometric involution of M preserving its r-fold VCP χ. If the fixed point set M^σ has dimension $r+1$ then it is an instanton in M.*

Proof. Given any tangent vectors $v_1, ..., v_r$ to M^σ at any point x, we write

$$v_1 \times \cdots \times v_r = t + n,$$

where t (resp. n) is tangent (resp. normal) to M^σ. Since M^σ is the fixed point set of an involution, we have $\sigma_* (t+n) = t - n$. On the other hand,

$$\begin{aligned} \sigma_* (v_1 \times \cdots \times v_r) &= \sigma_* (v_1) \times \cdots \times \sigma_* (v_r) \\ &= v_1 \times \cdots \times v_r \end{aligned}$$

because σ preserves χ. This implies $n = 0$, thus M^σ is preserved by χ, namely an instanton in M. ∎

Remark. Given any r-dimensional analytic submanifold S in M, an r-fold VCP χ on M determines a unique normal direction on S. Using Cartan-Kähler theory, we can always integrate out this direction and obtain an instanton in M containing S (see e.g. [8]).

2.3. Deformations of instantons. In order to describe instantons and their deformations effectively, we need to develop further the linear algebra of an inner product space V with an r-fold VCP

$$\chi : \wedge^r V \to V,$$

or their associated VCP form $\phi \in \Lambda^{r+1} V^*$. We will show that instantons can be given by the solutions of the equation $\tau|_A = 0$ for an appropriate $\tau : \wedge^{r+1} V \to \mathbf{g}^\perp$. Such a description is very useful in describing the deformations of instantons.

Definition 11. *Let V be an inner product space with an r-fold VCP χ. We define a homomorphism*

$$\tau : \wedge^{r+1} V \to \wedge^2 V$$

as the composition of the following homomorphisms:

$$\wedge^{r+1} V \overset{(i)}{\to} V \otimes \wedge^r V \overset{(ii)}{\to} V \otimes V \overset{(iii)}{\to} \wedge^2 V,$$

where these maps are (i) the natural inclusion, (ii) $id \otimes \chi$, (iii) the natural projection. Explicitly we have

$$\tau(v_1, ..., v_{r+1}) = \frac{1}{\sqrt{r+1}} \sum_{k=1}^{r+1} (-1)^{k-1} v_k \wedge \chi(v_1, ..., \widehat{v_k}, ..., v_{r+1}).$$

As a matter of fact, the image of τ lies inside a much smaller subspace in $\wedge^2 V$. We define $\mathbf{g} \subset \mathbf{so}(V) \cong \wedge^2 V^*$ to be the space of infinitesimal isometries of V preserving the VCP χ. Namely $\zeta \in \mathbf{so}(V) \subset End(V)$ lies inside $\mathbf{g}$ if

$$\zeta(v_1 \times v_2 \times ... \times v_r) = \zeta(v_1) \times v_2 \times ... \times v_r + ... + v_1 \times v_2 \times ... \times \zeta(v_r),$$

for any $v_1, v_2, ..., v_r$ in V. This is equivalent to

$$\sum_{i=1}^{r+1} \phi(v_1, v_2, ..., \zeta(v_i), ..., v_{r+1}) = 0,$$

for any $v_1, v_2, ..., v_r$ in V. The next proposition says that the image of τ is orthogonal to $\mathbf{g} \subset \Lambda^2 V^*$ with respect to the natural pairing between $\Lambda^2 V$ and $\Lambda^2 V^*$.

Proposition 12. *Let V be an inner product space with an r-fold VCP χ, we have*

$$\tau : \wedge^{r+1} V \to \mathbf{g}^{\perp} \subset \wedge^2 V.$$

Proof. Given any $\zeta \in \mathbf{g} \subset \mathbf{so}(V) \subset End(V)$, we denote the corresponding two form in $\wedge^2 V^*$ as $\bar{\zeta}$. For any v_i's in V, we compute

$$\begin{aligned}
&\left\langle \tau(v_1, ..., v_{r+1}), \bar{\zeta} \right\rangle \\
&= \bar{\zeta}\left(\frac{1}{\sqrt{r+1}} \sum_{i=1}^{r+1} (-1)^{i-1} v_i \wedge (v_1 \times ... \times \widehat{v_i} \times ... \times v_{r+1}) \right) \\
&= \frac{1}{\sqrt{r+1}} \left\langle \sum_{i=1}^{r+1} (-1)^{i-1} (v_1 \times ... \times \widehat{v_i} \times ... \times v_{r+1}), \zeta(v_i) \right\rangle \\
&= \frac{(-1)^r}{\sqrt{r+1}} \sum_{i=1}^{r+1} \phi(v_1, ..., \zeta(v_i), ..., v_{r+1}) \\
&= 0.
\end{aligned}$$

Hence the result is proved. □

The main property of τ is given by the following theorem.

Theorem 13. *Let V be an inner product space with an r-fold VCP χ and ϕ its associated VCP form. We have*

$$|\phi(v_1, ..., v_{r+1})|^2 + |\tau(v_1, ..., v_{r+1})|^2_{\wedge^2} = |v_1 \wedge ... \wedge v_{r+1}|^2_{\Lambda^{r+1}}$$

Proof. It suffices to assume that v_i's are orthogonal to each other. Note that when $l \neq m$, we have v_l perpendicular to both v_m and $\chi(v_1, ..., \widehat{v_m}, ..., v_{r+1})$ and therefore,

$$\langle v_l \wedge \chi(v_1, ..., \widehat{v_l}, ..., v_{r+1}), v_m \wedge \chi(v_1, ..., \widehat{v_m}, ..., v_{r+1}) \rangle = 0.$$

We compute

$$
\begin{aligned}
&\left|\tau\left(v_1,...,v_{r+1}\right)\right|^2\\
&=\frac{1}{r+1}\left|\sum_{k=1}^{r+1}\left(-1\right)^{k-1}v_k\wedge\chi\left(v_1,...,\widehat{v_k},...,v_{r+1}\right)\right|^2\\
&=\frac{1}{r+1}\sum_{k=1}^{r+1}\left|v_k\wedge\chi\left(v_1,...,\widehat{v_k},...,v_{r+1}\right)\right|^2\\
&=\frac{1}{r+1}\sum_{k=1}^{r+1}\left(\left|v_1\right|^2\cdots\left|v_{r+1}\right|^2-\left\langle v_k,\chi\left(v_1,...,\widehat{v_k},...,v_{r+1}\right)\right\rangle^2\right)\\
&=\frac{1}{r+1}\sum_{k=1}^{r+1}\left(\left|v_1\right|^2\cdots\left|v_{r+1}\right|^2-\left|\phi\left(v_1,...,v_{r+1}\right)\right|^2\right)\\
&=\left|v_1\wedge...\wedge v_{r+1}\right|^2-\left|\phi\left(v_1,...,v_{r+1}\right)\right|^2.
\end{aligned}
$$

Hence the result is proved. □

As an immediate corollary of the above theorem, we have

Corollary 14. *Suppose that χ is an r-fold VCP on V. An $(r+1)$-dimensional linear subspace $P\subset V$ is preserved by χ, i.e. an instanton, if and only if*

$$\tau\left(v_1,...,v_{r+1}\right)=0$$

for any basis $v_1,...,v_{r+1}$ of P.

Remark. These two results were first obtained by Harvey and Lawson in the G_2- and $Spin\left(7\right)$-manifolds cases using special structures of the octonions [8].

We also denote the above condition as $\tau\left(P\right)=0$. Assuming this is the case, we denote the orthogonal decomposition of V as

$$V=P\oplus N.$$

Similarly we have the following decomposition,

$$\wedge^2V\cong\wedge^2P\oplus\wedge^2N\oplus\left(P\otimes N\right).$$

Proposition 15. *Suppose that χ is an r-fold VCP on V with an orthogonal decomposition*

$$V=P\oplus N$$

where P is an $(r+1)$-dimensional linear subspace of V preserved by χ. We have

$$\tau\left(p_1,...,p_r,n\right)\in P\otimes N$$

for any $p_1,...,p_r$ in P and $n\in N$.

Proof. Note that P being an instanton in V implies that

$$\left\langle\chi\left(p_1,...,p_{r-1},n\right),p_r\right\rangle=\pm\left\langle\chi\left(p_1,...,p_{r-1},p_r\right),n\right\rangle=0.$$

That is $\chi(p_1,...,p_{r-1},n) \in N$, or equivalently, $p_r \wedge \chi(p_1,...,p_{r-1},n) \in P \otimes N$. However $\tau(p_1,...,p_r,n)$ is a linear combination of terms of this form and therefore $\tau(p_1,...,p_r,n) \in P \otimes N$. This proves the proposition. □

We are going to use these linear algebra results to study the deformations of instantons. Given any VCP χ on a Riemannian manifold M, the spaces of infinitesimal automorphisms of (T_M, χ) on various fibers glue together to form a subbundle

$$\mathbf{g}_M \subset \Lambda^2 T_M^*.$$

Similarly the vector cross product χ determines a tensor

$$\tau \in \Omega^{r+1}\left(M, \mathbf{g}_M^{\perp}\right).$$

On the global level, the above corollary 14 is equivalent to the following result.

THEOREM 16. *If M is a Riemannian manifold with an r-fold VCP χ, then an $(r+1)$-dimensional submanifold A is an instanton if and only if*

$$\tau|_A = 0 \in \Omega^{r+1}\left(A, \mathbf{g}_M^{\perp}\right).$$

This theorem is useful in studying the deformation of instantons. We first recall that any nearby submanifold to A is the image of the exponential map

$$\exp_v : A \to M,$$

for some small normal vector field $v \in \Gamma\left(A, N_{A/M}\right)$. Therefore, given any instanton A in M, we can describe its nearby instantons as the zeros of the following map,

$$F : \Gamma\left(A, N_{A/M}\right) \to \Gamma\left(A, \mathbf{g}_M^{\perp}\right)$$

defined as $F(v) := *_A \exp_v^*(\tau)$ where $*_A$ is the Hodge star operator of A. Suppose that A_t is a family of submanifolds in M with $A = A_0$ an instanton and we denote its variation normal vector field as

$$v = \left.\frac{dA_t}{dt}\right|_{t=0} \in \Gamma\left(A, N_{A/M}\right).$$

By proposition 15 and A being an instanton, we have

$$\left.\frac{d\tau|_{A_t}}{dt}\right|_{t=0} \in \Gamma\left(A, T_A^* \otimes N_{A/M} \cap \mathbf{g}_M^{\perp}\right).$$

In particular the derivative of F is given by

$$\begin{gathered} F'(0) : \Gamma\left(A, N_{A/M}\right) \to \Gamma\left(A, T_A^* \otimes N_{A/M} \cap \mathbf{g}_M^{\perp}\right) \\ F'(0)\left(\left.\frac{dA_t}{dt}\right|_{t=0}\right) = \left.\frac{d\tau|_{A_t}}{dt}\right|_{t=0}. \end{gathered}$$

By studying individual cases, we find out that $N_{A/M}$ is always a twisted spinor bundle over A and the linearization of F coincides with a twisted Dirac operator, i.e. $F'(0) = \mathcal{D}$.

2.4. Branes for vector cross products. In symplectic geometry, the natural free boundary condition for holomorphic curves requires their boundaries lie inside a Lagrangian submanifold. The analog of a Lagrangian submanifold for VCP forms of higher degree is called a brane. In this section we derive some basic properties of branes and show that the brane does not exist in the *Spin* (7) case.

DEFINITION 17. *Suppose M is an n-dimensional manifold with a closed VCP form ϕ of degree $r+1$. A submanifold C is called a* ***brane*** *if*

$$\begin{cases} \phi\mid_C = 0 \\ dimC = (n+r-1)/2. \end{cases}$$

Remark on the dimension of a brane : Branes have the largest possible dimension among submanifolds C satisfying $\phi|_C = 0$. To see this, it suffices to consider the linear case. Taking any $(r-1)$-dimensional linear subspace W in C, the interior product of ϕ by any orthonormal basis of W determines a symplectic form on the orthogonal complement of W in M, which we denote as M/W. Furthermore C/W is an isotropic subspace in M/W and therefore $\dim C/W \leq (\dim M/W)/2$. The equality sign holds exactly when $\dim C = (n+r-1)/2$.

As we recall in a symplectic manifold, a holomorphic disk intersects perpendicularly a Lagrangian submanifold along the boundary. We have the following lemma for intersection of an instanton and branes along the boundaries of the instanton in a manifold with a closed VCP form.

LEMMA 18. *Let A be an instanton in an n-dimensional manifold M with a closed VCP form ϕ of degree $r+1$. Suppose the boundary of A lies in a brane C, then A intersects C perpendicularly along ∂A.*

Proof. For $x \in \partial A \subset C$, consider $u \in T_x A$ perpendicular to ∂A and any $v \in T_x C$. Observe that there are $u_1, ..., u_r \in T_x(\partial A)$ such that $u = u_1 \times u_2 \times ... \times u_r$ since $\phi|_A$ is the volume form on A. Then,

$$\begin{aligned} g(u,v) &= g(u_1 \times u_2 \times ... \times u_r, v) \\ &= \phi(u_1, u_2, ..., u_r, v) = 0 \end{aligned}$$

because $u_1, u_2, ..., u_r$ and v lie in $T_x C$ and $\phi|_C = 0$. That is, u is perpendicular to C. □

Note that we only need the assumption $\phi|_C = 0$ on C in the above lemma. The condition $\phi|_C = 0$ also implies that $[\phi] \in H^{r+1}(M, C)$. Any such instanton A minimizes volume within the relative homology class $[A] \in H_{r+1}(M, C)$, and the volume equals to the pairing of $[\phi]$ and $[A]$. Furthermore any submanifold $(A', \partial A') \subset (M, C)$ with $[A'] = [A]$ and $vol(A') = vol(A)$ is also an instanton. However, if $\dim C < (n+r-1)/2$, then finding instantons with boundaries lying on C is an overdetermined system of equations.

Since the definition of branes depends only on the closed VCP form ϕ instead of χ, the image of any brane under an ϕ-preserving diffeomorphism $f \in Diff(M, \phi)$ is again a brane. Infinitesimally, $v = df_t/dt|_{t=0} \in Vect(M, \phi)$ satisfies $L_v \phi = 0$. This implies that $\iota_v \phi$ is a closed form because ϕ is closed.

Definition 19. *Suppose that ϕ is a closed VCP form on M. A ϕ-preserving vector field $v \in Vect(M, \phi)$ is called a ϕ-**Hamiltonian vector field** if $\iota_v \phi$ is exact. That is*

$$\iota_v \phi = d\eta$$

*for some degree r differential form η, which we call a ϕ-**Hamiltonian differential form**.*

We will discuss the ϕ-Hamiltonian equivalence of branes in the next section.

Examples of branes. (i) Branes in an oriented n-dimensional Riemannian manifold (i.e. VCP form of degree n) are hypersurfaces. (ii) Branes in a Kähler manifold (i.e. parallel VCP form of degree 2) are Lagrangian submanifolds. (iii) Branes in a G_2-manifold (i.e. parallel VCP form of degree 3) can be identified as those four-dimensional submanifolds calibrated by $*\phi$ (see [8][10]) and they are called coassociative submanifolds. (iv) The next result shows that there is no brane in any *Spin* (7)-manifold.

Theorem 20. *A brane does not exist in any Spin (7)-manifold. That is, there is no 5-dimensional submanifold where the Cayley form vanishes.*

Proof. Suppose that C is any submanifold in a *Spin* (7)-manifold M where the Cayley form vanishes. This implies that $\chi(e_i, e_j, e_k)$ (denoted as χ_{ijk}) is perpendicular to C for any orthonormal tangent vectors e_l on C. Notice that these unit vectors satisfy

$$\chi_{ijp} \perp \chi_{ijq}.$$

This is because

$$\|\chi(e_i, e_j, e_p + e_q)\| = \|e_p + e_q\|,$$

which implies that

$$g(\chi(e_i, e_j, e_p), \chi(e_i, e_j, e_q)) = g(e_p, e_q) = 0.$$

If $\dim C = 5$, i.e. C is a brane in M, then its normal bundle has rank three. However, by the above property, $\chi_{123}, \chi_{124}, \chi_{134}$ and χ_{234} are four orthonormal vectors normal to C, which is a contradiction. □

Remark on 0-fold VCPs. Even though we usually assume r is positive and exclude 0-fold the *VCP* in the classification, such a *VCP* or its corresponding VCP form is simply given by a closed one form ϕ with unit pointwise length. When ϕ has integral period, we can integrate it to obtain a function,

$$f : M \to S^1 \text{ and } \phi = f^* d\theta.$$

Instantons are gradient flow lines for the Morse function f on M. Branes are middle-dimensional submanifolds in fibers of f.

Manifolds M ($\dim M$)	*VCP* form ϕ (degree of ϕ)	Instanton A ($\dim A$)	Brane C ($\dim C$)
Oriented Riem. mfd. (n)	Volume form (n)	Open Subset (n)	Hypersurface ($n-1$)
Kähler mfd. ($2m$)	Kähler form (2)	Holomorphic Curve (2)	Lagrangian Submanifold (m)
G_2-manifold (7)	G_2-form (3)	Associative Submanifold (3)	Coassociative Submanifold (4)
Spin (7)-mfd. (8)	Cayley form (4)	Cayley submfd. (4)	N/A

Table 2
Classification of instantons and branes

2.5. Deformation theory of branes. The intersection theory of branes plays an important role in describing the geometry of the vector cross product; this is analogous to the role of the Floer's Lagrangian intersection theory in symplectic geometry. Deformation theory of branes is essential in understanding both the intersection theory of branes and the moduli space of branes.

First we need to identify the normal bundle to any brane.

Proposition 21. *Let C be a brane in M. The VCP χ induces a surjective homomorphism*

$$\chi : \Lambda^r T_C \to N_{C/M}.$$

Proof. Note that $\phi|_C = 0$ implies

$$\chi : \Lambda^r T_C \to N_{C/M} \subset T_M.$$

When C has the maximum possible dimension, i.e. a brane, this is a surjective homomorphism onto $N_{C/M}$. If this were not the case there would exist $\nu \in N_{C/M}$ perpendicular to the image of $\chi(\Lambda^r T_C)$, thus ϕ would vanish on the linear span of T_C and ν, i.e. a bigger space containing C. This gives a contradiction. □

By taking the dual on $\chi : \Lambda^r T_C \to N_{C/M}$, we obtain an injective map,

$$t : N^*_{C/M} \to \Lambda^r T^*_C$$

defined by

$$t(\alpha)(u_1, ..., u_r) := \alpha(\chi(u_1, ..., u_r))$$

for $\alpha \in N^*_{C/M}$ and $u_1, ..., u_r \in T_C$.

Definition 22. *Let C be a brane in M with an r-fold VCP. We define $\Lambda^r_{VCP} T^*_C$ to be the image of the above injective map $t : N^*_{C/M} \to \Lambda^r T^*_C$.*

Observe that

$$\alpha(\chi(u_1, ..., u_r)) = g(\chi(u_1, ..., u_r), \breve{\alpha}) = \phi(u_1, ..., u_r, \breve{\alpha})$$

where $\breve{\alpha} \in N_{C/M}$ such that its dual is α. Then, for any $\alpha \in N^*_{C/M}$ with $|\alpha| = 1$, $t(\alpha)$ is a VCP form on T_C of degree r. This is because for any orthonormal vectors $e_1, ..., e_{r-1} \in T_{C,x}$ and $x \in C$,

$$\left|\iota_{e_1 \wedge \ldots \wedge e_{r-1}} (t(\alpha))\right| = \left|-\iota_{e_1 \wedge \ldots \wedge e_{r-1} \wedge \breve{\alpha}} (\phi)\right| = 1$$

because ϕ is a VCP form and $\breve{\alpha}$ is a unit normal vector. Therefore we proved the following proposition.

Proposition 23. *Suppose that C is a brane in a manifold M with a VCP form of degree $r+1$, then the image of the map*

$$t : N_{C/M} \to \Lambda^r T^*_C,$$

is the subbundle spanned by the VCP form of degree r on $T_{C,x}$ for all $x \in C$.

By using the classification results of branes below, we will see that $\Lambda^r_{VCP} T^*_C$ equals (i) T^*_C when $r = 1$, (ii) $\Lambda^2_+ T^*_C$ when $r = 2$, (iii) $\Lambda^{n-1} T^*_C$ when $r = n-1$. Note that a brane does not exist when $r = 3$ (Proposition 20).

Using the exponential map, small deformations of C correspond to sections of $N_{C/M}$ and the branes are the zeros of the following map,

$$F : \Gamma\left(N_{C/M}\right) \to \Omega^{r+1}(C)$$
$$F(v) = (\exp_v)^* \phi,$$

defined on a small neighborhood of the origin in $\Gamma\left(N_{C/M}\right)$. We are going to study the deformation theory of branes following the approach by McLean in [13]. Under the identification $t : \Gamma\left(N_{C/M}\right) \cong \Omega^r_{VCP}(C)$, the differential of F at 0 is given by the exterior derivative because

$$dF(0)(v) = \mathcal{L}_v(\phi)|_C = d(\iota_v \phi)|_C = d(t(v)).$$

Recall $F(0) = 0$, we obtain $[F(v)] = [F(0)] = 0 \in H^{r+1}(C)$ because C and $\exp_v(C)$ are homologous in M. Therefore we have

$$F : \Omega^r_{VCP}(C) \to d\Omega^r(C)$$
$$F(0) = 0,$$
$$dF(0) = d.$$

If we know that $d\Omega^r_{VCP}(C) = d\Omega^r(C)$, then using the implicit function theorem, we can show that $F^{-1}(0)$ is smooth near 0 and the tangent space is given by the kernel of $dF(0)$. The condition $d\Omega^r_{VCP}(C) = d\Omega^r(C)$ can be verified in each individual case, however the authors do not know of any general proof of this. In any case we have proved the following result.

Proposition 24. *Suppose that ϕ is a VCP form of degree $r+1$ on M. Then small deformations of any brane C are parametrized by closed form in $\Omega^r_{VCP}(C)$. In particular the space of branes in M is smooth.*

The space of branes is usually infinite dimensional. But quotienting out the equivalence relationship of ϕ-Hamiltonian, the moduli space of branes is finite dimensional.

Definition 25. *Suppose that C_1 and C_0 are two branes in a manifold M with a closed VCP form ϕ of degree $r+1$. They are called **ϕ-Hamiltonian equivalent*** *to each other if they are joined by a family of branes C_t such that their deformation vector fields $v_t = dC_t/dt \in \Gamma\left(N_{C_t/M}\right)$ satisfy*

$$\iota_{v_t}\phi = d\eta_t,$$

for some $\eta_t \in \Omega^{r-1}(C)$.

Using the Hodge theory and the previous proposition, we have the tangent space to the moduli space of branes at any point C equals $H^r_{VCP}(C)$, the space of harmonic forms in $\Omega^r_{VCP}(C)$. In particular, the moduli space is smooth and of finite dimension. Its tangent space is given by (i) $H^1(C,\mathbb{R})$ when $r=1$; (ii) $H^2_+(C,\mathbb{R})$ when $r=2$ and (iii) $H^{n-1}(C,\mathbb{R}) \cong \mathbb{R}$ when $r=n-1$. In the third case, i.e. ϕ is the volume form on M, two nearby hypersurfaces C and C' are ϕ-Hamiltonian equivalent if there is a singular chain B satisfying $\partial B = C - C'$ and $Vol(B) = 0$.

Lagrangian intersection theory in symplectic geometry plays the central role in the subject, and also plays a very essential role in mirror symmetry. Naively speaking we need to *count* the number of instantons bounding two Lagrangian submanifolds. It is natural to generalize this to other VCPs and count the number of instantons bounding two branes. This is a very difficult problem except when r equals zero. In this case, suppose that C_1 and C_2 are two branes in M, i.e. $C_i \subset f^{-1}(\theta_i)$ for $i=1,2$ are middle dimensional submanifolds. Here we continue the notations in the previous remark. Since f is a Riemannian submersion, M is a Riemannian mapping cylinder, i.e.

$$M = X \times [0,1] / \sim$$

for some isometry h on X, identifying $X \times \{0\}$ and $X \times \{1\}$. Thus both C_i's can be regarded as middle dimensional submanifolds in X. Then instantons in M bounding C_1 and C_2 correspond to intersection points between C_1 and $h^k(C_2)$ in X for any integer k. Therefore the generating function for the number of instantons is given explicitly by the following topological sum,

$$\sum_{k=-\infty}^{\infty} \#\left(C_1 \cap h^k(C_2)\right) t^k.$$

Remark. Instantons and branes are important classes of submanifolds in M, reflecting its geometry. For examples, they are usually supersymmetric cycles in various physical theories. In the next chapter we study the complex version of the VCP. In that case, the theory of instantons and branes is even more interesting.

3. Complex vector cross product.

3.1. Classification of complex vector cross products. In this section we study vector cross products on complex vector spaces, or Hermitian complex manifolds. For the sake of convenience, the complex vector cross product ($\mathbb{C}$-VCP) will be defined in terms of complex vector cross forms on a Hermitian complex manifold. Recall that a Hermitian complex manifold is a Riemannian manifold (M,g) with a Hermitian complex structure J, that is $g(Ju,Jv) = g(u,v)$ for any tangent vectors u and v.

Definition 26. *On a Hermitian complex manifold (M,g,J) of complex dimension n, an r-fold* ***complex vector cross product*** *(abbrev. $\mathbb{C}$-VCP) is a holomorphic* *form ϕ of degree $r+1$ satisfying*

$$\left|\iota_{e_1\wedge e_2..\wedge e_r}(\phi)\right| = 2^{(r+1)/2},$$

for any orthonormal tangent vectors $e_1, ..., e_r \in T_x^{1,0}M$, for any x in M.

A $\mathbb{C}$-VCP is called closed (resp. parallel) if ϕ is closed (resp. parallel with respect to the Levi-Civita connection) form.

Notice that if the manifold M is a closed Kähler manifold, then every holomorphic form in M is closed. For completeness we include the proof of this well-known fact.

Lemma 27. *Suppose M is a closed Kähler manifold. Then every holomorphic form is a closed differential form.*

Proof. Assume that ψ is any holomorphic form of degree k in M, that is $\psi \in \Omega^{k,0}(M)$ and $\bar{\partial}\psi = 0$. We need to show that $\partial\psi = 0 \in \Omega^{k+1,0}(M)$. By the Riemann bilinear relation, the pairing

$$\int \eta_1 \wedge \bar{\eta}_2 \wedge \omega^{n-k-1}$$

is definite on $\eta_i \in \Omega^{k+1,0}(M)$. That is,

$$\int_M |\partial\psi|^2 \omega^n = C \int_M \partial\psi \wedge \overline{\partial\psi} \wedge \omega^{n-k-1}.$$

Using integration by parts on closed manifolds and holomorphicity of ψ, we have

$$\int_M |\partial\psi|^2 \omega^n = -C \int_M \partial\bar{\partial}\psi \wedge \bar{\psi} \wedge \omega^{n-k-1} = 0.$$

This implies that $\partial\psi = 0$, that is, ψ is a closed form on M. □

We are going to see that there are exactly two classes of Kähler manifolds with $\mathbb{C}$-VCPs, namely Calabi-Yau manifolds and hyperkähler manifolds. Furthermore every $\mathbb{C}$-VCP is automatically parallel, in particular closed, provided that the manifold itself is closed.

Example. *Calabi-Yau manifold* (i.e. $(n-1)$-fold $\mathbb{C}$-VCP). A linear complex volume form ϕ on $\mathbb{C}^n$ is an element in $\Lambda^{n,0}(\mathbb{C}^n)$ with $\phi\bar{\phi}$ equals the Riemannian volume form on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. This is because of the equality $|\det_{\mathbb{C}}(A)|^2 = \det(A_{\mathbb{R}})$ between a complex matrix A and its realization $A_{\mathbb{R}}$. It is given as follows,

$$\phi = dz^1 \wedge dz^2 \wedge ... \wedge dz^n,$$

for a suitable choice of complex coordinates $z^1, z^2, ..., z^n$ on V. It is easy to see that ϕ defines a constant $(n-1)$-fold $\mathbb{C}$-VCP. Similarly an $(n-1)$-fold $\mathbb{C}$-VCP structure on a closed Kähler manifold (M, g) is a holomorphic volume form $\Omega \in \Omega^{n,0}(M)$,

$$\Omega\bar{\Omega} = C_n \omega^n,$$

where the constant C_n equals $i^n(-1)^{n(n-1)/2}2^{-n}/n!$. This implies that the Ricci curvature of M vanishes. Thus, using Bochner arguments, we can show that every holomorphic form on M is parallel. In particular Ω is a parallel complex volume form on M and therefore the holonomy group of M lies inside $SU(n)$, i.e. a Calabi-Yau manifold. A celebrated theorem of Yau [18] says that any closed Kähler manifold with trivial first Chern class $c_1(M)$ admits Kähler metric with vanishing Ricci curvature. This implies that M is a Calabi-Yau manifold if the canonical line bundle is trivial holomorphically.

Example. *Hyperkähler manifold* (i.e. 1-fold $\mathbb{C}$-VCP). A hyperkähler manifold is a Riemannian manifold (M, g) of dimension $n = 4m$ with its holonomy group lying inside $Sp(m) = GL(m, \mathbb{H}) \cap SO(4m)$. Namely it has parallel Hermitian complex structures I, J and K satisfying the Hamilton relation,

$$I^2 = J^2 = K^2 = IJK = -Id.$$

These complex structures define three different Kähler structures ω_I, ω_J and ω_K on (M, g) respectively. If we fix one of them, say J, then $\Omega = \omega_I - i\omega_K \in \Omega^{2,0}(M)$ is a parallel J-holomorphic symplectic form on M. These two descriptions of a hyperkähler manifold are equivalent and it is simply the global version of the isomorphism $Sp(m) = U(2m) \cap Sp(m, \mathbb{C})$. This form Ω is a parallel 1-fold $\mathbb{C}$-VCP form on M. The reasoning is the same as the one in the real case. In the linear case, this is given as follows,

$$\Omega = dz^1 \wedge dz^2 + ... + dz^{2m-1} \wedge dz^{2m},$$

for some suitable choice of coordinates on $\mathbb{C}^{2m}$. Conversely, if Ω is a 1-fold $\mathbb{C}$-VCP on a closed Kähler manifold (M, g, J), then it is a holomorphic symplectic form on M. Since $Sp(m) \subset SU(2m)$, any hyperkähler manifold is a Calabi-Yau manifold. This implies that Ω is indeed parallel as before. Therefore a hyperkähler structure is equivalent to a 1-fold $\mathbb{C}$-VCP on any closed Kähler manifold.

We remark that, as in the real setting, a constant r-fold $\mathbb{C}$-VCP on a complex vector space induces an $(r-1)$-fold $\mathbb{C}$-VCP on any of its complex hyperplanes.

We are going to show that there is no other complex vector cross product besides the holomorphic volume form and the holomorphic symplectic form as discussed above. In particular, there is no complex analog of the *VCP* for G_2-manifolds and $Spin(7)$-manifolds.

THEOREM 28. *On a complex vector space V of complex dimension n, there is an r-fold $\mathbb{C}$-VCP if and only if either (i) $r = 1$ and $n = 2m$ or (ii) $r = n - 1$ and n arbitrary. The corresponding $\mathbb{C}$-VCP form is a holomorphic symplectic form and a holomorphic volume form respectively*

Proof. From the above two examples, there is an $(n-1)$-fold $\mathbb{C}$-*VCP* on V, and moreover if n is an even number, a 1-fold $\mathbb{C}$-*VCP* exists on it. Now, we need to see there is no other type of $\mathbb{C}$-*VCP* on a complex vector space. For that matter, we claim that for $r \geq 2$ if there is an r-fold $\mathbb{C}$-*VCP* on a complex vector space V of complex dimension n, then r must be $n-1$. At first, observe that for $r \geq 2$, an r-fold $\mathbb{C}$-*VCP* on a vector space induces an $(r-1)$-fold $\mathbb{C}$-*VCP* on the complex hyperplane. Therefore, an r-fold $\mathbb{C}$-*VCP* on a complex vector space of complex dimension n is a reduced 2-fold $\mathbb{C}$-*VCP* on a complex $(n-r+2)$-dimensional vector space. Now, to prove our claim, it is enough to verify if there is a 2-fold $\mathbb{C}$-*VCP* on a complex vector space W, then its complex dimension must be three.

As in the first example of $\mathbb{C}$-*VCP*, when $\dim_{\mathbb{C}} W = 3$, there is a 2-fold $\mathbb{C}$-*VCP*. Now, we need to show there is no higher complex vector space with a 2-fold $\mathbb{C}$-*VCP*. Suppose $\dim_{\mathbb{C}} W \geq 4$ with a 2-fold $\mathbb{C}$-*VCP* ϕ, and by choosing any unit holomorphic vector z in W, consider a complex subspace Z spanned by z and $\bar{z}$. Then, $\iota_z\phi$ is a 1-fold $\mathbb{C}$-*VCP* on $Z^\perp$ and moreover $\dim_{\mathbb{C}} Z^\perp$ is at least four because $\dim_{\mathbb{C}} Z^\perp \geq 3$ and an even number so that $Z^\perp$ has a 1-fold $\mathbb{C}$-*VCP*.

Now, we may rewrite the 2-fold $\mathbb{C}$-*VCP* on W as

$$\phi = z^* \wedge \iota_z\phi + \phi_1,$$

where z^* is dual form of z and ϕ_1 is the sum of terms without z^*. Since $\dim_{\mathbb{C}} Z^\perp$ is at least 4, the 1-fold $\mathbb{C}$-*VCP* $\iota_z\phi$ on $Z^\perp$ has the form $a_1^* \wedge b_1^* + a_2^* \wedge b_2^*$....where a_1, a_2, b_1 and b_2 are orthonormal holomorphic vectors in $Z^\perp$. We consider the following,

$$\begin{aligned}\iota_{(b_1+b_2)}\iota_{(a_1+a_2)}\phi &= \iota_{b_1}\iota_{a_1}\phi + \iota_{b_2}\iota_{a_1}\phi + \iota_{b_1}\iota_{a_2}\phi + \iota_{b_2}\iota_{a_2}\phi \\ &= 2z^* + \iota_{b_2}\iota_{a_1}\phi + \iota_{b_1}\iota_{a_2}\phi,\end{aligned}$$

and

$$\iota_{\left(-\sqrt{-1}b_1+b_2\right)}\iota_{\left(-\sqrt{-1}a_1+a_2\right)}\phi = -\sqrt{-1}\iota_{b_2}\iota_{a_1}\phi - \sqrt{-1}\iota_{b_1}\iota_{a_2}\phi.$$

Note that $a_1 + a_2$, $b_1 + b_2$, $-\sqrt{-1}a_1 + a_2$ and $-\sqrt{-1}b_1 + b_2$ are holomorphic vectors with the same length.

The interior products of any terms from $\iota_z\phi$ and ϕ_1 are zero because they satisfy for example, $|\iota_{a_1\wedge b_1}\phi| = 1$. This implies that ϕ_1 does not have any terms in $\iota_z\phi$. Hence we have $\iota_{b_1}\iota_{a_1}\phi = z^*$ and $\iota_{b_2}\iota_{a_2}\phi = z^*$.

From this choice of orthonormal holomorphic vectors z, a_1, b_1, a_2 and b_2, holomorphic vectors $a_1 + a_2$ and $b_1 + b_2$ are orthogonal to each other, and this is true between holomorphic vectors $-\sqrt{-1}a_1 + a_2$ and $-\sqrt{-1}b_1 + b_2$. So $\iota_{(b_1+b_2)}\iota_{(a_1+a_2)}\phi$ and $\iota_{\left(-\sqrt{-1}b_1+b_2\right)}\iota_{\left(-\sqrt{-1}a_1+a_2\right)}\phi$ are supposed to produce the same length by definition of the $\mathbb{C}$-*VCP*, but it can be checked that this is impossible because z^* and $\iota_{b_2}\iota_{a_1}\phi + \iota_{b_1}\iota_{a_2}\phi$ are perpendicular to each other. From this contradiction, a complex vector space W is of complex dimension three so that it has a 2-fold $\mathbb{C}$-*VCP*. ☐

From this proposition and the examples of $\mathbb{C}$-*VCPs*, one can conclude the following theorem.

THEOREM 29. *(**Classification of** $\mathbb{C}$-VCPs) Suppose M is a closed Kähler manifold of complex dimension n, with an r-fold $\mathbb{C}$-VCP. Then either*
(i) $r = n - 1$ and M is a Calabi-Yau manifold, or
(ii) $r = 1$ and M is a hyperkähler manifold.

3.2. Instantons for complex vector cross products. In this section, we introduce and study instantons and branes on a Kähler manifold M with a closed $\mathbb{C}$-VCP $\phi \in \Omega^{r+1,0}(M)$. Recall that an instanton, in the real setting, is an $(r + 1)$-dimensional submanifold A preserved by χ, or equivalently A is calibrated by the VCP form. In the complex setting, the real and imaginary parts of the complex VCP form are always calibrating forms and we called such calibrated submanifolds instantons.

THEOREM 30. *Suppose ϕ is a closed $\mathbb{C}$-VCP form of degree $r + 1$ on a Kähler manifold M, then (i) $\mathrm{Re}\left(e^{i\theta}\phi\right)$ is a calibrating form for any real number θ, and (ii) an $(r + 1)$-dimensional submanifold A in M is calibrated by $\mathrm{Re}\left(e^{i\theta}\phi\right)$ only if*

$$\mathrm{Im}\left(e^{i\theta}\phi\right)|_A = \omega|_A = 0.$$

Proof. It suffices to check the linear case, namely $M = \mathbb{C}^n$ with the standard complex structure J. Consider any oriented orthonormal vectors $a_1,\ldots, a_{r+1}$ in $\mathbb{R}^{2n} = \mathbb{C}^n$ and denote $\xi = a_1 \wedge \ldots \wedge a_{r+1}$, then

$$\left\{\mathrm{Re}\left(e^{i\theta}\phi\right)(\xi)\right\}^2 + \left\{\mathrm{Im}\left(e^{i\theta}\phi\right)(\xi)\right\}^2 = \left|\left(e^{i\theta}\phi\right)(\xi)\right|^2 = \left|\phi(\xi)\right|^2.$$

Since ϕ is of type $(r+1,0)$, we have

$$\left|\phi(\xi)\right|^2 = 2^{-(r+1)}\left|\phi(\xi\otimes\mathbb{C})\right|^2.$$

where $\xi\otimes\mathbb{C} = \tilde{a}_1 \wedge\ldots\wedge \tilde{a}_{r+1}$, with $\tilde{a}_i = \left(a_i - \sqrt{-1}Ja_i\right)/\sqrt{2}$. This is because $dz_i(\tilde{a}_k) = \sqrt{2}dz_i(a_k)$.

If we denote the dual vector of any one form η as $\eta^{\#}$, then

$$\begin{aligned} 2^{-(r+1)}\left|\phi(\xi\otimes\mathbb{C})\right|^2 &= 2^{-(r+1)}\left|\phi(\tilde{a}_1,\ldots,\tilde{a}_{r+1})\right|^2 \\ &= 2^{-(r+1)}\left|\left\langle\left(\iota_{\tilde{a}_1\wedge\ldots\wedge\tilde{a}_r}\phi\right)^{\#}, \tilde{a}_{r+1}\right\rangle\right| \\ &\le 2^{-(r+1)}\left|\left(\iota_{\tilde{a}_1\wedge\ldots\wedge\tilde{a}_r}\phi\right)^{\#}\right|\left|\tilde{a}_{r+1}\right| \\ &= 2^{-(r+1)}\left|\iota_{\tilde{a}_1\wedge\ldots\wedge\tilde{a}_r}\phi\right| \le 1, \end{aligned}$$

because ϕ is a $\mathbb{C}$-VCP and $\tilde{a}_1,\ldots,\tilde{a}_{r+1}$ are elements in $T^{1,0}M$ of unit length. Therefore

$$\left|\mathrm{Re}\left(e^{i\theta}\phi\right)(\xi)\right| \le 1,$$

and when the equality sign holds, we have (i) $\left|\mathrm{Im}\left(e^{i\theta}\phi\right)(\xi)\right| = 0$, (ii) $\left(\iota_{\tilde{a}_1\wedge\ldots\wedge\tilde{a}_r}\phi\right)^{\#}$ parallel to $\tilde{a}_{r+1}$ and (iii) $\left|\iota_{\tilde{a}_1\wedge\ldots\wedge\tilde{a}_r}\phi\right| = 2^{r+1}$. Since $a_1,\ldots, a_{r+1}$ are orthonormal, condition (iii) is equivalent to $\tilde{a}_1,\ldots,\tilde{a}_{r+1}$ being orthonormal vectors in $T^{1,0}M$. This happens exactly when the linear span of $a_1,\ldots, a_{r+1}$ is isotropic with respect to ω. □

We remark that when ϕ is the holomorphic volume form of a Calabi-Yau manifold, then a middle dimensional submanifold A in M is calibrated by $\mathrm{Re}\left(e^{i\theta}\phi\right)$ if and only if it satisfies $\mathrm{Im}\left(e^{i\theta}\phi\right)|_A = \omega|_A = 0$, and it is called a special Lagrangian submanifold with phase angle θ.

Definition 31. *On a closed Kähler manifold M with an r-fold $\mathbb{C}$-VCP ϕ, an $(r+1)$-dimensional submanifold A is called an* ***instanton*** *with phase $\theta \in \mathbb{R}$ if it is calibrated by Re $\left(e^{i\theta}\phi\right)$, i.e.*

$$Re\left(e^{i\theta}\phi\right)|_A = vol_A.$$

Equivalently, $\mathrm{Im}\left(e^{i\theta}\phi\right)|_A = 0$ *and* $\left|\iota_{e_1\wedge\ldots\wedge e_{r+1}}(\phi|_A)\right| = 1$ *for any orthonormal tangent vectors $e_1,\ldots,e_{r+1}$ on A.*

Remark. Recall that the volume of a calibrated submanifold is topological. In this case, for the fundamental class $[A] \in H_{r+1}(A,\mathbb{Z})$ and $\left[Re\left(e^{i\theta}\phi\right)\right] \in H^{r+1}(M,\mathbb{R})$,

$$vol(A) = \int_A Re\left(e^{i\theta}\phi\right) = [A]\cdot\left[Re\left(e^{i\theta}\phi\right)\right]$$

Using the classification result of $\mathbb{C}$-VCPs in theorem 29, ϕ must be either a holomorphic volume form in a Calabi-Yau manifold or a holomorphic symplectic form in a hyperkähler manifold. In the former case, instantons are called special Lagrangian submanifolds (see e.g. [15]). In the latter case,

$$\operatorname{Re}\left(e^{i\theta}\left(\omega_I - i\omega_K\right)\right) = \cos\theta\omega_I + \sin\theta\omega_K = \omega_{\cos\theta I + \sin\theta K}$$

so instantons are J_θ-holomorphic curves where $J_\theta = \cos\theta I + \sin\theta K$.

3.3. Dirichlet and Neumann branes. Schoen's school studies ([4], [14], [15]) free boundary value problems for special Lagrangian submanifolds in a Calabi-Yau manifold M. Suppose that A is a special Lagrangian submanifold of zero phase, i.e. calibrated by $\operatorname{Re}\Omega_M$, in M. If the boundary of A is non-empty and lies on a submanifold C in M, then (i) C being a complex hypersurface in M corresponds to Neumann boundary condition on A and (ii) C being a special Lagrangian submanifold of phase $\pi/2$ corresponds to Dirichlet boundary condition on A. Motivated from these, we have the following definitions of branes in Hermitian complex manifolds.

Definition 32. *On a Hermitian complex manifold (M, ω) of complex dimension n with an r-fold $\mathbb{C}$-VCP $\phi \in \Omega^{r+1,0}(M)$,*

(i) a submanifold C is called a **Neumann brane** *(abbrev. N-brane) if* $\dim(C) = n + r - 1$ *and*

$$\phi|_C = 0,$$

(ii) an n-dimensional submanifold C is called a **Dirichlet brane** *(abbrev. D-brane) with phase $\theta \in \mathbb{R}$ if*

$$\omega|_C = 0\ ,\ Re\left(e^{i\theta}\phi\right)|_C = 0$$

Even though branes are defined in Hermitian complex manifolds, for simplicity we will focus on branes in a closed Kähler manifold.

As in the real setting, N-branes are submanifolds in M with the biggest dimension on which ϕ vanishes. Furthermore N-branes are complex submanifolds in M because of the following proposition.

Proposition 33. *Suppose that M is an n-dimensional Kähler manifold with an r-fold $\mathbb{C}$-VCP. Assume that S is a submanifold in M such that $\phi|_S = 0$, then* $\dim S \leq n + r - 1$.

When the equality sign holds, i.e. S is an N-brane, then it is a complex submanifold in M.

Proof. It suffices to check the linear case, i.e. T_xS, $x \in S$. Consider a tangent vector $a \in T_xS$ and $\tilde{a} = \left(a - \sqrt{-1}Ja\right)/\sqrt{2}$. Since $\phi \in \Omega^{r+1,0}$, $\iota_{\tilde{a}}\phi = \sqrt{2}\iota_a\phi$. Therefore, for $b_1, ..., b_r$ in T_xS, we have

$$\sqrt{2}\phi\left(a, b_1, ..., b_r\right) = \phi\left(\tilde{a}, b_1, ..., b_r\right).$$

So if $\phi(a, b_1, ..., b_r) = 0$, then $\phi(Ja, b_1, ..., b_r) = 0$, namely, $\phi|_S = 0$ implies $\phi|_{S+JS} = 0$. And more if S has the maximal dimension, then T_xS is a complex linear subspace. In that case, as in the real setting, we have $\dim_{\mathbb{C}} S = (n + r - 1)/2$. □

The above proposition can also be verified case by case using the classification result of $\mathbb{C}$-VCPs. The following classification of instantons and branes for the $\mathbb{C}$-*VCP* is presented with the classification of $\mathbb{C}$-*VCPs* on a closed Kähler manifold.

Instantons and branes in Calabi-Yau manifold : (n − 1)-fold ℂ- ***VCP***

A closed Kähler manifold of complex dimension n with an $(n-1)$-fold $\mathbb{C}$- VCP ϕ is a Calabi-Yau n-fold.

An instanton in the Calabi-Yau n-fold is a special Lagrangian submanifold with phase θ since it is calibrated by $\mathrm{Re}\left(e^{i\theta}\phi\right)$. As in the previous proposition, an N-brane in the Calabi-Yau n-fold is a complex hypersurface, and a D-brane is a special Lagrangian submanifold with phase $\theta-\pi/2$, because it is calibrated by $\mathrm{Re}\left(e^{i(\theta-\pi/2)}\phi\right) = \mathrm{Im}\left(e^{i\theta}\phi\right)$.

Instantons and branes in hyperkähler manifold : 1-fold ℂ- ***VCP***

A closed Kähler manifold of complex dimension $2n$ with a Kähler form ω_J and a 1-fold $\mathbb{C}$- VCP ϕ is a hyperkähler manifold. Denote $\phi =: \omega_I - \sqrt{-1}\omega_K$ and J, I and K as the complex structures corresponding to Kähler structures ω_J, ω_I and ω_K , respectively. Moreover, by putting $e^{i\theta}\phi$ in place of ϕ, one can observe $\mathrm{Re}\, e^{i\theta}\phi$ is another Kähler structure with a complex structure $J_\theta = \cos\theta I + \sin\theta K$.

Now, an instanton in a hyperkähler manifold is a J_θ-holomorphic curve since it is calibrated by $\mathrm{Re}\, e^{i\theta}\phi$, namely preserved by J_θ. An N-brane in a hyperkähler manifold is a real $2n$-dimensional submanifold where ϕ vanishes, and as in the previous proposition, it is equivalently a J-complex Lagrangian which is a complex submanifold preserved by a complex structure J with complex dimension n. A D-brane is a real $2n$-dimensional submanifold where ω and $\mathrm{Re}\left(e^{i\theta}\phi\right)$ vanish. One can show that a D-brane is preserved by $J_{\theta+\pi/2}$, the almost complex structure corresponding to $-\mathrm{Im}\left(e^{i\theta}\phi\right)$, since $e^{i\theta}\phi = \omega_{J_\theta} - \sqrt{-1}\omega_{J_{\theta+\pi/2}}$. So a D-brane is equivalently a $J_{\theta+\pi/2}$-complex Lagrangian.

The above classification of instantons, N-branes and D-branes in manifolds with ℂ-VCPs is summarized in the table on page 122.

4. Final remarks. In this paper we study both real and complex vector cross products. Instantons in either setting are calibrated submanifolds. This gives a unified way to explain the calibrating property of many such examples, as studied by Harvey and Lawson in [8]. It is desirable to study further the calibration geometry from this point of view. On the other hand, understanding the mean curvature flow(MCF) for branes would give us a unified treatment for hypersurfaces MCF and Lagrangian MCF.

Manifolds with real VCPs include symplectic/Kähler and G_2-manifolds. In [11], we relate the geometry of the VCP on the manifolds M to the symplectic geometry of their knot spaces $\mathcal{K}_\Sigma M$. Motivated from this relationship, it is natural to study the intersection theory of branes and count the number of instantons bounding them, similar to the Floer's homology theory of Lagrangian intersections. For example, in the case of G_2-manifolds, counting associative submanifolds bounding nearby coassociative submanifolds is closely related to the Seiberg-Witten invariants of the four dimensional coassociative submanifolds [12]. Results along this line should be useful in understanding the M-theory in Physics.

Manifolds with ℂ-VCPs are either Calabi-Yau manifolds or hyperkähler manifolds. In [11] we construct a complex analog of knot spaces for Calabi-Yau manifolds and show that they behave like hyperkähler manifolds. Furthermore both complex hypersurfaces and special Lagrangians in a Calabi-Yau manifold give complex Lagrangians in this infinite dimensional hyperkähler manifold, but with respect to different complex structures in the twistor family. Therefore one would hope that the mirror symmetry transformation for Calabi-Yau manifolds can be explained as a twistor rotation. Further studies along this direction will be quite interesting.

Acknowledgments. The research of the second author is partially supported by NSF/DMS-0103355, Direct Grant from CUHK and RGC Earmarked Grants of Hong Kong. Authors express their gratitude to S.L. Kong, X.W. Wang for useful discussions.

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