Available online at www.sciencedirect.com
ScienceDirect
Advances in Mathematics 213 (2007) 140–164
ADVANCES IN Mathematics
www.elsevier.com/locate/aim

Higher dimensional knot spaces for manifolds with vector cross products

Jae-Hyouk Lee [a,*], Naichung Conan Leung [b]

[a] *Department of Mathematics, Washington University in St. Louis, USA*
[b] *Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Hong Kong*

Received 4 March 2006; accepted 12 December 2006

Available online 22 December 2006

Communicated by Michael J. Hopkins

Abstract

Vector cross product structures on manifolds include symplectic, volume, G_2- and *Spin*(7)-structures. We show that the knot spaces of such manifolds have natural symplectic structures, and relate instantons and branes in these manifolds to holomorphic disks and Lagrangian submanifolds in their knot spaces.

For the complex case, the holomorphic volume form on a Calabi–Yau manifold defines a complex vector cross product structure. We show that its isotropic knot space admits a natural holomorphic symplectic structure. We also relate the Calabi–Yau geometry of the manifold to the holomorphic symplectic geometry of its isotropic knot space.

Keywords: Vector cross product; Branes; Instantons; Symplectic structures on knot spaces; Holomorphic symplectic spaces on isotropic knot spaces

1. Introduction

In our earlier paper [10] we studied the geometry of manifolds with vector cross products (abbreviated as VCP) and manifolds with complex vector cross products (abbreviated as $\mathbb{C}$-VCP). For example manifolds with closed 1-fold (respectively 2-fold, 3-fold) VCPs include Kähler manifolds (respectively G_2-manifolds, *Spin*(7)-manifolds). They play an important role in String

* Corresponding author.
E-mail addresses: jhlee@math.wustl.edu (J.-H. Lee), leung@ims.cuhk.edu.hk (N.C. Leung).

0001-8708/$ – see front matter
doi:10.1016/j.aim.2006.12.003

theory (respectively M-theory, F-theory). For instance, instantons and branes for VCP geometry are supersymmetric cycles in the corresponding physical theories. For closed $\mathbb{C}$-VCPs we proved that they are either Calabi–Yau geometry or hyperkähler geometry. These geometries have special duality symmetries, called the Mirror Symmetry. From a holonomy point of view, this gives a coherent description of all possible special holonomy structures on Riemannian manifolds, as classified by Berger.

In this article we will show that (i) the geometry of an r-fold VCP on M can be realized as the symplectic geometry of the knot space $\mathcal{K}_\Sigma M$ of $(r-1)$-dimensional knots in M and (ii) the geometry of an $(n-1)$-fold $\mathbb{C}$-VCP geometry on a Calabi–Yau manifold M can be realized as the complex symplectic geometry of the isotropic knot space $\hat{\mathcal{K}}_\Sigma M$.

The idea of using infinite dimensional mapping spaces to study the geometry of the underlying manifold M has proved to be very useful in many situations. For instance, Morse used the loop space to study the topology of M [1], Witten used the loop space to interpret the Atiyah–Singer index theorem [15] and the elliptic genus [14], Floer used the Morse theory on the loop space to study the Lagrangian intersection theory of a symplectic manifold M and defined his Floer homology group [5], and Hitchin used the moment map on the mapping space of a Calabi–Yau manifold M to study the moduli space of special Lagrangian submanifolds in M [9]. The space of knots in three manifolds was used very successfully by Brylinski [3] to study the knot theory and Chern–Simons theory. Millson and Zombro found a Kähler structure on this knot space when $M=\mathbb{R}^3$ [12].

For a VCP on a manifold M, we prove that its knot space $\mathcal{K}_\Sigma M$ has a natural symplectic structure. Furthermore branes in M correspond to Lagrangian submanifolds in $\mathcal{K}_\Sigma M$. There is also a similar correspondence between instantons in M and holomorphic curves in $\mathcal{K}_\Sigma M$. Unlike the first correspondence whose objects are defined by the vanishing of certain differential forms, the latter correspondence is obtained by applying a dimension reduction argument on a system of differential equations. Thus it can only apply to a restricted class of instantons as we will explain fully in Section 3.

For instance, quantum intersection theory of coassociative submanifolds in a 7-dimensional G_2-manifold M, as studied by Wang and the second author in [11], would correspond to Floer's Lagrangian intersection theory in $\mathcal{K}_\Sigma M$. Such interpretation is useful in understanding the geometry of G_2-manifolds and ultimately the M-theory. We summarize our results on knot spaces below.

Theorem 1. *Suppose M is a Riemannian manifold with a closed differential form ϕ. Then we have*

(1) *ϕ is a VCP form on M if and only if its transgression defines an almost Kähler structure on $\mathcal{K}_\Sigma M$;*
(2) *$\mathcal{K}_\Sigma C$ is a Lagrangian submanifold in $\mathcal{K}_\Sigma M$ if and only if C is a brane in M;*
(3) *For a normal disk D in $Map(\Sigma, M)_{\mathrm{emb}}$, $\hat{D}$ is a holomorphic disk in $\mathcal{K}_\Sigma M$ if and only if $D \tilde{\times} \Sigma$ gives a tight instanton in M.*

When we consider the complex analog of the above theorem, a holomorphic volume form on a Kähler manifold M defines a $\mathbb{C}$-VCP on M, thus M is a Calabi–Yau manifold. Naively, one would expect that $Map(\Sigma, M)_{\mathrm{emb}}/\mathit{Diff}(\Sigma) \otimes \mathbb{C}$ is hyperkähler. Since the complexification of $\mathit{Diff}(\Sigma)$ does not exist, we need to consider the symplectic reduction of $Map(\Sigma, M)_{\mathrm{emb}}$ and restrict our attention to the isotropic mapping space because of the moment condition. We will explain in Section 5 how to resolve these issues and prove the following theorem.

Theorem 2. *Suppose M is a Calabi–Yau n-fold and Σ is a closed manifold of dimension $n-2$. Then the isotropic knot space $\hat{\mathcal{K}}_\Sigma M$ has a natural holomorphic symplectic structure.*

Notice that $\hat{\mathcal{K}}_\Sigma M$ consists only of *isotropic* knots because of the moment condition, which was needed in the symplectic reduction construction.

Furthermore, the Calabi–Yau geometry on M can be interpreted as the holomorphic symplectic geometry on $\hat{\mathcal{K}}_\Sigma M$. Namely, we prove the following theorem in Section 6.

Theorem 3. *Suppose M is a Calabi–Yau n-fold. We have*

(1) *$\hat{\mathcal{K}}_\Sigma C$ is a J-complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$ if and only if C is a complex hypersurface in M;*
(2) *$\hat{\mathcal{K}}_\Sigma C$ is a K-complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$ if and only if C is a special Lagrangian submanifold in M with phase $-\pi/2$.*

Even though complex hypersurfaces and special Lagrangian submanifolds look very different inside a Calabi–Yau manifold, their isotropic knot spaces are both complex Lagrangian submanifolds in $\hat{\mathcal{K}}_\Sigma M$. One reason is that any knot inside a special Lagrangian submanifold is automatically isotropic. To prove this theorem, we need to carefully construct certain appropriate deformations of isotropic knots inside C so that $\hat{\mathcal{K}}_\Sigma C$ being a J-complex Lagrangian (respectively I-complex Lagrangian) implies that the dimension of C is at least $2n-2$ (respectively at most n).

For example, when M is a Calabi–Yau threefold, $\Sigma \subset M$ is simply a loop and the isotropic condition is automatic. In this case $\hat{\mathcal{K}}_\Sigma M$ is the space of loops in M modulo the equivalence relations generated by deforming Σ along *complex* curves in M.

The organization of this paper is as follows: In Section 2, we recall some basic results from [10] on the geometry of manifolds M with VCPs. In Section 3, we show that the knot space of M admits a natural symplectic structure. Moreover branes and instantons in M correspond to Lagrangian submanifolds and holomorphic curves in $\mathcal{K}_\Sigma M$. In Section 4, we recall some basic results from [10] on the geometry of manifolds M with $\mathbb{C}$-VCPs. In Section 5, we construct the isotropic knot space $\hat{\mathcal{K}}_\Sigma M$ using a modified version of the symplectic quotient construction. Moreover, we prove that it has a natural complex symplectic structure and a compatible almost hyperkähler structure. In Section 6, we prove a correspondence between both special Lagrangian submanifolds and complex hypersurfaces in M and complex Lagrangian submanifolds in $\hat{\mathcal{K}}_\Sigma M$, but with respect to different complex structures in the twistor family.

2. Geometry of vector cross product

The VCP structure on a manifold was introduced by Gray [2,6] and it is a natural generalization of the vector product, or sometimes called the cross product, on $\mathbb{R}^3$.

Definition 4. Suppose that M is an n-dimensional manifold with a Riemannian metric g.

(1) An r-fold vector cross product (VCP) is a smooth bundle map,

$$\chi : \wedge^r TM \to TM$$

satisfying

$$\begin{cases} g(\chi(v_1, \ldots, v_r), v_i) = 0 \quad (1 \leqslant i \leqslant r), \\ g(\chi(v_1, \ldots, v_r), \chi(v_1, \ldots, v_r)) = \|v_1 \wedge \cdots \wedge v_r\|^2 \end{cases}$$

where $\|\cdot\|$ is the induced metric on $\wedge^r TM$.

(2) We define the VCP form $\phi \in \Omega^{r+1}(M)$ as

$$\phi(v_1, \ldots, v_{r+1}) = g\big(\chi(v_1, \ldots, v_r), v_{r+1}\big).$$

(3) The VCP χ is called closed (respectively parallel) if ϕ is a closed (respectively parallel) differential form.

The classification of the linear VCPs on a vector space V with positive definite inner product g, by Brown and Gray [2], can be summarized in the following.

(i) $r = 1$: A 1-fold VCP is a Hermitian complex structure on V and its VCP form is the Kähler form. For $V = \mathbb{C}^m$ we have

$$\phi = dx^1 \wedge dy^1 + \cdots + dx^m \wedge dy^m.$$

(ii) $r = n - 1$: The VCP form is the induced volume form Vol_V on V. For $V = \mathbb{R}^n$ we have

$$\phi = dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

(iii) $r = 2$: A 2-fold VCP on a 7-dimensional vector space $\mathbb{R}^7$ is a cross product defined as $a \times b = \operatorname{Im}(ab)$ for any a, b in $\mathbb{R}^7 = \operatorname{Im}\mathbb{O}$, where $\mathbb{O}$ is the Octonion numbers, or the Cayley numbers. For coordinates $(x_1, \ldots, x_7)$ on $\operatorname{Im}\mathbb{O}$, the VCP form Ω is

$$\Omega = dx^{123} - dx^{167} + dx^{145} + dx^{257} + dx^{246} - dx^{356} + dx^{347}$$

where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$.

(iv) $r = 3$: A 3-fold VCP on an 8-dimensional vector space $\mathbb{R}^8 = \mathbb{O}$ is a cross product defined as $a \times b \times c = \frac{1}{2}(a(\bar{b}c) - c(\bar{b}a))$ for any a, b and c in $\mathbb{O}$. Its VCP form Θ is

$$\begin{aligned} \Theta = {} & -dx^{1234} - dx^{5678} - \big(dx^{21} + dx^{34}\big)\big(dx^{65} + dx^{78}\big) \\ & - \big(dx^{31} + dx^{42}\big)\big(dx^{75} + dx^{86}\big) - \big(dx^{41} + dx^{23}\big)\big(dx^{85} + dx^{67}\big). \end{aligned}$$

Remark that the existence of a VCP on a Riemannian manifold M is equivalent to the reduction of the structure group of the frame bundle from $O(n)$ to the groups of real-linear transformation preserving g and a VCP which are $U(m)$, $SO(n)$, G_2 and $Spin(7)$, for $r = 1$, $n - 1$, 2 and 3 respectively. For simplicity we will assume that the VCP structure on M is always parallel in this paper.

Next we introduce the natural class of submanifolds in M, called instantons.

Definition 5. Let M be a Riemannian manifold with a closed r-fold VCP χ. An $(r+1)$-dimensional submanifold A is called an *instanton* if it is preserved by χ.

Instantons are always calibrated submanifolds of M. In particular, they are absolute minimal submanifolds in M. It is not difficult to identify instantons for the r-fold VCP with various r; When $r=1$, instantons are holomorphic curves in Kähler manifolds. When $r=n-1$, instantons are open domains in M. When $r=2$, instantons are associative submanifolds in G_2-manifolds. When $r=3$, instantons are Cayley submanifolds in $Spin(7)$-manifolds (see [7]). In [10] we also discuss different characterizations of instantons and the deformation theory of them.

The natural boundary condition for an instanton is to require its boundary to lie on a *brane*.

Definition 6. Suppose M is an n-dimensional manifold with a closed VCP form ϕ of degree $r+1$. A submanifold C is called a *brane* if

$$\begin{cases} \phi|_C = 0, \\ \dim C = (n+r-1)/2. \end{cases}$$

Remark. In fact, the branes are of maximal dimension among the submanifolds where the closed VCP form ϕ vanishes. To see this, we consider an n-dimensional vector space V with a VCP form ϕ of degree $r+1$. For each oriented orthonormal vectors $v_1, \ldots, v_{r-1}$, it is easy to see $\iota_{v_{r-1}} \ldots \iota_{v_1}\phi$ is a VCP form of degree 2, i.e. a symplectic form, on the orthogonal complement of the subspace W_1 spanned by $v_1, \ldots, v_{r-1}$. Therefore ϕ vanishes on a vector subspace W_2 containing W_1 if $W_2 \cap W_1^{\perp}$ is isotropic for $\iota_{v_{r-1}} \ldots \iota_{v_1}\phi$, and the possible maximal dimension of such a W_2 is obtained when $W_2 \cap W_1^{\perp}$ is Lagrangian.

When $r=1$, branes are Lagrangian submanifolds in Kähler manifolds. When $r=n-1$, branes are hypersurfaces. When $r=2$, branes are coassociative submanifolds in G_2-manifolds. When $r=3$, branes do not exist at all. We also discuss the deformations of branes in [10]. In particular, they are also smooth manifolds.

The intersection theory of branes with instanton corrections is an important topic. When $r=1$ this is the theory of Fukaya–Floer category. When $r=2$, the second author and X. Wang [11] found a relationship between this and the Seiberg–Witten theory.

We summarize various structures associated to the VCP in the following table.

Manifolds M ($\dim M$)	VCP form ϕ (degree of ϕ)	Instanton A ($\dim A$)	Brane C ($\dim C$)
Oriented Riem. mfd. (n)	Volume form (n)	Open subset (n)	Hypersurface ($n-1$)
Kähler manifold ($2m$)	Kähler form (2)	Holomorphic curve (2)	Lagrangian submanifold (m)
G_2-manifold (7)	G_2-form (3)	Associative submanifold (3)	Coassociative submanifold (4)
$Spin(7)$-manifold (8)	Cayley form (4)	Cayley submanifold (4)	N/A

3. Symplectic geometry on knot spaces

Recall that a 1-fold VCP on M is a symplectic structure. In general an r-fold VCP form on M induces a symplectic structure on $\mathcal{K}_\Sigma M = Map(\Sigma, M)/Diff(\Sigma)$, the space of embedded submanifolds Σ of dimension $r-1$. When M has dimension three, Brylinski [3] showed that $\mathcal{K}_\Sigma M$ has a natural Kähler structure and used it to study geometric quantization.

3.1. Symplectic structure on knot spaces

Let (M, g) be an n-dimensional Riemannian manifold M with a closed VCP form ϕ of degree $r+1$. Supposing Σ is any $(r-1)$-dimensional oriented closed manifold Σ, we consider the mapping space of embedding from Σ to M,

$$Map(\Sigma, M) = \{f : \Sigma \to M \mid f \text{ is an embedding}\}.$$

Let

$$ev : \Sigma \times Map(\Sigma, M) \to M$$

be the evaluation map $ev(x, f) = f(x)$ and p_1, p_2 be the projection maps from $\Sigma \times Map(\Sigma, M)$ to its first and second factor respectively. We define a two form ω_{Map} on $Map(\Sigma, M)$ by taking the transgression of the VCP form ϕ,

$$\omega_{Map} = (p_2)_*(ev)^*\phi = \int_\Sigma ev^*\phi.$$

Explicitly, supposing u and v are tangent vectors to $Map(\Sigma, M)$ at f, that is $u, v \in \Gamma(\Sigma, f^*(T_M))$, we have

$$\omega_{Map}(u, v) = \int_\Sigma \iota_{u\wedge v}\phi.$$

Since $(\iota_{u\wedge v}\phi)|_\Sigma$ can never be a top degree form if u is tangent to Σ, ω_{Map} degenerates along tangent directions to the orbits of the natural action of $Diff(\Sigma)$ on $Map(\Sigma, M)$. Thus it descends to a two form $\boldsymbol{\omega}^{\mathcal{K}}$ on the quotient space

$$\mathcal{K}_\Sigma M = Map(\Sigma, M)/Diff(\Sigma),$$

the space of submanifolds in M. For simplicity, we call it a (multi-dimensional) *knot space*. Note that tangent vectors of $\mathcal{K}_\Sigma M$ are sections of the normal bundle of Σ in M.

On $\mathcal{K}_\Sigma M$ there is a natural L^2-metric given as follows: suppose $[f] \in \mathcal{K}_\Sigma M$ and $u, v \in \Gamma(\Sigma, N_{\Sigma/M})$ are tangent vectors to $\mathcal{K}_\Sigma M$ at $[f]$, then the inner product is

$$\mathbf{g}^{\mathcal{K}}(u, v) = \int_\Sigma g(u, v)\nu_\Sigma,$$

where ν_Σ is the volume form of Σ with respect to the induced metric on Σ. We define an endomorphism $\mathbf{J}^{\mathcal{K}}$ on the tangent space of $\mathcal{K}_\Sigma M$ as follows: suppose $[f] \in \mathcal{K}_\Sigma M$ and $u \in \Gamma(\Sigma, N_{\Sigma/M})$ is a tangent vector to $\mathcal{K}_\Sigma M$ at $[f]$. Letting $e_1, \ldots, e_{r-1}$ be an oriented orthonormal basis of $T\Sigma$ at a point x, we define

$$\mathbf{J}^{\mathcal{K}}(u) = \chi(e_1, \ldots, e_{r-1}, u).$$

It is not difficult to see that if χ is an r-fold VCP on T_xM, then $\chi(e_1, \ldots, e_{r-1,} \cdot)$ defines a 1-fold VCP on the orthogonal complement to any oriented orthonormal vectors $e_1, \ldots, e_{r-1}$ at T_xM. Thus $\mathbf{J}^{\mathcal{K}}$ is an almost complex structure on $\mathcal{K}_\Sigma M$.

Lemma 7. *Let (M, g) be a Riemannian manifold with a VCP. Suppose $[f] \in \mathcal{K}_\Sigma M$ and $u, w \in \Gamma(\Sigma, N_{\Sigma/M})$ are tangent vectors to $\mathcal{K}_\Sigma M$ at $[f]$. Then*

$$\boldsymbol{\omega}^{\mathcal{K}}(u, w) = \mathbf{g}^{\mathcal{K}}\big(\mathbf{J}^{\mathcal{K}}(u), w\big).$$

Proof. Let ϕ be the VCP form on M. From the definition of $\boldsymbol{\omega}^{\mathcal{K}}$, we have

$$\boldsymbol{\omega}^{\mathcal{K}}(u, w) = \int_\Sigma \iota_{u\wedge w}\phi.$$

Letting $e_1, \ldots, e_{r-1}$ be an oriented orthonormal basis of $T\Sigma$ at a point x, we have

$$\begin{aligned} \iota_{u\wedge w}(ev^*\phi) &= \phi(e_1, \ldots, e_{r-1}, u, w)v_\Sigma \\ &= g\big(\chi(e_1, \ldots, e_{r-1}, u), w\big)v_\Sigma \\ &= g\big(\mathbf{J}^{\mathcal{K}}(u), w\big)v_\Sigma. \end{aligned}$$

Integrating over Σ, we obtain $\boldsymbol{\omega}^{\mathcal{K}}(u, w) = \mathbf{g}^{\mathcal{K}}(\mathbf{J}^{\mathcal{K}}(u), w)$. □

As an immediate corollary, we have the following.

Corollary 8. *Suppose (M, g) is a Riemannian manifold with a VCP. Then $\mathbf{J}^{\mathcal{K}}$ is a Hermitian almost complex structure on $\mathcal{K}_\Sigma M$, i.e. a 1-fold VCP.*

Remark. The above proof actually shows that if $\mathbf{J}^{\mathcal{K}}$ on $\mathcal{K}_\Sigma M$ is induced by an arbitrary differential form ϕ on M, then $\mathbf{J}^{\mathcal{K}}$ is an Hermitian almost complex structure on $\mathcal{K}_\Sigma M$ if and only if ϕ is an r-fold VCP form on M.

When the VCP form ϕ on M is closed and Σ is a closed manifold, then ω_{Map} on $Map(\Sigma, M)$ is also closed because

$$d\omega_{Map} = d\int_\Sigma ev^*\phi = \int_\Sigma ev^*\, d\phi = 0.$$

Therefore ω_{Map} descends to a closed 1-fold VCP form on $\mathcal{K}_\Sigma M$.

Conversely the closedness of ω_{Map} on $Map(\Sigma, M)$, or on $\mathcal{K}_\Sigma M$, implies the closedness of ϕ on M. Since this type of the localization argument will be used several times in this paper, we include the proof of the following standard lemma.

Lemma 9 *(Localization). Let Σ be an s-dimensional manifold without boundary. A form η of degree $k > s$ on a manifold M vanishes if the corresponding $(k-s)$-form on $Map(\Sigma, M)$ obtained by transgression vanishes.*

Proof. We need to show that

$$\eta(p)(v_1, v_2, \dots, v_k) = 0$$

for any fixed $p \in M$ and any fixed $v_i \in T_p M$. For simplicity, we may choose $v_1, v_2, \dots, v_k$ to be orthonormal vectors. We can find $f \in Map(\Sigma, M)$ with $p \in f(\Sigma)$ such that $v_1, \dots, v_s$ are along the tangential directions, and $v_{s+1}, \dots, v_k$ are along the normal directions at p in $f(\Sigma)$. Moreover, we can choose sections $\check{v}_{s+1}, \tilde{v}_{s+2}, \dots, \tilde{v}_k \in \Gamma(\Sigma, f^*(T_M))$ which equal $v_{s+1}, \dots, v_k$ at p respectively. By multiplying $\check{v}_{s+1}$ with a sequence of functions on Σ approaching the delta function at p, we obtain sections $(\check{v}_{s+1})_\varepsilon$ which approach $\delta(p)v_{s+1}$ as $\varepsilon \to 0$ where $\delta(p)$ is Dirac delta function. Therefore,

$$\eta(p)(v_1, v_2, \dots, v_k) = \lim_{\varepsilon \to 0} \left(\int_\Sigma ev^* \eta \right) \left((\check{v}_{s+1})_\varepsilon, \tilde{v}_{s+2}, \dots, \tilde{v}_k \right).$$

From the given condition $\int_\Sigma ev^* \eta = 0$, we conclude that

$$\eta(p)(v_1, v_2, \dots, v_k) = 0.$$

Hence the lemma is proved. □

As a corollary of the above lemma and discussions, we have the following result.

Theorem 10. *Suppose (M, g) is a Riemannian manifold with a differential form ϕ of degree $r+1$. Then ϕ is a closed r-fold VCP form on M if and only if $\boldsymbol{\omega}^{\mathcal{K}}$ is an almost Kähler structure on $\mathcal{K}_\Sigma M$, i.e. a closed 1-fold VCP form.*

Remark. In general the transgression of any closed form ϕ always gives a closed form on the knot space $\mathcal{K}_\Sigma M$ of degree $r+1-s$ where $\dim \Sigma = s$. However this can never be a VCP form unless $s = r-1$. To see this, we can use the above localization method and the fact that there is no VCP form of degree greater than two on any vector space with sufficiently large dimension.

Remark. Given any ϕ-preserving vector field $v \in Vect(M, \phi)$, it induces a vector field V of $\mathcal{K}_\Sigma M$ preserving $\boldsymbol{\omega}^{\mathcal{K}}$. Furthermore, if v is ϕ-Hamiltonian, that is

$$\iota_v \phi = d\eta,$$

for some $\eta \in \Omega^r(M)$, then we can define a function on the knot space,

$$F_\eta : \mathcal{K}_\Sigma M \to \mathbb{R},$$
$$F_\eta(f) = \int_\Sigma f^*\eta.$$

It is easy to see that F_η is a Hamiltonian function on the symplectic manifold $\mathcal{K}_\Sigma M$ whose Hamiltonian vector field equals V.

3.2. Holomorphic curves and Lagrangians in knot spaces

In this subsection, we are going to show that Lagrangian submanifolds in the knot space $\mathcal{K}_\Sigma M$ of M correspond to branes in M. Furthermore, holomorphic disks in the knot space $\mathcal{K}_\Sigma M$ of M correspond to instantons in M in a certain restricted sense.

Because $\mathcal{K}_\Sigma M$ is infinite dimensional, a Lagrangian submanifold is defined as a subspace in $\mathcal{K}_\Sigma M$ where the restriction of $\boldsymbol{\omega}^{\mathcal{K}}$ vanishes and with the property that any vector field $\boldsymbol{\omega}^{\mathcal{K}}$-orthogonal to L is a tangent vector field along L, see [13] and [3]. In fact, it is easy to see that the latter condition for a submanifold L to be Lagrangian is equivalent to the statement that $\boldsymbol{\omega}^{\mathcal{K}}$ will not vanish on any bigger space containing L. Some people refer to this condition as the maximally self $\omega^{\mathcal{K}}$-perpendicular condition.

Theorem 11. *Suppose that M is an n-dimensional manifold with a closed VCP form ϕ of degree $r+1$ and C is a submanifold in M. Then $\mathcal{K}_\Sigma C$ is a Lagrangian submanifold in $\mathcal{K}_\Sigma M$ if and only if C is a brane in M.*

Proof. ($\Rightarrow$) We suppose that $\mathcal{K}_\Sigma C$ is a Lagrangian in $\mathcal{K}_\Sigma M$ and we want to show that ϕ vanishes along C and $\dim C = (n+r-1)/2$. For any fixed $[f] \in \mathcal{K}_\Sigma C$, we have

$$0 = \boldsymbol{\omega}^{\mathcal{K}}_{[f]}(u, v) = \int_\Sigma \iota_{u\wedge v}(ev^*\phi)$$

where $u, v \in \Gamma(\Sigma, N_{\Sigma/C})$. By applying the localization method as in Lemma 9 along C, we obtain the following: for any x in Σ,

$$\phi\big(u(x), v(x), e_1, \ldots, e_{r-1}\big) = 0,$$

where $e_1, \ldots, e_{r-1}$ are any oriented orthonormal vectors of $T_x\Sigma$. By varying u, v, f to cover T_C, one can show $\phi|_C = 0$.

Moreover, $\mathcal{K}_\Sigma C$ being $\boldsymbol{\omega}^{\mathcal{K}}$-perpendicular in $\mathcal{K}_\Sigma M$ implies that C has the biggest possible dimension with $\phi|_C = 0$. As explained in the remark following the definition of branes, this gives $\dim C = (n+r-1)/2$.

($\Leftarrow$) We assume that C is any brane in M. The condition $\phi|_C = 0$ implies that $\boldsymbol{\omega}^{\mathcal{K}}|_{\mathcal{K}_\Sigma C} = 0$. Recall that the tangent space at any point $[f] \in \mathcal{K}_\Sigma C$ is $\Gamma(\Sigma, N_{\Sigma/C})$. Suppose there is a section v in $\Gamma(\Sigma, N_{\Sigma/M})$ but not in $\Gamma(\Sigma, N_{\Sigma/C})$ such that it is $\boldsymbol{\omega}^{\mathcal{K}}$-perpendicular to $\Gamma(\Sigma, N_{\Sigma/C})$. By the localization arguments as in Lemma 9, given any point $x \in f(\Sigma) \subset C$, ϕ vanishes on the linear space spanned by $v(x)$ and T_xC. This contradicts the fact that C is a brane. Hence, the theorem is proved. □

Remark. A ϕ-Hamiltonian deformation of a brane C in M gives a Hamiltonian deformation of the corresponding Lagrangian submanifold $\mathcal{K}_\Sigma C$ in the symplectic manifold $\mathcal{K}_\Sigma M$. More precisely, if v is a normal vector field to C satisfying

$$\iota_v \phi = d\eta,$$

for some $\eta \in \Omega^{r-1}(C)$, then the transgression of η defines a function on $\mathcal{K}_\Sigma C$ which generates a Hamiltonian deformation of $\mathcal{K}_\Sigma C$. For the converse to hold true, we need the Hamiltonian function H on $\mathcal{K}_\Sigma C$ to be local in the sense that $H(f) = \int_\Sigma f^*\eta$ for some differential form η on C.

Next we discuss holomorphic disks, i.e. instantons, in $\mathcal{K}_\Sigma M$. We consider a 2-dimensional disk D in $Map(\Sigma, M)$ such that for each tangent vector $v \in T_{[f]}D$, the corresponding vector field in $\Gamma(\Sigma, f^*T_M)$ is normal to Σ. We call such a disk D a *normal disk*. For simplicity we assume that the $(r+1)$-dimensional submanifold

$$A = \bigcup_{f \in D} f(\Sigma) \subset M,$$

is an embedding. This is always the case if D is small enough. Notice that A is diffeomorphic to $D \times \Sigma$. We will denote the corresponding disk in $\mathcal{K}_\Sigma M$ as $\hat{D} =: \pi(D)$. We remark that the principal fibration

$$Diff(\Sigma) \to Map(\Sigma, M) \xrightarrow{\pi} \mathcal{K}_\Sigma M$$

has a canonical connection (see [3]) and D being a normal disk is equivalent to it being an integral submanifold for the horizontal distribution of this connection.

In the following theorem, we describe the relationship between a disk $\hat{D}$ in $\mathcal{K}_\Sigma M$ given above and the corresponding $(r+1)$-dimensional subspace A in M.

Theorem 12. *Suppose that M is a manifold with a closed r-fold VCP form ϕ and $\mathcal{K}_\Sigma M$ is its knot space as before. For a normal disk D in $Map(\Sigma, M)$, $\hat{D} =: \pi(D)$ is a $J^{\mathcal{K}}$-holomorphic disk in $\mathcal{K}_\Sigma M$, i.e. calibrated by $\omega^{\mathcal{K}}$, if and only if A is an instanton in M and $A \to D$ is a Riemannian submersion.*

We call such an A a *tight instanton.*

Proof. For a fixed $[f] \in \hat{D}$, we consider $\nu, \mu \in T_{[f]}(\hat{D}) \subset \Gamma(\Sigma, N_{\Sigma/A})$. Since ϕ is a calibrating form, we have,

$$\phi(\nu, \mu, e_1, \ldots, e_{r-1}) \leqslant Vol_A(\nu, \mu, e_1, \ldots, e_{r-1}) = |\nu \wedge \mu|$$

where $e_1, \ldots, e_{r-1}$ is any orthonormal frame on $f(\Sigma)$. In particular we have

$$\int_{f(\Sigma)} \iota_{\nu\wedge\mu}(ev^*\phi) \leqslant \int_{f(\Sigma)} |\nu \wedge \mu| vol_\Sigma,$$

and the equality sign holds for every $[f]\in\hat{D}$ if and only if A is an instanton in M. We will simply denote $\int_{f(\Sigma)}$ by $\int_\Sigma$. Notice that the symplectic form on $\mathcal{K}_\Sigma M$ is given by,

$$\boldsymbol{\omega}^{\mathcal{K}}_{[f]}(\nu,\mu)=\int_\Sigma \iota_{\nu\wedge\mu}(ev^*\phi).$$

Since $\boldsymbol{\omega}^{\mathcal{K}}$ is a 1-fold VCP form on $\mathcal{K}_\Sigma M$, we have

$$\boldsymbol{\omega}^{\mathcal{K}}_{[f]}(\nu,\mu)\leqslant\left(|\nu|^2_{\mathcal{K}}|\mu|^2_{\mathcal{K}}-\langle\nu,\mu\rangle^2_{\mathcal{K}}\right)^{1/2}$$

where $\langle\nu,\mu\rangle_{\mathcal{K}}=: g_{\mathcal{K}}(\nu,\mu)$ and $|a|^2_{\mathcal{K}}=: g_{\mathcal{K}}(a,a)$. Furthermore the equality sign holds when $\hat{D}$ is a $J^{\mathcal{K}}$-holomorphic disk in $\mathcal{K}_\Sigma M$.

($\Rightarrow$) We suppose that $\hat{D}$ is a $J^{\mathcal{K}}$-holomorphic disk in $\mathcal{K}_\Sigma M$. From the above discussions, we have

$$\begin{aligned}\int_\Sigma|\nu\wedge\mu|\geqslant\int_\Sigma\iota_{\nu\wedge\mu}(ev^*\phi)&=\left(|\nu|^2_{\mathcal{K}}|\mu|^2_{\mathcal{K}}-\langle\nu,\mu\rangle^2_{\mathcal{K}}\right)^{1/2}\\&=\left(\int_\Sigma|\nu|^2\int_\Sigma|\mu|^2-\left(\int_\Sigma\langle\nu,\mu\rangle\right)^2\right)^{1/2}.\end{aligned}$$

Combining with the Hölder inequality,

$$\left(\int_\Sigma|\nu\wedge\mu|\right)^2+\left(\int_\Sigma\langle\nu,\mu\rangle\right)^2\leqslant\int_\Sigma|\nu|^2\int_\Sigma|\mu|^2,$$

we obtain

$$\text{(i)}\quad\int_\Sigma\iota_{\nu\wedge\mu}(ev^*\phi)=\int_\Sigma|\nu\wedge\mu|vol_\Sigma\quad\text{and}$$

$$\text{(ii)}\quad\left(\int_\Sigma|\nu\wedge\mu|\right)^2+\left(\int_\Sigma\langle\nu,\mu\rangle\right)^2=\int_\Sigma|\nu|^2\int_\Sigma|\mu|^2.$$

Condition (i) says that A is an instanton in M. Condition (ii) implies that given any $[f]$, there exist constants C_1 and C_2 such that for any $x\in\Sigma$,

$$\left|\nu(x)\right|=C_1\left|\mu(x)\right|,\qquad\angle\left(\nu(x),\mu(x)\right)=C_2.$$

This implies that $A\to D$ is a *Riemannian submersion.*

($\Leftarrow$) We notice that A being an instanton in M implies that

$$\int_{\Sigma} |\nu \wedge \mu| = \int_{\Sigma} \iota_{\nu\wedge\mu}(ev^*\phi) = \boldsymbol{\omega}^{\mathcal{K}}_{[f]}(\nu, \mu) \leqslant \left(|\nu|^2_{\mathcal{K}}|\mu|^2_{\mathcal{K}} - \langle \nu, \mu\rangle^2_{\mathcal{K}}\right)^{1/2}$$
$$= \left(\int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2 - \left(\int_{\Sigma} \langle \nu, \mu\rangle\right)^2\right)^{1/2}.$$

Recall that the Riemannian submersion condition implies the equality

$$\left(\int_{\Sigma} |\nu \wedge \mu|\right)^2 + \left(\int_{\Sigma} \langle \nu, \mu\rangle\right)^2 = \int_{\Sigma} |\nu|^2 \int_{\Sigma} |\mu|^2.$$

So, the above inequality turns into an equality, giving

$$\boldsymbol{\omega}^{\mathcal{K}}_{[f]}(\nu, \mu) = \int_{\Sigma} \iota_{\nu\wedge\mu}(ev^*\phi) = \left(|\nu|^2_{\mathcal{K}}|\mu|^2_{\mathcal{K}} - \langle \nu, \mu\rangle^2_{\mathcal{K}}\right)^{1/2},$$

i.e. $\hat{D}$ is $J^{\mathcal{K}}$ holomorphic in $\mathcal{K}_{\Sigma}M$. □

4. Geometry of complex vector cross product

In this section we review the *complex* version of the VCP geometry from [10]. The definition of instantons depends on a phase θ and there are two types of branes corresponding to the Dirichlet and Neumann boundary conditions for instantons. In the case of M being a Calabi–Yau manifold, D-branes (respectively N-branes) are special Lagrangian submanifolds (respectively complex hypersurfaces) in M. We will see in the next two sections that all these geometric structures will have a beautiful interpretation in terms of the hyperkähler geometry of the isotropic knot space of M.

Definition 13. On a Kähler manifold (M, g, J) of complex dimension n, an r-fold *complex vector cross product* (abbreviated as $\mathbb{C}$-VCP) is a holomorphic form ϕ of degree $r+1$ satisfying

$$\left|\iota_{e_1\wedge e_2\wedge\cdots\wedge e_r}(\phi)\right| = 2^{(r+1)/2},$$

for any orthonormal tangent vectors $e_1, \ldots, e_r \in T^{1,0}_x M$ and for any x in M.

A $\mathbb{C}$-VCP is called closed (respectively parallel) if ϕ is closed (respectively parallel with respect to the Levi-Civita connection) form.

Remark. One can use the above to define a $\mathbb{C}$-VCP on Hermitian manifolds and this is true for the following definitions of instantons and branes, too.

There are only two classes of $\mathbb{C}$-VCPs corresponding to the holomorphic volume form ($r = n - 1$) and the holomorphic symplectic form ($r = 1$). The corresponding closed Kähler manifolds M are Calabi–Yau manifolds and hyperkähler manifolds. In particular, the analogs of G_2- and $Spin(7)$-manifolds do not exist.

For the VCP, instantons are submanifolds calibrated by the VCP form. In the complex case, the $\mathbb{C}$-VCP form is a complex differential form and we have a circle family of calibrated forms $\mathrm{Re}(e^{i\theta}\phi)$ parametrized by the phase θ.

Definition 14. On a Kähler manifold M with an r-fold $\mathbb{C}$-VCP ϕ, an $(r+1)$-dimensional submanifold A is called an *instanton* with phase $\theta \in \mathbb{R}$ if it is calibrated by $\mathrm{Re}(e^{i\theta}\phi)$, i.e.

$$\mathrm{Re}\big(e^{i\theta}\phi\big)\big|_A = vol_A.$$

As explained in [10] we can impose either the Dirichlet or the Neumann boundary conditions for instantons. Thus we have the following definitions.

Definition 15. On a Kähler manifold (M,ω) of complex dimension n with an r-fold $\mathbb{C}$-VCP $\phi \in \Omega^{r+1,0}(M)$,

(i) a submanifold C is called a *Neumann brane* (abbreviated as N-brane) if $\dim(C) = n + r - 1$ and

$$\phi|_C = 0,$$

(ii) an n-dimensional submanifold C is called a *Dirichlet brane* (abbreviated as D-brane) with phase $\theta \in \mathbb{R}$ if

$$\omega|_C = 0, \quad \mathrm{Re}\big(e^{i\theta}\phi\big)\big|_C = 0.$$

Instantons and branes in CY manifold: $(n-1)$-fold $\mathbb{C}$-VCP. A Kähler manifold of complex dimension n with an $(n-1)$-fold $\mathbb{C}$-VCP ϕ is a Calabi–Yau n-fold M. An instanton in M is a special Lagrangian submanifold with phase θ since it is calibrated by $\mathrm{Re}(e^{i\theta}\phi)$ and a D-brane is a special Lagrangian submanifold with phase $\theta - \pi/2$, because the definition of D-brane implies it is calibrated by $\mathrm{Re}(e^{i(\theta-\pi/2)}\phi) = \mathrm{Im}(e^{i\theta}\phi)$. An N-brane in M is a complex hypersurface.

Instantons and branes in hyperkähler manifold: 1-fold $\mathbb{C}$-VCP. A Kähler manifold of complex dimension $2n$ with a Kähler form ω_J and a 1-fold $\mathbb{C}$-VCP $\phi := \omega_I - \sqrt{-1}\omega_K$ is a hyperkähler manifold M. Here, J, I and K are the complex structure corresponding to Kähler structures ω_J, ω_I and ω_K, respectively. Note that by putting $e^{i\theta}\phi := \omega_{J_\theta} - \sqrt{-1}\omega_{J_{\theta+\pi/2}}$ in place of ϕ, one can obtain another Kähler structure $\mathrm{Re}(e^{i\theta}\phi)$ with a complex structure $J_\theta := \cos\theta I + \sin\theta K$.

An instanton in M is a J_θ-holomorphic curve since it is calibrated by $\mathrm{Re}(e^{i\theta}\phi)$; namely, preserved by J_θ. A D-brane is a real $2n$-dimensional submanifold where ω and $\mathrm{Re}(e^{i\theta}\phi)$ vanish, which implies it is preserved by $J_{\theta+\pi/2}$ the almost complex structure corresponding to $-\mathrm{Im}(e^{i\theta}\phi)$, i.e. $\omega_{J_{\theta+\pi/2}}$. Therefore, a D-brane is a $J_{\theta+\pi/2}$-complex Lagrangian. An N-brane in M is a real $2n$-dimensional submanifold where $\phi = \omega_I - \sqrt{-1}\omega_K$ vanishes, i.e. calibrated by ω_J. So it is a J-complex Lagrangian which is a complex submanifold preserved by a complex structure J with complex dimension n.

The above classification of instantons, N- and D-branes in Kähler manifolds with $\mathbb{C}$-VCPs is summarized in the following table.

Manifolds w/$\mathbb{C}$-VCP	Calabi–Yau manifolds	Hyperkähler manifolds
Instantons	Special Lagrangian$_{\theta=0}$	I-holomorphic curves
N-branes	Complex hypersurfaces	J-complex Lagrangians
D-branes	Special Lagrangian$_{\theta=-\pi/2}$	K-complex Lagrangians

5. Isotropic knot spaces of CY manifolds

Recall that any volume form on a Riemannian manifold M of dimension n determines a closed 1-fold VCP on the knot space $\mathcal{K}_\Sigma M = \mathit{Map}(\Sigma, M)/\mathit{Diff}(\Sigma)$ where Σ is any closed manifold of dimension $n-2$. It is natural to guess that the holomorphic volume form on any Calabi–Yau n-fold M would determine a closed 1-fold $\mathbb{C}$-VCP on the symplectic quotient $\mathit{Map}(\Sigma, M)//\mathit{Diff}(\Sigma)$. In particular we obtain a hyperkähler knot space.

There are some complications to this naive approach. First we need to choose a background volume form σ on Σ to construct a symplectic structure in $\mathit{Map}(\Sigma, M)$ which is only invariant under $\mathit{Diff}(\Sigma, \sigma)$, the group of volume preserving diffeomorphism of Σ [4,9]. Therefore we can only construct the symplectic quotient $\mathit{Map}(\Sigma, M)//\mathit{Diff}(\Sigma, \sigma) = \mu^{-1}(0)/\mathit{Diff}(\Sigma, \sigma)$. Second, this larger space is not hyperkähler because the holomorphic two form $\int_\Sigma \Omega_M$ degenerates. We will show that a further symplectic reduction will produce a Hermitian integrable complex manifold $\hat{\mathcal{K}}_\Sigma M$ with a 1-fold $\mathbb{C}$-VCP, in particular a holomorphic symplectic structure. This may not be hyperkähler because although the Hermitian complex structure is integrable on $\hat{\mathcal{K}}_\Sigma M$, its Kähler form may not be closed. $\hat{\mathcal{K}}_\Sigma M$ should be regarded as a modified construction for the non-existing space $\mathit{Map}(\Sigma, M)//\mathit{Diff}(\Sigma)$. We call this an *isotropic knot space* of M for reasons which will become clear later.

Furthermore we will relate instantons, N- and D-branes in a Calabi–Yau manifold M to holomorphic curves and complex Lagrangian submanifolds in the holomorphic symplectic manifold $\hat{\mathcal{K}}_\Sigma M$. These constructions are of particular interest when M is a Calabi–Yau threefold (see Remark 19 for details).

5.1. Holomorphic symplectic structures on isotropic knot spaces

Let M be a Calabi–Yau n-fold with a holomorphic volume form Ω_M, i.e. a closed $(n-1)$-fold $\mathbb{C}$-VCP. To obtain a 1-fold $\mathbb{C}$-VCP on a certain knot space by transgression, we first fix an $(n-2)$-dimensional manifold Σ without boundary and we let $\mathit{Map}(\Sigma, M)$ be the space of embedding from Σ to M as before. For simplicity we assume that the first Betti number of Σ is zero, $b_1(\Sigma) = 0$.

If we fix a background volume form σ on Σ, then the Kähler form ω on M induces a natural symplectic form on $\mathit{Map}(\Sigma, M)$ as follows: for any tangent vectors X and Y at a point f in $\mathit{Map}(\Sigma, M)$, we define

$$\omega_{\mathit{Map}}(X, Y) = \int_\Sigma \omega(X, Y)\sigma$$

where $X, Y \in \Gamma(\Sigma, f^*(TM))$. Note that this symplectic structure on $\mathit{Map}(\Sigma, M)$ is not invariant under general diffeomorphisms of Σ. Instead, it is preserved by the natural action of $\mathit{Diff}(\Sigma, \sigma)$, the group of volume preserving diffeomorphism on (Σ, σ).

As studied by Donaldson in [4] and Hitchin in [9], the action of $\mathit{Diff}(\Sigma,\sigma)$ is Hamiltonian on the components of $\mathit{Map}_0(\Sigma,M)$ consisting of those f's satisfying

$$f^*\big([\omega]\big)=0\in H^2(\Sigma,\mathbb{R}).$$

To see this, we first recall that the dual of the Lie algebra of $\mathit{Diff}(\Sigma,\sigma)$ can be naturally identified with $\Omega^1(\Sigma)\big/d\Omega^0(\Sigma)$. The moment map of the $\mathit{Diff}(\Sigma,\sigma)$ action is given by

$$\mu:\mathit{Map}_0(\Sigma,M)\to\Omega^1(\Sigma)\big/d\Omega^0(\Sigma),$$
$$f\mapsto\mu(f)=[\alpha]$$

for any one form $\alpha\in\Omega^1(\Sigma)$ satisfying $d\alpha=f^*\omega$.

In particular, $\mu^{-1}(0)$ consists of isotropic embeddings of Σ in M. The reason is $f\in\mu^{-1}(0)$ if and only if $\alpha=dh$ for some function h on Σ. Thus

$$f^*\omega=d\alpha=d(dh)=0.$$

Namely $f:\Sigma\to M$ is an isotropic embedding.

Therefore, the symplectic quotient

$$\mathit{Map}(\Sigma,M)//\mathit{Diff}(\Sigma,\sigma)=\mu^{-1}(0)/\mathit{Diff}(\Sigma,\sigma)$$

is almost the same as the moduli space of isotropic submanifolds in M, which is $\mu^{-1}(0)/\mathit{Diff}(\Sigma)$. Observe that the definition of the map μ is independent of the choice of σ and therefore $\mu^{-1}(0)$ is preserved by the action of $\mathit{Diff}(\Sigma)$.

Remark. For the moment map μ to be well defined, the condition $b_1(\Sigma)=0$ is necessary. However, even in the case of $b_1(\Sigma)\neq 0$, which always happens when M is a Calabi–Yau threefold, there is a modification of the symplectic quotient construction and we can obtain a symplectic manifold which is a torus bundle over the moduli space of isotropic submanifolds in M with fiber dimension $b_1(\Sigma)$ [4,9].

In order to construct a holomorphic symplectic manifold, we first consider a complex closed 2-form Ω_{Map} on $\mathit{Map}(\Sigma,M)$ induced from the holomorphic volume form Ω_M on M by transgression:

$$\Omega_{\mathit{Map}}=\int_\Sigma ev^*\Omega_M=\int_\Sigma ev^*\,\mathrm{Re}\;\Omega_M+\sqrt{-1}\int_\Sigma ev^*\,\mathrm{Im}\,\Omega_M=\omega_I-\sqrt{-1}\omega_K,$$

where $\omega_I=\int_\Sigma ev^*(\mathrm{Re}\,\Omega_M)$ and $\omega_K=-\int_\Sigma ev^*(\mathrm{Im}\,\Omega_M)$. We also define endomorphisms I and K on $T_f(\mu^{-1}(0))$ as follows:

$$\omega_I(A,B)=g(IA,B),\qquad \omega_K(A,B)=g(KA,B)$$

where $A,B\in T_f(\mu^{-1}(0))$ and g is the natural L^2-metric on $\mu^{-1}(0)$. Both ω_I and ω_K are degenerate two forms. We will define the isotropic knot space as the symplectic reduction of $\mu^{-1}(0)$ with respect to either ω_I or ω_K and show that it has a natural holomorphic symplectic structure.

To do that, we define a distribution $\mathcal{D}$ on $\mu^{-1}(0)$ by

$$\mathcal{D}_f = \{X \in T_f(\mu^{-1}(0)) \subset \Gamma(\Sigma, f^*(T_M)) \mid \iota_X \omega_{I,f} = 0\} \subset T_f(\mu^{-1}(0)),$$

for any $f \in \mu^{-1}(0)$.

Lemma 16. *For any $f \in \mu^{-1}(0) \subset Map(\Sigma, M)$, we have,*

$$\mathcal{D}_f = \Gamma(\Sigma, T_\Sigma + JT_\Sigma).$$

Proof. First, it is easy to see that $\mathcal{D}_f \supset \Gamma(\Sigma, T_\Sigma)$ since for any X in $\Gamma(\Sigma, T_\Sigma)$ and any Y in $\Gamma(\Sigma, T_M)$, $\iota_{X \wedge Y} \operatorname{Re} \Omega_M$ cannot be a top degree form on Σ. By similar reasons, $\mathcal{D}_f$ is preserved by the Hermitian complex structure J on M. Because f is isotropic, we have $\mathcal{D}_f \supset \Gamma(\Sigma, T_\Sigma + JT_\Sigma)$. Now, as in Lemma 9, we consider the localization of

$$0 = \iota_X \omega_{I,f} = \int_\Sigma \iota_X ev^* \operatorname{Re} \Omega_M$$

at x in Σ by varying Σ, and we obtain

$$0 = \iota_{X(x) \wedge E_1 \wedge \cdots \wedge E_{n-2}} \operatorname{Re} \Omega_M$$

where $E_1, \ldots, E_{n-2}$ is an orthonormal oriented basis of $(T_\Sigma)_x$. This implies that

$$\mathcal{D}_f = \Gamma(\Sigma, T_\Sigma + JT_\Sigma),$$

because for any X in $T_M \backslash (T_\Sigma + JT_\Sigma)$, there is an x in Σ such that

$$\iota_{X(x) \wedge E_1 \wedge \cdots \wedge E_{n-2}} \operatorname{Re} \Omega_M \neq 0.$$

Note that the same construction applied to ω_K, instead of ω_I, will give another distribution which is identical to $\mathcal{D}_f$ because f is isotropic. □

Observe that, for each $f \in \mu^{-1}(0)$, the rank of the subbundle $T_\Sigma + JT_\Sigma$ in f^*T_M is $2(n-2)$, i.e. constant rank on $\mu^{-1}(0)$. That is, ω_I is a closed two form of constant rank on $\mu^{-1}(0)$. From the standard theory of symplectic reduction, the distribution $\mathcal{D}$ is integrable and the space of leaves has a natural symplectic form descended from ω_I. We call this space as the *isotropic knot space* of M and we denote it by

$$\hat{\mathcal{K}}_\Sigma M = \mu^{-1}(0)/\langle \mathcal{D} \rangle$$

where $\langle \mathcal{D} \rangle$ are equivalence relations generated by the distribution $\mathcal{D}$.

Remark. The isotropic knot space $\hat{\mathcal{K}}_\Sigma M$ is a quotient space of $\mu^{-1}(0)/Diff(\Sigma)$, the space of isotropic submanifolds in M. If $f : \Sigma \to M$ parametrizes an isotropic submanifold in M, then deforming Σ along T_Σ directions simply changes the parametrization of the submanifold $f(\Sigma)$; namely, the equivalence class in $\mu^{-1}(0)/Diff(\Sigma)$ remains unchanged. However, if we deform Σ

along $T_\Sigma \otimes \mathbb{C}$ directions, then the equivalence classes in $\hat{\mathcal{K}}_\Sigma M$ remain constant. Notice that the isotropic condition on Σ implies that $T_\Sigma \otimes \mathbb{C} \cong T_\Sigma \oplus J(T_\Sigma) \subset f^*T_M$. Therefore, roughly speaking, $\hat{\mathcal{K}}_\Sigma M$ is the space of isotropic submanifolds in M divided by $\mathit{Diff}(\Sigma) \otimes \mathbb{C}$. In particular, the tangent space of $\hat{\mathcal{K}}_\Sigma M$ is given by

$$T_{[f]}(\hat{\mathcal{K}}_\Sigma M) = \Gamma\bigl(\Sigma, f^*(TM)/(T_\Sigma + JT_\Sigma)\bigr)$$

for any $[f]$ in $\hat{\mathcal{K}}_\Sigma M$. Note that a leaf is preserved by the induced almost Hermitian structure on $\mu^{-1}(0)$ from the Hermitian complex structure J on M, because a tangent vector on it is a section to the subbundle $T_\Sigma + JT_\Sigma$. However $\hat{\mathcal{K}}_\Sigma M$ may not be symplectic, because the symplectic form ω_{Map} on $\mu^{-1}(0)$ descends to a 2-form $\omega^{\mathcal{K}}$ on $\hat{\mathcal{K}}_\Sigma M$ which may not be closed.

We will show that $\hat{\mathcal{K}}_\Sigma M$ admits three almost complex structures $I^{\mathcal{K}}$, $K^{\mathcal{K}}$ and $J^{\mathcal{K}}$ satisfying the Hamilton relation

$$\bigl(I^{\mathcal{K}}\bigr)^2 = \bigl(J^{\mathcal{K}}\bigr)^2 = \bigl(K^{\mathcal{K}}\bigr)^2 = I^{\mathcal{K}} J^{\mathcal{K}} K^{\mathcal{K}} = -id.$$

Furthermore the associated Kähler forms $\omega_I^{\mathcal{K}}$ and $\omega_K^{\mathcal{K}}$ are closed. In [8], Hitchin showed that the existence of such structures on any finite dimensional Riemannian manifold implies that its almost complex structure $J^{\mathcal{K}}$ is *integrable*. Namely we obtain a 1-fold $\mathbb{C}$-VCP on a Hermitian complex manifold. If, in addition, $\omega_J^{\mathcal{K}}$ is closed then $I^{\mathcal{K}}$, $J^{\mathcal{K}}$ and $K^{\mathcal{K}}$ are all integrable complex structures and we have a hyperkähler manifold.

We begin with the following lemma which holds true both on $\mu^{-1}(0)$ and on $\hat{\mathcal{K}}_\Sigma M$.

Lemma 17. *On $\mu^{-1}(0) \subset Map(\Sigma, M)$, the endomorphisms I, J and K satisfy the following relations,*

$$IJ = -JI \quad \text{and} \quad KJ = -JK,$$
$$I = -KJ \quad \text{and} \quad K = IJ.$$

Proof. The formula $IJ = -JI$ can be restated as follows: for any $f \in \mu^{-1}(0)$ and for any tangent vectors of $Map(\Sigma, M)$ at f, $A, B \in \Gamma(\Sigma, f^*(T_M))$ we have

$$g_f\bigl(I_f J_f(A), B\bigr) = g_f\bigl(-J_f I_f(A), B\bigr).$$

For simplicity, we ignore the subscript f. Since $J^2 = -id$, we have

$$\begin{aligned}
&g\bigl(IJ(A), B\bigr) - g\bigl(-JI(A), B\bigr)\\
&\quad = g\bigl(IJ(A), B\bigr) - g\bigl(I(A), JB\bigr)\\
&\quad = \omega_I(JA, B) - \omega_I(A, JB)\\
&\quad = \int_\Sigma \iota_{JA\wedge B} \operatorname{Re} \Omega_M - \int_\Sigma \iota_{A\wedge JB} \operatorname{Re} \Omega_M\\
&\quad = \int_\Sigma \iota_{(JA\wedge B - A\wedge JB)} \frac{1}{2}(\Omega_M + \bar{\Omega}_M) = 0.
\end{aligned}$$

The last equality follows from the fact that Ω_M and $\bar{\Omega}_M$ vanish when each is contracted by an element in $\wedge^{1,1}T_M$. By replacing $\operatorname{Re}\Omega_M$ with $\operatorname{Im}\Omega_M$ in the above calculations, we also have $KJ = -JK$.

To prove the others formulas, we consider

$$\begin{aligned} & g(IA, B) - g(-KJA, B) \\ &= \omega_I(A, B) + \omega_K(JA, B) \\ &= \int_\Sigma \iota_{A\wedge B} \operatorname{Re}\Omega_M + \int_\Sigma \iota_{JA\wedge B}(-\operatorname{Im}\Omega_M) \\ &= \int_\Sigma \iota_{A\wedge B} \frac{1}{2}(\Omega_M + \bar{\Omega}_M) - \int_\Sigma \iota_{JA\wedge B} \frac{-\sqrt{-1}}{2}(\Omega_M - \bar{\Omega}_M) \\ &= \frac{1}{2}\int_\Sigma \iota_{(A+\sqrt{-1}JA)\wedge B}\Omega_M + \frac{1}{2}\int_\Sigma \iota_{(A-\sqrt{-1}JA)\wedge B}\bar{\Omega}_M = 0 \end{aligned}$$

since $i_{(A+\sqrt{-1}JA)}\Omega_M = 0$ and taking complex conjugates. This implies that $g(IA, B) - g(-KJA, B) = 0$ and therefore $I = -KJ$. Finally, $J^2 = -Id$ and $I = -KJ$ imply that $K = IJ$. Hence, the lemma follows. □

Theorem 18. *Suppose that M is a Calabi–Yau n-fold M. For any $(n-2)$-dimensional closed manifold Σ with $b_1(\Sigma) = 0$, the isotropic knot space $\hat{\mathcal{K}}_\Sigma M$ is an infinite dimensional integrable complex manifold with a natural 1-fold $\mathbb{C}$-VCP structure, in particular a natural holomorphic symplectic structure.*

Proof. From the construction of $\hat{\mathcal{K}}_\Sigma M$, it has a Hermitian Kähler form $\omega^{\mathcal{K}}$ induced from that of M and a closed holomorphic symplectic form $\Omega^{\mathcal{K}}$ given by the transgression of the closed $(n-1)$-fold $\mathbb{C}$-VCP form Ω_M on M. As we have seen above, the induced holomorphic symplectic form is closed but the induced Hermitian Kähler form may not be closed. If their corresponding almost complex structures satisfy the Hamilton relation then this implies that $J^{\mathcal{K}}$ is integrable [8]. Namely $\hat{\mathcal{K}}_\Sigma M$ is a Hermitian integrable complex manifold with a 1-fold $\mathbb{C}$-VCP structure. In order to verify the Hamilton relation,

$$\left(I^{\mathcal{K}}\right)^2 = \left(J^{\mathcal{K}}\right)^2 = \left(K^{\mathcal{K}}\right)^2 = I^{\mathcal{K}}J^{\mathcal{K}}K^{\mathcal{K}} = -id,$$

we only need to show that $(I^{\mathcal{K}})^2 = -Id$ and $(K^{\mathcal{K}})^2 = -Id$ because of the previous lemma. Namely, $I^{\mathcal{K}}$ and $K^{\mathcal{K}}$ are almost complex structures on $\hat{\mathcal{K}}_\Sigma M$.

We consider a fixed $[f]$ in $\hat{\mathcal{K}}_\Sigma M$ and by the localization method as in the proof of Lemma 9, we can reduce the identities to the tangent space of a point x in Σ. The transgression

$$\int_\Sigma ev^*\Omega_M = \omega_I^{\mathcal{K}} - \sqrt{-1}\omega_K^{\mathcal{K}}$$

is descended to

$$\iota_{E_1\wedge\cdots\wedge E_{n-2}}\Omega_M=\omega_I^{\mathcal{K}}-\sqrt{-1}\omega_K^{\mathcal{K}}$$

where $E_1,\ldots,E_{n-2}$ is an orthonormal oriented basis $(T_\Sigma)_x$. Since f is isotropic, the complexified vectors $E_i-\sqrt{-1}JE_i$ of E_i can be defined over $(T_\Sigma+JT_\Sigma)_x$. Therefore the above equality is equivalent to

$$\iota_{(E_1-\sqrt{-1}JE_1)/2\wedge\cdots\wedge(E_{n-2}-\sqrt{-1}JE_{n-2})/2}\Omega_M=\omega_I^{\mathcal{K}}-\sqrt{-1}\omega_K^{\mathcal{K}},$$

i.e. a 1-fold $\mathbb{C}$-VCP on $f^*(TM)/(T_\Sigma+JT_\Sigma)|_x$ which is $T_{[f],x}\hat{\mathcal{K}}_\Sigma M$. Since this 1-fold $\mathbb{C}$-VCP gives a hyperkähler structure on $T_{[f],x}\hat{\mathcal{K}}_\Sigma M$, $I^{\mathcal{K}}$ and $K^{\mathcal{K}}$ satisfy the Hamilton relation at x in Σ, that is,

$$\left(I^{\mathcal{K}}\right)^2=-Id\quad\text{and}\quad\left(K^{\mathcal{K}}\right)^2=-Id.$$

Therefore, we have $(I^{\mathcal{K}})^2=-Id$ and $(K^{\mathcal{K}})^2=-Id$. Hence, the result is proved. □

Remark 19. In String theory we need to compactify a 10-dimensional space–time on a Calabi–Yau threefold. When M is a Calabi–Yau threefold, then Σ is an 1-dimensional circle and therefore $b_1(\Sigma)$ is nonzero. In general, as discussed in [4] and [9], the symplectic quotient construction for $Map(\Sigma,M)//Diff(\Sigma,\sigma)$ can be modified to obtain a symplectic structure on a rank $b_1(\Sigma)$ torus bundle over the space of isotropic submanifolds in M. Roughly speaking this torus bundle is the space of isotropic submanifolds coupled with flat rank one line bundles (or gerbes) in M. In the Calabi–Yau threefold case, every circle Σ in M is automatically isotropic. Therefore $\hat{\mathcal{K}}_\Sigma M$ is the space of loops (or *string*) coupled with flat line bundles in M, up to deformations of strings along their complexified tangent directions. We wonder whether this infinite dimensional holomorphic symplectic manifold $\hat{\mathcal{K}}_\Sigma M$ has any natural physical interpretations.

6. Complex Lagrangians in isotropic knot spaces

In this section we relate the geometry of the $\mathbb{C}$-VCP of a Calabi–Yau manifold M to the holomorphic symplectic geometry of its isotropic knot space $\hat{\mathcal{K}}_\Sigma M$. For example, both N-branes (i.e. complex hypersurfaces) and D-branes (i.e. special Lagrangian submanifold with phase $-\pi/2$) in M correspond to complex Lagrangian submanifolds in $\hat{\mathcal{K}}_\Sigma M$, but for different almost complex structures in the twistor family. First, we discuss the correspondence for instantons.

In the following proposition, we use $e^{i\theta}\Omega_M$ instead of Ω_M to get a 1-fold $\mathbb{C}$-VCP on $\hat{\mathcal{K}}_\Sigma M$, and also $\mathrm{Re}(e^{i\theta}\Omega_M)$ gives another symplectic form $\omega_{I,\theta}^{\mathcal{K}}$ and corresponding Hermitian almost complex structure $J_\theta^{\mathcal{K}}$ on $\hat{\mathcal{K}}_\Sigma M$ defined as $J_\theta^{\mathcal{K}}=\cos\theta I^{\mathcal{K}}+\cos\theta K^{\mathcal{K}}$.

Proposition 20. *Suppose that M is a Calabi–Yau n-fold. Let D be a normal disk in $\mu^{-1}(0)$ with an n-dimensional submanifold A defined as $A=D\tilde{\times}\Sigma$, and assume $A\to D$ is a Riemannian submersion. We denote the reduction of D in $\hat{\mathcal{K}}_\Sigma M$ as $\hat{D}$.*

Then $\hat{D}$ is an instanton, i.e. a $J_\theta^{\mathcal{K}}$-holomorphic curve in $\hat{\mathcal{K}}_\Sigma M$ if and only if A is an instanton, i.e. a special Lagrangian with phase θ in M.

Proof. In Proposition 12, the VCP form ϕ plays the role of calibrating form rather than that of a VCP form. So by replacing ϕ by $\mathrm{Re}(e^{i\theta}\Omega_M)$ in Proposition 12, this theorem can be proved in essentially the same manner. But, since Proposition 12 is given for the parametrized knot space and this theorem is given by a reduction, we need to check $\hat{D}$ is a disk in $\hat{\mathcal{K}}_\Sigma M$. Let f be the center of disk D. Since D is a normal disk, a tangent vector v at f along $D \subset \mu^{-1}(0)$ is in $\Gamma(\Sigma, N_{\Sigma/M})$. And since f is isotropic, Jv is also in $\Gamma(\Sigma, N_{\Sigma/M})$ equivalently v is in $\Gamma(\Sigma, N_{J\Sigma/M})$. So $v \in \Gamma(\Sigma, f^*(TM)/(T_\Sigma + JT_\Sigma))$. This implies $T_f D$ can be identified with $T_{[f]}\hat{D}$, i.e. $\hat{D}$ is a disk in $\hat{\mathcal{K}}_\Sigma M$. □

Next we are going to relate N- and D-branes in Calabi–Yau manifolds M to complex Lagrangian submanifolds in $\hat{\mathcal{K}}_\Sigma M$.

Definition 21. For any submanifold C in a Calabi–Yau manifold M, we define a subspace $\hat{\mathcal{K}}_\Sigma C$ in $\hat{\mathcal{K}}_\Sigma M = \mu^{-1}(0)/\langle\mathcal{D}\rangle$ as follows: the equivalent relation $\langle\mathcal{D}\rangle$ on $\mu^{-1}(0)$ restricts to one on $Map(\Sigma, C) \cap \mu^{-1}(0)$ and we define

$$\hat{\mathcal{K}}_\Sigma C = \{Map(\Sigma, C) \cap \mu^{-1}(0)\}/\langle\mathcal{D}\rangle.$$

We want to show that a submanifold C in a Calabi–Yau manifold M is a complex hypersurface if $\hat{\mathcal{K}}_\Sigma C$ is a $J^{\mathcal{K}}$-complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$. What we will actually prove is that for any isotropic submanifold $f : \Sigma \hookrightarrow C$, its open neighborhood in C is a complex hypersurface in M. Therefore, if C is also a real analytic submanifold in M, then C is a complex hypersurface in M everywhere. We can also replace the analyticity assumption of C by the density of isotropic submanifolds in C. The same remark also applies to D-branes in M.

Theorem 22. *Let C be a connected analytic submanifold in a Calabi–Yau manifold M. Then the following two statements are equivalent*:

(i) *C is an N-brane (i.e. a complex hypersurface) in M*;
(ii) *$\hat{\mathcal{K}}_\Sigma C$ is a $J^{\mathcal{K}}$-complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$.*

Proof. (ii) ⇒ (i). Assume that $\hat{\mathcal{K}}_\Sigma C$ is a $J^{\mathcal{K}}$-complex Lagrangian in $\hat{\mathcal{K}}_\Sigma M$. First we need to show that the dimension of C is at least $2n - 2$ where n is the complex dimension of M.

Before proving this claim for the general case, let us discuss a simplified linear setting where some of the key arguments become more transparent. Suppose that M is a *linear* Calabi–Yau manifold, that is $M \cong \mathbb{C}^n$ with the standard Kähler structure and $\Omega_M = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n$. Let Σ be an $(n-2)$-dimensional isotropic linear subspace in M lying inside another linear subspace C in M. For simplicity we assume that Σ is the linear span of $x^1, x^2, \ldots, x^{n-2}$. Of course $M/\Sigma \oplus J\Sigma \cong \mathbb{C}^2$ is a linear holomorphic symplectic manifold with

$$\int_\Sigma \Omega_M = dz^{n-1} \wedge dz^n,$$

which is the standard holomorphic symplectic form on $\mathbb{C}^2$. Suppose that

$$C/(\Sigma \oplus J\Sigma) \cap C \subset M/\Sigma \oplus J\Sigma$$

is a complex Lagrangian subspace. Then, there is a vector in C perpendicular to both Σ and $J\Sigma$, say x^{n-1}. If we denote the linear span of $x^2, \dots, x^{n-2}, x^{n-1}$ as Σ', then Σ' is another $(n-2)$-dimensional isotropic linear subspace in M lying inside C. Furthermore, x^1 is a normal vector in C perpendicular to $\Sigma' \oplus J\Sigma'$. This implies that $y^1 = Jx^1$ also lies in C. This is because $C/(\Sigma' \oplus J\Sigma') \cap C \subset M/\Sigma' \oplus J\Sigma'$ being a complex Lagrangian subspace implies that it is invariant under the complex structure on $M/\Sigma' \oplus J\Sigma'$ induced by J on M. By the same reasoning, y^j also lies in C for $j = 1, 2, \dots, n-2$. That is C contains the linear span of $\{x^j, y^j\}_{j=1}^{n-2}$. On the other hand, it also contains x^{n-1} and y^{n-1} and therefore $\dim C \geqslant 2n-2$.

We come back to the general situation where $\hat{\mathcal{K}}_\Sigma C$ is a $J^{\mathcal{K}}$-complex Lagrangian in $\hat{\mathcal{K}}_\Sigma M$. One difficulty is to rotate the isotropic submanifold Σ to Σ' inside M.

We observe that the tangent space to $\hat{\mathcal{K}}_\Sigma C$ at any point $[f]$ is given by

$$T_{[f]}(\hat{\mathcal{K}}_\Sigma C) = \Gamma\left(\Sigma, \frac{f^* T_C}{T_\Sigma \oplus JT_\Sigma \cap f^* T_C}\right).$$

Since any $J^{\mathcal{K}}$-complex Lagrangian submanifold is indeed an integrable complex submanifold, $T_{[f]}(\hat{\mathcal{K}}_\Sigma C)$ is preserved by $J^{\mathcal{K}}$. This implies that the complex structure on T_M induces a complex structure on the quotient bundle $f^*T_C/[T_\Sigma \oplus JT_\Sigma \cap f^*T_C]$. For any $v \in T_{[f]}(\hat{\mathcal{K}}_\Sigma C)$ we can regard it as a section of f^*T_C over Σ, perpendicular to T_Σ and $JT_\Sigma \cap f^*T_C$. In particular, Jv is also such a section.

We need the following lemma which will be proven later.

Lemma 23. *In the above situation, we have $JT_\Sigma \subset f^*T_C$.*

This lemma implies that the restriction of T_C on Σ contains the linear span of T_Σ, JT_Σ, v and Jv, which has rank $2n-2$. (More precisely we consider the restriction to the complement of the zero set of v in Σ.) Therefore $\dim C \geqslant 2n-2$.

On the other hand, if $\dim C > 2n-2$, then by using the localizing method as in Lemma 9, $\hat{\mathcal{K}}_\Sigma C$ would be too large to be a complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$. Thus, $\dim C = 2n-2$.

In particular, this implies that f^*T_C is isomorphic to the linear span of T_Σ, JT_Σ, v and Jv outside the zero set of v. Therefore, T_C is preserved by the complex structure of M along Σ. By varying the isotropic submanifold Σ in C, we can cover an open neighborhood of Σ in C (for example using the gluing arguments as in the proof of the above lemma). This implies that an open neighborhood of Σ in C is a complex submanifold in M. By the analyticity of C, the submanifold C is a complex hypersurface in M.

(i) $\Rightarrow$ (ii). Suppose C is complex hypersurface in M. Then, it is clear that $\omega_I^{\mathcal{K}}$ and $\omega_K^{\mathcal{K}}$ vanish along $\hat{\mathcal{K}}_\Sigma C$ since Ω_M vanishes along C. Using similar arguments as in the proof of Proposition 11, it is not difficult to verify that $\hat{\mathcal{K}}_\Sigma C$ is maximally self $\omega_I^{\mathcal{K}}$-perpendicular and maximally self $\omega_K^{\mathcal{K}}$-perpendicular in $\hat{\mathcal{K}}_\Sigma M$, that is, $\hat{\mathcal{K}}_\Sigma C$ is a $J^{\mathcal{K}}$-complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$. Hence the theorem is proved. □

We suspect that the analyticity assumption on C is unnecessary. All we need is to deform the isotropic submanifold Σ of M inside C to cover every point in C.

Proof of Lemma 23. For any tangent vector u of Σ at a point p, i.e. $u \in T_p\Sigma \subset T_pC$, we need to show that $Ju \in T_pC$. We can assume that $\nu(p)$ has unit length. First, we choose local holomorphic coordinates $z^1, z^2, \dots, z^n$ near p satisfying the following properties at p:

$$z^i(p) = 0 \quad \text{for all } i = 1, \dots, n,$$
$$\Omega_M(p) = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n,$$
$$T_p\Sigma \text{ is spanned by } \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^{n-2}},$$
$$\nu(p) = \frac{\partial}{\partial x^{n-1}} \quad \text{and} \quad u = \frac{\partial}{\partial x^1}$$

where $z^j = x^j + iy^j$ for each j. We could use $x^1, \dots, x^{n-2}$ to parametrize Σ near p.

Recall that

$$\nu \in T_{[f]}(\hat{\mathcal{K}}_\Sigma C) = \Gamma\left(\Sigma, \frac{f^*T_C}{T_\Sigma \oplus JT_\Sigma \cap f^*T_C}\right).$$

For any smooth function $\alpha(x^1, \dots, x^{n-2})$ on Σ, $\alpha\nu$ is again a tangent vector to $\hat{\mathcal{K}}_\Sigma C$ at $[f]$. We are going to construct a particular α supported on a small neighborhood of p in Σ satisfying

$$\frac{\partial\alpha}{\partial x^1}(0) = \infty,$$
$$\frac{\partial\alpha}{\partial x^j}(0) = 0, \quad \text{for } j = 2, \dots, n-2.$$

For simplicity we assume that this small neighborhood contains the unit ball in Σ. To construct α, we fix a smooth even cutoff function $\eta : \mathbb{R} \to [0, 1]$ with $\operatorname{supp}(\eta) \in (-1, 1)$, $\eta(0) = 1$ and $\eta'(0) = 0$. We define $\alpha : \Sigma \to \mathbb{R}$ as follows:

$$\alpha\left(x^1, x^2, \dots, x^{n-2}\right) = \left(x^1\right)^{1/3} \cdot \eta\left(x^1\right) \cdot \eta\left(\sum_{j=2}^{n-2}\left(x^j\right)^2\right).$$

For any small real number ε, we write

$$\nu_\varepsilon = \varepsilon\alpha\nu \in T_{[f]}(\hat{\mathcal{K}}_\Sigma C)$$

and we denote the corresponding family of isotropic submanifolds of M in C by $f_\varepsilon : \Sigma_\varepsilon \to M$. From $\alpha(0) = 0$, there is a family of points $p_\varepsilon \in \Sigma_\varepsilon$ with the property that

$$|p_\varepsilon - p| = O\left(\varepsilon^2\right).$$

Using the property $\partial\alpha/\partial x^1(0) = \infty$, we can find a family of normal vectors at p_ε,

$$u_\varepsilon \in N_{\Sigma_\varepsilon/C},$$

with the property

$$\left| u_\varepsilon - \frac{\partial}{\partial x^1} \right| = O\bigl(\varepsilon^2\bigr).$$

Since $T_{[f_\varepsilon]}(\hat{\mathcal{K}}_\Sigma C)$ is preserved by $J^{\mathcal{K}}$ and $f_\varepsilon^* T_C/[T_{\Sigma_\varepsilon} \oplus JT_{\Sigma_\varepsilon} \cap f_\varepsilon^* T_C]$ is preserved by J, we have $Ju_\varepsilon \in T_{p_\varepsilon} C$. By letting ε go to zero, we have $Ju \in T_p C$. Hence, the lemma is proved. □

In the following theorem, we see the relationship between a D-brane L in M with respect to $e^{i\theta}\Omega_M$ and a $J^{\mathcal{K}}_{\theta+\pi/2}$ complex Lagrangian $\hat{\mathcal{K}}_\Sigma L$ in $\hat{\mathcal{K}}_\Sigma M$, i.e. it is maximally self $\omega^{\mathcal{K}}_{I,\theta}$-perpendicular and maximally self $\omega^{\mathcal{K}}$-perpendicular.

Theorem 24. *Let L be a connected analytic submanifold of a Calabi–Yau manifold M, then the following two statements are equivalent*:

(i) *L is a D-brane with phase θ (i.e. a special Lagrangian with phase $\theta - \pi/2$)*;
(ii) *$\hat{\mathcal{K}}_\Sigma L$ is a $J^{\mathcal{K}}_{\theta+\pi/2}$-complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$.*

Proof. By replacing the holomorphic volume form on M from Ω_M to $e^{i\theta}\Omega_M$ if necessary, we can assume that θ is zero.

(i) ⇒ (ii). We assume that L is a special Lagrangian submanifold in M with phase $-\pi/2$. Because the Kähler form ω and $\operatorname{Re}\Omega_M$ of M vanish along L, it is clear that $\omega^{\mathcal{K}}$ and $\omega^{\mathcal{K}}_{I,0}$ vanish along $\hat{\mathcal{K}}_\Sigma L$. Notice that L being a Lagrangian submanifold in M implies that any submanifold in L is automatically isotropic in M, and moreover the equivalence relation $\langle\mathcal{D}\rangle$ on $\mu^{-1}(0)$ restricted to $Map(\Sigma, L)$ is trivial locally. That is, $\hat{\mathcal{K}}_\Sigma L$ is the same as $\mathcal{K}_\Sigma L = Map(\Sigma, L)/Diff(\Sigma)$ at least locally.

We claim that $\hat{\mathcal{K}}_\Sigma L$ is maximally self $\omega^{\mathcal{K}}$-perpendicular in $\hat{\mathcal{K}}_\Sigma M$. Otherwise there is normal vector field $\nu \in \Gamma(\Sigma, N_{\Sigma/M})$ not lying in $\Gamma(\Sigma, N_{\Sigma/L})$ such that $\omega(\nu, u) = 0$ for any $u \in \Gamma(\Sigma, f^* T_L)$. Suppose that $\nu(p) \notin N_{\Sigma/L,p}$. Then, this implies that ω vanishes on the linear span of $T_p L$ and $\nu(p)$ inside $T_p M$. This is impossible because the dimension of the linear span is bigger than n.

Similarly $\hat{\mathcal{K}}_\Sigma L$ is maximally self $\omega^{\mathcal{K}}_{I,0}$-perpendicular in $\hat{\mathcal{K}}_\Sigma M$. Otherwise $\operatorname{Re}\Omega_M$ vanishes on the linear span of $T_p L$ and $\nu(p)$ inside $T_p M$. This is again impossible because $\Omega_{M,p}$ is a complex volume form on $T_p M \cong \mathbb{C}^n$ and therefore cannot vanish on any coisotropic subspaces other than Lagrangians. Hence $\hat{\mathcal{K}}_\Sigma L$ is a $J^{\mathcal{K}}_{\pi/2}$-complex Lagrangian in $\hat{\mathcal{K}}_\Sigma M$.

(ii) ⇒ (i). We assume that $\hat{\mathcal{K}}_\Sigma L$ is a $K^{\mathcal{K}}$-complex Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$. For any $[f] \in \hat{\mathcal{K}}_\Sigma L$, the tangent space of $\hat{\mathcal{K}}_\Sigma L$ (respectively $\hat{\mathcal{K}}_\Sigma M$) at $[f]$ is the section of the bundle $f^* T_L / T_\Sigma \oplus JT_\Sigma \cap f^* T_L$ (respectively $f^*(T_M)/(T_\Sigma + JT_\Sigma)$) over Σ. Letting $v \in T_{[f]}(\hat{\mathcal{K}}_\Sigma L)$, we can regard v as a section of $f^* T_L$ over Σ, orthogonal to $T_\Sigma \oplus JT_\Sigma \cap f^* T_L$.

Since $f^*(T_M)/(T_\Sigma + JT_\Sigma)$ is a rank four bundle and $\hat{\mathcal{K}}_\Sigma L$ is a Lagrangian in $\hat{\mathcal{K}}_\Sigma M$, this implies that $f^* T_L / T_\Sigma \oplus JT_\Sigma \cap f^* T_L$ must be a rank two bundle over Σ. Therefore

$$\dim L \geqslant n,$$

with equality if and only if $JT_\Sigma \cap f^* T_L$ is trivial. Suppose that ν is a tangent vector of $\hat{\mathcal{K}}_\Sigma L$ at f.

Assume that $JT_\Sigma \cap f^*T_L$ is not trivial. We can find a tangent vector u to $f(\Sigma)$ at a point p such that $Ju \in T_pL$. For simplicity we assume that ν has unit length at the point p.

As in the proof of the previous theorem, we can choose local holomorphic coordinates $z^1, z^2, \dots, z^n$ of M around p such that

$$z^i(p) = 0 \quad \text{for all } i = 1, \dots, n,$$
$$\Omega_M(p) = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n,$$
$$T_p\Sigma \text{ is spanned by } \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \dots, \frac{\partial}{\partial x^{n-2}},$$
$$\nu(p) = \frac{\partial}{\partial x^{n-1}}, \quad u = \frac{\partial}{\partial x^1} \quad \text{and} \quad Ju = \frac{\partial}{\partial y^1}.$$

We choose a function $\alpha(x)$ as in the proof of the previous theorem, write

$$\nu_\varepsilon = \varepsilon\alpha\nu \in T_{[f]}(\hat{\mathcal{K}}_\Sigma L)$$

for any small real number ε, and denote the corresponding family of isotropic submanifolds of M in L by $f_\varepsilon : \Sigma_\varepsilon \to M$ as before. From $\alpha(0) = 0$, there is a family of points $p_\varepsilon \in \Sigma_\varepsilon$ with the property that

$$|p_\varepsilon - p| = O(\varepsilon^2).$$

Using the property $\partial\alpha/\partial x^1(0) = \infty$, we can find two families of normal vectors at p_ε,

$$u_\varepsilon, w_\varepsilon \in N_{\Sigma_\varepsilon/L},$$

with the property

$$\left| u_\varepsilon - \frac{\partial}{\partial x^1} \right| = O(\varepsilon^2) \quad \text{and} \quad \left| w_\varepsilon - \frac{\partial}{\partial y^1} \right| = O(\varepsilon^2).$$

However, using localization arguments as before, $\hat{\mathcal{K}}_\Sigma L$ being an $\omega^{\mathcal{K}}$-Lagrangian submanifold in $\hat{\mathcal{K}}_\Sigma M$ implies that $\omega(u_\varepsilon, w_\varepsilon) = 0$. By letting ε go to zero, we have $0 = \omega(\partial/\partial x^1, \partial/\partial y^1) = -1$, a contradiction. Hence $JT_\Sigma \cap f^*T_L$ is trivial and $\dim L = n$.

Since $\Gamma(\Sigma, f^*T_L/T_\Sigma \oplus JT_\Sigma \cap f^*T_L)$ is preserved by $K^{\mathcal{K}}$, we have $K^{\mathcal{K}}\nu(p) \in f^*T_L$. Note that $K^{\mathcal{K}}\nu(p)$ is the tangent vector of M which is the metric dual of the one form,

$$\nu \lrcorner \omega_K^{\mathcal{K}}(p) = \left(\frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \cdots \wedge \frac{\partial}{\partial x^{n-2}} \wedge \nu \right) \lrcorner \operatorname{Im} \Omega_M(p) = \pm dy^n,$$

since $\nu(p) = \partial/\partial x^{n-1}$. This implies that T_pL is spanned by $\partial/\partial x^1, \partial/\partial x^2, \dots, \partial/\partial x^{n-1}$ and $\partial/\partial y^n$; that is, a special Lagrangian subspace of phase $-\pi/2$.

As in the proof of the previous theorem, by deforming the isotropic submanifold Σ in L and using the fact that L is an analytic submanifold of M, we conclude that L is a special Lagrangian submanifold in M with zero phase.

Hence the theorem is proved. □

Acknowledgments

The research of the first author is partially supported by NSF/DMS-0103355, Direct Grant from CUHK and RGC Earmarked Grants of Hong Kong. Authors express their gratitude to the referee for useful comments to improve the presentation of this article.

References

[1] R. Bott, Marston Morse and his mathematical works, Bull. Amer. Math. Soc. (N.S.) 3 (3) (1980) 907–950.
[2] R.B. Brown, A. Gray, Vector cross products, Comment. Math. Helv. 42 (1967) 222–236.
[3] J.L. Brylinski, Loop Spaces, Characteristic Classes and Geometric Quantization, Progr. Math., vol. 107, Birkhäuser Boston, Boston, MA, 1993.
[4] S.K. Donaldson, Moment maps and diffeomorphisms, Asian J. Math. 2 (1) (1999) 1–16.
[5] A. Floer, Morse theory for Lagrangian intersections, J. Differential Geom. 28 (3) (1988) 513–547.
[6] A. Gray, Vector cross products on manifolds, Trans. Amer. Math. Soc. 141 (1969) 465–504.
[7] R. Harvey, H.B. Lawson Jr., Calibrated geometries, Acta Math. 148 (1982) 47.
[8] N.J. Hitchin, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) 55 (1) (1987) 59–126.
[9] N.J. Hitchin, Lectures on special Lagrangian submanifold, in: Winter School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds, Cambridge, MA, 1999, in: AMS/IP Stud. Adv. Math., vol. 23, Amer. Math. Soc., Providence, RI, 2001, pp. 151–182.
[10] J.H. Lee, N.C. Leung, Instantons and branes in manifolds with vector cross product, preprint.
[11] N.C. Leung, X.W. Wang, Intersection theory of coassociative submanifolds in G_2-manifolds and Seiberg–Witten invariants, math.DG/0401419.
[12] J. Millson, B. Zombro, A Kähler structure on the moduli space of isometric maps of a circle into Euclidean space, Invent. Math. 123 (1) (1996) 35–59.
[13] M.M. Movshev, The structure of a symplectic manifold on the space of loops of 7-manifold, preprint, math.SG/9911100.
[14] E. Witten, The index of the Dirac operator in loop space, in: Elliptic Curves and Modular Forms in Algebraic Topology, Princeton, NJ, 1986, in: Lecture Notes in Math., vol. 1326, Springer, Berlin, 1988, pp. 161–181.
[15] E. Witten, Index of Dirac operators, in: Quantum Fields and Strings: A Course for Mathematicians, Vols. 1, 2, Princeton, NJ, 1996/1997, Amer. Math. Soc., Providence, RI, 1999, pp. 475–511.