J. ALGEBRAIC GEOMETRY
15 (2006) 591–601
S 1056-3911(06)00439-5
Article electronically published on May 2, 2006

COUNTING ELLIPTIC CURVES IN K3 SURFACES

JUNHO LEE AND NAICHUNG CONAN LEUNG

Abstract

We compute the genus $g = 1$ family GW-invariants of K3 surfaces for non-primitive classes. These calculations verify the Göttsche-Yau-Zaslow formula for non-primitive classes with index two. Our approach is to use the genus two topological recursion formula and the symplectic sum formula to establish relationships among various generating functions.

The number of genus g curves in K3 surfaces X representing a homology class $A \in H_2(X, \mathbb{Z})$ and pass through a g generic point depends only on the self-intersection number $A\cdot A = 2d-2$ and the index[1] r of the class A ([1], [2]). We denote it as $N_g(d, r)$ which is certain virtual counting of curves in X. The conjectural formulas of Göttsche [4], which generalize the Yau-Zaslow formula [14], then assert that the generating function for those numbers $N_g(d, r)$s is given by

$$\sum_{d\geq 0} N_g(d,r)\, t^d \;=\; (t\, G_2'(t))^g \prod_{d\geq 1} \left(\frac{1}{1-t^d}\right)^{24} \tag{0.1}$$

where $G_2(t)$ is the Eisenstein series of weight 2, i.e.,

$$G_2(t) \;=\; \sum_{d\geq 0} \sigma(d)\, t^d \quad \text{where} \quad \sigma(d) \;=\; \sum_{k|d} k,\; d\geq 1 \;\text{ and }\; \sigma(0) \;=\; -\tfrac{1}{24}\,.$$

In particular, $N_g(d, r)$ should be independent of the index of the homology class. In [1, 8], this formula was verified for any genus and primitive classes $(r = 1)$. It was also verified for genus $g = 0$ and index $r = 2$ classes [10]. In this article, we verify the formula (0.1) for genus $g = 1$ and index $r = 2$ classes by computing the family GW-invariants defined in [7]. Our main theorem is the following result.

Theorem 0.1. *Let X be a K3 surface and $A/2 \in H_2(X;\mathbb{Z})$ be a primitive class. Then, the genus $g = 1$ family GW-invariant of X for the class A is*

Received April 29, 2004 and, in revised form, October 9, 2005. The second author is partially supported by NSF/DMS-0103355, CUHK/2060275, and CUHK/2160256.

[1] The index of A is the largest positive integer r such that $r^{-1}A$ is integral. An index one class is called primitive.

given by

$$GW^{\mathcal{H}}_{A,1}(pt) \;=\; GW^{\mathcal{H}}_{B,1}(pt) \;+\; 2\,GW^{\mathcal{H}}_{A/2,1}(pt) \tag{0.2}$$

where B is any primitive class with $B^2 = A^2$.

To explain the equivalence between the above theorem and the Göttsche-Yau-Zaslow formula, we first notice that the *family Gromov-Witten* invariant counts the number of *J-holomorphic maps* for any complex structure J in the twistor family of the K3 surface X. As explained in [1], there is a unique J in the twistor family which supports holomorphic curves representing A and this justifies the use of our family invariant. An important issue is the distinction between holomorphic maps to X and holomorphic curves in X. This is because a multiple curve in X can be the image of different holomorphic maps. This issue does not arise when the homology class $A/2$ they represent is primitive and therefore $N_1(d', 1) = GW^{\mathcal{H}}_{A/2,1}(pt)$, where $(A/2)^2 = 2d' - 2$. Similarly, we have $N_1(d, 1) = GW^{\mathcal{H}}_{B,1}(pt)$ where $A^2 = B^2 = 2d - 2$. The number of genus one holomorphic maps covering a fixed elliptic curve with degree r equals the sum of divisors $\sigma(r)$, for instance $\sigma(2) = 1 + 2 = 3$. Each primitive elliptic curve representing the class $A/2$ in X contributes 3 to $GW^{\mathcal{H}}_{A,1}(pt)$, which counts maps to X. On the other hand, the same curve contributes 1 to $N_1(d, 2)$, which counts curves in X; that is, we have

$$GW^{\mathcal{H}}_{A,1}(pt) \;-\; N_1(d, 2) \;=\; 2N_1(d', 1) \;=\; 2GW^{\mathcal{H}}_{A/2,1}(pt).$$

Therefore the above theorem is equivalent to,

$$N_1(d, 2) \;=\; GW^{\mathcal{H}}_{B,1}(pt) \;=\; N_1(d, 1).$$

Together with the validation of the formula for primitive classes, this implies the Göttsche-Yau-Zaslow formula for the number of elliptic curves in K3 surfaces representing index two homology classes.

The organization of this paper is as follows: The construction of family GW-invariants is briefly described in Section 1. This section also contains a family version of composition law. In Section 2, using the composition law and the genus $g = 2$ TRR (Topological Recursion Relation) formula [3] we establish the $g = 2$ TRR formula for family GW-invariants. In Section 3, we prove Theorem 0.1 by combining this TRR formula with the symplectic sum formulas of [10].

1. Composition law for family GW-invariants

This section briefly describes family GW-invariants defined in [7]. Let X be a Kähler surface with a Kähler structure (ω, J, g). For each 2-form α in the linear space

$$\mathcal{H} \;=\; \mathrm{Re}\big(H^{2,0} \oplus H^{0,2}\big)$$

we define an endomorphism K_α of TX by the equation $\langle u, K_\alpha v\rangle = \alpha(u,v)$. Since $Id + JK_\alpha$ is invertible,

$$J_\alpha \;=\; (Id + JK_\alpha)^{-1} J\,(Id + JK_\alpha)$$

is an almost complex structure on X.

Denote by $\overline{\mathcal{F}}_{g,k}(X,A)$ the space of all stable maps $f:(C,j)\to X$ of genus g with k-marked points which represent homology class A. For each such map, collapsing unstable components of the domain determines a point in the Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$ and evaluation of marked points determines a point in X^k. Thus we have a map

$$st \times ev \;:\; \overline{\mathcal{F}}_{g,k}(X,A) \;\to\; \overline{\mathcal{M}}_{g,k} \times X^k \tag{1.1}$$

where st and ev denote the stabilization map and the evaluation map, respectively. On the other hand, there is a *generalized orbifold bundle* E over $\overline{\mathcal{F}}_{g,k}(X,A)\times\mathcal{H}$ whose fiber over $\big((f,j),\alpha\big)$ is $\Omega^{0,1}_{jJ_\alpha}(f^*TX)$. This bundle has a section Φ defined by

$$\Phi\big(f,j,\alpha\big) \;=\; df + J_\alpha\, df\, j\,.$$

When X is a K3 surface and $A\neq 0$, the moduli space $\Phi^{-1}(0) = \overline{\mathcal{M}}^{\mathcal{H}}_{g,k}\big(X,A\big)$ is compact. By the same manner as in the theory of the ordinary GW-invariants [11], this section then gives rise to a well-defined rational homology class

$$GW^{\mathcal{H}}_{g,k}(X,A) \;\in\; H_{2r}\big(\overline{\mathcal{F}}_{g,k}(X,A);\mathbb{Q}\big) \quad \text{where} \;\; r = g+k.$$

The family of GW invariants of (X,J) are then defined by

$$GW^{\mathcal{H}}_{g,k}(X,A)\big(\beta;\gamma_1,\cdots,\gamma_k\big) \;=\; GW^{\mathcal{H}}_{g,k}(X,A) \cap \big(st^*(\beta^*)\cup ev^*(\gamma_1^*\cup\cdots\cup\gamma_k^*)\big)$$

where β^* and γ_i^* are Poincaré dual of $\beta\in H_*\big(\overline{\mathcal{M}}_{g,k};\mathbb{Q}\big)$ and $\gamma_i \in H_*\big(X^k;\mathbb{Q}\big)$, respectively.

In [7] the first author proved that the above family of GW-invariants of K3 surfaces are the same as the invariants defined by Bryan and Leung [1] using the twistor family. In particular, they are independent of complex structures and for any two classes A, B of the same index with $A^2 = B^2$, there exists an orientation preserving diffeomorphism $h: X\to X$ such that $h_*(A) = B$ and

$$GW^{\mathcal{H}}_{g,k}(X,A)(\beta;\gamma) \;=\; GW^{\mathcal{H}}_{g,k}(X,B)(\beta;h_*\gamma). \tag{1.2}$$

Below we will often denote the above family of GW-invariants simply as $GW^{\mathcal{H}}_{A,g}$.

The family of GW-invariants has a property analogous to the composition law of ordinary GW-invariants (cf. [13]). Consider a node of a stable curve

C in the Deligne-Mumford space $\overline{\mathcal{M}}_{g,k}$. When the node is separating, the normalization of C has two components. The genus and the number of marked points decompose as $g = g_1 + g_2$ and $k = k_1 + k_2$ and there is a natural map

$$\sigma : \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \ \to \ \overline{\mathcal{M}}_{g,k}$$

defined by identifying $(k_1 + 1)$th marked points of the first component to the first marked point of the second component. We denote by $PD(\sigma)$ the Poincaré dual of the image of this map σ. For the non-separating node, there is another natural map

$$\theta : \overline{\mathcal{M}}_{g-1,k+2} \to \overline{\mathcal{M}}_{g,k}$$

defined by identifying the last two marked points. We also write $PD(\theta)$ for the Poincaré dual of the image of this map θ.

Recall that the ordinary GW-invariants of K3 surfaces are all zero except for the trivial homology class. The proposition below thus follows from Proposition 3.7 of [7].

Proposition 1.1. *Let $\{H_a\}$ and $\{H^a\}$ be bases of $H_*(X;\mathbb{Z})$ dual by the intersection form.*

(a) *For the gluing map σ as above, we have*

$$\begin{aligned} &GW^{\mathcal{H}}_{A,g}(PD(\sigma);\gamma_1,\cdots,\gamma_k) \\ &\quad = \sum_a GW^{\mathcal{H}}_{A,g_1}(\gamma_1,\cdots,\gamma_{k_1},H_a)\, GW_{0,g_2}(H^a,\gamma_{k_1+1},\cdots,\gamma_k) \\ &\qquad + \sum_a GW_{0,g_1}(\gamma_1,\cdots,\gamma_{k_1},H_a)\, GW^{\mathcal{H}}_{A,g_2}(H^a,\gamma_{k_1+1},\cdots,\gamma_k) \end{aligned}$$

where GW_{0,g_1} and GW_{0,g_2} denotes the ordinary GW invariants of K3 *surfaces.*

(b) *For the gluing map θ as above, we have*

$$GW^{\mathcal{H}}_{A,g}(PD(\theta);\gamma_1,\cdots,\gamma_k) \ = \ \sum_a GW^{\mathcal{H}}_{A,g-1}(\gamma_1,\cdots,\gamma_k,H_a,H^a).$$

2. Topological recursion relations

The first composition law, Proposition 1.1(a) relates family invariants and ordinary invariants (for trivial homology class) of K3 surfaces. In this section, we first recall the ordinary GW-invariants of a closed symplectic 4-manifold for trivial homology class. Similarly as in [8], we then combine the composition law with the genus $g = 2$ TRR formula [3] to establish a family version of $g = 2$ TRR formula.

Let ψ_i be the first Chern class of the line bundle $\mathcal{L}_i \to \overline{\mathcal{M}}_{g,k}\big(X, A\big)$ whose geometric fiber at the point $\big(C; x_1, \cdots, x_k, f\big)$ is $T^*_{x_i}C$. One can use ψ_i to impose constraints on the ordinary GW invariants as follows:

$$\begin{aligned} &GW_{g,k}(X,A)(\tau_{m_1}(\gamma_1), \cdots, \tau_{m_k}(\gamma_k)) \\ &\quad = \; GW_{g,k}(X,A) \cap \big(\psi_1^{m_1} \cdots \psi_k^{m_k} \cup ev^*(\gamma_1 \cup \cdots \cup \gamma_k)\big) \end{aligned}$$

where $GW_{g,k}(X,A)$ is the rational homology class in $H_*(\overline{\mathcal{F}}_{g,k}(X,A);\mathbb{Q})$ that defines the ordinary GW invariants. We also denote by ϕ_i the first Chern class of the line bundle $L_i \to \overline{\mathcal{M}}_{g,k}$ whose geometric fiber at the point $\big(C; x_1, \cdots, x_k\big)$ is $T^*_{x_i}C$.

Lemma 2.1. *Let X be a closed symplectic 4-manifold and $c_1 = c_1(TX)$.*

(a) $GW_{0,0}\big(\gamma_1, \cdots, \gamma_k\big) = 0$ *unless $k = 3$ and $\sum \deg\alpha_i = 4$. In that case,*

$$GW_{0,0}\big(\gamma_1, \gamma_2, \gamma_3\big) \;=\; \int_X \gamma_1^* \, \gamma_2^* \, \gamma_3^* \, .$$

(b) $GW_{0,1}\big(\gamma_1, \cdots, \gamma_k\big) = 0$ *unless $k = 1$. In that case,*

$$GW_{0,1}\big(\gamma\big) = -\tfrac{1}{24}\, c_1(\gamma)\, .$$

(c) $GW_{0,2}\big(\tau(\gamma_1), \gamma_2\big) = 0$. *When $c_1 = 0$, we also have $GW_{0,2}\big(\tau(\gamma)\big) = 0$.*

Proof. (a) and (b) directly follow from Proposition 1.4.1 of [6]. On the other hand, the formula (7) of [6] says that

$$\begin{aligned} GW_{0,2}\big(\tau(\gamma_1), \gamma_2\big) \;&=\; \big(\gamma_1 \cdot \gamma_2\big) \int_{\overline{\mathcal{M}}_{2,2}} \lambda_2^2 \, \phi_1, \\ GW_{0,2}\big(\tau(\gamma)\big) \;&=\; -c_1(\gamma) \int_{\overline{\mathcal{M}}_{2,1}} \lambda_1 \, \lambda_2 \, \phi_1, \end{aligned}$$

where $\lambda_i = c_i(E)$ is the Chern class of the Hodge bundle E. Thus, the first invariant in (c) is zero by Mumford's relation $\lambda_2^2 = 0$ (cf. [12]) and the second vanishes when $c_1 = 0$. □

Let φ_i be the first Chern class of the line bundle $\mathcal{L}_i^{\mathcal{H}} \to \overline{\mathcal{M}}^{\mathcal{H}}_{g,k}\big(X, A\big)$ whose geometric fiber at the point $\big(C; x_1, \cdots, x_k, f, \alpha\big)$ is $T^*_{x_i}C$. Similarly as for the ordinary GW invariants, one can also use φ_i to impose constraints on $GW^{\mathcal{H}}$ invariants as follows:

$$\begin{aligned} &GW^{\mathcal{H}}_{g,k}(X,A)(\tau_{m_1}(\gamma_1), \cdots, \tau_{m_k}(\gamma_k)) \\ &\quad = \; GW^{\mathcal{H}}_{g,k}(X,A) \;\cap\; \big(\varphi_1^{m_1} \cdots \varphi_k^{m_k} \cup ev^*(\gamma_1 \cup \cdots \cup \gamma_k)\big). \end{aligned}$$

Let $E(2) \to \mathbb{P}^1$ be an elliptic K3 surface with a section of self-intersection number -2. Denote by s and f the section class and the fiber class, respectively.

Proposition 2.2. *Let s and f be as above and (S, F) denote either $(2s, f)$ or $(s-3f, 2f)$. Then, the family of GW invariants of an elliptic* K3 *surface $E(2)$ satisfies the following formula,*

$$\begin{aligned} GW^{\mathcal{H}}_{S+dF,2}\big(\tau(F), \tau(F)\big) \;=\; & \tfrac{1}{9}\, d(d-1)\, GW^{\mathcal{H}}_{S+dF,0} \;-\; \tfrac{1}{3}\, d\, GW^{\mathcal{H}}_{S+dF,0} \\ & +\; \tfrac{4}{9}\, GW^{\mathcal{H}}_{S+dF,0} \;-\; \tfrac{2}{3}\, GW^{\mathcal{H}}_{S+dF,1}\big(pt\big). \end{aligned}$$

Proof. Similarly as for the ordinary GW moduli spaces, the family of GW moduli spaces can be stratified by the dual graph of the domain of maps. The family of GW invariants with $\varphi_1\varphi_2$ constraint is then equal to the invariant with the constraint $\phi_1\phi_2$ plus the contribution of some boundary strata of $\overline{\mathcal{M}}^{\mathcal{H}}_{2,2}(E(2), S+dF)$. Those boundary strata consist of maps with domains containing unstable genus 0 components and hence the dual graphs of them correspond to the first three terms and the fifth term in (17) of [9]. Applying Proposition 1.1 thus gives the contribution of the boundary strata corresponding to $\varphi_1\varphi_2 - st^*(\phi_1\phi_2)$:

$$\begin{aligned} & \sum_a 2\, GW^{\mathcal{H}}_{S+dF,2}\big(\tau(F), H_a\big)\, GW_{0,0}\big(H^a, F\big) \\ & \quad + \sum_a 2\, GW_{0,2}\big(\tau(F), H_a\big)\, GW^{\mathcal{H}}_{S+dF,0}\big(H^a, F\big) \\ & \quad - \sum_{a,b} GW^{\mathcal{H}}_{S+dF,0}\big(F, H_a\big)\, GW_{0,0}\big(F, H_b\big)\, GW_{0,2}\big(H^a, H^b\big) \\ & \quad - \sum_{a,b} GW_{0,0}\big(F, H_a\big)\, GW^{\mathcal{H}}_{S+dF,0}\big(F, H_b\big)\, GW_{0,2}\big(H^a, H^b\big) \\ & \quad - \sum_{a,b} GW_{0,0}\big(F, H_a\big)\, GW_{0,0}\big(F, H_b\big)\, GW^{\mathcal{H}}_{S+dF,2}\big(H^a, H^b\big) \\ & \quad - \sum_{a,b} 3\, GW^{\mathcal{H}}_{S+dF,0}\big(F, F, H_a\big)\, GW_{0,0}\big(H^a, H_b\big)\, GW_{0,2}\big(H^b\big) \\ & \quad - \sum_{a,b} 3\, GW_{0,0}\big(F, F, H_a\big)\, GW^{\mathcal{H}}_{S+dF,0}\big(H^a, H_b\big)\, GW_{0,2}\big(H^b\big) \\ & \quad - \sum_{a,b} 3\, GW_{0,0}\big(F, F, H_a\big)\, GW_{0,0}\big(H^a, H_b\big)\, GW^{\mathcal{H}}_{S+dF,2}\big(H^b\big) \end{aligned} \tag{2.1}$$

where $\{H_a\}$ and $\{H^a\}$ are bases of $H^*\big(E(2);\mathbb{Z}\big)$ dual by the intersection form. By Lemma 2.1, this contribution (2.1) vanishes.

On the other hand, the class $\phi_1\phi_2$ is a boundary class in $\overline{\mathcal{M}}_{2,2}$. One can thus apply Proposition 1.1 and (5) of [3] together to compute the invariant with $\phi_1\phi_2$ constraint — this computation is analogous to the case of the ordinary

GW invariants (cf. (17) of [9]). Consequently, the family of GW invariants with $\phi_1\phi_2$ constraint can be written as a sum of 24 terms each of which is a product of ordinary GW invariants and family GW invariants. Among them the following terms correspond to the fourth, sixth, seventh, eighth, ninth, tenth, eleventh, and twelfth terms in (17) of [9]:

$$
\begin{aligned}
&\sum_a 3\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a\big)\, GW_{0,2}\big(\tau(H^a)\big)\\
&\qquad + \sum_a 3\, GW_{0,0}\big(F,F,H_a\big)\, GW^{\mathcal{H}}_{S+dF,2}\big(\tau(H^a)\big)\\
&\qquad + \sum_{a,b} \tfrac{13}{10}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H_b\big)\, GW_{0,1}\big(H^a\big)\, GW_{0,1}\big(H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{13}{10}\, GW_{0,0}\big(F,F,H_a,H_b\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^a\big)\, GW_{0,1}\big(H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{13}{10}\, GW_{0,0}\big(F,F,H_a,H_b\big)\, GW_{0,1}\big(H^a\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{8}{5}\, GW^{\mathcal{H}}_{S+dF,1}\big(F,H_a\big)\, GW_{0,0}\big(H^a,F,H_b\big)\, GW_{0,1}\big(H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{8}{5}\, GW_{0,1}\big(F,H_a\big)\, GW^{\mathcal{H}}_{S+dF,0}\big(H^a,F,H_b\big)\, GW_{0,1}\big(H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{8}{5}\, GW_{0,1}\big(F,H_a\big)\, GW_{0,0}\big(H^a,F,H_b\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^b\big)\\
&\qquad - \sum_{a,b} \tfrac{4}{5}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a\big)\, GW_{0,1}\big(H^a,H_b\big)\, GW_{0,1}\big(H^b\big)\\
&\qquad - \sum_{a,b} \tfrac{4}{5}\, GW_{0,0}\big(F,F,H_a\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^a,H_b\big)\, GW_{0,1}\big(H^b\big)\\
&\qquad - \sum_{a,b} \tfrac{4}{5}\, GW_{0,0}\big(F,F,H_a\big)\, GW_{0,1}\big(H^a,H_b\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{23}{240}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H^a,H_b\big)\, GW_{0,1}\big(H_b\big)\\
&\qquad + \sum_{a,b} \tfrac{23}{240}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H^a,H_b\big)\, GW_{0,1}\big(H_b\big)\\
&\qquad + \sum_{a,b} \tfrac{2}{48}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,H_a,H^a,H_b\big)\, GW_{0,1}\big(H_b,F\big)\\
&\qquad + \sum_{a,b} \tfrac{2}{48}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,H_a,H^a,H_b\big)\, GW_{0,1}\big(H_b,F\big).
\end{aligned}
$$

These terms all vanish by Lemma 2.1. The following terms correspond to the fourteenth and seventeenth terms in (17) of [9]:

$$\begin{aligned}
&\sum_{a,b} \tfrac{7}{30}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H_b\big)\, GW_{0,1}\big(H^a,H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{7}{30}\, GW_{0,0}\big(F,F,H_a,H_b\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^a,H^b\big)\\
&\qquad - \sum_{a,b} \tfrac{1}{30}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a\big)\, GW_{0,1}\big(H^a,H_b,H^b\big)\\
&\qquad - \sum_{a,b} \tfrac{1}{30}\, GW_{0,0}\big(F,F,H_a\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^a,H_b,H^b\big).
\end{aligned}$$

These terms also vanish by Lemma 2.1. Now, the remaining terms are:

$$\begin{aligned}
&- \sum_{a,b} \tfrac{1}{80}\, GW^{\mathcal{H}}_{S+dF,1}\big(F,F,H_a\big)\, GW_{0,0}\big(H^a,H_b,H^b\big)\\
&- \sum_{a,b} \tfrac{1}{80}\, GW_{0,1}\big(F,F,H_a\big)\, GW^{\mathcal{H}}_{S+dF,0}\big(H^a,H_b,H^b\big)\\
&+ \sum_{a,b} \tfrac{2}{30}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,H_a,H_b\big)\, GW_{0,1}\big(H^a,H^b,F\big)\\
&+ \sum_{a,b} \tfrac{2}{30}\, GW_{0,0}\big(F,H_a,H_b\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^a,H^b,F\big)\\
&+ \sum_{a,b} \tfrac{1}{576}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H^a,H_b,H^b\big).
\end{aligned} \tag{2.2}$$

Since the contribution of the boundary strata (2.1) vanishes, applying Lemma 2.1 to (2.2) then gives

$$\begin{aligned}
&GW^{\mathcal{H}}_{S+dF,2}\big(\tau(F),\tau(F)\big)\\
&\qquad = \sum_{a,b} \big(-\tfrac{1}{80}\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(F,F,H_a\big)\, GW_{0,0}\big(H^a,H_b,H^b\big)\\
&\qquad + \sum_{a,b} \tfrac{1}{15}\, GW_{0,0}\big(F,H_a,H_b\big)\, GW^{\mathcal{H}}_{S+dF,1}\big(H^a,H^b,F\big)\\
&\qquad + \sum_{a,b} \tfrac{1}{576}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H^a,H_b,H^b\big).
\end{aligned} \tag{2.3}$$

This can be further simplified by using Lemma 2.1(a), Divisor Axiom, and the facts

$$\sum_{b} GW_{0,0}([E(2)],H_b,H^b) \;=\; 24 \quad \text{and} \quad \sum_{a} (H_a\cdot F)(H^a\cdot(S+dF)) \;=\; 2.$$

The right-hand side of (2.3) becomes

$$-\tfrac{2}{3}\, GW^{\mathcal{H}}_{S+dF,1}\big(pt\big) \;+\; \sum_{a,b}\, \tfrac{1}{576}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H^a,H_b,H^b\big). \tag{2.4}$$

One the other hand, the genus $g=0$ invariants with point constraint vanish by dimensional reason. This observation combined with $\sum_a (H_a\cdot A)(A\cdot H^a) = A^2$ shows

$$\sum_{a,b}\, \tfrac{1}{576}\, GW^{\mathcal{H}}_{S+dF,0}\big(F,F,H_a,H^a,H_b,H^b\big) \;=\; \tfrac{4}{576}\,(4d-8)^2\, GW^{\mathcal{H}}_{S+dF,0}\,. \tag{2.5}$$

Then, the proof follows from (2.3), (2.4) and (2.5). □

3. Proof of Theorem 0.1

Our goal is to compute the genus $g=1$ family GW-invariants of K3 surfaces for the classes A of index 2. By (1.2), it suffices to compute the family of GW-invariants of $E(2)$ for the classes $2(s+df)$. Introduce three generating functions by the following formulas:

$$\begin{aligned} M_g(t) &= \sum GW^{\mathcal{H}}_{2s+df,g}\big(pt^g\big)\, t^d, \\ P_g(t) &= \sum GW^{\mathcal{H}}_{(s-3f)+d(2f),g}\big(pt^g\big)\, t^d, \\ N_g(t) &= \sum GW^{\mathcal{H}}_{s+df,g}\big(pt^g\big)\, t^d. \end{aligned}$$

Notice that the coefficients of the even terms of $M_1(t)$ give the invariants $GW^{\mathcal{H}}_{A,1}$ for all index two classes. Therefore, Theorem 0.1 in the introduction is equivalent to the following proposition by restricting only to even terms.

Proposition 3.1. *The above generating functions satisfy the following relation,*

$$M_1(t) \;=\; P_1(t) \;+\; 2\, N_1(t^2).$$

Proof. Introduce further generating functions by the formulas:

$$\begin{aligned} H_2\big(\tau(F),\tau(F)\big)(t) \;&=\; \sum GW^{\mathcal{H}}_{S+dF,g}\big(\tau(F),\tau(F)\big)\, t^d, \\ H_g(t) \;&=\; \sum GW^{\mathcal{H}}_{S+dF,g}\big(pt^g\big)\, t^d. \end{aligned}$$

It then immediately follows from Proposition 2.2 that

$$H_2\big(\tau(F),\tau(F)\big)(t) \;=\; \tfrac{1}{9}\, t^2\, H_0''(t) \;-\; \tfrac{1}{3}\, t\, H_0'(t) \;+\; \tfrac{4}{9}\, H_0(t) - \;\tfrac{2}{3}\, H_1(t). \tag{3.1}$$

In [10] we used the symplectic sum formula of [5] to show that

$$\begin{aligned} &H_2\big(\tau(F),\tau(F)\big)(t) \;-2\, H_1(t) \\ &\quad = \tfrac{20}{3}\, G_2(t)\, t\, H_0'(t) \;-\; \Big(64\, G_2^2(t) \;+\; \tfrac{40}{3}\, G_2(t) \;-\; 8\, t\, G_2'(t)\Big) H_0(t). \end{aligned} \tag{3.2}$$

The equations (3.1) and (3.2) then yield

$$\begin{aligned}(3.3)\qquad &\tfrac{1}{9}\,t^2\,H_0''(t)\;-\;\tfrac{1}{3}\,t\,H_0'(t)\;+\;\tfrac{4}{9}\,H_0(t)\;-\;\tfrac{8}{3}\,H_1(t)\\ &\quad=\;\tfrac{20}{3}\,G_2(t)\,t\,H_0'(t)\;-\;\Big(64\,G_2^2(t)\;+\;\tfrac{40}{3}\,G_2(t)\;-\;8\,t\,G_2'(t)\Big)H_0(t).\end{aligned}$$

On the other hand, we have

$$\tfrac{1}{8}\,N_0(t^2)\;=\;M_0(t)-P_0(t), \tag{3.4}$$

$$t\,\frac{d}{dt}\,N_0(t)\;=\;24\,G_2(t)\,N_0(t)\;+\;N_0(t), \tag{3.5}$$

$$N_1(t)\;=\;\Big(t\,\frac{d}{dt}\,G_2(t)\Big)N_0(t) \tag{3.6}$$

(cf. [10, 1, 8]). It then follows from (3.4) and (3.5) that

$$t\,\frac{d}{dt}\big(\,M_0(t)\;-\;P_0(t)\,\big)\;=\;\big(\,48\,G_2(t^2)\;+\;2\,\big)\,\big(\,M_0(t)\;-\;P_0(t)\,\big). \tag{3.7}$$

Recalling (S, F) is either $(2s, f)$ or $(s-3f, 2f)$, we combine (3.3) and (3.7) to show

$$M_1(t)\;-\;P_1(t)\;=\;\big(\,4\,t^2\,G_2'(t^2)\;+\;3\,E(t)\,\big)\big(\,M_0(t)\;-\;P_0(t)\,\big) \tag{3.8}$$

where $E(t)\;=\;32\,G_2^2(t^2)-40\,G_2(t^2)\,G_2(t)+8\,G_2^2(t)-t\,G_2'(t)$. Then, we have

$$\begin{aligned}M_1(t)\;-\;P_1(t)\;&=\;16\,t^2\,G_2'(t^2)\,\big(\,M_0(t)\;-\;P_0(t)\,\big)\\ &=\;2\,t^2\,G_2'(t^2)\,N_0(t^2)\;=\;2\,N_1(t^2)\end{aligned}$$

where the first equality follows from (3.8) and the fact $E(t)\;=\;4\,t^2\,G_2'(t^2)$ [10], the second equality follows from (3.4), and the last equality follows from (3.6). □

Acknowledgment

The first author would like to thank Thomas Parker for his extremely helpful discussions. He is also grateful to Eleny Ionel, Bumsig Kim, and Ionut Ciocan-Fontaine for their useful comments. In addition, the first author wishes to thank Ronald Fintushel for his interest in this work and especially for his encouragement. We also thank the referee for useful comments.

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921 D Cherry Lane, East Lansing, Michigan 48823
E-mail address: leejunho@msu.edu

Institute of Mathematical Sciences, The Chinese University of Hong Kong, Shatin, NT, Hong Kong
E-mail address: leung@ims.cuhk.edu.hk