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# Mirror Symmetry Without Corrections

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We give geometric explanations and proofs of various mirror symmetry conjectures for $T^n$-invariant Calabi-Yau manifolds when instanton corrections are absent. This uses a fiberwise Fourier transformation together with a base Legendre transformation.
We discuss mirror transformations of
(i) moduli spaces of complex structures and complexified symplectic structures, $H^{p,q}$'s, Yukawa couplings;
(ii) $\mathbf{sl}(2) \times \mathbf{sl}(2)$-actions;
(iii) holomorphic and symplectic automorphisms and
(iv) A- and B-connections, supersymmetric A- and B-cycles, correlation functions.
We also study (ii) for $T^n$-invariant hyperkahler manifolds.

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The mirror symmetry conjecture predicts that there is a transformation from the complex (resp. symplectic) geometry of one Calabi-Yau manifold $M$ to the symplectic (resp. complex) geometry of another Calabi-Yau manifold $W$ of the same dimension. Such pairs of manifolds are called mirror manifolds. This transformation should also has the inversion property, namely if we take the transformation twice, we recover the original geometry.

It is expected that such transformation exists for Calabi-Yau manifolds near a *large complex structure limit* point. Such point in the moduli space

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should correspond to the existence of a semi-flat Calabi-Yau metric, possibly highly singular.

To understand *why and how* these two different kinds of geometry were interchanged between mirror manifolds, we study the $T^n$-invariant case in details. The importance of the $T^n$-invariant (or more generally semi-flat) case is first brought up by Strominger, Yau and Zaslow in their foundational paper [SYZ] which explains mirror symmetry from a physical/geometric viewpoint. This is now called the SYZ mirror conjecture. The $T^n$-invariant case is then studied by Hitchin in [H1], Yau, Zaslow and the author in [LYZ] and it is also an important part of this paper. The main advantage here is the absence of holomorphic disks, the so-called instantons.

We start with an affine manifold $D$, which we assume to be a domain in $\mathbb{R}^n$ in this introduction. Let $\phi$ be an elliptic solution to the real Monge-Ampère equation on $D$:

$$\det \nabla^2 \phi = 1,$$
$$\nabla^2 \phi > 0.$$

Then it determines two noncompact Calabi-Yau manifolds, $TD$ and $T^*D$. Notice that $T^*D$ carries a canonical symplectic structure and $TD$ carries a canonical complex structure because $D$ is affine. We can also compactify the fiber directions by quotienting $TD$ and $T^*D$ with a lattice $\Lambda$ in $\mathbb{R}^n$ and its dual lattice $\Lambda^*$ in $\mathbb{R}^{n*}$ respectively and obtain mirror manifolds $M$ and $W$. The natural fibrations on $M$ and $W$ over $D$ are both special Lagrangian fibrations.

The mirror transform from $M$ to $W$, and vice versa, is a combination of (i) the Fourier transformation on fibers of $M \to D$ together with (ii) the Legendre transformation on the base $D$. The Calabi-Yau manifold $W$ can also be identified as the moduli space of flat $U(1)$ connections on special Lagrangian tori on $M$ with its $L^2$ metric. We are going to explain how the mirror transformation exchanges the complex geometry and the symplectic geometry between $M$ and $W$:

(1) The identification between moduli spaces of complex structures on $M$ and complexified symplectic structures on $W$, moreover this map is both a biholomorphism and an isometry;

(2) The identification of $H^{p,q}(M)$ and $H^{n-p,q}(W)$;

(3) The mirror transformation of certain A-cycles in $M$ to B-cycles in $W$. We also identify their moduli spaces and correlation functions (this is partly borrowed from [LYZ]). In fact the simplest case here is the classical Blaschke connection and its conjugate connection, they correspond to each other under the Legendre transformation;

(4) There is an $\mathbf{sl}(2)$-action on the cohomology of $M$ induced from variation of Hodge structures. Together with the $\mathbf{sl}(2)$-action given by the hard Lefschetz theorem, we obtain an $\mathbf{sl}(2) \times \mathbf{sl}(2)$-action on the cohomology of $M$. Under the mirror transformation from $M$ to $W$, these two $\mathbf{sl}(2)$-actions interchange their roles;

(5) Transformations of holomorphic automorphisms of $M$ to symplectic automorphisms of $W$, in fact its preserves a naturally defined two tensor on $W$, not just the symplectic two form.

In the last section we study $T^n$-invariant hyperkähler manifolds. That is when the holonomy group of $M$ is inside $Sp(n/2) \subset SU(n)$. The cohomology of a hyperkähler manifold admits a natural $\mathbf{so}(4,1)$-action. In the $T^n$-invariant case, we show that our $\mathbf{sl}(2) \times \mathbf{sl}(2) = \mathbf{so}(3,1)$-action on the cohomology is part of this hyperkähler $\mathbf{so}(4,1)$-action.

In [KS] Kontsevich and Soibelman also study mirror symmetry for these $T^n$-invariant Calabi-Yau manifolds, their emphasis is however very different from ours.

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## 1. $T^n$-invariant Calabi-Yau and their mirrors.

A Calabi-Yau manifold $M$ of real dimension $2n$ is a Riemannian manifold with $SU(n)$ holonomy, or equivalently a Kähler manifold with zero Ricci curvature. We can reduce this condition to a complex Monge-Ampère equation provided that $M$ is compact. Yau proved that this equation is always solvable as long as $c_1(M) = 0$, vanishing of the first Chern class of $M$.

Even though it is easy to construct Calabi-Yau manifolds, it is extremely difficult to write down their Ricci flat metrics. When Calabi-Yau manifolds have $T^n$ symmetry, we can study translation invariant solution to the complex Monge-Ampère equation and reduce the problem to finding a solution of a real Monge-Ampère equation.

These $T^n$-invariant Calabi-Yau manifolds form a natural class of semi-flat Calabi-Yau manifolds. Recall that a Calabi-Yau manifold is called semi-flat if it admits a fibration by flat special Lagrangian tori. Such manifolds are introduced into mirror symmetry in [SYZ] and then further studied in [H1], [Gr] and [LYZ].

**The real Monge-Ampère equation.** First we consider the dimension reduction of the complex Monge-Ampère equation to the real Monge-Ampère equation. The resulting Ricci flat metric would be a $T^n$-invariant Calabi-Yau metric: Let $M$ be a tubular domain in $\mathbb{C}^n$ with complex coordinates $z^j = x^j + iy^j$'s,

$$M = D \times i\mathbb{R}^n \subset \mathbb{C}^n,$$

where $D$ is a convex domain in $\mathbb{R}^n$. The holomorphic volume form on $M$ is given by

$$\Omega_M = dz^1 \wedge dz^2 \wedge \cdots \wedge dz^n.$$

Let $\omega_M$ be the Kähler form of $M$, then the complex Monge-Ampère equation for the Ricci flat metric is the following:

$$\Omega_M \bar{\Omega}_M = C\omega_M^n.$$

We assume that the Kähler potential $\phi$ of the Kähler form $\omega_M = i\partial\bar{\partial}\phi$ is invariant under translations along imaginary directions. That is,

$$\phi\left(x^j, y^j\right) = \phi\left(x^j\right)$$

is a function of the $x^j$'s only. In this case the complex Monge-Ampère equation becomes the real Monge-Ampère equation. Cheng and Yau [CY] proved that there is a unique elliptic solution $\phi(x)$ to the corresponding boundary value problem

$$\det\left(\frac{\partial^2\phi}{\partial x^j \partial x^k}\right) = C,$$
$$\phi|_{\partial D} = 0.$$

Ellipticity of a solution $\phi$ is equivalent to the convexity of $\phi$, i.e.

$$\left(\frac{\partial^2\phi}{\partial x^j \partial x^k}\right) > 0.$$

We can compactify imaginary directions by taking a quotient of $i\mathbb{R}^n$ by a lattice $i\Lambda$. That is we replace the original $M$ by $M = D \times iT$ where $T$ is the torus $\mathbb{R}^n/\Lambda$ and the above Kähler structure $\omega_M$ descends to $D \times iT$. If we write

$$\phi_{jk} = \frac{\partial^2\phi}{\partial x^j \partial x^k},$$

then the Riemannian metric on $M$ is

$$g_M = \Sigma\phi_{jk}\left(dx^j \otimes dx^k + dy^j \otimes dy^k\right)$$

and the symplectic form $\omega_M$ is

$$\omega_M = \frac{i}{2}\Sigma\phi_{jk}dz^j \wedge d\bar{z}^k.$$

Notice that $\omega_M$ can also be expressed as

$$\omega_M = \Sigma\phi_{jk}dx^j \wedge dy^k$$

because of $\phi_{jk} = \phi_{kj}$. The closedness of $\omega_M$ follows from $\phi_{ijk} = \phi_{kji} = \partial_i\partial_j\partial_k\phi$.

Remark: It is easy to see that $D \times i\Lambda \subset TD$ is a special Lagrangian submanifold (for the definition of a special Lagrangian, readers can refer to later part of this section.)

**Affine manifolds and complexifications.** Notice that the real Monge-Ampère equation

$$\det\left(\frac{\partial^2\phi}{\partial x^j\partial x^k}\right) = const,$$

is invariant under any affine transformation

$$\left(x^j\right) \to \left(\bar{x}^j\right) = \left(A^j_k x^k + B^j\right).$$

This is because

$$\frac{\partial^2\phi}{\partial x^j\partial x^k} = A^l_j A^m_k \frac{\partial^2\phi}{\partial\bar{x}^l\partial\bar{x}^m},$$

and

$$\det\left(\frac{\partial^2\phi}{\partial x^j\partial x^k}\right) = \det\left(A\right)^2 \det\left(\frac{\partial^2\phi}{\partial\bar{x}^l\partial\bar{x}^m}\right).$$

The natural spaces to study such equation are affine manifolds. A manifold $D$ is called an affine manifold if there exists local charts such that transition functions are all affine transformations as above. Over $D$, there is a natural real line bundle whose transition functions are given by $\det A$. We denote it by $\mathbb{R} \to L \to D$. Now suppose $\phi\left(x\right)$ is a solution to the above equation with $const = 1$ on the coordinate chart with local coordinates $x^j$'s. Under the affine coordinate change $\bar{x}^j = A^j_k x^k + B^j$, the function $\bar{\phi} = \left(\det A\right)^2\phi$ satisfies

$$\det\left(\frac{\partial^2\bar{\phi}}{\partial\bar{x}^j\partial\bar{x}^k}\right) = 1.$$

Therefore on a general affine manifold $D$, a solution to the real Monge-Ampère equation (with $const = 1$) should be considered as a section of $L^{\otimes 2}$.

It is not difficult to see that the tangent bundle of an affine manifold is naturally an affine complex manifold: If we write a tangent vector of $D$ as $\Sigma y^j \frac{\partial}{\partial x^j}$ locally, then $z^j = x^j + iy^j$'s are local holomorphic coordinates of $TD$. The transition function for $TD$ becomes $\left(z^j\right) \to \left(A^j_k z^k + B^j\right)$, hence $TD$ is an affine complex manifold.

We want to patch the $T^n$-invariant Ricci flat metric on each coordinate chart of $TD$ to the whole space and thus obtaining a $T^n$-invariant Calabi-Yau manifold $M = TD$ (or $TD/\Lambda$). To do this we need to assume that $\det A = 1$ for all transition functions, such $D$ is called a special affine manifold. Then

$$g_M = \Sigma \phi_{jk} \left(dx^j \otimes dx^k + dy^j \otimes dy^k\right)$$
$$\omega_M = \Sigma \phi_{jk}(x)\, dx^j \wedge dy^k = \frac{i}{2} \Sigma \phi_{jk} dz^j \wedge d\bar{z}^k.$$

are well-defined Kähler metric and Kähler form over the affine complex manifold $M$, which has a fibration over the real affine manifold $D$. Moreover

$$g_D = \Sigma \phi_{jk}(x)\, dx^j \otimes dx^k$$

defines a Riemannian metric of Hessian type on $D$.

**Legendre transformation.** All our following discussions work for $D$ being a special orthogonal affine manifold. For simplicity we assume that $D$ is simply a convex domain in $\mathbb{R}^n$ and $M = TD = D \times i\mathbb{R}^n$.

It is well-known that one can produce another solution to the real Monge-Ampère equation from any given one via the so-called Legendre transformation: We consider a change of coordinates $x_k = x_k\left(x^j\right)$ given by

$$\frac{\partial x_k}{\partial x^j} = \phi_{jk},$$

thanks to the convexity of $\phi$. Then we have

$$\frac{\partial x^j}{\partial x_k} = \phi^{jk},$$

where

$$\left(\phi^{jk}\right) = \left(\phi_{jk}\right)^{-1}.$$

Since $\phi^{jk} = \phi^{kj}$, locally there is a function $\psi(x_k)$ on the dual vector space $\mathbb{R}^{n*}$ such that

$$x^j(x_k) = \frac{\partial \psi(x_k)}{\partial x_j}.$$

Therefore,

$$\phi^{jk} = \frac{\partial^2 \psi}{\partial x_j \partial x_k}.$$

This function $\psi(x_k)$ is called the Legendre transformation of the function $\phi(x^j)$. It is obvious that the convexity of $\phi$ and $\psi$ are equivalent to each other. Moreover

$$\det\left(\frac{\partial^2 \phi}{\partial x^j \partial x^k}\right) = C,$$

is equivalent to

$$\det\left(\frac{\partial^2 \psi}{\partial x_j \partial x_k}\right) = C^{-1}.$$

Furthermore the Legendre transformation has the inversion property, namely the transformation of $\psi$ is $\phi$ again.

**Dual tori fibration - fiberwise Fourier transformation.** This construction works for any $T^n$-invariant Kähler manifold $M$, not necessary a Calabi-Yau manifold. On $M = D \times iT$ there is a natural torus fibration structure given by the projection to the first factor,

$$M \to D,$$
$$\left(x^j, y^j\right) \to \left(x^j\right).$$

Instead of performing the Legendre transformation to the base of this fibration, we are going to replace the fiber torus $T = \mathbb{R}^n/\Lambda$ by the dual torus $T^* = \mathbb{R}^{n*}/\Lambda^*$, where $\Lambda^* = \{v \in \mathbb{R}^{n*} : v(u) \in \mathbb{Z} \text{ for any } u \in \Lambda\}$ is the dual lattice to $\Lambda$.

In dimension one, taking the dual torus is just replacing a circle of radius $R$ to one with radius $1/R$. In general, if $y^j$'s are the coordinates for $T$ and $y_j$'s their dual coordinates. Then a flat metric on $T$ is given by $\Sigma\phi_{jk} dy^j \otimes dy^k$ for some constant positive definite symmetric tensor $\phi_{jk}$. As usual we write

$$\left(\phi^{jk}\right) = \left(\phi_{jk}\right)^{-1},$$

then $\Sigma\phi^{jk} dy_j \otimes dy_k$ is the dual flat metric on $T^*$.

Now we write $W = D \times iT^*$, the fiberwise dual torus fibration to $M = D \times iT$. Since the metric $g_M$ on $M$ is $T^n$-invariant, its restriction to each torus $\{x\} \times iT$ is the flat metric $\Sigma \phi_{jk}(x)\, dy^j \otimes dy^k$. The dual metric on the dual torus $\{x\} \times iT^*$ is $\phi^{jk}(x)\, dy_j \otimes dy_k$. So the natural metric on $W$ is given by

$$g_W = \Sigma \phi_{jk} dx^j \otimes dx^k + \phi^{jk} dy_j \otimes dy_k.$$

If we view $T^*$ as the moduli space of flat $U(1)$ connections on $T$, then it is not difficult to check that the Weil-Petersson $L^2$ metric on $T^*$ is also $\phi^{jk} dy_j \otimes dy_k$.

If we ignore the lattice structure for the moment, then $M$ is the tangent bundle $TD$ of an affine manifold and

$$W = T^* D,$$

moreover, $g_W$ is just the induced Riemannian metric on the cotangent bundle from the Riemannian metric $g_D = \Sigma \phi_{jk} dx^j \otimes dx^k$ on $D$.

Even though $T^*D$ does not have a natural complex structure like $TD$, it does carry a natural symplectic structure:

$$\omega_W = \Sigma dx^j \wedge dy_j,$$

which is well-known and plays a fundamental role in symplectic geometry. $\omega_W$ and $g_W$ together determine an almost complex structure $J_W$ on $W$ as follow,

$$\omega_W(X, Y) = g_W(J_W X, Y).$$

In fact this almost complex structure is integrable and the holomorphic coordinates are given by $z_j = x_j + iy_j$'s where $x_j(x)$ is determined by the Legendre transformation $\frac{\partial x_j}{\partial x^k} = \phi_{jk}$ as before. In terms of this coordinate system, we can rewrite $g_W$ and $\omega_W$ as follows

$$\begin{aligned} g_W &= \Sigma \phi^{jk} (dx_j \otimes dx_k + dy_j \otimes dy_k) \\ \omega_W &= \frac{i}{2} \Sigma \phi^{jk} dz_j \wedge d\bar{z}_k. \end{aligned}$$

Suppose that $g_M$ is a Calabi-Yau metric on $M$, namely $\phi\left(x^j\right)$ satisfies the real Monge-Ampère equation, then $\psi(x_j)$ also satisfies the real Monge-Ampère equation because of

$$\phi^{jk} = \frac{\partial^2 \psi}{\partial x_j \partial x_k}.$$

Therefore the metric $g_W$ on $W$ is again a $T^n$-invariant Calabi-Yau metric.

We call this combination of the Fourier transform on fibers and the Legendre transform on the base of a $T^n$-invariant Kähler manifold the *mirror transformation.*

The similarities between $g_M, \omega_M$ and $g_W, \omega_W$ are obvious. In particular, the mirror transformation has the inversion property, namely the transform of $W$ is $M$ again.

Here is an important observation: On the tangent bundle $M = TD$, suppose we vary its symplectic structure while keeping its natural complex structure fixed. We would be looking at a family of solutions to the real Monge-Ampère equation. On the $W = T^*D$ side, the corresponding symplectic structure is unchanged, namely $\omega_W = \Sigma dx^j \wedge dy_j$. But the complex structure on $W$ varies because the complex coordinates on $W$ are given by $dz_j = \phi_{jk} dx^k + i dy_j$ which depends on particular solutions of the real Monge-Ampère equation.

By the earlier remark about the symmetry between $M$ and $W$, changing the complex structures on $M$ is also equivalent to changing the symplectic structures on $W$. To make this precise, we need to consider complexified symplectic structures by adding B-fields as we will explain later.

In fact the complex geometry and the symplectic geometry of $M$ and $W$ are indeed interchangeable! String theory predicts that such phenomenon should hold for a vast class of pairs of Calabi-Yau manifolds. This is the famous Mirror Symmetry Conjecture.

General Calabi-Yau manifolds do not admit $T^n$-invariant metrics, therefore we want to understand the process of constructing $W$ from $M$ via a geometric way. To do this we need to introduce A- and B-cycles.

**Supersymmetric A- and B-cycles.** It was first argued by Strominger, Yau and Zaslow [SYZ] from string theory considerations that the mirror manifold $W$ should be identified as the moduli space of special Lagrangian tori together with flat $U(1)$ connections on them. These objects are called supersymmetric A-cycles (see for example [MMMS], [L1]). Let us recall the definitions of A-cycles and B-cycles (we also include the B-field in these definitions, see the next section for discussions on the B-field).

**Definition 1.** *Let $M$ be a Calabi-Yau manifold of dimension $n$ with a complexified Kähler form $\omega^{\mathbb{C}} = \omega + i\beta$ and a holomorphic volume form $\Omega$. We called a pair $(C, E)$ a supersymmetric A-cycle (or simply A-cycle), if (i) $C$*

*is a special Lagrangian submanifold in $M$, namely $C$ is a real submanifold of dimension $n$ with*

$$\omega|_C = 0,$$

*and*

$$\operatorname{Im} e^{i\theta}\Omega|_C = 0,$$

*for some constant angle $\theta$ which is called the phase angle.*

*(ii) $E$ is a unitary vector bundle on $C$ whose curvature tensor $F$ satisfies the deformed flat condition,*

$$\beta|_C + F = 0.$$

Note that the condition (ii) implies that the restriction of $\beta$ to $C$ represents an integral cohomology class. Also the Lagrangian condition and the deformed flat equation can be combined into one complex equation on $C$:

$$\omega^{\mathbb{C}} + F = 0.$$

**Definition 2.** *Let $M$ be a Kähler manifold with a complexified Kähler form $\omega^{\mathbb{C}}$, we called a pair $(C, E)$ a supersymmetric B-cycle (or simply B-cycle), if $C$ is a complex submanifold in $M$ of dimension $m$, $E$ is a holomorphic vector bundle on $C$ with a Hermitian metric whose curvature tensor $F$ satisfies the following deformed Hermitian-Yang-Mills equations on $C$:*

$$\operatorname{Im} e^{i\theta}\left(\omega^{\mathbb{C}} + F\right)^m = 0,$$

*for some constant angle $\theta$ which is called the phase angle.*

**Constructing the mirror manifold.** Now we consider the moduli space of A-cycles $(C, E)$ on $M$ with $C$ a torus and the rank of $E$ equals one. In [SYZ] Strominger, Yau and Zaslow conjectured that $W$ is the mirror manifold of $M$. The $L^2$ metric on this moduli space is expected to coincide with the Calabi-Yau metric on $W$ after suitable *corrections* which comes from contributions from holomorphic disks in $M$ whose boundaries lie on these A-cycles, these are called instantons.

When $M$ is a $T^n$-invariant Calabi-Yau manifold with fibration $\pi : M \to D$ as before. Then each fiber of $\pi$ is indeed a special Lagrangian torus and $D$ is their moduli space. Each fiber together with the restricted metric is a flat torus. Its dual torus can be naturally identified with the moduli space of flat $U(1)$ connections on it. Therefore the space $W$, obtained by replacing each fiber torus in $M$ by its dual, can be naturally identified as the moduli space of A-cycles in this case.

Furthermore the $L^2$ metric on this moduli space coincides with the dual metric $g_W$ up to a constant multiple. Physically this is because of the absence of instanton in this case. We have the following simple result.

**Theorem 3.** *Under the natural identification of $W$ with the moduli space of flat $U(1)$ connections on special Lagrangian tori in $M$, the metric $g_W$ equals the $L^2$ metric on the moduli space multiply with the volume of the fiber.*

Proof: Recall that $D$ is the moduli space of special Lagrangian tori. Let $\frac{\partial}{\partial x^j}$ be a tangent vector at a point in $D$, say the origin. This corresponds to a harmonic one form on the central fiber $C \subset M$. This harmonic one form on $C$ is $\Sigma\phi_{jk}(0)\,dy^k$. Now the moduli space L$^2$ inner product of $\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial x^l}$ equals

$$\begin{aligned}
&\ll \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l} \gg \\
&= \int_C \left\langle \phi_{jk} dy^k, \phi_{lm} dy^m \right\rangle dv_C \\
&= \int_C \phi_{jk}\phi_{lm}\phi^{km} dv_C \\
&= \phi_{jl}(0)\, vol(C)\,.
\end{aligned}$$

Note that the volume of special Lagrangian fibers in a $T^n$-invariant Calabi-Yau manifold is constant.

On the other hand

$$g_W\left(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^l}\right) = \phi_{jl}(0)\,.$$

Similarly we can identify metrics along fiber directions of $\pi : W \to D$. By definition the L$^2$ metric has no mixed terms involving both the base and fiber directions. Hence we have the theorem. □

**Shrinking the torus fibers.** Now we fix the symplectic form on $M$ as $\omega_M = \Sigma\phi_{jk}dx^j \wedge dy^k$ and vary the complex structures. Instead of using holomorphic coordinates $z^j = x^j + iy^j$'s, we define the new complex structure on $M$ using the following holomorphic coordinates,

$$z_t^j = \frac{1}{t}x^j + iy^j,$$

for any $t \in \mathbb{R}_{>0}$. The corresponding Calabi-Yau metric becomes

$$g_t = \Sigma\phi_{jk}\left(\frac{1}{t}dx^j \otimes dx^k + tdy^j \otimes dy^k\right).$$

The same fibration $\pi : M \to D$ is a special Lagrangian fibration for each $t$. Moreover the volume form on $M$ is independent of $t$, namely $dv_M = \omega_M^n/n!$. As $t$ goes to zero, the size of the fibers shrinks to zero while the base gets infinitely large.

If we rescale the metric to $tg_t$, then the diameter of $M$ stays bound and $(M, tg_t)$'s converge in the Gromov-Hausdorff sense to the real $n$ dimension manifold $D$ with the metric $g_D = \Sigma\phi_{jk}dx^j dx^k$ as $t$ approaches zero. It is expected that similar behaviors hold true for Calabi-Yau metrics near the large complex structure limit, as least over a large portion of $M$. This prediction is verified by Gross and Wilson when $M$ is a K3 surface [GW].

**B-fields.** The purpose of introducing $B$-fields is to complexify the space of symplectic structures on $M$, the conjectural mirror object to the space of complex structures on $W$ which is naturally a complex space. Readers could skip this part for the first time.

The usual definition of a B-field $\beta$ is a harmonic two form of type $(1,1)$ on $M$, i.e. $\beta \in \Omega^{1,1}(M, \mathbb{R}/\mathbb{Z})$ with $d\beta = 0$ and $d^*\beta = 0$. These are equivalent to the following conditions,

$$d\beta = 0$$
$$\beta \wedge \omega_M^{n-1} = c'\omega_M^n.$$

It is shown by Gross [Gr] that if we consider a closed form $\beta$, then the modified Legendre transformation, as we will describe later, does preserve the Calabi-Yau condition on the $W$ side. However the harmonicity will be lost. To remedy this problem, we need to deform the harmonic equation. There are two natural way to do this, depending on whether we prefer the complex polarization or the real polarization. We will first discuss the one using the real polarization, namely the special Lagrangian fibration. When we are in the large complex and Kähler structure limit, the complex conjugation is the same as the real involution which sends the fiber directions to its negative, namely $dx^j \to dx^j$ and $dy^j \to -dy^j$ in our previous coordinates. In general the complex conjugation and the real involution are different. We denote the holomorphic volume form of $M$ under this real involution (resp. complex conjugation) by $\widehat{\Omega}$ (resp. $\bar{\Omega}$). The author thanks Gross for pointing out an earlier mistake about $\widehat{\Omega}$. Just like the distinction between Kähler metrics and those satisfying the Monge-Ampère equations, namely Calabi-Yau metrics, we need the following definitions.

**Definition 4.** *Let $M$ be a Calabi-Yau manifold with holomorphic volume form $\Omega$. Suppose that $\omega$ is a Kähler form on $M$ and $\beta$ is a closed real two from on $M$ of type $(1,1)$. Then $\omega^{\mathbb{C}} = \omega + i\beta$ is called a complexified Calabi-Yau Kähler form on $M$ if $\left(\omega^{\mathbb{C}}\right)^n$ is a nonzero constant multiple of $i^n\Omega \wedge \widehat{\Omega}$.*

$$\left(\omega^{\mathbb{C}}\right)^n = ci^n\Omega \wedge \widehat{\Omega}.$$

*We call this the complexified complex Monge-Ampère equation.*

The second definition of a B-field is to use $\bar{\Omega}$ and require that the Calabi-Yau manifold $M$ satisfies

$$\begin{aligned} \omega^n &= ci^n\Omega\bar{\Omega} \\ \operatorname{Im} e^{i\theta}\left(\omega + i\beta\right)^n &= 0 \\ \operatorname{Im} e^{i\phi}\Omega &= 0 \text{ on the zero section.} \end{aligned}$$

If we expand the second equation near the large Kähler structure limit, namely we replace $\omega$ by a large multiple of $\omega$, or equivalently we replace $\beta$ by a small multiple of it, we have

$$\left(\omega + i\varepsilon\beta\right)^n = \omega^n + i\varepsilon n\beta\omega^{n-1} + O\left(\varepsilon^2\right).$$

So if we linearize this equation, by deleting terms of order $\varepsilon^2$ or higher, then it becomes

$$\beta\omega^{n-1} = c'\omega^n.$$

That is $\beta$ is a harmonic real two form. This approximation is in fact the usual convention for a B-field.

**Including $B$-fields in the $T^n$-invariant case.** We first consider the case when $\omega^{\mathbb{C}} = \omega + i\beta$ satisfies the complexified Monge-Ampère equation.

We suppose $\pi : M \to D$ is a $T^n$-invariant Calabi-Yau manifold as before and $\omega_M = \Sigma\phi_{jk}(x)\,dx^j dy^k$ is a $T^n$-invariant Kähler form on it. As usual we will include a $B$-field on $M$ which is invariant along fiber directions of $\pi$, namely $\beta_M = i\partial\bar{\partial}\eta(x)$. It is easy to see that

$$\beta_M = \frac{i}{2}\Sigma\eta_{jk}(x)\,dz^j \wedge d\bar{z}^k = \Sigma\eta_{jk}(x)\,dx^j \wedge dy^k$$

with

$$\eta_{jk} = \eta_{kj} = \frac{\partial^2 \eta}{\partial x^j \partial x^k}.$$

Then the complexified Kähler form $\omega_M^{\mathbb{C}} = \omega_M + i\beta_M$ is a complexified Calabi-Yau Kähler form if and only if the complex valued function $\phi(x) + i\eta(x)$ satisfies the following complexified real Monge-Ampère equation,

$$\det(\phi_{jk} + i\eta_{jk}) = C,$$

for some nonzero constant $C$. If we write

$$\theta_{jk}(x) = \phi_{jk}(x) + i\eta_{jk}(x),$$

then the above equation becomes $\det(\theta_{jk}) = C$. In these notations, the complexified Kähler metric and complexified Kähler form on $M$ are

$$g_M^{\mathbb{C}} = \Sigma\theta_{jk}(x)\left(dx^j \otimes dx^k + dy^j \otimes dy^k\right) \text{ and}$$
$$\omega_M^{\mathbb{C}} = \frac{i}{2}\Sigma\theta_{jk}(x)\, dz^j \wedge d\bar{z}^k,$$

respectively.

Now we consider the dual $T^n$-invariant manifold $W$ as before. Instead of the Legendre transformation $dx_j = \Sigma\phi_{jk}dx^k$, we need to consider a complexified version of it. Symbolically we should write

$$dx_j = \Sigma\theta_{jk}dx^k = \Sigma(\phi_{jk} + i\eta_{jk})\, dx^k.$$

The precise meaning of this is the complex coordinates $dz_j$'s on $W$ is determined by $\operatorname{Re} dz_j = \phi_{jk}dx^k$ and $\operatorname{Im} dz_j = dy_j + \eta_{jk}dx^k$. That is $dz_j = dx_j + idy_j$.

As before we define

$$\left(\theta^{jk}\right) = (\theta_{jk})^{-1}.$$

It is easy to check directly that the canonical symplectic form on $W$ can be expressed as follow

$$\begin{aligned}\omega_W^{\mathbb{C}} &= \Sigma dx^j \wedge dy_j, \\ &= \frac{i}{2}\Sigma\theta^{jk}dz_j \wedge d\bar{z}_k.\end{aligned}$$

Similarly the corresponding complexified Kähler metric is given by

$$g_W^{\mathbb{C}} = \Sigma\theta^{jk}\left(dx_j \otimes dx_k + dy_j \otimes dy_k\right),$$
$$= \Sigma\theta^{jk} dz_j \otimes d\bar{z}_k.$$

After including the B-fields, we can argue using the same reasonings as before and conclude: If we varies the complexified symplectic structures on $M$ while keeping its complex structure fixed, then under the Fourier transformation along fibers and the Legendre transformation on the base, it corresponds to varying the complex structures on $W$ while keeping its complexified symplectic structure fixed. And the reverse also hold true.

**Theorem 5.** *Let $M$ be a $T^n$-invariant Calabi-Yau manifold and $W$ is its mirror. Then the moduli space of complex structures on $M$ (resp. on $W$) is identified with the moduli space of complexified symplectic structures on $W$ (resp. on $M$) under the above mirror transformation.*

Remark: In order to have the above mirror transformation between complex structures and symplectic structures, it is important that the B-fields satisfy the complexified Monge-Ampère equation instead of being a harmonic two form.

Next we use the second definition of a B-field, namely $\omega^n = ci^n\Omega\bar{\Omega}$, $\operatorname{Im} e^{i\theta}(\omega + i\beta)^n = 0$ and $\operatorname{Im} e^{i\phi}\Omega = 0$ on the zero section. We still use the Fourier and Legendre transformation, $\operatorname{Re} dz_j = \phi_{jk}dx^k$ and $\operatorname{Im} dz_j = dy_j + \eta_{jk}dx^k$. Then Gross observed that [Gr],

$$\Omega_W\bar{\Omega}_W = \prod\left(\phi_{jk}dx^k + idy_j + i\eta_{jk}dx^k\right)\left(\phi_{jk}dx^k - idy_j - i\eta_{jk}dx^k\right)$$
$$= \prod\left(\phi_{jk}dx^k + idy_j\right)\left(\phi_{jk}dx^k - idy_j\right).$$

So we still have $\omega_W^n = ci^n\Omega_W\bar{\Omega}_W$, as if $\beta$ has no effect.

If we restrict $\Omega_W$ to the zero section of $W$, which is defined by $y_j = 0$ for all $j$, then

$$\operatorname{Im} e^{i\theta}\Omega_W = \operatorname{Im} e^{i\theta}\prod\left(\phi_{jk}dx^k + idy_j + i\eta_{jk}dx^k\right)$$
$$= \operatorname{Im} e^{i\theta}\prod\left(\phi_{jk}dx^k + i\eta_{jk}dx^k\right)$$
$$= \operatorname{Im} e^{i\theta}\det\left(\phi_{jk} + i\eta_{jk}\right)dx^1\cdots dx^n.$$

Hence the equation $\operatorname{Im} e^{i\theta}(\omega + i\beta)^n = 0$ for $\beta$ on the $M$ side is equivalent to the zero section of $W$ being a special Lagrangian submanifold $\operatorname{Im} e^{i\theta}\Omega_W = 0$.

Hence under the mirror transformation, the following conditions on $M$,

$$\omega_M^n = ci^n \Omega_M \bar{\Omega}_M$$
$$\operatorname{Im} e^{i\theta} \left(\omega_M + i\beta_M\right)^n = 0$$
$$\operatorname{Im} e^{i\phi} \Omega_M = 0 \text{ on the zero section.}$$

becomes the corresponding conditions on $W$:

$$\omega_W^n = ci^n \Omega_W \bar{\Omega}_W$$
$$\operatorname{Im} e^{i\theta} \Omega_W = 0 \text{ on the zero section.}$$
$$\operatorname{Im} e^{i\phi} \left(\omega_W + i\beta_W\right)^n = 0.$$

In the following discussions, we will always use the first definition of the B-field.

## 2. Transforming $\Omega^{p,q}$, $H^{p,q}$ and Yukawa couplings.

**Transformation on moduli spaces: A holomorphic isometry.** Continuing from above discussions, we are going to analyze the mirror transformation from the moduli space of complexified symplectic structures on $M$ to the moduli space of complex structures on $W$. We will see that this map is both a holomorphic map and an isometry.

To do this, we need to study this transformation on the infinitesimal level. Since infinitesimal deformation of Kähler structures on $M$ (resp. complex structures on $W$) is parametrized by $H^1\left(M, T_M^*\right)$ (resp. $H^1\left(W, T_W\right)$)[1], we should have a homomorphism

$$T : H^1\left(M, T_M^*\right) \to H^1\left(W, T_W\right).$$

Suppose we vary the $T^n$ symplectic form on $M$ to

$$\omega_M^{new} = \omega_M + \varepsilon \Sigma \xi_{jk} dx^j dy^k = \Sigma\left(\phi_{jk} + \varepsilon \xi_{jk}\right) dx^j dy^k.$$

Here

$$\xi_{jk} = \frac{\partial^2 \xi(x)}{\partial x^j \partial x^k},$$

and $\varepsilon$ is the deformation parameter. Then

$$\Sigma \xi_{jk} dx^j dy^k = \frac{i}{2} \Sigma \xi_{jk} dz^j d\bar{z}^k$$

[1]Cohomology groups are interpreted as spaces of $T^n$-invariant harmonic forms.

represents an element in $\Omega^{0,1}(M, T_M^*)$ which parametrizes deformations of Kähler forms. This form is harmonic, namely it defines an element in $H^1(M, T_M^*)$, if and only if $\Sigma_k \xi_{jkk} = 0$ for all $j$. If we assume every member of the family of $T^n$-invariant Kähler forms is Calabi-Yau, then the infinitesimal variation $\xi$ satisfies a linearization of the Monge-Ampère equation. This implies that $\xi$ is harmonic.

Then the new complex structure on $W$ is determined by its new complex coordinates

$$\begin{aligned} dz_j^{new} &= \Sigma\left(\phi_{jk} + \varepsilon \xi_{jk}\right) dx^k + i dy_j \\ &= dz_j + \varepsilon \Sigma \xi_{jk} dx^k \\ &= \Sigma\left(\delta_j^l + \frac{\varepsilon}{2} \phi^{lk} \xi_{jk}\right) dz_l + \frac{\varepsilon}{2} \phi^{kl} \xi_{jk} d\bar{z}_l. \end{aligned}$$

Therefore if we project the new $\bar{\partial}$-operator on $W$ to the old $\Omega^{0,1}(W)$, we have

$$\bar{\partial}^{new} = \bar{\partial} - \frac{\varepsilon}{2} \Sigma \phi^{jk} \xi_{kl} \frac{\partial}{\partial z_l} \otimes d\bar{z}_j + O\left(\varepsilon^2\right).$$

It gives an element

$$-\frac{1}{2} \Sigma \phi^{jk} \xi_{kl} \frac{\partial}{\partial z_l} \otimes d\bar{z}_j \in \Omega^{0,1}(W, T_W),$$

that determines the infinitesimal deformation of corresponding complex structures on $W$. This element is harmonic, namely it defines an element in $H^1(W, T_W)$, if and only if $\frac{\partial}{\partial x_j}\left(\xi_{lk} \phi^{jl}\right) = 0$. This is equivalent to $\xi_{jjk} = 0$. Hence we have obtained explicitly the homomorphism

$$\begin{aligned} H^1(M, T_M^*) &\to H^1(W, T_W) \\ i \Sigma \xi_{jk} dz^j d\bar{z}^k &\to -\Sigma \xi_{jk} \phi^{kl} \frac{\partial}{\partial z_j} \otimes d\bar{z}_l. \end{aligned}$$

Notice that these infinitesimal deformations are $T^n$-invariant, $\xi_{jk} = \xi_{jk}(x)$. Therefore we can use $\xi_{jk}$'s to denote both a tensor in $M$ and its transformation in $W$.

We should also include the B-fields and use the complexified symplectic forms on $M$, however the formula is going to be the same (with $\theta$ replacing $\phi$). From this description, it is obvious that the transformation from the moduli space of complexified symplectic forms on $M$ to the moduli space of complex structures on $W$ is holomorphic.

Next we are going to verify that this mirror map between the two moduli spaces is an isometry. We take two such deformation directions $i\Sigma\xi_{jk}dz^j d\bar{z}^k$ and $i\Sigma\zeta_{jk}dz^j d\bar{z}^k$, their L$^2$-inner product is given by

$$\left\langle i\Sigma\xi_{jk}dz^j d\bar{z}^k, i\Sigma\zeta_{jk}dz^j d\bar{z}^k \right\rangle_M = 2V \int_D \phi^{jl}\phi^{km}\xi_{jk}\zeta_{lk}dv_D.$$

While the L$^2$-inner product of their image on the $W$ side is given by

$$\begin{aligned}
&\left\langle -\Sigma\xi_{jk}\phi^{kl}\frac{\partial}{\partial z_j}\otimes d\bar{z}_l, -\Sigma\zeta_{jk}\phi^{kl}\frac{\partial}{\partial z_j}\otimes d\bar{z}_l \right\rangle_W \\
&= 2V^{-1}\int_D \phi^{jp}\phi_{lq}\left(\zeta_{jk}\phi^{kl}\xi_{pm}\phi^{mq}\right) dv_D \\
&= 2V^{-1}\int_D \phi^{jl}\phi^{km}\xi_{jk}\zeta_{lk}dv_D.
\end{aligned}$$

Here $V$ (resp. $V^{-1}$) is the volume of the special Lagrangian fiber in $M$ (resp. $W$). Therefore up to a overall constant, this transformation between the two moduli spaces is not just holomorphic, it is an isometry too. We conclude that

**Theorem 6.** *The above explicit mirror map from the moduli space of complex structures on $M$ (resp. on $W$) to the moduli space of complexified symplectic structures on $W$ (resp. on $M$) is a holomorphic isometry.*

**Transforming differential forms.** Next we transform differential forms of higher degrees from $M$ to $W$:

$$T : \Omega^{0,q}\left(M, \Lambda^p T_M^*\right) \to \Omega^{0,q}\left(W, \Lambda^p T_W\right).$$

Using the triviality of the canonical line bundle of $W$, this is the same as

$$T : \Omega^{p,q}\left(M\right) \to \Omega^{n-p,q}\left(W\right).$$

Readers are reminded that we are discussing only $T^n$-invariant differential forms.

First we give the motivations for this homomorphism. Since $M$ and $W$ are related by fiberwise dual torus construction, the obvious transformation for their tensors would be

$$\begin{cases} dx^j \to dx^j = \Sigma\phi^{jk}dx_k \\ dy^j \to \frac{\partial}{\partial y_j}. \end{cases}$$

In symplectic language, such transformation uses the real polarizations on $M$ and $W$. To transform $(p,q)$ forms, we want to map this real polarization to the complex polarization. The real polarization is defined by the vertical tangent bundle $V \subset T_M$ and the complex polarization is defined by $T_M^{1,0} \subset T_M \otimes \mathbb{C}$. So it is natural to carry $V \otimes \mathbb{C}$ to $T_M^{1,0}$ and its complement to $T_M^{0,1}$. That is $dz^j \to dy^j$ and $d\bar{z}^j \to dx^j$ on the $M$ side. By doing the same identification on the $W$ side and compose with the above transformation, we have

$$T : \Omega^{0,q}\left(M, \Lambda^p T_M^*\right) \to \Omega^{0,q}\left(W, \Lambda^p T_W\right),$$

with

$$T\left(dz^j\right) = \frac{\partial}{\partial z_j}$$
$$T\left(d\bar{z}^j\right) = \Sigma \phi^{jk} d\bar{z}_k.$$

This homomorphism obviously coincides with the previous identification between infinitesimal deformation of symplectic structures on $M$ and complex structures on $W$ up to a constant factor $i$.

Using the holomorphic volume form $\Omega_W = dz_1 dz_2 \cdots dz_n$ on $W$, we can identify $\wedge^p T_W^{1,0}$ with $\Lambda^{n-p} T_W^{1,0*}$, so we obtain a homomorphism

$$T : \Omega^{p,q}(M) \to \Omega^{n-p,q}(W).$$

Explicitly if

$$\alpha = \Sigma \alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} dz^{i_1} \cdots dz^{i_p} d\bar{z}^{j_1} \cdots d\bar{z}^{j_q} \in \Omega^{p,q}(M)$$

then

$$T(\alpha) = \Sigma \alpha_{i_1 \ldots i_p \bar{j}_1 \ldots \bar{j}_q} \phi^{k_1 \bar{j}_1} \cdots \phi^{k_q \bar{j}_q} dz_1 \cdots \widehat{dz_{i_1}} \cdots \widehat{dz_{i_p}} \cdots dz_n d\bar{z}_{k_1} \cdots d\bar{z}_{k_q}.$$

**Transforming $\mathbf{H}^{p,q}(M)$ to $\mathbf{H}^{n-p,q}(W)$.** If $\alpha \in \Omega^{p,q}(M)$ is a $T^n$-invariant form, then we claim that the above transformation of differential forms from $M$ to $W$ does commute with the $\bar{\partial}$-operator and also $\bar{\partial}^*$-operator. Therefore it descends to the Hodge cohomology (also Dolbeault cohomology) level:

$$T : H^{p,q}(M) \to H^{n-p,q}(W).$$

To simplify our notations we assume that $\alpha$ is of type $(1,1)$. That is

$$\alpha = \Sigma \alpha_{jk} dz^j \wedge d\bar{z}^k.$$

The form $\alpha$ being $T^n$-invariant means that $\alpha_{jk} = \alpha_{jk}(x)$ depends on the $x$ variables only. We have

$$\bar{\partial}\alpha = \frac{1}{2}\Sigma\left(\frac{\partial\alpha_{jk}}{\partial x^p} - \frac{\partial\alpha_{jp}}{\partial x^k}\right)dz^j d\bar{z}^p d\bar{z}^k$$

Their transformations are

$$T(\alpha) = \Sigma\alpha_{jk}dz_1\cdots\widehat{dz_j}\cdots dz_n d\bar{z}_l\phi^{kl},$$
$$T(\bar{\partial}\alpha) = \frac{1}{2}\Sigma\left(\frac{\partial\alpha_{jk}}{\partial x^p} - \frac{\partial\alpha_{jp}}{\partial x^k}\right)dz_1\cdots\widehat{dz_j}\cdots dz_n d\bar{z}_q d\bar{z}_l\phi^{kl}\phi^{pq}.$$

Now

$$\bar{\partial}T(\alpha) = \frac{1}{2}\Sigma\left(\frac{\partial}{\partial x_q}\left(\alpha_{jk}\phi^{kl}\right) - \frac{\partial}{\partial x_l}\left(\alpha_{jk}\phi^{kq}\right)\right)dz_1\cdots\widehat{dz_j}\cdots dz_n d\bar{z}_q d\bar{z}_l.$$

Using the Legendre transformation

$$\frac{\partial}{\partial x_q} = \Sigma\phi^{pq}\frac{\partial}{\partial x^p},$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial x_q}\left(\alpha_{jk}\phi^{kl}\right) - \frac{\partial}{\partial x_l}\left(\alpha_{jk}\phi^{kq}\right)\\
&= \Sigma\phi^{pq}\frac{\partial\alpha_{jk}}{\partial x^p}\phi^{kl} + \phi^{pq}\alpha_{jk}\frac{\partial\phi^{kl}}{\partial x^p} - \phi^{pl}\frac{\partial\alpha_{jk}}{\partial x^p}\phi^{kq} - \phi^{pl}\alpha_{jk}\frac{\partial\phi^{kq}}{\partial x^p}\\
&= \Sigma\left(\frac{\partial\alpha_{jk}}{\partial x^p} - \frac{\partial\alpha_{jp}}{\partial x^k}\right)\phi^{kl}\phi^{pq} + \alpha_{jk}\left(-\phi^{pq}\phi^{ks}\frac{\partial\phi_{st}}{\partial x^p}\phi^{tl} + \phi^{pl}\phi^{ks}\frac{\partial\phi_{st}}{\partial x^p}\phi^{tq}\right).
\end{aligned}$$

The second bracket vanishes because $\frac{\partial}{\partial x^p}\phi_{st}$ is symmetric with respect to $s, t$ and $p$. Hence we have

$$\bar{\partial}T(\alpha) = T(\bar{\partial}\alpha).$$

We can also verify

$$\bar{\partial}^*T(\alpha) = T(\bar{\partial}^*\alpha)$$

in the same way, and is left to our readers. Therefore the transformation $T$ descends to both the Hodge cohomology and Dolbeault cohomology. Moreover if we go the other direction, namely from $W$ to $M$, then the corresponding transformation is the inverse of $T$. That is $T$ is an isomorphism. So we have the following result.

**Theorem 7.** *The above mirror transformation $T$ identifies $\Omega^{p,q}(M)$ (resp. $H^{p,q}(M)$) with $\Omega^{n-p,q}(W)$ (resp. $H^{n-p,q}(W)$).*

**Transforming Yukawa couplings.** Next we compare Yukawa couplings on these moduli spaces of complex and symplectic structures; they are first computed by Mark Gross in [Gr]. We choose any $n$ closed differential forms of type $(1,1)$ on $M$: $\alpha, \beta, \cdots, \gamma$. We write $\alpha = \Sigma \alpha_{ij} dz^i \wedge d\bar{z}^j$ and so on. The Yukawa coupling in the A side on $M$ is defined and computed as follows,

$$\begin{aligned} {}_AY_M(\alpha, \beta, ..., \gamma) &= \int_M \alpha \wedge \beta \wedge \cdots \wedge \gamma \\ &= \int_M \pm \alpha_{i_1 j_1} \beta_{i_2 j_2} \cdots \gamma_{i_n j_n} dv_M \\ &= V \int_x \sum \pm \alpha_{i_1 j_1} \beta_{i_2 j_2} \cdots \gamma_{i_n j_n} dx^1 dx^2 \cdots dx^n \end{aligned}$$

where the summation is such that $\{i_1, i_2, ..., i_n\} = \{j_1, j_2, ..., j_n\} = \{1, 2, ..., n\}$. The constant $V$ is the volume of a special Lagrangian fiber in $M$.

For the Yukawa coupling in the $B$ side on $W$ we have the following definition: For $\alpha' = T(\alpha), \beta' = T(\beta), \cdots, \gamma' = T(\gamma) \in \Omega^{0,1}(W, T_W)$, we have

$${}_BY_W(\alpha', \beta', ..., \gamma') = \int_W \Omega \wedge \delta_{\alpha'} \delta_{\beta'} ... \delta_{\gamma'} \Omega.$$

Since $\Omega = dz_1 dz_2 \cdots dz_n$ with $dz_j = \Sigma \phi_{jk} dx^k + i dy_j$, we have

$$\delta_\alpha \Omega = \Sigma \alpha_{1k} dx^k dz_2 ... dz_n + dz_1 \alpha_{2k} dx^k ... dz_n + ... + dz_1 dz_2 ... \alpha_{nk} dx^k.$$

Similarly we obtain

$$\delta_{\alpha'} \delta_{\beta'} ... \delta_{\gamma'} \Omega = \sum \pm \alpha_{i_1 j_1} \beta_{i_2 j_2} \cdots \gamma_{i_n j_n} dx^1 dx^2 \cdots dx^n$$

and therefore, up to an overall constant, we have

$${}_AY_M(\alpha, \beta, ..., \gamma) = {}_B Y_W(\alpha', \beta', ..., \gamma').$$

**Theorem 8.** *[Gr] The above mirror transformation identifies the Yukawa coupling on the moduli spaces of complexified symplectic structures on $M$ (resp. on $W$) with the Yukawa coupling on the moduli space of complex structures on $W$ (resp. on $M$).*

Remark: The Yukawa coupling is the $n^{th}$ derivative of a local holomorphic function on the moduli space, called the prepotential $\mathcal{F}$. On the A-side, this is given by

$${}_A\mathcal{F}(M) = \int_M \omega^n.$$

On the B-side, we need to specify a holomorphic family of the holomorphic volume form locally on the moduli space of complex structures on $W$. Then the prepotential function is given by

$$ {}_B\mathcal{F}(W) = \int_W \Omega \wedge \bar{\Omega}. $$

Similarly we can identify these two prepotentials by this transformation.

In fact we can express this identification of the two moduli spaces, together with identifications of all these structures on them, namely $\Omega^{*,*}, H^{*,*}, \mathcal{Y}$ and $\mathcal{F}$, as an isomorphism of two Frobenius manifolds.

## 3. $\mathbf{sl}_2 \times \mathbf{sl}_2$-action on cohomology and their mirror transform.

In this section we show that on the levels of differential forms and cohomology of $M$, there are two commuting $\mathbf{sl}(2)$ Lie algebra actions. Moreover the mirror transformation between $M$ and $W$ interchanges them. The first $\mathbf{sl}(2)$-action exists for all Kähler manifolds. We should note that results of this section depend only on the $T^n$-invariant condition but not the Calabi-Yau condition.

This type of structure is first proposed by Gopakumar and Vafa in [GV2] on the moduli space of flat $U(1)$ bundles over curves in $M$, that is B-cycles. They conjectured that this $\mathbf{sl}(2) \times \mathbf{sl}(2)$-representation determines all Gromov-Witten invariants in every genus in a Calabi-Yau manifold. In fact we conjecture that such $\mathbf{sl}(2) \times \mathbf{sl}(2)$-action on cohomology groups should exist for every moduli space of A- or B-cycles (with the rank of the bundle equals one) on mirror manifolds $M$ and $W$ [L1].

**Hard Lefschetz $\mathbf{sl}(2)$-action.** Recall that the cohomology of any Kähler manifold admits an $\mathbf{sl}(2)$-action. Let us recall its construction: Let $M$ be a Kähler manifold with its Kähler form $\omega_M$. Wedging with $\omega_M$ gives a homomorphism $L_A$:

$$ L_A : \Omega^k(M) \to \Omega^{k+2}(M). $$

Let

$$ \Lambda_A : \Omega^{k+2}(M) \to \Omega^k(M) $$

be its adjoint homomorphism. Then we have the following relation,

$$ [L_A, \Lambda_A] = H_A, $$

where $H_A = (n-k)\, I$ is the multiplication endomorphism on $\Omega^k(M)$. Moreover we have

$$[L_A, H_A] = 2L_A,$$
$$[\Lambda_A, H_A] = -2\Lambda_A.$$

These commutation relations determine an $\mathbf{sl}(2)$-action on $\Omega^*(M)$. We call it the hard Lefschetz $\mathbf{sl}(2)$-action.

These operations commute with $\bar{\partial}$ and $\bar{\partial}^*$ because $\omega_M$ is a parallel form on $M$. Therefore this $\mathbf{sl}(2)$-action descends to the cohomology group $H^{*,*}(M)$.

**Variation of Hodge structures $\mathbf{sl}(2)$-action.** Suppose $M$ is a $T^n$-invariant manifold. It comes with a natural family of deformation of complex structures whose complex coordinates are given by $z^j = \frac{1}{t}x^j + iy^j$.

Recall from the standard deformation theory that a deformation of complex structures determines a variation of Hodge structures. Infinitesimally the variation of Hodge filtration $F^p(H^*(M,\mathbb{C}))$ lies insides $F^{p-1}(H^*(M,\mathbb{C}))$: If we write the infinitesimal variation of complex structure as $\frac{dM_t}{dt} \in H^1(M, T_M)$. Then the variation of Hodge structures is determined by taking the trace of the cup product with $\frac{dM_t}{dt}$ which sends $H^q(M, \Omega_M^p)$ to $H^{q+1}\left(M, \Omega_M^{p-1}\right)$. We denote this homomorphism by $L_B$. That is

$$L_B = \frac{dM_t}{dt} : H^{p,q}(M) \to H^{p-1,q+1}(M).$$

For the $T^n$-invariant Kähler manifold $M$, it turns out that $L_B$ determines an $\mathbf{sl}(2)$-action on $H^*(M,\mathbb{C}) = \oplus H^{p,q}(M)$.

To describe this $\mathbf{sl}(2)$-action explicitly, first we need to describe the adjoint of $L_B$ which we will call $\Lambda_B$. In general if

$$\frac{dM_t}{dt} = \Sigma a_{\bar{k}}^j(z,\bar{z}) \frac{\partial}{\partial z^j} \otimes d\bar{z}^k$$

on a Kähler manifold with metric $\Sigma g_{j\bar{k}} dz^j \otimes d\bar{z}^k$, then the adjoint of $L_B$ on the level of differential forms is just

$$\Lambda_B = \Sigma b_j^{\bar{k}} \frac{\partial}{\partial \bar{z}^k} \otimes dz^j$$

where $b_j^{\bar{k}} = \overline{g^{k\bar{l}} a_{\bar{l}}^m g_{m\bar{j}}}$. Since $\bar{\partial}\left(\frac{dM_t}{dt}\right) = 0$, $L_B$ commutes with the $\bar{\partial}$-operator. However $\Lambda_B$ might not commutes with $\bar{\partial}$ and therefore would not descend to the level of cohomology in general.

For the $T^n$-invariant case, it is not difficult to check directly that $\frac{dM_t}{dt} = \frac{t-1}{2}\Sigma\frac{\partial}{\partial z^j}\otimes d\bar{z}^j$. That is the whole family of complex structures on $M$ is along the same direction. We can rescale and assume

$$\frac{dM_t}{dt} = \Sigma\frac{\partial}{\partial z^j}\otimes d\bar{z}^j.$$

That is $a^j_{\bar{k}}(z,\bar{z}) = \delta_{jk}$. Hence

$$\begin{aligned} b^{\bar{k}}_j &= \overline{g^{k\bar{l}} a^m_{\bar{l}} g_{m\bar{j}}} \\ &= \phi^{kl}\delta_{ml}\phi_{mj} \\ &= \delta_{jk}. \end{aligned}$$

That is

$$\Lambda_B = \Sigma\frac{\partial}{\partial \bar{z}^j}\otimes dz^j.$$

Moreover their commutator $H_B = [L_B, \Lambda_B]$ is the multiplication of $(p-q)$ on forms in $\Omega^{p,q}(M)$. We have the following result:

**Theorem 9.** *On a $T^n$-invariant manifold $M$ as before, if we define*

$$\begin{aligned} L_B &= \Sigma\frac{\partial}{\partial z^j}\otimes d\bar{z}^j : \Omega^{p,q}(M) \to \Omega^{p-1,q+1}(M) \\ \Lambda_B &= \Sigma\frac{\partial}{\partial \bar{z}^j}\otimes dz^j : \Omega^{p,q}(M) \to \Omega^{p+1,q-1}(M) \\ H_B &= [L_B, \Lambda_B] = (p-q) : \Omega^{p,q}(M) \to \Omega^{p,q}(M). \end{aligned}$$

*Then they satisfy $\Lambda_B = (L_B)^*$ and*

$$\begin{aligned} [L_B, \Lambda_B] &= H_B \\ [H_B, L_B] &= -2L_B \\ [H_B, \Lambda_B] &= 2\Lambda_B. \end{aligned}$$

*Hence they define an* **sl**$(2)$*-action on $\Omega^{*,*}(M)$. Moreover these operators commute with $\bar{\partial}$ and $\bar{\partial}^*$ and descend to give an* **sl**$(2)$*-action on $H^*(M,\mathbb{C})$.*

As a corollary we have the following.

**Corollary 10.** *On any $T^n$-invariant manifold $M$ with $\frac{dM_t}{dt}$ as before. The operators $L_B$ defined by the variation of Hodge structures, its adjoint operator $\Lambda_B$ and their commutator $H_B = [L_B, \Lambda_B]$ together defines an* **sl**$(2)$*-action on the cohomology of $M$.*

We call this the variation of Hodge structures **sl**$(2)$-action, or simply VHS **sl**$(2)$-action.

**An $\mathbf{sl}(2) \times \mathbf{sl}(2)$-action on cohomology.** We already have two $\mathbf{sl}(2)$-actions on $H^*(M)$, we want to show that they commute with each other.

**Lemma 11.** *On a $T^n$-invariant manifold $M$ as above, we have*

$$[L_A, L_B] = 0,$$
$$[L_A, \Lambda_B] = 0.$$

Proof of lemma: We verify this lemma by direct calculations. Let us consider

$$\begin{aligned}
&L_A L_B \left(dz^{j_1} \cdots dz^{j_p} d\bar{z}^{k_1} \cdots d\bar{z}^{k_q}\right) \\
&= L_A \Sigma (-1)^{p-s} dz^{j_1} \cdots \widehat{dz^{j_s}} \cdots dz^{j_p} d\bar{z}^{j_s} d\bar{z}^{k_1} \cdots d\bar{z}^{k_q} \\
&= \Sigma (-1)^{p-s} (-1)^{p-1} \phi_{jk} dz^j dz^{j_1} \cdots \widehat{dz^{j_s}} \cdots dz^{j_p} d\bar{z}^k d\bar{z}^{j_p} d\bar{z}^{k_1} \cdots d\bar{z}^{k_q}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&L_B L_A \left(dz^{j_1} \cdots dz^{j_p} d\bar{z}^{k_1} \cdots d\bar{z}^{k_q}\right) \\
&= L_B \Sigma (-1)^p \phi_{jk} dz^j dz^{j_1} \cdots dz^{j_p} d\bar{z}^k d\bar{z}^{k_1} \cdots d\bar{z}^{k_q}.
\end{aligned}$$

If $j$ is not any of the $j_r$'s, then

$$\begin{aligned}
&L_B (-1)^p \phi_{jk} dz^j dz^{j_1} \cdots dz^{j_p} d\bar{z}^k d\bar{z}^{k_1} \cdots d\bar{z}^{k_q} \\
&= \Sigma \phi_{jk} dz^{j_1} \cdots dz^{j_p} d\bar{z}^j d\bar{z}^k dz^{k_1} \cdots dz^{k_q} \\
&+ \Sigma (-1)^p (-1)^{p-s} \phi_{jk} dz^j dz^{j_1} \cdots \widehat{dz^{j_s}} \cdots dz^{j_p} d\bar{z}^{j_s} d\bar{z}^k d\bar{z}^{k_1} \cdots d\bar{z}^{k_q}.
\end{aligned}$$

However the first term on the right hand side is zero because $\phi_{jk} = \phi_{kj}$ and $d\bar{z}^j d\bar{z}^k = -d\bar{z}^k d\bar{z}^j$. If $j$ is one of the $j_r$'s, it turns out we have the same result. This verifies $L_A L_B = L_B L_A$ on such forms. However forms of this type generate all differential form and therefore we have

$$[L_A, L_B] = 0.$$

If we replace $L_B = \Sigma \frac{\partial}{\partial z^j} \otimes d\bar{z}^j$ by $\Lambda_B = \Sigma \frac{\partial}{\partial \bar{z}^j} \otimes dz^j$, it is not difficult to check that the same argument works and give us

$$[L_A, \Lambda_B] = 0.$$

Hence we have the lemma. □

**Corollary 12.** *On the cohomology of $M$ as above, the hard Lefschetz* $\mathbf{sl}(2)$*-action and the VHS* $\mathbf{sl}(2)$*-action commute.*

*In other words, we have an* $\mathbf{sl}(2) \times \mathbf{sl}(2)$*-action on* $H^*(M, \mathbb{C})$.

Proof of corollary: From the lemma we have $[L_A, L_B] = 0$ and $[L_A, \Lambda_B] = 0$. Taking adjoint, we obtain the other commutation relations. Hence the result. □

Remark: The hard Lefschetz $\mathbf{sl}(2)$-action is a vertical action and the one from the variation of Hodge structure is a horizontal action with respect to the Hodge diamond in the following sense: $L_A(H^{p,q}) \subset H^{p+1,q+1}$ and $L_B(H^{p,q}) \subset H^{p-1,q+1}$.

Remark: Notice that $\mathbf{so}(3,1) = \mathbf{sl}(2) \times \mathbf{sl}(2)$. Later we will show that when $M$ is a hyperkähler manifold, this $\mathbf{so}(3,1)$-action embeds naturally inside the canonical hyperkähler $\mathbf{so}(4,1)$-action on its cohomology group.

**Transforming the $\mathbf{sl}(2) \times \mathbf{sl}(2)$-action.** First we recall that the variation of complex structures $dz^j = \frac{1}{t}dx^j + idy^j$ on $M$ was carried to the variation of symplectic structures $\omega = \frac{1}{t}dx^j dy_j$ on $W$.

**Theorem 13.** *Let $M$ and $W$ be mirror $T^n$-invariant Kähler manifolds to each other. Then the mirror transformation $T$ carries the hard Lefschetz* $\mathbf{sl}(2)$*-action on $M$ (resp. on $W$) to the variation of Hodge structure* $\mathbf{sl}(2)$*-action on $W$ (resp. on $M$).*

Proof: Let us start by comparing $H_A$ and $H_B$. On $\Omega^{p,q}(M)$, $H_B$ is the multiplication by $p-q$. On $\Omega^{n-p,q}(W)$, $H_A$ is the multiplication by $n-(n-p)-q = p-q$. On the other hand, $T$ carries $\Omega^{p,q}(M)$ to $\Omega^{n-p,q}(W)$. Therefore

$$H_A T = T H_B.$$

Next we compare $L_A$ and $L_B$ for one forms on $M$. For the $(0,1)$ form $d\bar{z}^j$, we have

$$\begin{aligned} T\left(d\bar{z}^j\right) &= \Sigma \phi^{jk} dz_1 \cdots dz_n d\bar{z}_k, \\ L_A T\left(d\bar{z}^j\right) &= 0. \end{aligned}$$

The last equality follows from type considerations. On the other hand $L_B\left(d\bar{z}^j\right) = 0$, therefore

$$L_A T\left(d\bar{z}^j\right) = T L_B\left(d\bar{z}^j\right).$$

For the $(1,0)$ form $dz^j$, we have

$$\begin{aligned} T\left(dz^j\right) &= (-1)^{n-j}\, dz_1 \cdots \widehat{dz_j} \cdots dz_n \\ L_A T\left(dz^j\right) &= \Sigma\, (-1)^{n-j}\, (-1)^{n+j}\, \phi^{jk} dz_1 \cdots dz_n d\bar{z}_k \\ &= \Sigma \phi^{jk} dz_1 \cdots dz_n d\bar{z}_k. \end{aligned}$$

On the other hand

$$\begin{aligned} L_B\left(dz^j\right) &= d\bar{z}^j \\ T L_B\left(dz^j\right) &= \Sigma \phi^{jk} dz_1 \cdots dz_n d\bar{z}_k. \end{aligned}$$

That is

$$L_A T\left(dz^j\right) = T L_B\left(dz^j\right).$$

Similarly we can argue for other forms in the same way and obtain

$$L_A T = T L_B.$$

We can also compare $\Lambda_A$ and $\Lambda_B$ in the same way to obtain

$$\Lambda_A T = T \Lambda_B.$$

Hence the variation of Hodge structure $\mathbf{sl}\,(2)$-action on $M$ was carried to the hard Lefschetz $\mathbf{sl}\,(2)$-action on $W$. By symmetry, the two actions flip under the mirror transformation $T$. □

## 4. Holomorphic vs symplectic automorphisms.

**Induced holomorphic automorphisms.** For any diffeomorphism $f$ of the affine manifold $D$, its differential $df$ is a diffeomorphism of $TD$ which is linear along fibers, for simplicity we ignore the lattice $\Lambda$ in this section and write $M = TD$. We write $f_B = df : M \to M$ explicitly as

$$f_B\left(x^j + iy^j\right) = f^k\left(x^j\right) + i\Sigma \frac{\partial f^k}{\partial x^j} y^j.$$

We want to know when $f_B$ is a holomorphic diffeomorphism of $M$. We compute

$$\begin{aligned}
&\frac{\partial}{\partial \bar{z}^l}\left(f^k + i\Sigma \frac{\partial f^k}{\partial x^j} y^j\right) \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x^l} + i\frac{\partial}{\partial y^l}\right)\left(f^k + i\Sigma \frac{\partial f^k}{\partial x^j} y^j\right) \\
&= \frac{1}{2}\left(\frac{\partial f^k}{\partial x^l} + i\Sigma \frac{\partial^2 f^k}{\partial x^j \partial x^l} y^j - \Sigma \frac{\partial f^k}{\partial x^j}\delta^j_l\right) \\
&= \frac{i}{2}\Sigma \frac{\partial^2 f^k}{\partial x^j \partial x^l} y^j.
\end{aligned}$$

Therefore $f_B$ is holomorphic on $M$ if and only if $f$ is an affine diffeomorphism on $D$. Moreover

$$(f \circ g)_B = f_B \circ g_B,$$

that is $f \to f_B$ is a covariant functor.

Even though $\bar{\partial} f_B$ does not vanish in general, its real part does. To understand what this implies, we recall that a $T^n$-invariant Calabi-Yau manifold has a natural deformation of complex structure towards its large complex structure limit point. Its complex coordinates are given by $dz^j(t) = t^{-1}dx^j + idy^j$'s with $t$ approaches 0. Therefore we would have

$$f_B\left(\frac{1}{t}x^j + iy^j\right) = \frac{1}{t}f^k\left(x^j\right) + i\Sigma \frac{\partial f^k}{\partial x^j} y^j,$$

and

$$\frac{\partial}{\partial \bar{z}^l(t)}\left(f^k + i\Sigma \frac{\partial f^k}{\partial x^j} y^j\right) = t\left(\frac{i}{2}\Sigma \frac{\partial^2 f^k}{\partial x^j \partial x^l} y^j\right).$$

Namely (1) the function $f_B$ is holomorphic at the large complex structure limit point $J_\infty$; (2) If $f$ is an affine diffeomorphism of $D$, then $f_B$ is holomorphic with respect to $J_t$ for all $t$. We denote this functor $f \to f_B$ in these two cases as follows:

$$(\bullet)_B : Diff(D) \to Diff(M, J_\infty),$$

and

$$(\bullet)_B : Diff(D, affine) \to Diff(M, J).$$

**Induced symplectic automorphisms.** On the other hand, any diffeomorphism $f : D \to D$ induces a diffeomorphism $\hat{f} : D^* \leftarrow D^*$ going the other direction. Here $D^* \subset \mathbb{R}^{n*}$ denote the image of the Legendre tranformation of $\phi$. Pulling back one forms defines a symplectic automorphism on the total space $T^*D^*$ which is just $M$ again (see below for explicit formula). We denote this functor as

$$\begin{array}{rccc} (\bullet)_A : & Diff\,(D) & \to & Diff\,(M, \omega) \\ & f & \to & f_A. \end{array}$$

Again

$$(f \circ g)_A = f_A \circ g_A.$$

That is $f \to f_A$ is also a covariant functor.

What if $f$ also preserves the affine structure on $D$?

The map $\hat{f}^* : M \to M$ is given by

$$\hat{f}^* \left( \hat{f}_j \left( x_k \right), y^j \right) = \left( x_j, \Sigma \frac{\partial \hat{f}_k}{\partial x_j} y^k \right),$$

for $\left( \hat{f}_j \left( x_k \right), y^j \right) \in T^*D^* = M$.

Since $M$ is the total space of a cotangent bundle $T^*D^*$, it has a canonical symplectic form, namely $\omega = \Sigma dx_j \wedge dy^j$, where $dx_j = \Sigma \phi_{jk} dx^k$. When the base space $D^*$ is an affine manifold, then there is a degree two tensor $\varpi$ on its cotangent bundle $M = T^*D^*$ whose antisymmetric part is $\omega$. It is given by

$$\varpi = \Sigma dx_j \otimes dy^j.$$

It is easy to see that $\varpi$ is well-defined on $M$.

**Lemma 14.** *If $f$ is a diffeomorphism of $D$, then $f$ preserves the affine structure on $D$ if and only if $f_A$ preserves $\varpi$ on $M$.*

Proof: Consider the inverse of $f$ as before, $\hat{f} : D^* \to D^*$. The pullback map it induced, $\hat{f}^* : M \to M$, is given by

$$\hat{f}^* \left( \hat{f}_j \left( x_k \right), y^j \right) = \left( x_j, \Sigma \frac{\partial \hat{f}_k}{\partial x_j} y^k \right),$$

for $\left(\hat{f}_j(x_k), y^j\right) \in T^*D^* = M$. We compute

$$\begin{aligned}\hat{f}^*(\varpi) &= \hat{f}^*\left(\Sigma dx_j \otimes dy^j\right) \\ &= \Sigma dx_j \otimes d\left(\frac{\partial \hat{f}_k}{\partial x_j} y^k\right) \\ &= \Sigma dx_j \otimes \left(\frac{\partial \hat{f}_k}{\partial x_j} dy^k + y^k \frac{\partial^2 \hat{f}_k}{\partial x_j \partial x_l} dx_l\right) \\ &= \varpi + \Sigma y^k \frac{\partial^2 \hat{f}_k}{\partial x_j \partial x_l} dx_j \otimes dx_l.\end{aligned}$$

Therefore $\hat{f}^*(\varpi) = \varpi$ if and only if $\hat{f}$ is an affine transformation on $D^*$. And this is equivalent to $f$ being an affine transformation on $D$. □

We denote this functor $f \to f_A$ in these two cases as follows:

$$(\bullet)_A : Diff(D) \to Diff(M, \omega),$$

and

$$(\bullet)_A : Diff(D, affine) \to Diff(M, \varpi).$$

**Transforming symplectic and holomorphic automorphisms.** We denote the spaces of those automorphisms $Diff(M, *)$ which are linear along fibers of the Lagrangian fibration by $Diff(M, *)_{lin}$. Here $*$ may stand for $J, J_\infty, \omega$ or $\varpi$. Notice that for any diffeomorphism $f$ of $D$, its induced diffeomorphisms $f_A$ and $f_B$ of $M$ are always linear along fibers of the Lagrangian fibration $\pi : M \to D$. In fact the converse is also true.

**Proposition 15.** *(i) The map $f \to f_B$ induces an isomorphism,*

$$(\bullet)_B : Diff(D) \overset{\cong}{\to} Diff(M, J_\infty)_{lin},$$

*and similarly*

$$(\bullet)_B : Diff(D, affine) \overset{\cong}{\to} Diff(M, J)_{lin}.$$

*(ii) Moreover the map $f \to f_A$ induces an isomorphism,*

$$(\bullet)_A : Diff(D) \overset{\cong}{\to} Diff(M, \omega)_{lin},$$

*and similarly,*

$$(\bullet)_A : Diff(D, affine) \overset{\cong}{\to} Diff(M, \varpi)_{lin}.$$

Proof of proposition: All these homomorphisms are obviously injective. To prove surjectivity, we let $F$ be any diffeomorphism of $M$ which is linear along fibers. We can write

$$F = \left(F^1, ..., F^n\right)$$
$$F^k = f^k(x) + i\Sigma g_l^k(x)\, y^l,$$

for some functions $f^k(x)$ and $g_l^k(x)$'s. We have

$$\begin{aligned}\frac{\partial}{\partial \bar{z}^j} F^k &= \left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j}\right)\left(f^k(x) + i\Sigma g_l^k(x)\, y^l\right) \\ &= \frac{\partial f^k}{\partial x^j} - g_l^k \delta_{jk} + i\frac{\partial g_l^k}{\partial x^j} y^k.\end{aligned}$$

So if $F$ preserves $J_\infty$, then

$$\operatorname{Re} \frac{\partial}{\partial \bar{z}^j} F^k = 0,$$

for all $j$ and $k$. That is $g_j^k = \frac{\partial f^k}{\partial x^j}$, or equivalently $F = f_B$. If $F$ preserves $J$, additionally we have

$$\begin{aligned}0 &= \frac{\partial g_l^k}{\partial x^j} \\ &= \frac{\partial}{\partial x^j}\frac{\partial f^k}{\partial x^l}.\end{aligned}$$

That is $f$ is an affine function of $x^j$'s. This proves part (i). The proof for the second part is similar. Hence we have the proposition. □

Remark: Paul Yang proved that every biholomorphism of $M = TD$ with $D$ convex is induced from an affine transformation on $D$, namely the assumption on the function $F$ being linear along fibers is automatic. That is $Diff(M, J) \cong Diff(M, J)_{lin} \cong Diff(D, affine)$.

Next we are going to show that the mirror transformation interchanges these two type of automorphisms. Recall that $W$ is the moduli space of special Lagrangian tori in $M$ together with flat $U(1)$ connections on them. That is the moduli space of $A$-cycles $(C, L)$ with $C$ a topological torus. Given any diffeomorphism $F : M \to M$ which is linear along fibers of $\pi$, $F$ carries a special Lagrangian torus $C$ in $M$, which is a fiber to $\pi$, to another special Lagrangian torus in $M$. The flat $U(1)$ connection over $C$ will be carried along under $F$. Therefore $F$ induces a diffeomorphism of $W$, this is the mirror transformation of $F$ and we call it $\hat{F}$ or $T(F)$.

**Theorem 16.** *For $T^n$-invariant Calabi-Yau mirror manifolds $M$ and $W$, the above mirror transformation $T$ induces isomorphisms : (i)*

$$T : Diff\,(M, J)_{lin} \overset{\cong}{\to} Diff\,(W, \varpi)_{lin}\,,$$
$$T : Diff\,(M, \varpi)_{lin} \overset{\cong}{\to} Diff\,(W, J)_{lin}\,.$$

*and (ii)*

$$T : Diff\,(M, J_\infty)_{lin} \overset{\cong}{\to} Diff\,(W, \omega)_{lin}\,,$$
$$T : Diff\,(M, \omega)_{lin} \overset{\cong}{\to} Diff\,(W, J_\infty)_{lin}\,.$$

*Moreover the composition of two mirror transformations is the identity.*

Proof of theorem: Given any $F \in Diff\,(M, J)_{lin}$ there is a unique diffeomorphism $f \in Diff\,(D, affine)$ such that $F = f_B$. Let $\hat{f}$ be the inverse of $f$ which we consider as an affine diffeomorphism of $D^*$. It is not difficult to verify that

$$\hat{F} = \left(\hat{f}\right)_A .$$

In particular $\hat{F} \in Diff\,(W, \omega)_{lin}$. Clearly

$$\hat{\hat{F}} = F.$$

Other isomorphisms can be verified in the same way. Hence we have the theorem. □

**Isometries of $M$.** Recall that a diffeomorphism of a Kähler manifold preserving both the complex structure and the symplectic structure is an isometry. Suppose $F$ is such an isometry of a $T^n$-invariant Calabi-Yau manifold $M$, and we assume that $F$ is also linear along fibers of the special Lagrangian fibration. Then it induces a diffeomorphism $f$ of $D$ which preserves $g_D = \Sigma\phi_{jk} dx^j \otimes dx^k$. That is $f \in Diff\,(D, g_D)$. In this case we have $F = f_A = f_B$. Hence we have

$$\begin{aligned} Diff\,(D, g_D) &= Diff\,(M, g)_{lin} \\ &= Diff\,(M, J)_{lin} \cap Diff\,(M, \omega)_{lin}\,. \end{aligned}$$

By the above theorem, this implies that the mirror transform $\hat{F}$ lies inside,

$$\hat{F} \in Diff\,(W, \varpi)_{lin} \cap Diff\,(W, J_\infty)_{lin}\,.$$

In fact one can also show that this common intersection is simply $Diff\,(W,g)_{lin}$. A different way to see this is to observe that the Legendre transformation from $D$ to $D^*$ preserves the corresponding metrics $g_D$ and $g_{D^*}$ because

$$\begin{aligned}&\Sigma\phi^{jk}dx_j\otimes dx_k\\&=\Sigma\phi^{jk}\left(\phi_{jl}dx^l\right)\otimes\left(\phi_{km}dx^m\right)\\&=\Sigma\delta_l^k\phi_{km}dx^l\otimes dx^m\\&=\Sigma\phi_{jk}dx^j\otimes dx^k.\end{aligned}$$

Therefore if $f\in Diff\,(D,g_D)$ then $\hat{f}\in Diff\,(D^*,g_{D^*})$. Hence $\hat{F}$ is an isometry of $W$,

$$\hat{F}\in Diff\,(W,g)\,.$$

Thus we have proved the following theorem.

**Theorem 17.** *For $T^n$-invariant Calabi-Yau mirror manifolds $M$ and $W$, the mirror transformation induces an isomorphism,*

$$T:Diff\,(M,g_M)_{lin}\overset{\cong}{\to}Diff\,(W,g_W)_{lin}\,.$$

*Moreover the composition of two mirror transformations is the identity.*

## 5. A- and B-connections.

Mirror transformations of other A- and B-cycles can be interpreted as generalization of the classical duality between Blaschke connection and its conjugate connection via the Legendre transformation. We first recall these classical geometries.

**Blaschke connection and its conjugate connection.** On a special affine manifold $D$, $\phi$ is a section of a trivial real line bundle over $D$. For simplicity we assume $D\subset\mathbb{R}^n$ and $\phi$ is a convex function on $D$. If $G\subset D\times\mathbb{R}$ denote the graph of $\phi$. Using the affine structure on $D\times\mathbb{R}$ one can define an affine normal $\nu$ which is a transversal vector field along $G$ (see for example [CY]): If we parallel translate the tangent plane of $G$, its intersection with $G$ determines a small convex domain. Its center of gravity then traces out a curve in the space whose initial direction is the affine normal direction.

Using $\nu$, we can decompose the restriction of the standard affine connection on $\mathbb{R}^n\times\mathbb{R}$ to $G$ into tangent directions and normal direction. So we obtain an induced torsion free connection on $G$, called the Blaschke connection [Bl], or the B-connection, ${}_B\nabla$. The convexity of $\phi$ implies that the second fundamental form is a positive definite symmetric two tensor $g_G$ on $G$. Its Levi-Civita connection is denoted as $\nabla^{LC}$. We define a conjugate connection ${}_A\nabla$ by

$$Xg_G(Y,Z) = g_G({}_B\nabla_X Y, Z) + g_G(Y, {}_A\nabla_X Z).$$

We call it an A-connection. The two connections ${}_A\nabla$ and ${}_B\nabla$ on $D$ induce torsion free connections on $M = TD$ by pullback. We continue to call them the A-connection ${}_A\nabla$ and the B-connection ${}_B\nabla$.

When the function $\phi$ satisfying the real Monge-Ampère equation, then $G$ is a parabolic affine sphere in $\mathbb{R}^n \times \mathbb{R}$. Namely the affine normal $\nu$ of $G$ in $\mathbb{R}^n \times \mathbb{R}$ is the unit vector along the last direction, that is the fiber direction of the real line bundle over $D$. By abuse of notations, we identify $G$ with $D$ via the projection to the first factor in $D \times \mathbb{R}$. These two torsion free connections ${}_A\nabla$ and ${}_B\nabla$ on $D$ are flat in this case. In term of the affine coordinates $x^j$'s on $D$, the B-connection ${}_B\nabla$ is just given by the exterior differentiation $d$. The A-connection is

$${}_A\nabla = d + \Sigma\phi^{jk}\phi_{klm}.$$

One can check directly that it has zero curvature. We will see later that this also follows from the Legendre transformation (or the mirror transformation).

**A- and B-connections on $T^n$-invariant Calabi-Yau manifolds.** From above, we have two torsion free flat connections ${}_A\nabla$ and ${}_B\nabla$ on $M = TD/\Lambda$. Recall that the complex structure on $M$ is given $z^j = x^j + iy^j$ and its symplectic form is $\omega_M = \Sigma\phi_{jk}(x)\,dx^j \wedge dx^k$. Since ${}_B\nabla$ is the same as the exterior differentiation on the affine coordinates $x^j$'s on $D$, it preserves the complex structure on $M$. In fact ${}_A\nabla$ preserves the symplectic structure on $M$.

**Proposition 18.** *Let $M = TD/\Lambda$ be a $T^n$-invariant Calabi-Yau manifold as before. Then its A-connection ${}_A\nabla$ and B-connection ${}_B\nabla$ satisfies*

$$\begin{aligned} {}_A\nabla\omega_M &= 0, \\ {}_B\nabla J &= 0. \end{aligned}$$

Proof of proposition: We have seen that ${}_B\nabla J = 0$. From previous discussions, ${}_A\nabla$ is a torsion free flat connection on $M$. To check that it preserves the symplectic form, we recall that $\omega_M = \Sigma\phi_{jk}(x)\,dx^j \wedge dy^k$ and ${}_A\nabla = d + \Gamma^j_{lm}dx^m$ where $\Gamma^j_{lm} = \phi^{jk}\phi_{klm}$. Note that ${}_A\nabla\left(dy^k\right) = 0$ because ${}_A\nabla$ is induced from the base $D$. Therefore

$$\begin{aligned} {}_A\nabla\omega_M &= \Sigma\phi_{jkl}dx^l \otimes \left(dx^j \wedge dy^k\right) - \Sigma\phi_{jk}\Gamma^j_{pq}dx^q \otimes \left(dx^p \wedge dy^k\right) \\ &= \Sigma\phi_{jkl}dx^l \otimes \left(dx^j \wedge dy^k\right) - \Sigma\phi_{jk}\phi^{jl}\phi_{lpq}dx^q \otimes \left(dx^p \wedge dy^k\right) \\ &= \Sigma\phi_{jkl}dx^l \otimes \left(dx^j \wedge dy^k\right) - \Sigma\phi_{kpq}dx^q \otimes \left(dx^p \wedge dy^k\right) \\ &= 0. \end{aligned}$$

We have use the symmetry of $\phi_{jkl}$ with respect to its indices. □

In section 6, we will see that the mirror transformation of flat connection ${}_A\nabla$ (resp. ${}_B\nabla$) on $M$ is the flat connection ${}_B\nabla$ (resp. ${}_A\nabla$) on the zero section in $W$ and vice versa. In particular the Levi-Civita connection $\nabla^{LC} = \left({}_A\nabla + {}_B\nabla\right)/2$ is preserved under the mirror transformation. In fact this is a special case of the mirror transformation between A- and B-cycles on $M$ and $W$.

## 6. Transformation of A- and B-cycles.

In this section we discuss how certain A-cycles on $M$ will transform to B-cycles on $W$. This materials is largely borrowed from [LYZ].

**Transforming A- and B-connections.** Recall the A-connection ${}_A\nabla$ on $M$ is $d + \Sigma\Gamma^j_{kl}dx^l$, where $\Gamma^j_{kl} = \Sigma\phi^{jm}\phi_{mkl}$, in the affine coordinate system. Let us consider $M$ and $W$ with their dual special Lagrangian tori fibrations. The restriction of ${}_A\nabla$ on each fiber in $M$ is trivial because $dx^l$'s vanish along fiber directions. Since the dual torus $T^*$ parametrizes flat $U(1)$ connections on $T$, the restriction of ${}_A\nabla$ corresponds to the origin of the corresponding dual torus. Putting all fibers on $M$ together, we obtain the zero section in $W$. This is the Fourier transformation.

However this is not the end of the story, the second fundamental form of ${}_A\nabla$ on each fiber in $M$ is non-trivial. This induces a connection on the zero section in $W$. To determine this connection, we need to perform the

Legendre transformation on $M$,

$$
\begin{aligned}
{}_A\nabla_{\frac{\partial}{\partial x^j}}\left(\tfrac{\partial}{\partial x^k}\right) &= \Gamma^l_{jk}\frac{\partial}{\partial x^l} \\
\nabla_{\frac{\partial}{\partial x^j}}\left(\Sigma\phi_{kq}\tfrac{\partial}{\partial x^q}\right) &= \Sigma\phi^{lm}\phi_{mjk}\frac{\partial}{\partial x^l} \\
\Sigma\phi_{kqj}\tfrac{\partial}{\partial x_q} + \Sigma\phi_{jp}\phi_{kq}\nabla_{\frac{\partial}{\partial x_p}}\left(\tfrac{\partial}{\partial x_q}\right) &= \Sigma\phi_{mjk}\frac{\partial}{\partial x_m} \\
\Sigma\phi_{jp}\phi_{kq}\nabla_{\frac{\partial}{\partial x_p}}\left(\tfrac{\partial}{\partial x_q}\right) &= 0.
\end{aligned}
$$

That is

$$\nabla_{\frac{\partial}{\partial x_p}}\left(\tfrac{\partial}{\partial x_q}\right) = 0,$$

or equivalently the induced connection on the zero section of $W$ is $d$ in the affine coordinate system of $W$. This is exactly the B-connection ${}_B\nabla$.

Conversely if we start with the B-connection ${}_B\nabla$ on the whole manifold $M$, its mirror transformation will be the A-connection ${}_A\nabla$ on the zero section of $W$. In particular we recover the classical duality between the Blaschke connection and its conjugate connection for the parabolic affine sphere. Such duality is in fact more interesting for other affine hypersurfaces (see for example [Lo]). Summarizing we have the following theorem.

**Theorem 19.** *For a $T^n$-invariant manifold $M$, the above mirror transformation take the A-connection (resp. B-connection) on the whole space $M$ to the B-connection (resp. A-connection) on the zero section of $W$.*

**Transforming special Lagrangian sections.** Now we are going to generalize the previous picture to duality between other supersymmetric cycles. Let $(C, E)$ be an A-cycle in $M$ such that $C$ is a section of the special Lagrangian fibration $\pi : M \to D$.

Note that $C = \{y = y(x)\} \subset M$ being Lagrangian with respect to $\omega_M = \Sigma\phi_{jk}dx^j dy^k$ is equivalent to

$$\frac{\partial}{\partial x^j}(y^l\phi_{lk}) = \frac{\partial}{\partial x^k}(\phi_{lj}y^l).$$

Therefore locally there is a function $f$ on $D$ such that

$$y^j = \Sigma\phi^{jk}\frac{\partial f}{\partial x^k}.$$

Next we want to understand the special condition on $C$. Namely

$$\operatorname{Im} e^{i\theta}\Omega_M|_C = 0.$$

Recall that the holomorphic volume form on $M$ equals $\Omega_M = dz^1 \wedge dz^2 \wedge ... \wedge dz^n$.

On the Lagrangian section $C$ we have

$$\begin{aligned} dy^j &= d\left(\Sigma\phi^{jk}\frac{\partial f}{\partial x^k}\right) \\ &= \Sigma\phi^{jl}\left(\frac{\partial^2 f}{\partial x^l \partial x^k} - \phi^{pq}\phi_{lkp}\frac{\partial f}{\partial x^q}\right) dx^k \\ &= \Sigma\phi^{jl}\, {}_A Hess\,(f)_{lk}\, dx^k. \end{aligned}$$

Here ${}_A Hess\,(f)$ denote the Hessian of $f$ with respect to the restriction of the torsion free A-connection ${}_A\nabla$ and we use the affine coordinate on $D$ to parametrize the section $C$. We have

$$\begin{aligned} dz^j &= dx^j + i dy^j \\ &= \Sigma\left(\delta_{jk} + i\phi^{jl}\left(\frac{\partial^2 f}{\partial x^l \partial x^k} - \phi^{pq}\phi_{lkp}\frac{\partial f}{\partial x^q}\right)\right) dx^k, \end{aligned}$$

and

$$\begin{aligned} \Omega_M|_C &= \det\left(I + ig^{-1}\, {}_A Hess\,(f)\right) dx^1 \wedge ... \wedge dx^n \\ &= \det(g)^{-1} \det\left(g + i\, {}_A Hess\,(f)\right) dx^1 \wedge ... \wedge dx^n, \end{aligned}$$

Hence $C$ is a special Lagrangian section if and only if

$$\operatorname{Im} e^{i\theta} \det\left(g + i\, {}_A Hess\,(f)\right) = 0.$$

Now we perform the fiberwise Fourier transformation on $M$. On each torus fiber $T$, the special Lagrangian section $C$ determines a point $y = (y^1, ..., y^n)$ on it, and therefore a flat $U(1)$ connection $D_y$ on its dual torus $T^*$. Explicitly, we have

$$D_y = d + i\Sigma y^j dy_j.$$

By putting all these fibers together, we obtain a $U(1)$ connection $\nabla_A$ on the whole $W$,

$$\nabla_A = d + i\Sigma y^j dy_j.$$

Its curvature two form is given by,

$$F_A = (\nabla_A)^2 = \Sigma i \frac{\partial y^j}{\partial x_k} dx_k \wedge dy_j.$$

The $(2,0)$ component of the curvature equals

$$F_A^{2,0} = \frac{1}{2}\Sigma \left( \frac{\partial y^k}{\partial x_j} - \frac{\partial y^j}{\partial x_k} \right) dz_j \wedge dz_k.$$

Therefore $\nabla_A$ gives a holomorphic line bundle on $W$ if and only if

$$\frac{\partial y^k}{\partial x_j} = \frac{\partial y^j}{\partial x_k},$$

for all $j,k$. This is equivalent to the existence of a function $f = f(x_j)$ on $D$ such that

$$y^j = \frac{\partial f}{\partial x_j}.$$

Therefore we can rewrite the curvature tensor as

$$F_A = i\Sigma \, {}_B Hess\,(f)_{jk}\, dx_k \wedge dy_j.$$

Here ${}_B Hess\,(f)$ is the Hessian of $f$ with respect to the B-connection ${}_B\nabla$ on $W$.

To compare with the $M$ side, we use the Legendre transformation to write

$$y^j = \Sigma \phi^{jk} \frac{\partial f}{\partial x^k}.$$

Then ${}_B Hess\,(f)$ on $W$ becomes ${}_A Hess\,(f)$ on $M$. Therefore the cycle $C \subset M$ being a special Lagrangian is equivalent to

$$F_A^{2,0} = 0,$$
$$\operatorname{Im} e^{i\theta} \left(\omega_W + F_A\right)^n = 0.$$

Next we bring back the flat $U(1)$ connection on $E$ over $C$ to the picture. We still use the affine coordinates on $D$ to parametrize $C$ because it is a section. We can express the flat connection on $C$ as

$$d + ide = d + i\Sigma \frac{\partial e}{\partial x^k} dx^k$$

for some function $e = e(x)$ on $C$. Now this connection will be added to the previous one on $W$ as the second fundamental form along fibers. We still call this connection $\nabla_A$. We have

$$\begin{aligned}\nabla_A &= d + i\Sigma y^j dy_j + ide\\ &= d + i\Sigma\phi^{jk}\frac{\partial f}{\partial x^k}dy_j + i\Sigma\frac{\partial e}{\partial x_j}dx_j.\end{aligned}$$

It is easy to see that the curvature form of this new connection is the same as the old one. In particular the transformed connection $\nabla_A$ on $W$ continues to satisfy the deformed Hermitian-Yang-Mills equations.

$$\begin{aligned}F_A^{0,2} &= 0,\\ \operatorname{Im} e^{i\theta}\left(\omega_W + F_A\right)^m &= 0,\end{aligned}$$

Therefore the mirror transformation of the A-cycle $(C, E)$ on $M$ produces a B-cycle on $W$. The same approach work for higher rank unitary bundle over the section $C$. This transformation is explained with more details in [LYZ].

**Transforming graded tangent spaces.** Recall from [L1] that the tangent space of the moduli space of A-cycle $(C, E)$ in $M$ is the space of complex harmonic one form with valued in the adjoint bundle. That is

$$T\left({}_A\mathcal{M}(M)\right) = H^1\left(C, ad(E)\right)\otimes\mathbb{C}.$$

And the tangent space of the moduli space of B-cycle $(C, E) = (W, E)$ in $W$ is the space of deformed $\bar\partial$-harmonic one form with valued in the adjoint bundle.

$$T\left({}_B\mathcal{M}(W)\right) = QH^1\left(C, End(E)\right).$$

A form $B \in \Omega^{0,q}\left(C, End(E)\right)$ is called a deformed $\bar\partial$-harmonic form if it satisfies the following deformation of the harmonic form equations:

$$\begin{aligned}\bar\partial B &= 0,\\ \operatorname{Im} e^{i\theta}\left(\omega + F\right)^{m-q}\wedge\partial B &= 0.\end{aligned}$$

Here $m$ is the complex dimension of $C$.

The graded tangent spaces are given by

$$\begin{aligned}T^{graded}\left({}_A\mathcal{M}(M)\right) &= \oplus_k H^k\left(C, ad(E)\right)\otimes\mathbb{C},\\ T^{graded}\left({}_B\mathcal{M}(W)\right) &= \oplus_k QH^k\left(C, End(E)\right).\end{aligned}$$

Now we identify these two spaces when $C \subset M$ is a special Lagrangian section. It is easy to see that the linearization of the above transformation of A-cycles on $M$ to B-cycles on $W$ is the following homomorphism

$$\Omega^1 (C, ad(E)) \otimes \mathbb{C} \to \Omega^{0,1} (W, End(E))$$
$$dx^j \to \Sigma \frac{i}{2} \phi^{jk} d\bar{z}_k .$$

We extend that homomorphism to higher degree forms, in the obvious way,

$$\Omega^q (C, ad(E)) \otimes \mathbb{C} \to \Omega^{0,q} (W, End(E)) .$$

It is verified in [LYZ] that the harmonic form equation on $\Omega^q (C, ad(E)) \otimes \mathbb{C}$ is transformed to the deformed harmonic form equation on $\Omega^{0,q} (W, End(E))$. Namely the image of $H^q (C, ad(E)) \otimes \mathbb{C}$ under the above homomorphism is inside $QH^q (W, End(E))$. In fact the image is given precisely by those forms which are invariant along fiber directions.

As a corollary of this identification, we can also see that the mirror transformation between moduli space of cycles, ${}_A\mathcal{M}(M) \to_B \mathcal{M}(W)$, is a holomorphic map.

**Identifying correlation functions.** The correlation functions on these moduli spaces of cycles are certain n-forms on them (see for example [L1] for the intrinsic definition). On the $M$ side, it is given by

$${}_A\Omega (C, E) (\alpha_1, ..., \alpha_n) = \int_C Tr_E [\alpha_1 \wedge ... \wedge \alpha_n]_{sym} ,$$

for $\alpha_j \in \Omega^1 (C, ad(E)) \otimes \mathbb{C}$ at a A-cycle $(C, E)$. On the $W$ side, it is given by

$${}_B\Omega (C, E) (\beta_1, ..., \beta_n) = \int_W \Omega_W Tr_E [\beta_1 \wedge \cdots \wedge \beta_n]_{sym} ,$$

for $\beta_j \in \Omega^{0,1} (W, End(E))$ at a B-cycle $(C, E) = (W, E)$. If $C \neq W$, then the formula is more complicated (see [L1]).

One can verify directly that the n-form ${}_B\Omega$ on the $W$ side is pullback to ${}_A\Omega$ on the $M$ side under the above mirror transformation (see [LYZ] for details). This verifies Vafa conjecture for rank one bundles in the $T^n$-invariant Calabi-Yau case. His conjecture says that the moduli spaces of A- and B-cycles, together with their correlation functions, on mirror manifolds should be identified. In general this identification should require instanton corrections.

## 7. $T^n$-invariant hyperkähler manifolds.

A Riemannian manifold $M$ of dimension $4n$ with holonomy group equals $Sp(n) \subset SU(2n)$ is called a hyperkähler manifold.

**$T^n$-invariant hyperkähler manifolds.** Let $D$ be an affine manifold with local coordinates $x^j$'s and $\phi(x)$ be a solution to the real Monge-Ampère equation $\det\left(\frac{\partial^2\phi}{\partial x^j \partial x^k}\right) = 1$. Then both its tangent bundle $TD$ and cotangent bundle $T^*D$ are naturally $T^n$-invariant Calabi-Yau manifolds. Moreover they are mirror to each other. If we denote the local coordinate of $TD$ as $x^j$ and $y^j$'s. Then the complex structure of $TD$ is determined by $dx^j + idy^j$'s as being $(1,0)$ forms and we call this complex structure $J$. Its symplectic form is given by $\omega = \Sigma\phi_{jk}(x)\, dx^j \wedge dy^k$ and its Ricci flat metric is $g = \Sigma\phi_{jk}\left(dx^j \otimes dx^k + dy^j \otimes dy^k\right)$.

Now we consider its cotangent bundle $M = T^*(TD)$ and denote the dual coordinates for $x^j$ and $y^j$ as $u_j$ and $v_j$ respectively. Therefore the induced metric on $M$ is given by

$$g_M = \Sigma\phi_{jk}\left(dx^j \otimes dx^k + dy^j \otimes dy^k\right) + \Sigma\phi^{jk}\left(du_j \otimes du_k + dv_j \otimes dv_k\right),$$

and its induced complex structure $J$ is determined by $dx^j + idy^j$'s and $du_j - idv_j$'s as being $(1,0)$ forms. Its corresponding symplectic form $\omega_J$ is given by

$$\omega_J = \Sigma\phi_{jk} dx^j \wedge dy^k - \Sigma\phi^{jk} du_j \wedge dv_k.$$

Since $M$ is the cotangent bundle of a complex manifold, it has a natural holomorphic symplectic form which we denote as $\eta_J$ and it is given by

$$\eta_J = \Sigma\left(dx^j + idy^j\right) \wedge \left(du_j - idv_j\right).$$

Notice that the projection $\pi : M \to TD$ is a holomorphic Lagrangian fibration with respect to $\eta_J$.

We are going to see that $M$ carries a natural hyperkähler structure. If we denote the real and imaginary part of $\eta_J$ by $\omega_I$ and $\omega_K$ respectively, then they are both real symplectic forms on $M$. Explicitly we have

$$\omega_I = \operatorname{Re}\eta_J = \Sigma\left(dx^j \wedge du_j + dy^j \wedge dv_j\right),$$
$$\omega_K = \operatorname{Im}\eta_J = \Sigma\left(dx^j \wedge dv_j - dy^j \wedge du_j\right).$$

They determine almost complex structures $I$ and $K$ on $M$ respectively. In fact these are both integrable complex structures. If we use the following change of variables, $du^j = \phi^{jk} du_k$ and $dv^j = \phi^{jk} dv_k$ then the complex structure of $I$ is determined by $dx^j + idu^j$ and $dy^j + idv^j$ as being $(1,0)$ forms. Similarly the complex structure of $K$ is determined by $dx^j + idv^j$ and $dy^j - idu^j$ as being $(1,0)$ forms. It follows from direct calculations that both $(M, g, I, \omega_I)$ and $(M, g, K, \omega_K)$ are Calabi-Yau structures on $M$. We can easily verify the following lemma.

**Lemma 20.** *$I^2 = J^2 = K^2 = IJK = -id$. Namely $(M, g)$ is a hyperkähler manifold.*

Remark: We call such $M$ a $T^n$-invariant hyperkahler manifold. Instead of $T^*(TD)$, we can also consider $T(T^*D)$ and it also has a natural hyperkähler structure constructed in a similar way. In fact these two are isomorphic hyperkähler manifolds.

**An $\mathbf{so}(4,1)$-action on cohomology.** For a hyperkähler manifold $M$, there is a $S^2$-family of Kähler structures $\omega_t$ on it: For any $t = (a,b,c) \in \mathbb{R}^3$ with $a^2+b^2+c^2 = 1$, $\omega_t = a\omega_I + b\omega_J + c\omega_K$ is a Kähler metric on $M$. For each $\omega_t$, there is a corresponding hard Lefschetz $\mathbf{sl}(2)$-action on its cohomology group $H^*(M, \mathbb{R})$. It is showed by Verbitsky in [Ve] that this $S^2$ family of $\mathbf{sl}(2)$-actions on $H^*(M, \mathbb{R})$ in fact determines an $\mathbf{so}(4,1)$-action on cohomology. It is interesting to compare the $\mathbf{so}(3,1)$-action from Gopakumar-Vafa conjecture with this $\mathbf{so}(4,1)$-action when $M$ admits a holomorphic Lagrangian fibration.

Note that $\mathbf{sl}(2) = \mathbf{so}(2,1)$ and $\mathbf{sl}(2) \times \mathbf{sl}(2) = \mathbf{so}(3,1)$. Therefore the cohomology group of Kähler manifolds admit $\mathbf{so}(2,1)$-actions, the cohomology of $T^n$-invariant Calabi-Yau manifolds admit $\mathbf{so}(3,1)$-actions and the cohomology of hyperkähler manifolds admit $\mathbf{so}(4,1)$-actions. We are going to show that the $\mathbf{so}(3,1)$-action we constructed in the $T^n$-invariant Calabi-Yau case is naturally embedded inside this $\mathbf{so}(4,1)$-action for hyperkähler manifolds. This is analogous to the statement that the hard Lefschetz $\mathbf{so}(2,1)$-action for Kähler manifolds is part of the $\mathbf{so}(3,1)$-action for Calabi-Yau manifolds, at least in the $T^n$-invariant case.

**Embedding $\mathbf{sl}(2) \times \mathbf{sl}(2)$ inside hyperkähler $\mathbf{so}(4,1)$-action.** As we discussed before, besides the hard Lefschetz $\mathbf{sl}(2)$-action on $\Omega^{*,*}(M)$, the other $\mathbf{sl}(2)$-action comes from a variation of complex structures on $M$. For our $T^n$-invariant hyperkähler manifold $M$ as above with the complex structure $I$, the Kähler structure $\omega_I$ and special Lagrangian fibration $\pi : M \to TD$, the second $\mathbf{sl}(2)$-action on $M$ can be expressed using $\omega_J$ and

$\omega_K$. That is we have a natural embedding of the $\mathbf{sl}(2) \times \mathbf{sl}(2)$-action into the hyperkähler $\mathbf{so}(4,1)$-action on $M$.

To verify this, we recall that the operator $L_B$ in the $\mathbf{sl}(2)$-action coming from the VHS will send $dx^j + idu^j$ (we write $du^j = \phi^{jk} du_k$) to $dx^j - idu^j$. On the other hand, for the operators $L_J, \Lambda_K$ in the hyperkähler $\mathbf{so}(4,1)$-action, we have

$$
\begin{aligned}
&[L_J, \Lambda_K] \left(dx^j + idu^j\right) \\
&= -\Lambda_K L_J \left(dx^j + idu^j\right) \\
&= -\Lambda_K \left(dx^j + idu^j\right) \left(\Sigma \phi_{kl} dx^k dy^l - \phi^{kl} du_k dv_l\right) \\
&= -\left(\phi^{kj} du_k - idx^k\right) \\
&= i\left(dx^j - idu^j\right),
\end{aligned}
$$

because $\omega_K = \Sigma\left(dx^j dv_j - dy^j du_j\right)$. The same holds true for all other forms. Thus we have the following theorem.

**Theorem 21.** *For any $T^n$-invariant hyperkähler manifold $M$, its Calabi-Yau $\mathbf{so}(3,1) = \mathbf{sl}(2) \times \mathbf{sl}(2)$-action on the cohomology of $M$ embeds naturally inside the hyperkähler $\mathbf{so}(4,1)$-action.*

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