# Geometric Aspects of Mirror Symmetry (with SYZ for Rigid CY manifolds)

Naichung Conan Leung
School of Mathematics,
University of Minnesota,
Minneapolis, MN 55455, U.S.A..

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**Abstract**

In this article we discuss the geometry of moduli spaces of (1) flat bundles over special Lagrangian submanifolds and (2) deformed Hermitian-Yang-Mills bundles over complex submanifolds in Calabi-Yau manifolds.

These moduli spaces reflect the geometry of the Calabi-Yau itself like a mirror. Strominger, Yau and Zaslow conjecture that the mirror Calabi-Yau manifold is such a moduli space and they argue that the mirror symmetry duality is a Fourier-Mukai transformation. We review various aspects of the mirror symmetry conjecture and discuss a geometric approach in proving it.

The existence of rigid Calabi-Yau manifolds poses a serious challenge to the conjecture. The proposed mirror partners for them are higher dimensional generalized Calabi-Yau manifolds. For example, the mirror partner for a certain K3 surface is a cubic fourfold and its Fano variety of lines is birational to the Hilbert scheme of two points on the K3. This hyperkähler manifold can be interpreted as the SYZ mirror of the K3 by considering *singular* special Lagrangian tori.

We also compare the geometries between a CY and its associated generalized CY. In particular we present a new construction of Lagrangian submanifolds.

# 1 Introduction

The mirror symmetry conjecture was proposed more than ten years ago from string theorists. From a mathematical viewpoint, it is an amazing conjecture about the geometry of Calabi-Yau manifolds.

Complex Kähler geometry studies manifolds $M$ with $U(n)$ holonomy. Using the complex structure $J$ and the Riemannian metric $g$, we define a parallel non-degenerate two form $\omega$,

$$\omega(X,Y) = g(JX,Y).$$

This is called the Kähler form and it defines a symplectic structure on $M$.

In complex geometry we study objects such as complex submanifolds and holomorphic vector bundles, or more generally coherent sheaves. Riemannian metric on $M$ is then used to rigidify these objects. By the Wirtinger's theorem, complex submanifolds are already rigidified, or calibrated. In particular, they are absolute volume minimizers. To rigidify a holomorphic vector bundle in geometry, we often look for a Hermitian-Yang-Mills connection, namely its curvature tensor satisfies the equation

$$F \wedge \omega^{n-1} = 0.$$

A famous theorem of Donaldson, Uhlenbeck and Yau says that on a Mumford stable bundle, there exists a unique Hermitian-Yang-Mills connection. In geometric invariant theory, we need Gieseker stable bundle to construct algebraic moduli space. On such bundles we have connections which satisfy a deformation of the Hermitian-Yang-Mills equations [L1]:

$$\left[e^{\frac{i}{2\pi}F+k\omega}Td(M)\right]^{2n} = C(k)\,\omega^n I_E,$$

for large enough $k$. We are going to absorb the factor $2\pi$ in the definition of $F$. If the Todd class of $M$ is trivial and $k$ equals one, then this equation becomes

$$(\omega + iF)^n = \text{constant}.$$

Note that both $\omega$ and $iF$ are real forms. From string theoretical considerations, the following deformed Hermitian-Yang-Mills equation is introduced in [MMMS] in order to preserve supersymmetry:

$$\mathrm{Im}\,(\omega + F)^n = 0.$$

A supersymmetry B-cycle is defined to be a pair $(C,E)$ where $C$ is a complex submanifold of $M$ and $E$ is a deformed Hermitian-Yang-Mills bundle over $C$. In section 3 and the appendix we will discuss the deformation theory of these B-cycles and introduce the correlation function which is the following n-form on their moduli space

$$_B\Omega = \int_C Tr\mathbb{F}^m \wedge ev^*\Omega.$$

On the other hand we can also study the symplectic geometry of $M$ using its Kähler form $\omega$ as the symplectic form. Natural geometric objects are Lagrangian submanifolds and their unitary flat bundles. To rigidify Lagrangian submanifolds we need a parallel n-form. This requires $M$ to be a Calabi-Yau manifold, namely its Riemannian metric has holonomy group in $SU(n)$. We denote its parallel holomorphic volume form as $\Omega$, it satisfies

$$\Omega\bar{\Omega} = (-1)^{n(n+1)/2}\, 2^n i^n \frac{\omega^n}{n!}.$$

This equation is equivalent to the Ricci flat condition for Calabi-Yau manifolds. Now we define a supersymmetry A-cycle as a pair $(C, E)$ where $C$ is a special Lagrangian submanifold in $M$ and $E$ is unitary flat bundle over $C$. A Lagrangian submanifold $C$ is called special [HL] if it is calibrated by $\operatorname{Re}\Omega$, namely

$$\operatorname{Re}\Omega|_C = vol_C.$$

This is also equivalent to

$$\operatorname{Im}\Omega|_C = 0.$$

Special Lagrangian submanifolds minimize volume functional among submanifolds representing the same homology class. In section 3, we discuss their deformation theory (after [Mc]) and introduce the correlation function which is also an n-form on the moduli space, namely

$${}_A\Omega = \int_C Tr\left(ev^*\omega + \mathbb{F}\right)^n.$$

On the symplectic side we have the Gromov-Witten theory which, roughly speaking, counts the number of holomorphic curves of various genera in $M$. In fact one should allow holomorphic curves to have real boundary whose images lie on $C$ (see for example [FOOO]). We call such holomorphic curves instantons.

Mirror symmetry conjectures predict that the complex geometry and the symplectic geometry on Calabi-Yau manifolds are essentially equivalent. More precisely for $M$ in a rather large class of Calabi-Yau manifolds, there should exist another Calabi-Yau manifold $W$ in the same class called the mirror manifold, this relation is reflexive; The complex geometry of $M$ should be equivalent to the symplectic geometry of $W$, with suitable instanton corrections. Conversely the symplectic geometry of $M$ should be equivalent to the complex geometry of $W$.

To understand *why and how* these two kinds of geometry are interchanged between mirror manifolds, we should first look at the semi-flat case. The importance of the semi-flat case is first brought up by Strominger, Yau and Zaslow in their paper [SYZ], which explains the mirror symmetry from a physical/geometric viewpoint. This is now called the SYZ mirror conjecture. The semi-flat case is then studied by Hitchin [H1], Gross [Gr2], Yau, Zaslow and the author in [LYZ] and also [L3]. The main advantage here is the absent of instantons:

We start with an affine manifold $D$, for simplicity we assume $D$ is just a domain in $\mathbb{R}^n$. Let $\phi$ be a solution to the real Monge-Ampère equation

$$\det\left(\nabla^2\phi\right) = 1,$$
$$\nabla^2\phi > 0.$$

Here $\nabla^2\phi$ is the Hessian of $\phi$ and it defines a Riemannian metric on $D$ which we call a Cheng-Yau manifold because of their fundamental result on the existence of such structure. Any such solution determines two open Calabi-Yau manifolds, $TD$ and $T^*D$. Notice that $T^*D$ carries a canonical symplectic structure and $TD$ carries a canonical complex structure because $D$ is affine. We can also compactify the fiber directions by quotienting $TD$ and $TD^*$ with a lattice $\Lambda$ in $\mathbb{R}^n$ and its dual lattice $\Lambda^*$ in $\mathbb{R}^{n*}$ respectively and obtains mirror manifolds $M$ and $W$. The natural fibrations on $M$ and $W$ over $D$ are both special Lagrangian fibrations.

The mirror transformation from $M$ to $W$, and vice versa, is a generalization of (i) the Fourier transformation on fibers of $M \to D$ together with (ii) the Legendre transformation on the base $D$. In section 5 we will explain how the mirror transformation exchanges the complex geometry and the symplectic geometry between $M$ and $W$.

The moduli space of complexified symplectic structures on $M$ is canonically identified with the moduli space of complex structures on $W$; moreover this map identifies various geometric structures on these moduli spaces, including the two Yukawa couplings. Moduli spaces of certain supersymmetric A-cycles on $M$ are canonically identified with moduli spaces of supersymmetric B-cycles on $W$, moreover this map identifies various geometric structures on these moduli spaces, including the two correlation functions. Holomorphic automorphisms of $M$ are transformed to symplectic automorphisms of $W$ when they are linear along fibers. On $M$ (and also $W$) there is an $\mathbf{sl}(2) \times \mathbf{sl}(2)$ action on its cohomology groups, these two $\mathbf{sl}(2)$ actions are interchanged under the mirror transformation from $M$ to $W$.

For general Calabi-Yau mirror manifolds $M$ and $W$, it is conjectured that above relationships should continue to hold after including instanton corrections on the symplectic side. In section 4 we explain various aspects of mirror symmetry conjectures. We also include a discussion on a closely related conjecture of Gopakumar and Vafa.

Even though the Fourier/Legendre transformation approach provides a good conceptual explanation to the mirror symmetry conjecture for Calabi-Yau *hypersurfaces* in toric varieties, there is a big puzzle when a Calabi-Yau manifold is only a complete intersection in a toric variety because such a Calabi-Yau may be a *rigid* manifold, namely $H^1(M, T_M) = 0$. Its mirror, if exists, would have $H^1(W, T_W^*) = 0$ and therefore $W$ cannot be Kähler, a contradiction! Nevertheless physicists still predict $M$ to have mirror duality and its mirror partner is a higher dimensional Fano manifold of a special kind.

In section 6, we provide a possible explanation of this mirror symmetry conjecture for (possibly rigid) Calabi-Yau manifolds using the SYZ approach. The basic idea is the complete intersection Calabi-Yau manifold should be interpreted as a moduli space of B-cycles in the higher dimensional Fano manifold. For example when $M$ is a Kummer K3 surface associated to the product of two elliptic curves with complex multiplication, its physical mirror partner would be the Fermat cubic fourfold $\bar{W}$ in $\mathbb{CP}^5$. It is a classical fact that the space of lines in $\bar{W}$, called the Fano variety of lines, is birational to the Hilbert scheme of two points on $M$. This hyperkähler manifold $W$ could be interpreted as the SYZ mirror to $M$. It is because each fiber in the natural special Lagrangian fibration on the four torus becomes a singular torus with a node in $M$, a generic deformation of it in $M$ is a smooth surface of genus two! The corresponding moduli space of A-cycles in $M$ is the hyperkähler manifold $W$.

It is a very important and interesting question to determine the complex and symplectic geometry (or category) of the Calabi-Yau manifold $M$ from the one on its associated higher dimensional Fano manifold $\bar{M}$. For example we will present a new construction of a Lagrangian submanifold in $\bar{M}$ from a (lower dimensional) Lagrangian submanifold in $M$. We would like to understand how the Floer homology theory of Lagrangian intersections behaves under this construction.

We should remark that the mirror symmetry for three dimensional Calabi-Yau manifolds has much richer structure than their higher dimensional counterparts. For example a conjecture of Gopakumar and Vafa [GV2] computes higher genus Gromov-Witten invariants of Calabi-Yau threefolds and another conjecture of Ooguri and Vafa [OV2] relates the large N Chern-Simons theory of knots to counting holomorphic disks on a local Calabi-Yau threefold. Certain parts of these conjectures are verfied mathematically by Bryan and Pandharipande [Pa], [BP], Katz and Liu [KL], Li and Song [LS]. The basic reason behind these is the $G_2$-structure on the seven dimensional manifold $M \times S^1$ (see e.g. [LL] and [L4] and references therein).

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## 2 Geometry of Calabi-Yau manifolds

Recall that a $2n$ dimensional compact Riemannian manifold $(M, g)$ with holonomy group inside $U(n)$ is a Kähler manifold. It has a parallel complex structure $J$ and a parallel symplectic structure $\omega$. By the classification result of Berger, possible special holonomy groups inside $U(n)$ include $SU(n)$ and $Sp(n/2)$. Corresponding manifolds are called Calabi-Yau manifolds and hyperkähler manifolds. They can be characterized by the existence of a parallel holomorphic volume form $\Omega$ and a parallel holomorphic symplectic form $\eta$ respectively. The following table gives a comparison among these geometries.

| | Kähler | Calabi-Yau | hyperkähler |
|---|---|---|---|
| Holonomy | $U(n)$ | $SU(n)$ | $Sp(n/2)$ |
| Parallel forms | $\omega \in \Omega^{1,1}$ | $\omega \in \Omega^{1,1}, \Omega \in \Omega^{n,0}$ | $\omega \in \Omega^{1,1}, \eta \in \Omega^{2,0}$ |
| Geometry | complex | cpx and sympl | $S^2$ family of cpx/sympl |
| Action on $H^*(M)$ | $\mathbf{so}(3)$ | $\mathbf{so}(4)$ | $\mathbf{so}(5)$ |

For the last row of the above table, the $\mathbf{so}(3)$ $(= \mathbf{sl}(2))$ action on $H^*(M)$ is the hard Lefschetz theorem. Notice that $\mathbf{so}(4) = \mathbf{sl}(2) \times \mathbf{sl}(2)$ and one of these two $\mathbf{sl}(2)$ actions in the Calabi-Yau case come from the Kähler geometry of $M$ as above. The existence of the other $\mathbf{sl}(2)$ action is conjectural, this will be explained in section 4. In the semi-flat case this second $\mathbf{sl}(2)$ action arises from a variation of complex structures toward the large complex limit point and it is described explicitly in [L3].

**Yau's theorem**

In general it is difficult to find manifolds with special holonomy other than $U(n)$. However in the $SU(n)$ case, we have the celebrated theorem of Yau: Any compact Kähler manifold with trivial canonical line bundle $K_M = \Lambda^n T_M^*$ admits a unique Kähler metric with $SU(n)$ holonomy inside any given Kähler class. Such a metric is called a Calabi-Yau metric.

**The complex Monge-Ampère equation**

This theorem is obtained by solving a fully nonlinear elliptic equation on $M$ with $c_1(M) = 0$: Suppose $\omega$ is any Kähler form on $M$, then its Ricci form can be expressed as $Rc(\omega) = i\partial\bar{\partial} \log \omega^n$ with respect to any local holomorphic coordinates on $M$. Since $c_1(M) = 0$, we must have $Rc(\omega) = i\partial\bar{\partial}F$ for some real valued function $F$ on $M$. By the $\partial\bar{\partial}$-lemma, any other Kähler form on $M$ in the same class must be of the form $\omega + i\partial\bar{\partial}f$ for some real valued function $f$ on

$M$. If this new Kähler form has zero Ricci curvature, $Rc\left(\omega+i\partial\bar{\partial}f\right)=0$, then

$$\begin{aligned}0&=i\partial\bar{\partial}\log\left(\omega+i\partial\bar{\partial}f\right)^n\\&=i\partial\bar{\partial}\log\frac{\left(\omega+i\partial\bar{\partial}f\right)^n}{\omega^n}+i\partial\bar{\partial}F.\end{aligned}$$

On a closed manifold $M$, this implies that $\log\frac{\left(\omega+i\partial\bar{\partial}f\right)^n}{\omega^n}+F=C$ for some constant $C$, which can be absorbed into $F$. Therefore the Kähler-Einstein equation becomes the following complex Monge-Ampère equation,

$$\left(\omega+i\partial\bar{\partial}f\right)^n=e^{-F}\omega^n.$$

By the work of Yau, this equation has a unique solution $f$, up to translation by a constant. Equivalently, there is a unique Ricci flat metric on each Kähler class if $c_1(M)=0$.

When the Ricci curvature is zero, $\int c_2(M)\,\omega^{n-2}$ becomes a positive multiple of the $\mathrm{L}^2$ norm of the Riemannian curvature tensor. Therefore if $M$ is a Calabi-Yau manifold, then

$$\int c_2(M)\,[\omega]^{n-2}\geq 0,$$

for any Kähler class $[\omega]$. Moreover it is zero if and only if $M$ is covered by a flat torus.

*SU* $(n)$ **holonomy**

Notice that $c_1(M)=0$ implies the canonical line bundle $K_M=\Lambda^n T_M^*$ of $M$ is topologically trivial. Suppose that $K_M$ is holomorphically trivial, which is automatic if $M$ is simply connected. Then there is a holomorphic section $\Omega$ of $K_M$. This is a holomorphic $n$-form on $M$.

To see that Ricci flat metric on such manifold has holonomy inside $SU(n)$, we need the following result which is proven by using standard Bochner type arguments: Every holomorphic $p$-form on a closed Ricci flat Kähler manifold is parallel. In particular $\Omega$ is a parallel holomorphic volume form. Using it, we can reduce the holonomy group from $U(n)$ to $SU(n)$. We can also normalize $\Omega$ so that

$$\Omega\bar{\Omega}=(-1)^{n(n+1)/2}\,2^n i^n\frac{\omega^n}{n!},$$

where $\omega$ is the Ricci flat Kähler form on $M$. In fact any Kähler form satisfying the above equation is automatically a Calabi-Yau metric since $Rc=i\partial\bar{\partial}\log\omega^n$.

**Examples of Calabi-Yau manifolds**

Obvious examples of Calabi-Yau manifolds are complex tori $\mathbb{C}^n/\Lambda$ with the flat metrics, however their holonomy groups are trivial. Smooth hypersurfaces in $\mathbb{CP}^{n+1}$ of degree $n+2$ have trivial first Chern class, by Yau's theorem they are Calabi-Yau manifolds and their holonomy groups equal $SU(n)$. When $n=2$, any smooth quartic surface is actually hyperkähler because $SU(2)=Sp(1)$. This is a K3 surface, namely a simply connected Kähler surface with trivial first

Chern class. It is known that all K3 surfaces form a connected (non-Hausdorff) moduli space. Moreover every Calabi-Yau surface is either a K3 surface or a complex torus. For $n = 3$ we have the quintic threefold in $\mathbb{CP}^4$ and it is the most studied Calabi-Yau threefold in mirror symmetry.

Among all Calabi-Yau hypersurfaces $M$ in $\mathbb{CP}^{n+1}$, there is a natural choice. Namely $M$ is the union of all coordinate hyperplanes in $\mathbb{CP}^{n+1}$:

$$M = \left\{ \left( z^0, z^1, ..., z^{n+1} \right) \in \mathbb{CP}^{n+1} : z^0 z^1 \cdots z^{n+1} = 0 \right\}.$$

It is singular! In a sense it is the most singular hypersurface and it is called the large complex structure limit by physicists. It can also be characterized mathematically by its mixed Hodge structure having maximal unipotent monodromy (see for example [CK]). Despite its non-smoothness, it is a semi-flat Calabi-Yau space. We will explain this and its importances in mirror symmetry in the following sections.

We can replace the projective space $\mathbb{CP}^{n+1}$ by a Fano toric variety to generate many more explicit examples of Calabi-Yau manifolds: Recall that a $(n+1)$ dimensional toric variety $X_\Delta$ is a Kähler manifold with an effective $(\mathbb{C}^\times)^{n+1}$-action. The analog of union of coordinate hyperplanes in $\mathbb{CP}^{n+1}$ would be the union of its lower dimensional $(\mathbb{C}^\times)^{n+1}$-orbits. Again this is a singular Calabi-Yau space and it can be deformed to a smooth, or mildly singular, Calabi-Yau manifold if $X_\Delta$ is Fano or equivalently its associated polytope $\Delta$ is reflexive.

One of many ways to define the polytope $\Delta$ for $X_\Delta$ is to use the moment map in symplectic geometry. If we restrict the $(\mathbb{C}^\times)^{n+1}$-action to its imaginary part, $\left(S^1\right)^{n+1}$, then it preserves a symplectic form on $X_\Delta$ and it has a moment map $\mu$,

$$\mu : X_\Delta \to \mathbb{R}^{n+1}.$$

The image $\mu(X_\Delta)$ is the associated polytope $\Delta$. The pre-image of $\mu$ at a point on a $k$ dimensional open facet of $\Delta$ is a $k$ dimensional torus $\left(S^1\right)^k$. Therefore the above singular Calabi-Yau hypersurface in $X_\Delta$ is just $\mu^{-1}(\partial\Delta)$ and it admits a fibration by $n$ dimensional tori. This torus fibration will play a crucial role in the SYZ mirror conjecture.

This construction has a natural generalization to Calabi-Yau complete intersections in toric varieties, especially if every hypersurface involved is semi-ample. Otherwise the Calabi-Yau manifold we construct as complete intersection may be rigid, see section 6 for its importance in mirror symmetry.

**Moduli space of Calabi-Yau manifolds and Tian-Todorov results**

Infinitesimal deformations of the complex structures on $M$ are parametrized by $H^1(M, T_M)$. After fixing a holomorphic volume form on the Calabi-Yau manifold $M$, we obtain a natural identification

$$H^1(M, T_M) = H^1\left(M, \Omega^{n-1}\right) = H^{n-1,1}(M).$$

Given an infinitesimal deformation of complex structures on $M$, there may be obstruction for it to come from a honest deformation. Various levels of obstructions all lie inside $H^2(M, T_M)$. Tian [Ti] and Todorov [Ti] show that all these obstructions vanish for Calabi-Yau manifolds. As a corollary, the moduli space of complex structures for Calabi-Yau manifolds is always smooth with tangent space equals $H^{n-1,1}(M)$.

Taking $L^2$ inner product of harmonic representatives in $H^{n-1,1}(M)$ with respect to the Calabi-Yau metric determines a Kähler metric on this moduli space and it is called the Weil-Petersson metric. Tian and Todorov show that its Kähler potential can be expressed in term of variation of complex structures on $M$ in a simple manner. This is later interpreted as a special geometry on the moduli space by Strominger. It also plays an important role in constructing a Frobenius structure on an extended moduli space (see for example [CK] for more details).

**The definition of $B$-fields**

A purpose of introducing $B$-fields is to complexify the space of symplectic structures on $M$, the conjectural mirror object to the space of complex structures on $W$ which has a natural a complex structure.

The usual definition of a $B$-field is a harmonic form of type $(1,1)$ with respect to the Calabi-Yau metric. The harmonicity condition does not respect mirror symmetry unless we are in the large complex and Kähler structure limit. In this limit, both the complex and real polarizations are essentially equivalent. Away from this limit, we define the B-field $\beta$ in two different ways depending one whether we use the real or complex polarization. In any case, $\beta$ would be a closed two form of type $(1,1)$. If we use the complex polarization, then we require $\beta$ to satisfy the equation $\operatorname{Im}(\omega+i\beta)^n=0$. We can show that if $M$ satisfies the following condition.

$$\omega^n = i^n \Omega\bar{\Omega}$$
$$\operatorname{Im} e^{i\theta}(\omega+i\beta)^n = 0$$
$$\operatorname{Im} e^{i\phi}\Omega = 0 \text{ on the zero section.}$$

Then in the semi-flat case, under the mirror transformation to $W$, they becomes

$$\omega_W^n = i^n \Omega_W \bar{\Omega}_W$$
$$\operatorname{Im} e^{i\theta}\Omega_W = 0 \text{ on the zero section.}$$
$$\operatorname{Im} e^{i\phi}(\omega_W+i\beta_W)^n = 0.$$

When we expand the second equation for small $\beta$ and assume the phase angle is zero, we have

$$(\omega+i\varepsilon\beta)^n = \omega^n + i\varepsilon n\beta\omega^{n-1} + O\left(\varepsilon^2\right).$$

If we linearize this equation, by deleting terms of order $\varepsilon^2$ or higher, then it becomes the harmonic equation for $\beta$:

$$\beta\omega^{n-1} = c'\omega^n.$$

That is (1) $\omega$ is a Calabi-Yau Kähler form and (2) $\beta$ is a harmonic real two form. This approximation is in fact the usual definition of a B-field.

**A remark**

Here we explain why we usually consider the complexified Kähler class to be a cohomology class in $H^{1,1}(M,\mathbb{C})/H^{1,1}(M,\mathbb{Z})$ rather than inside $H^{1,1}(M,\mathbb{C})$. Every element in $H^{1,1}(M,\mathbb{Z})=H^{1,1}(M)\cap H^2(M,\mathbb{Z})$ is the first Chern class of a holomorphic line bundle $L$ on $M$. As we will see in section 3, $L$ determines a supersymmetric $B$-cycle on $M$ if its curvature tensor $F\in\Omega^2(M,i\mathbb{R})$ satisfies the equation

$$\operatorname{Im} e^{i\theta}\left(\omega+i\left[\beta+\frac{i}{2\pi}F\right]\right)^n=0,$$

for some constant $C$. Therefore shifting the cohomology class of $-i(\omega+i\beta)$ by an element in $H^{1,1}(M,\mathbb{Z})$ is roughly the same as tensoring a supersymmetric cycle by a holomorphic line bundle. The correct way to look at this issue is to consider all B-cycles whose cohomology classes $\left[Tr\left(\omega/i+\left[\beta+\frac{i}{2\pi}F\right]\right)\right]$ are the same in $H^{1,1}(M,\mathbb{C})$ modulo $H^{1,1}(M,\mathbb{Z})$. In this respect the complexified Calabi-Yau metric on $M$ is the same as the trivial line bundle being a supersymmetric $B$-cycle in $M$. In fact more precisely a complexified Kähler class would be in $H^{1,1}(M,\mathbb{C}/\mathbb{Z})$.

Another approach is to the B-field is via the real polarization of $M$, namely a special Lagrangian fibration on $M$. The distinction here is the complex conjugation will be replaced by a real involution, by sending the fiber directions to its negative, namely $dx^j\to dx^j$ and $dy^j\to -dy^j$. In the large complex and Kähler structures limit, these two involutions are the same. We denote the real involution of $\Omega$ by $\widehat{\Omega}$. Suppose that $\omega$ is a Kähler form on $M$ and $\beta$ is a closed real two from on $M$ of type $(1,1)$. Then $\omega^{\mathbb{C}}=\omega+i\beta$ is called the complexified Calabi-Yau Kähler class of $M$ if $\left(\omega^{\mathbb{C}}\right)^n$ is a nonzero constant multiple of $i^n\Omega\wedge\widehat{\Omega}$.

$$(-1)^{n(n+1)/2}\,2^n i^n\left(\omega^{\mathbb{C}}\right)^n=\Omega\wedge\widehat{\Omega}.$$

We called this the complexified complex Monge-Ampère equation. We will see explicitly in the semi-flat case that we need these modified definitions of B-fields to exchange complex structures and complexified Kähler structures of $M$ and $W$ (see [L3] for details).

# 3 Supersymmetric cycles and their moduli

In this section we study geometric objects in $M$ and their moduli spaces in complex geometry and symplectic geometry.

In complex geometry we study geometric objects such as complex submanifolds and holomorphic vector bundles. In symplectic geometry we study Lagrangian submanifolds. It is also natural to include flat bundles on these Lagrangian submanifolds as an analog of holomorphic bundles on complex submanifolds on the complex side.

From string theory considerations, Marino, Minasian, Moore and Strominger argue in [MMMS] that supersymmetry imposes metric constraints on these geometric objects. On the complex side, they need to satisfy a deformation of Hermitian-Yang-Mills equations. Such deformations are similar to those studied by the author in [L1] and [L2] related to Gieseker stability of holomorphic vector bundles. On the symplectic side, Lagrangian submanifolds that preserve supersymmetry would be calibrated, the so-called special Lagrangian submanifolds. This type of submanifolds are introduced and studied by Harvey and Lawson in [HL] related to minimal submanifolds inside manifolds with special holonomy. These two types of objects are called B-cycles and A-cycles because of their relationships with B-model and A-model in string theory.

**B-cycles: Hermitian-YM bundles over complex submanifolds**

On B-side we study complex geometry of $M$. We do not need the Calabi-Yau assumption on $M$ until we define the correlation function on their moduli spaces.

**Definition 1** *Let $M$ be a Kähler manifold with complexified Kähler form $\omega^{\mathbb{C}}$, we call a pair $(C, E)$ a supersymmetry B-cycle, or simply B-cycle, if $C$ is a complex submanifold of $M$ of dimension $m$. $E$ is a holomorphic vector bundle on $C$ with a Hermitian metric whose curvature tensor $F$ satisfies the following deformed Hermitian-Yang-Mills equations on $C$:*

$$\operatorname{Im} e^{i\theta} \left(\omega^{\mathbb{C}} + F\right)^m = 0,$$

*for some constant angle $\theta$ which is called the phase angle.*

This equation is introduced in [MMMS]. Recall that $E$ being holomorphic is equivalent to its curvature tensor satisfies the integrability condition $F^{2,0} = 0$. From the complex geometry point of view, it is natural to include those pairs $(C, E)$'s which are singular, namely a coherent sheaf on $M$. Nevertheless it is unclear how to impose the deformed Hermitian-Yang-Mills equations on such singular objects.

If we replace $\omega^{\mathbb{C}}$ by a large multiple $N\omega^{\mathbb{C}}$, then the leading order term for the deformed Hermitian-Yang-Mills equation becomes

$$\operatorname{Im} e^{i\theta} F \wedge \left(\omega^{\mathbb{C}}\right)^{m-1} = 0.$$

If the B-field and the phase angle $\theta$ are both zero then this equation becomes the Hermitian-Yang-Mills equation for the vector bundle $E$:

$$\Lambda F = 0.$$

By the theorem of Donaldson, Uhlenbeck and Yau, an irreducible holomorphic bundle $E$ over a compact Kähler manifold $C$ admits a Hermitian-Yang-Mills connection if and only if the bundle is Mumford stable. Moreover such a connection is unique. We recall that a holomorphic bundle $E$ is called Mumford stable if for any proper coherent subsheaf $S$ of $E$, we have

$$\frac{1}{rank\,(S)}\int c_1\,(S)\,\omega^{n-1} < \frac{1}{rank\,(E)}\int c_1\,(E)\,\omega^{n-1}.$$

By the arguments in [L1], $E$ admits a solution to $\mathrm{Im}\,(k\omega + F)^m = 0$ for large enough $k$ is given by the following stability notion: For any proper coherent subsheaf $S$ of $E$, we require

$$\frac{1}{rk\,(S)}\,\mathrm{Im}\int Tr\,(k\omega + F_S)^m < \frac{1}{rk\,(E)}\,\mathrm{Im}\int Tr\,(k\omega + F_E)^m$$

for all large enough $k$.

The deformation theory of B-cycles and the geometry of their moduli space ${}_B\mathcal{M}\,(M)$ will be discussed in the appendix. For example the tangent space to ${}_B\mathcal{M}\,(M)$ is parametrized by deformed harmonic forms and there is a correlation function ${}_B\Omega$ on ${}_B\mathcal{M}\,(M)$ defined as

$${}_B\Omega = \int_C Tr\mathbb{F}^m \wedge ev^*\Omega.$$

They will play an important role in the mirror symmetry conjecture.

**A-cycles: Flat bundles on SLag submanifolds**

Special Lagrangian submanifolds are introduced by Harvey and Lawson in [HL] as calibrated submanifolds. Their original paper is an excellent reference for the subject. It turns out that such objects are supersymmetric cycles when coupled with deformed flat connections (see [BBS]).

**Definition 2** *Let $M$ be a Calabi-Yau manifold of dimension $n$ with complexified Kähler form $\omega^{\mathbb{C}}$ and holomorphic volume form $\Omega$. We called a pair $(C, E)$ a supersymmetry A-cycle, or simply A-cycle, if (i) $C$ is a special Lagrangian submanifold of $M$, namely $C$ is a real submanifold of dimension $n$ with*

$$\omega|_C = 0,$$

*and*

$$\mathrm{Im}\,e^{i\theta}\Omega|_C = 0.$$

*(ii) $E$ is a unitary vector bundle on $C$ whose curvature tensor $F$ satisfies the deformed flat condition,*

$$\beta|_C + F = 0.$$

Note that the Lagrangian condition and deformed flat equation can be combined into one complex equation on $C$:[1]

$$\omega^{\mathbb{C}} + F = 0.$$

These conditions are very similar to those defining a B-cycle. There is also a symplectic reduction picture on the A-side as introduced and studied by Donaldson in [D] and Hitchin in [H2]. Thomas has a different symplectic reduction picture and he compared it with the B-side story in [Th1]. The role of various objects are reversed - a mirror phenomenon.

**Moduli space of A-cycles and their correlation functions**

If we vary an A-cycle $(C, E)$ while keeping $C$ fixed in $M$, then the moduli space of these objects is isomorphic to the moduli space of flat connections on $C$. For simplicity we assume the B-field vanishes. However there is no canonical object analog to the trivial flat connection when the B-field is nonzero. Then this moduli space is independent of how $C$ sits inside $M$ because of its topological nature.

So we reduce the problem to understanding deformations of special Lagrangian submanifolds and this problem has been solved by McLean in [Mc]. Using the Lagrangian condition, a normal vector field of $C$ can be identified with an one form on $C$. McLean shows that infinitesimal deformations of $C$ are parametrized by the space of harmonic one forms on $C$, moreover, all higher order obstructions to deformations vanish. In particular every infinitesimal deformation comes from a honest family of special Lagrangian submanifolds in $M$. By Hodge theory the tangent space of moduli space of A-cycles, ${}_A\mathcal{M}(M)$, at $(C, E)$ is given by $H^1(C, \mathbb{R}) \times H^1(C, ad(E))$.

When $E$ is a line bundle, we have a natural isomorphism $H^1(C, ad(E)) = H^1(C, i\mathbb{R})$ and this tangent space is

$$H^1(C, \mathbb{R}) + H^1(C, i\mathbb{R}) = H^1(C, \mathbb{C}).$$

Therefore ${}_A\mathcal{M}(M)$ has a natural almost complex structure which is in fact integrable (see for example [H1]). By the symplectic reduction procedure mentioned earlier, it also carries a natural symplectic structure.

The correlation function is a degree $n$ form on this moduli space. First we define a degree-$n$ closed form on the whole configuration space $Map(C, M) \times \mathcal{A}_C(E)$ as follows,

$${}_A\Omega = \int_C Tr\,(ev^*\omega + \mathbb{F})^n.$$

Here $ev : C \times Map(C, M) \to M$ is the evaluation map and $\mathbb{F}$ is the curvature of the universal connection on $C \times \mathcal{A}_C(E)$.

Since the tangent spaces of the moduli of special Lagrangians and the moduli of flat connections can both be identified with the space of harmonic one forms,

[1] Since $\beta$ may represent a non-integral class, we need to use generalized connections on the bundle as studied in [L2].

a tangent vector of the moduli space ${}_A\mathcal{M}(M)$ is a complex harmonic one form $\eta + i\mu$. Then ${}_A\Omega$ can be written explicitly as follows

$$ {}_A\Omega\left(\eta_1 + i\mu_1, ..., \eta_m + i\mu_m\right) = \int_C Tr\left(\eta_1 + i\mu_1\right) \wedge ... \wedge \left(\eta_m + i\mu_m\right). $$

**Instanton corrections**

As we have seen the A-side story is much easier to describe so far because the equation involving $F$ is linear. However this correlation function is only valid in the so-called classical limit. To realize the mirror symmetry, we need to modify these structures by adding suitable contributions coming from holomorphic disks whose boundaries lie on $C$. These are called instanton corrections. In this paper we are not going to describe this aspect.

**A remark**

So far we have assumed $C$ and $E$ are always smooth objects. But if we want to study the whole moduli space, it is natural to look for a suitable compactification of it. In that case singular objects are unavoidable. On the A-side, this problem is fundamental for understanding mirror symmetry via special Lagrangian fibrations as proposed by Strominger, Yau and Zaslow [SYZ]. On the B-side, even when $C$ is a curve and $E$ is a line bundle, understanding how to compactify this moduli space is crucial in the study of the Gopakumar-Vafa conjecture [GV2].

# 4 Mirror symmetry conjectures

Roughly speaking the mirror symmetry conjecture says that for a Calabi-Yau manifold $M$, there is another Calabi-Yau manifold $W$ of the same dimension such that

| | | |
|---|---|---|
| Symplectic geometry on $M$ | $\cong$ | Complex Geometry on $W$ |

and

| | | |
|---|---|---|
| Complex geometry on $M$ | $\cong$ | Symplectic Geometry on $W$. |

The conjecture cannot be true as stated because rigid Calabi-Yau manifolds are counterexamples to it. From a string theory point of view, the conjecture should hold true whenever the Calabi-Yau manifold admits a deformation to a large complex structure limit. This includes Calabi-Yau hypersurfaces inside Fano toric varieties. From a mathematical point of view, this should correspond to the existence of a certain special Lagrangian fibration on $M$ as proposed in [SYZ]. The conjecture might hold true for an even larger class of Calabi-Yau manifolds. For example certain aspects of it is conjectured to hold true even for rigid Calabi-Yau manifolds (see section 6).

Other basic aspects of mirror symmetry are: (i) On the symplectic side, there are instanton corrections. These are holomorphic curves or holomorphic disks with boundaries on the special Lagrangian. (ii) On the complex side, the structures are nonlinear. For example the deformed Hermitian-Yang-Mills equations and the natural symplectic form on the moduli space of B-cycles are nonlinear in the curvature tensor. Since one of the main purposes of this paper is to understand how mirror symmetry works when there are no instantons present, we will not spend much effort describing instanton effects even though they are equally important in the whole subject.

In the early 90's, mirror symmetry mainly concern identification of the moduli spaces of symplectic structures on $M$ and complex structures on $W$. The information on the instanton corrections in this case predicts the genus zero Gromov-Witten invariants of $M$.

Kontsevich [K1] proposes a homological mirror symmetry conjecture. In 1996 Strominger, Yau and Zaslow [SYZ] proposed a geometric mirror symmetry conjecture (see also Vafa's paper [V2]). These conjectures relate the space of A-cycles in $M$ and the space of B-cycles in $W$. As an analogy of topological theories, the homological conjecture is about singular cohomology, or rational homotopy theory and the geometric conjecture is about harmonic forms. Singular cohomology class can always be represented by a unique harmonic form by Hodge theory. The corresponding results we need for mirror symmetry would roughly be Donaldson, Uhlenbeck and Yau theorem and Thomas conjecture ([Th2], [TY]).

The key breakthrough is realizing $W$ as a moduli space of supersymmetric A-cycles in $M$, as discovered in [SYZ]. Before SYZ, many mirror manifolds were constructed by Batyrev but this combinatorial construction does not give us any insight into why the geometries of $M$ and $W$ are related at all.

From this geometric point of view, we can now understand why mirror symmetry happens, at least when there are no instanton corrections (see section 5). We can also enlarge the mirror symmetry conjecture to include more geometric properties. For instance we will state a conjecture regarding an $sl(2) \times sl(2)$ action on cohomology groups of moduli spaces of supersymmetric cycles, which include in particular $M$ and $W$.

Now we start from the beginning of mirror symmetry.

**The paper by Candelas, de la Ossa, Green and Parkes**

The mirror symmetry conjecture in mathematics begins with the fundamental paper [COGP] by Candelas, de la Ossa, Green and Parkes. They analyze the mirror pair $M$ and $W$ constructed by Greene and Plesser. Here $M$ is a Fermat quintic threefold in $\mathbb{P}^4$ and $W$ is a Calabi-Yau resolution of a finite group quotient of $W$. In this paper they compare the moduli space of complexified Kähler structures on $M$ and the moduli space of complex structures on $W$. Both moduli spaces are one dimensional in this case. The key structure on these moduli spaces is the Yukawa coupling. On the complex side, the Yukawa coupling is determined by a variation of Hodge structures on $W$ and it can be written down explicitly by solving a hypergeometric differential equation. On the symplectic side, the Yukawa coupling consists of two parts: classical and quantum. The classical part is given simply by the cup product of the cohomology of $M$. The quantum part is a generating function of genus zero Gromov-Witten invariants of $M$.

Before we can compare these two Yukawa couplings, we first need an explicit map identifying the two moduli spaces. This mirror map cannot be arbitrary because it has to transform naturally under the monodromy action. In this case, this mirror map is determined in [COGP]. Therefore the mirror conjecture which identifies the two Yukawa couplings gives highly nontrivial predictions on the genus zero Gromov-Witten invariant of $M$ by solving a hypergeometric differential equation. The Gromov-Witten invariants of $M$ are very difficult to determine even in low degree. At that time, only those of degree not more than three were known mathematically. They are 2875, 609250, 317206375 etc.

**Gromov-Witten invariants**

Before stating the conjecture, let us briefly review Gromov-Witten theory. Here we will only consider three dimensional Calabi-Yau manifolds $M$. Roughly speaking the Gromov-Witten invariant $N_C^g(M)$, defined in [RT], counts the number of genus $g$ curves in $M$ representing the homology class $C \in H_2(M, \mathbb{Z})$. To handle the problem of possible non-reducedness and bad singularities of these curves, we need to study instead holomorphic maps from genus $g$ stable curves to $M$. The corresponding moduli space has expected dimension zero. If it is indeed a finite number of smooth points, then its cardinality is $N_C^g(M)$. Otherwise one can still define a virtual fundamental class (see [LT] for example) and obtain a symplectic invariant $N_C^g(M) \in \mathbb{Q}$[2]. This is nonetheless a very nontrivial result. On the other hand, computing these numbers $N_C^g(M)$ is a very difficult mathematical problem.

Recently Gopakumar and Vafa [GV2] proposed a completely different method to obtain these $N_C^g(M)$ using the cohomology of the moduli space of curves in the class $C \in H_2(M, \mathbb{Z})$ together with flat $U(1)$ bundles over these curves.

Counting the number of curves in $M$ is an old subject in algebraic geometry, called enumerative geometry. For example Clemens conjectured there are only finite number of rational curves of any fixed degree on a generic quintic threefold. However these $N_C^0(M)$'s do not really count the number of rational curves in $M$. The same also applies to the higher genus count too. For example, if there is only one curve in the class $C$ and it is a smooth rational curve with normal bundle $O(-1) \oplus O(-1)$. We have $N_C^0(M) = 1$ as expected; however, $N_{dC}^0(M) = \frac{1}{d^3}$ for every positive integer $d$. This is the so-called multiple cover formula, conjectured in [COGP], given a mathematical reasoning by Morrison and Plesser in [AM] and then proven rigorously by Voisin in [Vo].

So if all rational curves in $M$ are smooth and with normal bundle $O(-1) \oplus O(-1)$, then the genus zero Gromov-Witten invariants do determine the number of rational curves in $M$. However even for a generic quintic threefold in $\mathbb{P}^4$, this is not true. Vainsencher [Vai] shows that generic quintic threefold has degree five rational curves with six nodes. There are in fact 17601000 such nodal curves. Therefore to understand the enumerative meaning of the Gromov-Witten invariants, we need know how each rational curve contributes. In [BKL], Bryan, Katz and the author determine such contributions when the rational curve $C$ (i) has one node and it is superrigid (analog of $O(-1) \oplus O(-1)$ for nodal curves) or (ii) is a smooth contractible curve. In the second case, the normal bundle of $C$ must be either $O \oplus O(-2)$ or $O(1) \oplus O(-3)$.

**Candelas-de la Ossa-Green-Parkes mirror conjecture**

This famous conjecture has been discussed in details in many places (see e.g. [CK] for details) and our discussions will be very brief. Suppose that $M$ and $W$ are mirror manifolds. Then the variation of complexified symplectic structures of $M$ with instanton corrections should be equivalent to the variation of complex structures of $W$. The infinitesimal deformation spaces are given by $H^1(M, T_M^*) = H^{1,1}(M)$ and $H^1(W, T_W) = H^{n-1,1}(W)$ respectively. In particular we would have $\dim H^{1,1}(M) = \dim H^{n-1,1}(W)$. More generally, it is conjectured that

$$H^q(M, \Lambda^p T_M^*) = H^q(W, \Lambda^p T_W),$$

as they can be interpreted as tangent spaces of extended deformation problems [K1]. In particular we would have equalities for Hodge numbers

$$\dim H^{p,q}(M) = \dim H^{n-p,q}(W).$$

[2]These numbers can be fractional because of automorphisms of stable maps.

As a corollary, the conjecture implies equality of Euler characteristics up to sign,

$$\chi(M) = (-1)^n \chi(W).$$

The next step would be an identification of the moduli space of complexified symplectic structures on $M$ and the moduli space of complex structures on $W$, at least near a large complex structure limit point. This is called the mirror map in the literature.

On each moduli space, there is degree $n$ form called the Yukawa coupling. The mirror symmetry conjecture says that the Yukawa coupling on the moduli space of complexified symplectic structures on $M$ is pulled back to the Yukawa coupling on the moduli space of complex structures on $W$ by the mirror map.

On the $M$ side the classical part of Yukawa coupling is defined by the natural product structure on cohomology groups:

$$\begin{array}{rl} {}_A\mathcal{Y}_M^{cl} : & \Lambda^n H^1(M, T_M^*) \to H^n(M, \Lambda^n T_M^*) = \mathbb{C} \\ & {}_A\mathcal{Y}_M^{cl}(\zeta_1, \zeta_2, ..., \zeta_n) = \int_M \zeta_1 \wedge \zeta_2 \wedge ... \wedge \zeta_n. \end{array}$$

With instanton corrections by genus zero Gromov-Witten invariants $N_d^0$, the full Yukawa coupling when $n = 3$ is

$${}_A\mathcal{Y}_M^{cl}(\zeta_1, \zeta_2, \zeta_3) = \int_M \zeta_1 \cup \zeta_2 \cup \zeta_3 + \sum_{d \in H^2(M,\mathbb{Z}) \backslash 0} N_d^0(\zeta_1, \zeta_2, \zeta_3) \frac{q^d}{1 - q^d}.$$

On the $W$ side, the Yukawa coupling is again defined by the natural product structure on cohomology groups:

$$\begin{array}{rl} {}_B\mathcal{Y}_W : & \Lambda^n H^1(W, T_W) \to H^n(W, \Lambda^n T_W) = \mathbb{C} \\ & {}_B\mathcal{Y}_W(\zeta_1, \zeta_2, ..., \zeta_n) = \int_W \Omega \wedge (\zeta_1 \wedge \zeta_2 \wedge ... \wedge \zeta_n) \lrcorner \Omega. \end{array}$$

There is no instanton correction for the B-side. In particular identification of these two Yukawa couplings would determine all genus zero Gromov-Witten invariants of $M$. This is in fact the most well-known part of the mirror symmetry conjecture.

**Batyrev's construction and the mirror theorem**

Before anyone can prove this COGP mirror conjecture, we need to know how to construct $W$ from a given $M$. When $M$ is a Calabi-Yau hypersurface in a Fano toric variety $X_\Delta$ as discussed on page 8, Batyrev [Ba] proposes that $W$ can be constructed again as a hypersurface in another Fano toric variety $X_\nabla$, possibly with a crepant desingularization. In fact the two polytopes $\Delta$ and $\nabla$ are polar dual to each other (see [LV] for a physical explanation).

Batyrev verifies that the Hodge numbers for these pairs do satisfy the COGP mirror conjecture, namely $\dim H^{p,q}(M) = \dim H^{n-p,q}(W)$. To verify the rest of the COGP conjecture amounts to computing the genus zero Gromov-Witten invariants of $M$. This problem can be translated into computing certain coupled

Gromov-Witten invariants of $X_\Delta$. Since the toric variety $X_\Delta$ has a $T^n$ torus action, the moduli space of stable curves in $X_\Delta$ also inherit such a $T^n$ action. In the foundational paper [K2], Kontsevich developes the techniques needed to apply Bott's localization method to compute Gromov-Witten invariant using this torus action. In [Gi] Givental gives an argument to compute these invariants using localization. A complete and detailed proof of the COGP conjecture is given by Lian, Liu and Yau in [LLY1] and [LLY2]. This result is also proven by Bertram by a different method later.

**Bershadsky-Cecotti-Ooguri-Vafa conjecture**

In [BCOV1] and [BCOV2], the authors study higher genus Gromov-Witten invariants in a Calabi-Yau threefold. They conjecture that their generating functions satisfy a differential equation, so-called $tt^*$-equation. In particular in the quintic threefold case, this is enough to predict all Gromov-Witten invariants. The mathematical structure on the B-side is still not completely understood yet. For this amazing conjecture, the reader is referred to the original papers.

**Gopakumar-Vafa conjecture**

As we mentioned before, Gromov-Witten invariants for Calabi-Yau threefolds are very difficult to compute. One of the reasons is that it is very difficult to describe the moduli space of stable maps explicitly. For example the moduli space of stable maps can be very complicated even though their image is a single curve inside $M$. Moreover these invariants may not be integer and also there are infinite number of such invariants in each homology class of $M$, namely one for each genus.

Recently in [GV1], [GV2], Gopakumar and Vafa propose to study a rather different space whose cohomology admits an $sl\,(2) \times sl\,(2)$ action. Its multiplicities are called BPS numbers and they should determine all Gromov-Witten invariants. The origin of this conjecture comes from M-theory duality considerations in physics. First the space they consider is just the space of holomorphic curves, instead of stable maps, in $M$ together with flat $U\,(1)$ bundles over them. This is just the moduli space of B-cycles ${}_B\mathcal{M}\,(M)$ on $M$. Second, unlike Gromov-Witten invariants, these BPS numbers are always integers! Third, there are only a finite number of BPS numbers for a given homology class in $M$ because they are bounded by the dimension of the moduli space. In particular Gromov-Witten invariants for various genus are not independent! In a way the relationship between the BPS numbers and Gromov-Witten invariant for Calabi-Yau threefolds is like the relationship between Seiberg-Witten invariants and Donaldson invariants for four manifolds. Unlike the Seiberg-Witten invariants, the BPS numbers are still very difficult to compute. In fact they are not even well-defined at this moment.

The first issue is to compactify the moduli space ${}_B\mathcal{M}\,(M)$. When the curve $C$ in $M$ is smooth, then the space of flat $U\,(1)$ bundles on $C$ is its Jacobian $J\,(C)$. But when the curve becomes singular, especially nonreduced, there is no good notion of compactified Jacobian. Presumably it will also depend on how $C$ sits inside $M$. The compactified moduli space $\overline{{}_B\mathcal{M}\,(M)}$, if it exists, would admit an Abelian variety fibration obtained by forgetting the bundles. Roughly speaking the conjectured $sl(2) \times sl(2)$ action on the cohomology of $\overline{{}_B\mathcal{M}(M)}$ would correspond to the two $sl(2)$ actions from the hard Lefschetz actions on the base and the fiber of the Abelian variety fibration. It is still not clear how it works.

We now explain how to determine the Gromov-Witten invariants of curves in a class $\beta \in H_2(M, \mathbb{Z})$ in terms of the $\mathbf{sl}(2) \times \mathbf{sl}(2)$ action ([GV2]). Let us denote the standard two dimensional representation of $\mathbf{sl}(2)$ by $V_{1/2}$ and its $k^{th}$ symmetric power by $V_{k/2} = S^k V_{1/2}$. Similarly the $\mathbf{sl}(2) \times \mathbf{sl}(2)$ representation $V_j \otimes V_k$ is $V_{j,k}$ in our notation. We look at the moduli space of B-cycles $(C, E)$ with $[C] = \beta$ and $E$ a flat $U(1)$ bundle over $C$. Suppose that the conjectural $\mathbf{sl}(2) \times \mathbf{sl}(2)$ action exists on a suitable cohomology theory of ${}_B\mathcal{M}(M)$. We decompose this action as

$$H^*\left({}_B\mathcal{M}(M), \mathbb{C}\right) = \left[V_{1/2,0} + 2V_{0,0}\right] \otimes \sum_{j,k} N_{j,k}^{\beta} V_{j,k}.$$

We define integers $n_\beta^g(M)$ by the following,

$$\sum_{g \geq 0} n_\beta^g(M) \left[V_{1/2} + 2V_0\right]^{\otimes g} = \sum_{j,k} N_{j,k}^{\beta} (-1)^{2k} (2k+1) V_j.$$

These numbers are called the BPS numbers which count the numbers of BPS states.

The conjectural Gopakumar-Vafa formula expresses the Gromov-Witten invariants of various genera curves in class $\beta$ in terms of these integers $n_\beta^g(M)$'s. Their formula is

$$\sum_{\beta \neq 0} \sum_{r \geq 0} N_\beta^r(M) t^{2r-2} q^\beta = \sum_{\beta \neq 0} \sum_{r \geq 0} n_\beta^r(M) \sum_{k > 0} \frac{1}{k} \left(2 \sin \frac{kt}{2}\right)^{2r-2} q^{k\beta}.$$

For $\beta$ represented by a unique superrigid rational curve, namely a smooth rational curve with normal bundle $O(-1) \oplus O(-1)$, this formula reduces to the multiple cover formula $N_{d\beta}^0(M) = 1/d^3$ in the genus zero case. For a superrigid rational curve with one node, this formula is verified in [BKL]. The contribution of a super-rigid genus $g$ curve $C$ to $n_{d[C]}^r(M)$ has been computed through a certain range $r < R(d, g)$ by Bryan and Pandharipande in [BP]. They verify the integrality of these numbers which involves a non-trivial combinatorial analysis of the number of degree $d$, etale covers of a smooth curve.

**Kontsevich homological mirror conjecture**

In [K1] Kontsevich proposed a homological mirror symmetry conjecture: If $M$ and $W$ are mirror manifolds, then a derived category of Lagrangians with flat bundles in $M$ is isomorphic to the derived category of coherent sheaves on $W$. These derived categories contain a lot of informations of $M$ and $W$.

Unfortunately we will not discuss much of those here because our emphasis is on the geometric side; readers should consult his original paper [K1] for more details. This conjecture is verified by Polishchuk and Zaslow in [PZ] in dimension one, namely when $M$ and $W$ are elliptic curves. When the dimension is bigger than one, the derived category of Lagrangians is not even completely defined yet. One issue is about instanton corrections which are needed in the definition of product structure in this category. They involve a suitable count of holomorphic disks whose boundaries lie on Lagrangians (see [FOOO]). The COGP conjecture should follow from the homological conjecture by considering diagonals in the products $M \times \bar{M}$ and $W \times W$ as argued in [K1]. It is because a holomorphic disk in $M \times \bar{M}$ with boundary in the diagonal corresponds to two holomorphic disks in $M$ with the same boundary, thus producing genus zero holomorphic curves. This was explained to the author by Zaslow.

Another mirror conjecture of Kontsevich concerns symplectomorphism of $M$ and holomorphic automorphism of $W$. As he suggested, one should enlarge the collection of automorphisms of $W$ to automorphisms of the derived category of coherent sheaves. Such automorphisms usually come from monodromy on the moduli space of symplectic structures on $M$ and complex structures on $W$. And they should have a mirror correspondence them. This is analyzed by Seidel and Thomas [ST] and Horja [Ho] in some cases. We will explain how to transform certain symplectic diffeomorphisms of $M$ to holomorphic diffeomorphisms of $W$, and vice versa, in the semi-flat case in [L3].

Recently Kontsevich and Soibelman [KS] have important results. They argue that, under suitable assumptions, both the Fukaya category for $M$ and the derived category of coherent sheaves on $W$ reduce to one category, which should be realized as a Morse category for the base of a torus fibration in the large complex structure limit.

**Strominger-Yau-Zaslow conjecture**

Next we discuss various geometric aspects of mirror symmetry. This starts with the paper [SYZ] by Strominger, Yau and Zaslow. They argue physically that supersymmetric cycles on $M$ and $W$ should correspond to each other. The manifold $W$ itself is the moduli space of points in $W$! The mirror of each point, as a B-cycle in $W$, would be certain special Lagrangian with a flat $U(1)$ connection, as a A-cycle in $M$. Because $\dim W = n$, these special Lagrangian move in an $n$ dimensional family and therefore their first Betti number equals $n$ (see section 3). A natural choice would be an $n$ dimension torus which moves and forms a special Lagrangian torus fibration on $M$.

Consider at all those points in $W$ whose mirrors are the same special Lagrangian torus in $M$ but with different flat $U(1)$ connections over it. Then this is just the dual torus, because the moduli space of flat $U(1)$ connections on a torus is its dual. We conclude that if $M$ and $W$ are mirror manifolds, then $W$ should have special Lagrangian torus fibration and $M$ is the dual torus fibration.

It has been conjectured by various people including Kontsevich, Soibelman [KS], Gross and Wilson, Todorov and Yau, that as $M$ and $W$ approach certain large complex structure limit point, then they converge in the Gromov-Hausdorff sense to affine manifolds. In the surface case, this is verified by Gross and Wilson (see [GW]). As we will see in the semi-flat case, that the curvature is bounded in this limiting process and the affine manifolds are the same via a Legendre transformation. We expect that the same should hold true for general mirror pairs $M$ and $W$, at least away from singular fibers..

Fix $M$, a Calabi-Yau hypersurface in a Fano toric variety $X_\Delta$. As it approaches the large complex structure limit point, i.e. the union of toric divisors, then this Lagrangian torus fibration should be the one given by the toric structure on $X_\Delta$. In particular this is a semi-flat situation. The dual torus fibration is therefore the one given by $X_\nabla$ with the polytope $\nabla$ the polar dual to $\Delta$. This offers an explanation to why Batyrev construction should product mirror pairs. This (non-special) Lagrangian torus fibration picture for Calabi-Yau inside toric varieties has been verified by Gross [Gr3][3] and Ruan in [R1],[R3] (see also [LV] and [Z]).

In order to understand the geometry of this moduli space $W$ of special Lagrangians in $M$, for example to identify the $L^2$ metric with instanton corrections on this moduli space with the Calabi-Yau metric on $W$, we need to know how to count holomorphic disks. This point is still poorly understood.

**Vafa conjecture**

In [V2] Vafa extended the SYZ analysis to general supersymmetric A-cycles and B-cycles. Among other things, he wrote down a conjectural formula that identifies the two correlation functions with instanton correction terms included. This conjecture is verified in the special case when the B-cycle is a line bundle on a semi-flat manifold in [LYZ].

However it is still not known how to correct the $L^2$ metric on the moduli space to a Ricci flat metric, even conjecturally.

Roughly speaking, the mirror transformation between A-cycles and B-cycles should be given by fiberwise Fourier transformation on the special Lagrangian torus fibration, similar to the spectral cover construction in algebraic geometry.

For example if $M$ and $W$ are mirror manifolds with dual special Lagrangian torus fibrations,

$$\pi : M \to D,$$
$$\pi : W \to D.$$

Obviously $M \times M$ and $W \times W$ are also mirrors to each other. Now the diagonal $\Delta_W$ in $W \times W$ is a complex submanifold and thus determines a B-cycle in $W \times W$.

In Kontsevich picture, $\Delta_W$ corresponds to the identity functor, as a kernel of the Fourier-Mukai transform, on the derived category of sheaves on $W$. Therefore its mirror should correspond to the identity functor on the Fukaya category of Lagrangians in $M$. The kernel for this identity function is the diagonal $\Delta_M$ in $M \times \overline{M}$. It was explained to the author by Thomas.

[3] The torus fibration constructed by Gross is not Lagrangian.

What is the mirror to $\Delta_W$ in SYZ picture?

We have $\pi_{W\times W}(\Delta_W) = \Delta_B \subset B \times B$, the diagonal in $B$. Let $T_b = \pi_W^{-1}(b)$, then

$$\Delta_W \cap (T_b \times T_b) = \Delta_{T_b} \subset T_b \times T_b = \pi_{W\times W}^{-1}(b, b).$$

The Fourier transformation on this fiber $T_b \times T_b$ would be those flat $U(1)$ connections on $T_b \times T_b$ which are trivial along $\Delta_{T_b}$ and this is the off-diagonal

$$\widetilde{\Delta}_{T_b^*} = \{(y, -y)\} \subset T_b^* \times T_b^*.$$

So the mirror to $\Delta_W$ in $M \times M$ is the union of $\widetilde{\Delta}_{T_b^*}$'s over the diagonal $\Delta_B$ in $B \times B$.

If we map each point $y$ in the second factor of $T_b^* \times T_b^*$ to its inverse $-y$, then the symplectic structure on $M \times M$ would becomes the one in $M \times \overline{M}$. This can be verified rigorously in the semi-flat case. Moreover the mirror to the B-cycle $\Delta_W$ in $W \times W$ will be the A-cycle $\Delta_M$, the diagonal in $M \times \overline{M}$.

**Donaldson, Uhlenbeck-Yau theorem and Thomas-Yau conjecture**

The relationship between the Kontsevich homological mirror symmetry conjecture and SYZ geometric mirror symmetry conjecture is similar to the relationship between the deRham-Singular cohomology and the Hodge cohomology of harmonic forms for a manifold. To link the two, we need an analog of the Hodge theory which represents each cohomology class by a unique harmonic form.

On the B-side, we need to find a canonical Hermitian metric on holomorphic bundles over a complex submanifold in $W$, namely a deformed Hermitian-Yang-Mills metric. For the usual Hermitian-Yang-Mills equation, this is the result of Donaldson, Uhlenbeck and Yau which says that such equation has a unique solution on any Mumford stable bundle. For the deformed Hermitian-Yang-Mills equation near the large radius limit, i.e. $t\omega$ for large $t$, the author shows in [L1] that they can be solved on irreducible Gieseker type stable bundles.

On the A-side, in [Th2] Thomas formulates a stability notion for Lagrangian submanifolds and conjectures that it should be related to the existence of special Lagrangian in its Hamiltonian deformation class. Some special cases are verified by Thomas and Yau [TY] using mean curvature flow.

**Conjecture on $sl(2) \times sl(2)$ action on cohomology**

The hard Lefschetz theorem in Kähler geometry says that there is an $\mathbf{sl}(2)$ action on the cohomology group of any compact Kähler manifold. A generator for this $\mathbf{sl}(2)$ action is the hyperplane section on harmonic forms. Hubsch and Yau are the first to search for the mirror of this $\mathbf{sl}(2)$ action in [HY].

In the semi-flat case, there is another $\mathbf{sl}(2)$ action whose generator is the variation of Hodge structure along the deformation direction which shrinks the torus fiber (see [L3]). Moreover these two $\mathbf{sl}(2)$ actions commute with each other and thus determine an $\mathbf{sl}(2) \times \mathbf{sl}(2)$ action, or equivalently an $\mathbf{so}(4)$ action. Furthermore these two kinds of $\mathbf{sl}(2)$ actions get interchanged under the mirror transformation. When the semi-flat Calabi-Yau manifold is hyperkähler, this $\mathbf{so}(4)$ action imbeds into the natural hyperkähler $\mathbf{so}(5)$ action on its cohomology group [L3].

We conjecture: (1) if $M$ and $W$ are mirror manifolds, then their cohomology groups admit an $\mathbf{so}(4) = \mathbf{sl}(2) \times \mathbf{sl}(2)$ action, where one of them is given by the (possibly deformed) hard Lefschetz theorem, and the other one is related to the variation of Hodge structure for a complex deformation which shrinks the fibers of the special Lagrangian fibration. (2) Furthermore mirror transformation flips these two kinds of $\mathbf{sl}(2)$ action, possibly after instanton corrections. (3) When $M$ is also a hyperkähler manifold, then this $\mathbf{so}(4)$ action embeds inside the natural hyperkähler $\mathbf{so}(5)$ action on its cohomology groups.

More generally suitable cohomology groups of the compactified moduli space of supersymmetric A-cycles (or B-cycles) $(C, E)$ with $rank\,(E) = 1$ admits an $\mathbf{so}(4) = \mathbf{sl}(2) \times \mathbf{sl}(2)$ action. The mirror transformation between these moduli spaces for $M$ and $W$ flips the two $sl\,(2)$ factors. When these cycles are just points or special Lagrangian tori, this reduces to the previous part of the conjecture.

When the moduli space of B-cycles consists of curves, then this conjecture overlaps with the first part of the Gopakumar-Vafa conjecture. It would be interesting to know if there is an analog of the full Gopakumar-Vafa conjecture in the general case.

**How to prove the mirror conjectures?**

In the ground breaking paper of Strominger, Yau and Zaslow [SYZ], a geometric approach to proving the mirror conjecture is proposed. Namely the mirror manifold $W$ of $M$ should be the moduli space of certain supersymmetric cycles on $M$. Let me briefly explain their proposal here.

From string theory, the duality between $M$ and $W$ should have no instanton correction at the large complex and Kähler structure limit. For example, when $M$ is a Calabi-Yau hypersurface in projective space $\mathbb{P}^{n+1}$, the large complex structure limit point $M_0$ is just the union of $n+2$ hyperplanes inside $\mathbb{P}^{n+1}$. Each connected component of the smooth part of $M_0$ is a copy of $(\mathbb{C}^{\times})^n$ and it has a semi-flat Calabi-Yau metric.

The most important step is to understand what the Calabi-Yau metric on $M$ looks like when $M$ is close to such a large complex structure limit point. It is expected that on a large part on $M$, the Calabi-Yau metric on $M$ approximates the above semi-flat metric and therefore admits a special Lagrangian fibration. When these special Lagrangian tori approach singularities of $M_0$, it itself might develop singularities. In fact the Calabi-Yau metric on $M$ should be close to some model Calabi-Yau metric around singular special Lagrangian tori. Unfortunately the only such model metric is in dimension two, namely the Ooguri-Vafa metric [OV]. In general degeneration of Calabi-Yau metrics is a difficult analytic problem. However, in dimension two, Gross and Wilson [GW] verified such a prediction with the help of a hyperkähler trick.

The second step would be to explain the mirror conjecture in the case of semi-flat Calabi-Yau manifolds. This is partially done in [LYZ] and [L3]. In fact understanding this semi-flat case enable us to enrich the mirror conjecture as we saw in the last section. Notice that most of the predictions in the semi-flat case do not involve instanton corrections. Then we need to show that those duality properties in the semi-flat Calabi-Yau case continue to hold true when the Calabi-Yau metric is only close to a semi-flat metric.

The third step is to write down many model Calabi-Yau metrics for neighborhoods of singular special Lagrangian tori. By explicit calculations, we hope to show that mirror conjecture holds true for such models. The last step would be to glue the results from the previous two steps together. This would then imply the mirror conjecture for Calabi-Yau manifolds near a large complex structure limit point.

In conclusion, existence of certain types of special Lagrangian fibration is responsible for the duality behavior in mirror symmetry. For rigid Calabi-Yau manifolds, we know that traditional mirror conjecture cannot hold true for a simple reason. There are some proposals of using higher dimensional Fano manifolds as the mirror manifold. We will propose a link between these and the SYZ conjecture in section 6.

We should mention that there are other approaches to prove the homological mirror symmetry conjecture by Kontsevich and Soibelman [KS].

## 5 Mirror symmetry for semi-flat CY manifolds

**Dimension reduction and Cheng-Yau manifolds**

As we mentioned before, mirror symmetry concerns any Calabi-Yau manifold which admits a deformation into a large complex structure limit point. Near any such point, the Calabi-Yau metric is expected to approximate a semi-flat Calabi-Yau metric (see [KS] for an approach to prove the Kontsevich homological mirror symmetry conjecture in this case). In the semi-flat case, we can study translation invariant solution to the complex Monge-Ampère equation and reduce the problem to a real Monge-Ampère equation. Semi-flat Calabi-Yau manifolds are introduced into mirror symmetry in the foundational paper [SYZ] and then further studied in [H1], [Gr2] and [LYZ]. We start with a semi-flat Calabi-Yau manifold

$$M = D \times i\mathbb{R}^n \subset \mathbb{C}^n,$$

where $D$ is a convex domain in $\mathbb{R}^n$. More generally we can replace $D$ by an affine manifold and $M$ its tangent bundle. We only consider translation invariant quantity, for example, a translation invariant Kähler potential $\phi$ of the Kähler form $\omega = i\partial\bar{\partial}\phi$ determines a Calabi-Yau metric on $M$ if and only if $\phi$ is an elliptic solution of the real Monge-Ampère equation

$$\det\left(\frac{\partial^2\phi}{\partial x^j \partial x^k}\right) = const.$$

The deepest result is the theorem of Cheng and Yau in [CY] on the Dirichlet problem for this equation: Given any bounded convex domain $D$ in $\mathbb{R}^n$, there is a unique convex function $\phi$ defined on $D$ satisfying the real Monge-Ampère equation equation and $\phi|_{\partial D} = 0$. We will call such manifold $D$ a *Cheng-Yau manifold*. As we will see below, the Legendre transformation of $\phi$ produces another Cheng-Yau manifold $D^*$. This duality among Cheng-Yau manifolds is an important part of the mirror symmetry at the large complex structure limit.

**The Fourier transformation and the Legendre transformation**

We can compactify imaginary directions by taking a quotient of $i\mathbb{R}^n$ by a lattice $i\Lambda$. That is we replace the original $M$ by $M = D \times iT$ where $T$ is the torus $\mathbb{R}^n/\Lambda$ and the above Kähler structure $\omega$ descends to $D \times iT$ and $M$ has the following structures [L3] (see also [Gr2]):

| Structures on $M$: | |
|---|---|
| Riemannian metric | $g_M = \Sigma\phi_{jk}\left(dx^j \otimes dx^k + dy^j \otimes dy^k\right)$ |
| Complex structure | $\Omega_M = \Pi\left(dx^j + idy^j\right) = dz^1 \wedge ... \wedge dz^n$ |
| Symplectic structure | $\omega_M = \Sigma\phi_{jk} dx^j \wedge dy^k$. |

The projection map $\pi : M \to D$ is a special Lagrangian fibration on $M$. The moduli space of special Lagrangian tori with flat $U(1)$ connections is just the total space of the dual torus fibration $\pi : W \to D$. Moreover the $L^2$ metric on the moduli space is the same as putting the dual metric on each dual torus fiber (see for example [LYZ]). This is what we call a Fourier transformation:

$$dx^j \to dx^j \text{ and } dy^j \to dy_j.$$

Here $y_j$'s are the dual coordinates on the dual torus fiber. Since $M$ is the quotient of $TD$ by a lattice $\Lambda$, $W$ is naturally a quotient of $T^*D$ by the dual lattice $\Lambda$. In particular, it has a canonical symplectic form. Together with the metric, we obtain an almost complex structure on $W$. Then $W$ has the following structures

| Structures on $W$: | |
|---|---|
| Riemannian metric | $g_W = \Sigma \phi_{jk} dx^j \otimes dx^k + \phi^{jk} dy_j \otimes dy_k$ |
| Complex structure | $\Omega_W = \Pi \left(\phi_{jk} dx^k + i dy_j\right) = dz_1 \wedge ... \wedge dz_n$ |
| Symplectic structure | $\omega_W = \Sigma dx^j \wedge dy_j.$ |

Here $\left(\phi^{jk}\right) = \left(\phi_{jk}\right)^{-1}$. Next we perform a Legendre transformation along the base $D$, namely we consider a change of coordinates $x_k = x_k\left(x^j\right)$ given by

$$\frac{\partial x_k}{\partial x^j} = \phi_{jk},$$

thanks to the convexity of $\phi$. Combining both transformations we have

| | | | |
|---|---|---|---|
| Legendre transformation | $dx^j$ | $\to$ | $dx_j = \Sigma \phi_{jk} dx^k$ |
| Fourier transformation | $dy^j$ | $\to$ | $dy_j.$ |

and

| Structures on $W$: | |
|---|---|
| Riemannian metric | $g_W = \Sigma \phi^{jk} \left(dx_j \otimes dx_k + dy_j \otimes dy_k\right)$ |
| Complex structure | $\Omega_W = \Pi \left(dx_j + i dy_j\right) = dz_1 \wedge ... \wedge dz_n$ |
| Symplectic structure | $\omega_W = \Sigma \phi^{jk} dx_j \wedge dy_k.$ |

Moreover we have

$$\phi^{jk} = \frac{\partial^2 \psi}{\partial x_j \partial x_k},$$

for some function $\psi\left(x_k\right)$, the Legendre transformation of $\phi\left(x^j\right)$. It is obvious that the convexity of $\phi$ and $\psi$ are equivalent to each other. Moreover

$$\det\left(\frac{\partial^2 \phi}{\partial x^j \partial x^k}\right) = C \text{ is equivalent to } \det\left(\frac{\partial^2 \psi}{\partial x_j \partial x_k}\right) = C^{-1}.$$

Roughly speaking we have changed $T^*D$ to $TD^*$. Now the similarity between $g_M, \omega_M$ and $g_W, \omega_W$ are obvious. Therefore $W$ is also a semi-flat Calabi-Yau manifold. If we apply the same procedure to $W$, we are going to recover $M$ itself, an inversion property.

If we use the complex polarization of our B-field as follows,

$$\omega^n = i^n \Omega \bar{\Omega}$$
$$\operatorname{Im} e^{i\theta} (\omega + i\beta)^n = 0$$
$$\operatorname{Im} e^{i\phi} \Omega = 0 \text{ on the zero section.}$$

Then under the mirror transformation to $W$, they become

$$\omega_W^n = i^n \Omega_W \bar{\Omega}_W$$
$$\operatorname{Im} e^{i\theta} \Omega_W = 0 \text{ on the zero section.}$$
$$\operatorname{Im} e^{i\phi} (\omega_W + i\beta_W)^n = 0.$$

We come back and look at only the Fourier transformation which does not depend on the function $\phi$, but not the Legendre transformation which does depend on $\phi$. On the tangent bundle $M = TD$, if we vary its symplectic structures while keeping its natural complex structure fixed. This is equivalent to having a family of solutions to the real Monge-Ampère equation. Now on the $W = T^*D$ side, the corresponding symplectic structure is unchanged, namely $\omega_W = \Sigma dx^j \wedge dy_j$. But the complex structures on $W$ vary because the complex coordinates on $W$ are given by $dz_j = \Sigma \phi_{jk} dx^j + i dy_j$ which depends on particular solutions of the real Monge-Ampère equation.

To complete this identification, we need to complexified the symplectic structures, $\omega_M^{\mathbb{C}} = \omega_M + i\beta_M$, by including a B-field. We are using the real involution of $M$ to define a B-field on it. The complexified real Monge-Ampère equation is,

$$\det (\phi_{jk} + i\eta_{jk}) = C,$$

where $\beta_M = i\partial\bar{\partial}\eta(x)$ and $C$ is a non-zero constant. If we write

$$\theta_{jk}(x) = \phi_{jk}(x) + i\eta_{jk}(x),$$

then the above equation becomes $\det(\theta_{jk}) = C$. The above discussions can be extended to this case if we consider the following modified transformation:

$$\begin{array}{lccl} \text{Legendre transformation} & dx^j & \to & dx_j = \Sigma \phi_{jk} dx^k \\ \text{Fourier transformation} & dy^j & \to & dy_j + \Sigma \eta_{jk} dx^k. \end{array}$$

See [L3] for details. Now we do have an identification between the moduli space of complexified symplectic structures on $M$ with the moduli space of complex structures on $W$ [4]. This identification is both a holomorphic map and an isometry. Moreover the two Yukawa couplings are also identified under this transformation. How about instanton correction? There are neither holomorphic curves in $M$ or $W$, nor holomorphic disks whose boundaries lie on any fiber or any section. In particular this verifies the COGP mirror conjecture in this case.

[4]Notice that it is important that the B-fields satisfy the complexified Monge-Amperé equation instead of being a harmonic two form.

If we vary the complex structures on $M$ as follows: we define the new complex structure on $M$ using the following holomorphic coordinates,

$$z_t^j = x^j + ity^j,$$

for any $t \in \mathbb{R}_{>0}$. As $t$ goes to zero, this will approach a large complex structure limit for $M$. The corresponding Calabi-Yau metric for any $t$ is given by

$$g_t = \Sigma\phi_{jk}\left(\frac{1}{t}dx^j \otimes dx^k + tdy^j \otimes dy^k\right).$$

The given fibration $\pi : M \to D$ is always a special Lagrangian fibration for each $t$. Moreover the volume form on $M$ is independent of $t$, namely $dv_M = \omega^n/n!$. As $t$ goes to zero, the size of the fibers shrinks to zero while the base gets infinitely large.

If we rescale the metric to $tg_t$, then the diameter of $M$ stays bound and it converges in the Gromov-Hausdorff sense to the real $n$ dimension manifold $D$ with the metric $g_D = \Sigma\phi_{jk}dx^j dx^k$ as $t$ approaches zero. Moreover the sectional curvature stays bound in this limit. This is consistent with the above mentioned expectations of what happens near the large complex structure limit point.

In fact the variation of Hodge structures given by this family of variation of complex structures on $M$ determines an $\mathbf{sl}(2)$ action both on level of differential forms and on the level of cohomology groups of $M$ (see [L3]). Together with the hard Lefschetz $\mathbf{sl}(2)$ action, this produces an $\mathbf{sl}(2) \times \mathbf{sl}(2)$ action on $H^*(M, \mathbb{C})$. Under the above mirror transformation, the VHS $\mathbf{sl}(2)$ action on cohomology groups of $M$ (resp. $W$) will become the hard Lefschetz $\mathbf{sl}(2)$ action on cohomology groups of $W$ (resp. $M$). This confirms our $\mathbf{sl}(2) \times \mathbf{sl}(2)$ conjecture in the semi-flat Calabi-Yau case.

In [LYZ], Yau, Zaslow and the author identifies the moduli space of A-cycles on $M$ which are sections and B-cycles on $W$. We also identify their correlation functions, thus verifies the Vafa conjecture in this case.

In conclusion, we can understand and prove many mirror symmetry phenomenons in the semi-flat case via the modified Fourier transformation and the Legendre transformation.

## 6 Mirror symmetry for rigid CY manifolds

Since the beginning of the mirror symmetry era, it is well-known that the mirror symmetry conjecture cannot hold true for all Calabi-Yau manifolds. If $M$ is a Calabi-Yau manifold with $H^1(M, T_M) = 0$, then its complex structure has no infinitesimal deformation. We call such $M$ a rigid Calabi-Yau manifold. If $W$ is a mirror manifold of $M$, it would have $H^1(W, T_W^*) = 0$. In particular $W$ cannot be a Kähler manifold, a contradiction. Such failure is caused by the non-existence of a special Lagrangian fibration on $M$.

Nonetheless it is argued by Schimmrigk [Sc][Sc2], Candelas, Derrick and Parkes [CDP] and Vafa using conformal field theory that in certain cases there should still be mirror manifolds which are certain higher dimensional Fano manifolds. Batyrev and Borisov [BB] give a systematic way to construct candidates of such higher dimensional mirror manifolds using toric geometry.

In this section we will explain the construction of these higher dimension mirror manifolds and offer a partial answer to mirror symmetry phenomenons using ideas from [SYZ]. We begin with an example of rigid Calabi-Yau threefold.

**An example of rigid Calabi-Yau threefold**

Let $E_0$ be the Fermat cubic in $\mathbb{P}^2$:

$$E_0 = \left\{[z_0, z_1, z_2] \in \mathbb{P}^2 : z_0^3 + z_1^3 + z_2^3 = 0\right\} = \mathbb{C}/ <1, e^{\pi i/3}> .$$

This is the only elliptic curve with a nontrivial $\mathbb{Z}_3$ action. If $\omega$ is a nontrivial cubic root of unity, then the $\mathbb{Z}_3$ action of $E_0$ is generated by

$$[z_0, z_1, z_2] \to \left[z_0, \omega z_1, \omega^2 z_2\right].$$

This $\mathbb{Z}_3$ action has three fixed points: $[1, 0, 0]$, $[0, 1, 0]$ and $[0, 0, 1]$. The diagonal action of $\mathbb{Z}_3$ on $(E_0)^3$ preserves its holomorphic volume form and it has 27 fixed points. The quotient space

$$M_0 = (E_0)^3 / \mathbb{Z}_3,$$

is therefore a Calabi-Yau orbifold with 27 singular points, each model locally on $\mathbb{C}^3/\mathbb{Z}_3$. Resolving $M_0$ by replacing each singular point by a copy of $\mathbb{P}^2$ with normal bundle $O_{\mathbb{P}^2}(-3)$. We obtain a simply connected Calabi-Yau threefold $M$ with $H^1(M, T_M) = 0$, a rigid manifold.

It is easy to construct a special Lagrangian fibration on $(E_0)^3$. We start with one on the elliptic curve $E_0$ which is given by the projection to the imaginary part,

$$\begin{aligned} E_0 = \quad & \mathbb{C}/<1, e^{\pi i/3}> \to \mathbb{R} \\ & z \to \operatorname{Im} z. \end{aligned}$$

This is a special Lagrangian fibration on $E_0$ and similarly we have one on $(E_0)^3$ using the product fibration. However such fibration will not be invariant under the diagonal $\mathbb{Z}_3$ action and therefore it cannot descend to one on $M_0$. This is consistent with the non-existence of mirror for the rigid Calabi-Yau manifold $M$ from SYZ point of view.

Physicists argued from the point of view in the conformal field theory that there should still be a higher dimensional mirror manifold for $M$, say $\bar{W}$. Concretely $\bar{W}$ is the quotient of the Fermat cubic hypersurface in $\mathbb{P}^8$ by a $\mathbb{Z}_3$ action. For usual mirror manifolds $M$ and $W$, their Hodge diamonds flip, in the sense that $\dim H^{p,q}(M) = \dim H^{3-p,q}(W)$. In our present situation, $\bar{W}$ has bigger dimension than $M$ and the flipped Hodge diamond of $M$ sits in the middle of the Hodge diamond of $\bar{W}$, i.e.

$$\dim H^{p,q}(M) = \dim H^{3-p+2,q+2}\left(\bar{W}\right).$$

Explicitly we have $\dim H^{1,1}(M) = 36$, $\dim H^{2,1}(M) = 0$ and $\dim H^{4,3}\left(\bar{W}\right) = 36$ and $\dim H^{3,3}\left(\bar{W}\right) = 0$.

**A cubic fourfold as the mirror of a K3 surface**

To get a closer look at what happens, we consider the two dimensional case first. The simplest example is the Kummer K3 surface. The map which sends $x$ to $-x$ generates a $\mathbb{Z}_2$ action on $(E_0)^2$, or any complex two torus. The quotient $(E_0)^2/\mathbb{Z}_2$ has 16 singular points, each is an ordinary double point. Namely it is locally modelled on $\mathbb{C}^2/\mathbb{Z}_2$ and it is called an $A_1$ singularity. Blowing up will replace each singular point by a copy of $\mathbb{P}^1$ with normal bundle $O_{\mathbb{P}^1}(-2)$ and we obtain a smooth simply connected Calabi-Yau surface called the Kummer K3 surface.

To construct a special Lagrangian fibration on the Kummer K3 surface, we start with the one on $E_0$ and therefore $(E_0)^2$ as before. A simple observation: this special Lagrangian fibration on $(E_0)^2$ is invariant under the $\mathbb{Z}_2$ action and therefore descends to one on $(E_0)^2/\mathbb{Z}_2$. Using twistor transformation to an elliptic fibration problem, one can show that this special Lagrangian fibration structure continue to exist on the Kummer K3 surface.

Next we consider $M_0 = (E_0)^2/\mathbb{Z}_3$ where the first $\mathbb{Z}_3$ action is as before and the second one is its square. Now $M_0$ is a Calabi-Yau orbifold with 9 isolated singular points of type $A_2$. Resolving $M_0$ by blowing up these singularities will replace each singular point by two smooth rational curves meeting a point. The normal bundle of each rational curve is $O_{\mathbb{P}^1}(-2)$. This again produces a K3 surface $M$.

The natural special Lagrangian fibration on $(E_0)^2$ is not invariant under the $\mathbb{Z}_3$ action and therefore does not induce one on $M_0$ nor $M$. According to the recipe by Batyrev and Borisov, the mirror of such $M$ should be the Fermat cubic fourfold in $\mathbb{P}^5$:

$$\bar{W} = \left\{z_0^3 + z_1^3 + z_2^3 + z_3^3 + z_4^3 + z_5^3 = 0\right\} \subset \mathbb{P}^5.$$

We need to come back to understand why the special Lagrangian fibration on $(E_0)^2$ does not work. After twistor rotation, each fiber in $(E_0)^2$ is a genus one Riemann surface but its image in $(E_0)^2/\mathbb{Z}_3$ is no longer embedded. In fact it intersects itself transversely at one point. Namely it is a singular Riemann surface of genus two! A generic deformation of this supersymmetric cycle will smooth out the node. Such nearby cycles will no longer come from the special Lagrangian fibration on $(E_0)^2$.

The moduli space of these supersymmetric A-cycles is not going to produce us the usual mirror manifold of $M$, which is usually $M$ itself because it is hyperkähler. Instead it is the compactified Jacobian $\mathcal{J}$ for genus two curves in the K3 surface $M$. Birationally this hyperkähler manifold $\mathcal{J}$ is the Hilbert scheme of 2 points on $M$. In fact $\mathcal{J}$ is also birational to the Fano variety of lines on the cubic fourfold $\bar{W}$. In terms of our languages, this is a moduli space of B-cycles $(C, E)$ on $\bar{W}$ such that $C$ is a degree one curve in $\bar{W}$ and $E$ is a unitary flat line bundle over $C$ which is necessarily trivial. Therefore the Batyrev-Borisov mirror $\bar{W}$ and the SYZ mirror $\mathcal{J}$ of $M$ are related as one being the moduli space of B-cycles on another one. It is curious to know how much of these continue to happen for more general cases. The author thanks B. Hassett for many discussions on these matters.

**Generalized Calabi-Yaus and Batyrev-Borisov construction**

We are going to describe the Batyrev-Borisov construction of higher dimensional candidates of mirror manifolds to complete intersection Calabi-Yau manifolds inside toric varieties. In [Sc] Schimmrigk described a class of Fano manifolds, called generalized Calabi-Yau manifolds, that certain part of its conformal field theory behaves as if it comes from a Calabi-Yau manifold. For example it has a mirror which is again another generalized Calabi-Yau manifold. A systematic way to construct these mirrors of generalized Calabi-Yau manifolds using toric geometry is given by Batyrev and Borisov in [BB]. Moreover, given any Calabi-Yau manifold $M$ of dimension $n$ which is a complete intersection of $r$ hypersurfaces in a Fano toric variety, there is a generalized Calabi-Yau manifold $\bar{M}$ of dimension $n + 2(r-1)$ associated to it. When $M$ and $W$ are mirror manifolds in the usual sense, their associated generalized Calabi-Yau manifolds $\bar{M}$ and $\bar{W}$ will continue to be mirror manifolds in the generalized sense.

When the complete intersection Calabi-Yau manifold $M$ is rigid, its mirror manifold $W$ does not exist. However we still have $\bar{W}$, a mirror to the generalized Calabi-Yau manifold $\bar{M}$, which we regard as the higher dimensional mirror to $M$.

Now we describe the class of generalized Calabi-Yau manifolds $\bar{M}$, they are hypersurfaces in Fano toric varieties $X_\Delta$. Recall that the zero set of a section of $K_{X_\Delta}^{-1}$ is a Calabi-Yau manifold, possibly singular. Its mirror is another Calabi-Yau which is the zero set of a section of $K_{X_\nabla}^{-1}$, here the polytope $\nabla$ is the polar dual to $\Delta$. If we can write

$$K_{X_\Delta}^{-1} = O(r) = O(1)^{\otimes r}$$

the $r^{th}$ tensor power of a line bundle $O(1)$, then the zero set of a section of

$O(1)$ is called a generalized Calabi-Yau manifold $\bar{M}$, and we have

$$K_{\bar{M}}^{-1} = O(r-1).$$

In particular $\bar{M}$ is a Fano manifold. When this happens, the anti-canonical line bundle $K_{X_\nabla}^{-1}$ of $X_\nabla$ also have the same property, its corresponding section will determine another generalized Calabi-Yau manifold $\bar{W}$, we call it the mirror of $\bar{M}$. We define the integer $n$ as

$$\dim \bar{M} = \dim \bar{W} = n + 2(r-1).$$

To study mirror symmetry for these generalized Calabi-Yau manifolds, we should not include all complex submanifolds. For example points should not be considered as B-cycles. How about A-cycles? We should probably look at all Lagrangian submanifolds, but we do not have a good notion of a special Lagrangian submanifold. For the homological mirror symmetry, we can still discuss the Fukaya category of Lagrangians in $\bar{M}$, or $\bar{W}$. We would look for a category of certain coherent sheaves on $\bar{M}$, or $\bar{W}$ so that this category behaves as if it is a category of coherent sheaves on a Calabi-Yau manifold of dimension $n$. For instance $Ext^k(S_1, S_2)^* = Ext^{n-k}(S_2, S_1)$. The natural question would be to compare these two types of categories.

**From Calabi-Yaus to generalized Calabi-Yaus**

We use the above construction of mirror for generalized Calabi-Yau manifolds to obtain higher dimensional mirrors for rigid Calabi-Yau manifolds which are complete intersections in Fano toric varieties.

We first recall that when $M$ is a Calabi-Yau hypersurface in a Fano toric variety $X_\Delta$, it can be deformed to the most singular Calabi-Yau which is the union of toric divisors in $X_\Delta$. This is the large complex structure limit. It is expected that this is the origin of special Lagrangian fibration on $M$. The same idea works for Calabi-Yau complete intersections in $X_\Delta$ provided that there are enough deformations. This is the case if

$$M = \bigcap_{i=1}^{r} D_i$$

with each $D_i$ semi-ample and $\Sigma D_i = -K_{X_\Delta}$.

If some of the $D_i$'s are not semi-ample then the Calabi-Yau manifold $M$ may not be able to deform to a large complex structure limit point. And we do not expect to find a special Lagrangian fibration on $M$ coming from its embedding inside $X_\Delta$. This is precisely what happens for the rigid Calabi-Yau threefold $(E_0)^3/\mathbb{Z}_3$. Of course this will not happen if $r=1$ because $D = -K_{X_\Delta}$ is ample for Fano variety.

The collection of effective divisors $D_i$'s in $X_\Delta$ determines a hypersurface $\bar{M}$ in the projective bundle $\mathbb{P}(\oplus_{i=1}^r O(D_i))$ over $X_\Delta$.

$$\bar{M} \subset \mathbb{P}(\oplus_{i=1}^r O(D_i)).$$

This is because a section of $O(D_i)$ over $X_\Delta$ can be identified as a function on the total space of the line bundle $O(-D_i)$ which is linear along fibers. If we projectivize each fiber of the direct sum of $O(D_i)$'s, then this function will become a section of $O(1)$, the dual of the tautological line bundle. This gives all sections of $O(1)$, namely

$$\oplus_{i=1}^r H^0 (X_\Delta, O(D_i)) \cong H^0 (\mathbb{P}(\oplus_{i=1}^r O(D_i)), O(1)).$$

Note that $\bar{M}$ is a generalized Calabi-Yau manifold with $\dim \bar{M} = \dim M + 2r - 2$ and $h^{p,q}(M) \leq h^{p+r-1,q+r-1}(\bar{M})$. If we restrict the projective bundle morphism to $\bar{M}$, we obtain a surjective morphism

$$\pi : \bar{M} \to X_\Delta.$$

The fiber over a point in $M$ (resp. outside $M$) is a copy of $\mathbb{P}^{r-1}$ (resp. $\mathbb{P}^{r-2}$). Therefore $Y = \pi^{-1}(M) \subset \bar{M}$ is a $\mathbb{P}^{r-1}$-bundle over $M$,

$$\begin{array}{ccc} \mathbb{P}^{r-1} & \to & Y \\ & & \downarrow \pi_Y \\ & & M. \end{array}$$

In fact $M$ is the moduli space of $\mathbb{P}^{r-1}$ in the fiber homology class in $\bar{M}$.

**A construction of Lagrangian submanifolds in $\bar{M}$**

Since $\bar{M}$ has higher dimension than $M$, it is a very interesting problem to compare the symplectic geometries between $M$ and $\bar{M}$. The author benefits from discussions with Seidel.

Using the method of vanishing cycles, we give a new construction of a Lagrangian submanifold in $\bar{M}^{n+2r-2}$ from one in $M^n$: Recall that the hypersurface $\bar{M} \subset \mathbb{P}(\oplus_{i=1}^r O(D_i))$ is defined by $s_1 + \ldots + s_{r-1} + s_r = 0$. Notice each $s_j$ is only well-defined up to nonzero multiple. If we rescale $s_r$ to zero then

$$\bar{M}_0 = \{s_1 + \ldots + s_{r-1} + s_r = 0\} \subset \mathbb{P}(\oplus_{i=1}^r O(D_i))$$

is a cone over $X_\Delta$ of

$$\bar{N} = \{s_1 + \ldots + s_{r-1} = 0\} \subset \mathbb{P}\left(\oplus_{i=1}^{r-1} O(D_i)\right).$$

The fiber of $\bar{M}_0 \to X_\Delta$ over a point in $\bigcap_{i=1}^{r-1} D_i$ (resp. in $X_\Delta \backslash \bigcap_{i=1}^{r-1} D_i$) is a copy of $\mathbb{P}^{r-1}$ (resp. $\mathbb{P}^{r-2}$). Moreover $\bar{M}_0$ is singular along those cone points over $\bigcap_{i=1}^{r-1} D_i$. When $\bar{M}$ degenerates to $\bar{M}_0$ as we rescale $s_r$ to zero, the vanishing cycles is a family of spheres $S^{2r-3}$ parametrized by $\bigcap_{i=1}^{r-1} D_i = Sing(\bar{M}_0)$. However, over any point in $M \subset \bigcap_{i=1}^{r-1} D_i$, such a $S^{2r-3}$ in fact shrink to a point! This is because the degeneration from $\bar{M}$ to $\bar{M}_0$ is not semi-stable and the base locus is given by $M$. Being in the base locus means that such a point does not move during the degeneration process, thus we never create a vanishing cycle for it - the vanishing of the vanishing cycle.

Suppose $L$ is a Lagrangian submanifold in $M = \bigcap_{i=1}^{r} D_i$. We assume that there is a Lagrangian submanifold $\hat{L}$ in $\bigcap_{i=1}^{r-1} D_i$ with $\partial\hat{L} = L$. Such a $\hat{L}$ can be constructed, for example, when $L$ is a vanishing cycle for $M$ as $D_r$ deforms. The above family of $S^{2r-3}$ over $\hat{L}$ which shrink to points over its boundary $L$. The total space $\bar{L}$ is a *closed* submanifold of half dimensional in $\bar{M}$. With more care, one should be able to verify that $\bar{L}$ is a Lagrangian submanifold in $\bar{M}$.

A natural and important problem is to compare the Floer homology groups $HF(M; L, L')$ and $HF\left(\bar{M}'; \bar{L}, \bar{L}'\right)$.

**Complex geometries between $M$ and $\bar{M}$**

For any coherent sheaf $S$ on $M$ we can easily construct an another coherent sheaf on $\bar{M}$ as $\bar{S} = \iota_* \circ \pi_Y^* (S)$ where $\iota : Y \subset \bar{M}$ is the natural inclusion. An important problem is to recover the derived category of coherent sheaves on $M$ from the one on $\bar{M}$ together with $O(1) = K_{\bar{M}}^{1/(r-1)}$.

Similarly we can construct a homomorphism between their Chow groups,

$$\begin{aligned} CH_k(M) &\to CH_{k+r-1}(Y) \to CH_{k+r-1}\left(\bar{M}\right) \\ C &\to \pi_Y^{-1} C. \end{aligned}$$

Here the first morphism is the pullback map and the second morphism simply regards $\pi_Y^{-1}C$ as a algebraic cycle in $\bar{M}$ via the embedding $Y \subset \bar{M}$. In terms of cohomology groups, we have

$$H^l(M) \xrightarrow{\alpha} H^l(Y) \underset{\cong}{\xrightarrow{\beta}} H^{l+2r-2}\left(\bar{M}, \bar{M}\backslash Y\right) \xrightarrow{\gamma} H^{l+2r-2}\left(\bar{M}\right).$$

Here $\alpha = \pi_Y^*$ is the pullback morphism, $\beta$ is the Thom isomorphism and $\gamma$ is natural homomorphism in the long exact sequence for the pair $\left(\bar{M}, \bar{M}\backslash Y\right)$. We expect that these morphisms preserve Hodge structures, in particular, the image of $H^{p,q}(M)$ lies inside $H^{p+r-1,q+r-1}\left(\bar{M}\right)$.

It seems more natural to consider $H^{*,*}\left(\bar{M}, \bar{M}\backslash Y\right)$ than $H^{*,*}\left(\bar{M}\right)$. For an arbitrary generalized Calabi-Yau manifold $\bar{M}$ with $K_{\bar{M}}^{-1} = O(r-1)$, we define $M$ to be the moduli space of $\mathbb{P}^{r-1}$ representing the class $c_1(O(1))^{r-1}$ and $Y$ is the image of the universal variety inside $\bar{M}$. For example, in the case of cubic sevenfold (or cubic fourfold), we have $Y = \bar{M}$ and therefore $H^{*,*}\left(\bar{M}, \bar{M}\backslash Y\right) = H^{*,*}\left(\bar{M}\right)$.

When $\bar{M}$ is associated to a Calabi-Yau manifold $M$ as above, then we have the Thom isomorphism,

$$H^{l+2r-2}\left(\bar{M}, \bar{M}\backslash Y\right) \cong H^l(Y).$$

So we can simply look at $Y$, a $\mathbb{P}^{r-1}$-bundle over $M$, rather than the pair $\left(\bar{M}, \bar{M}\backslash Y\right)$. For example, if $E$ is a holomorphic vector bundle over $M$, then $\pi_Y^* E$ is a holomorphic vector bundle on $Y$ and we have an isomorphism,

$$H^k(Y, O(\pi_Y^* E)) \cong H^k(M, E),$$

using the Leray spectral sequence. We would like to extend $\pi_Y^* E$ over $Y$ to a holomorphic bundle $\bar{E}$ over the pair $\left(\bar{M}, \bar{M}\backslash Y\right)$ (up to quasi-isomorphism) and obtain an analog of the Thom isomorphism twisted by $\bar{E}$. More generally one might want to perform these comparisons on the derived categories of coherent sheaves on $M$ and $\left(\bar{M}, \bar{M}\backslash Y\right)$ (see the remark below).

We come back to discuss the mirror of generalized Calabi-Yau manifolds. If $D_i$'s are semi-ample so that $M$ has a mirror manifold $W$, then $W$ is also a complete intersection in a toric variety $X_\nabla$ where $\nabla$ is the polar dual of the polytope $\Delta$. So there is also a generalized Calabi-Yau manifold $\bar{W}$ associated to $W$. It can be shown that $\bar{W}$ coincides with the Batyrev-Borisov's construction of the mirror for the generalized Calabi-Yau manifold $\bar{M}$.

When $D_i$'s are not semi-ample, the mirror of $M$ might not exist but such higher dimensional mirror $\bar{W}$ always exists. The generalization of mirror symmetry conjecture to this situation should be the comparison of the symplectic geometry on $M$ to certain part of the complex geometry on $\bar{W}$.

**Rigid Calabi-Yau manifolds as moduli spaces**

A lesson we learned from the paper [SYZ] is that the mirror manifold of $M$ can be realized as a compactified moduli space of certain supersymmetric A-cycles on $M$. In this case they are special Lagrangian tori with flat connections on them. Roughly speaking these cycles sweep over $M$ once. This explains why the geometry of $M$ is reflected in the geometry of this moduli space $W$. A similar and simpler example is $M$ itself is a moduli space of supersymmetric B-cycles in $M$, namely the moduli space of points in $M$.

In general the complex (resp. symplectic) geometry of $M$ is closely related to the complex (resp. symplectic) geometry of any moduli space of B-cycles in $M$. And the symplectic (resp. complex) geometry of $M$ is closely related to the symplectic (resp. complex) geometry of any moduli space of A-cycles in $M$.

In fact any Calabi-Yau manifold $M$ in $X_\Delta$, which is a complete intersection of $r$ hypersurface, is itself a moduli space of supersymmetric B-cycles in $\bar{M}$. Each cycle $(C, E)$ in this moduli space consists of a projective space $\mathbb{P}^{r-1} = C$ and the trivial line bundle $E$ over it $C$. Moreover each of these $C$'s is a fiber of the projective bundle over $X_\Delta$. This suggests an explanation for why $M$ and $\bar{W}$ should be mirror manifolds. For example, for the particular K3 surface whose mirror is a cubic fourfold as before, we have $r = 2$ and that is why we looked at the moduli space of $\mathbb{P}^1$. If we look at the moduli space of those $\mathbb{P}^1$'s which are of degree one, namely the Fano variety of lines on the cubic fourfold then it is, at least birationally, the moduli space of those special Lagrangians on the K3 surface which descend from the special Lagrangian fibration of the four torus. Since this moduli space is a hyperkähler manifold, it is self-mirror. Hence this provides a possible explanation of mirror symmetry for generalized Calabi-Yau manifolds from the SYZ picture in this case.

**A remark on Yau's program on noncompact CY manifolds**

A generalized Calabi-Yau $\bar{M}$ might not look like a Calabi-Yau manifold, however its complement $X_\Delta\backslash\bar{M}$ does. Namely $X_\Delta\backslash\bar{M}$ admits a complete Ricci flat Kähler metric if $\bar{M}$ is a Calabi-Yau hypersurface or a generalized Calabi-Yau Kähler-Einstein manifold. We expect that this complement $X_\Delta \backslash \bar{M}$ also admit a special Lagrangian fibration and mirror symmetry. Yau first discusses the problem of constructing Kähler-Einstein metrics in the noncompact setting in [Y3] (see also [Y4]).

To begin, we suppose that $M$ is a Calabi-Yau manifold of dimension $n$ which is an anti-canonical divisor in a Fano toric variety $X_\Delta$. When $M$ approaches the large complex structure limit, which is the union of $n$ dimensional strata of $X_\Delta$, then we expect from SYZ proposal [SYZ] that $M$ should admit a special Lagrangian fibration which looks like the toric fibration on these strata in $X_\Delta$. The dual fibration would product the mirror manifold of $M$ (see also [LV]).

Not just $M$ admits a Ricci-flat Kähler metric, its complement $X_\Delta \backslash M$ also admits a complete Ricci-flat Kähler metric by the work of Tian and Yau [TY1] and also Bando and Kobayashi [BK] as the first step towards Yau's problem on noncompact CY manifolds [Y3]. When $M$ approaches the large complex structure limit, then $X_\Delta \backslash M$ would becomes $(\mathbb{C}^\times)^{n+1}$. This open manifold certainly admits a special Lagrangian fibration with respect to the standard flat metric. As an extension to the SYZ conjecture, we expect that $X_\Delta \backslash M$ would admit a special Lagrangian fibration. Moreover this fibration is compatible with the one on $M$, namely when the fiber torus $T^{n+1}$ in $X_\Delta \backslash M$ goes to infinity, which is $M$, then a circle direction collapses and it becomes a special Lagrangian torus $T^n$ for the fibration in $M$. The mirror conjectures for $M \subset X_\Delta$ and $W \subset X_\nabla$ should also be extended to $X_\Delta \backslash M$ and $X_\nabla \backslash W$.

Now we consider a generalized Calabi-Yau manifold $\bar{M}$, which is a Fano hypersurface in $X_\Delta$ with

$$K_{X_\Delta}^{-1} = O\left(r\bar{M}\right).$$

In [TY2] and also [BaK], the authors prove that $X_\Delta \backslash \bar{M}$ admits a complete Ricci-flat Kähler metric provided that $\bar{M}$ is Kähler-Einstein. It would be useful if we can construct a special Lagrangian fibration on $X_\Delta \backslash \bar{M}$ and understand its behaviors near infinity. We also expect that its dual fibration should coincide with the special Lagrangian fibration on $X_\nabla \backslash \bar{W}$. This is a very difficult analytic problem but its solution would provide us knowledge about mirror symmetry for generalized Calabi-Yau manifolds from SYZ point of view.

For example when $\bar{M}$ is a point, say the north pole, in $X_\Delta = \mathbb{P}^1$, then $X_\Delta \backslash \bar{M} = \mathbb{C}$ has a special Lagrangian fibration by lines with constant real part. Its dual fibration coincides with the corresponding fibration on $X_\nabla \backslash W = \mathbb{C}$. An anti-canonical Calabi-Yau hypersurface $M$ in $\mathbb{P}^1$ has two points, say the north pole and one other point $p$. Its complement $X_\Delta \backslash M$ is isomorphic to $\mathbb{C}^\times$ and it has a special Lagrangian fibration by $S^1$. If we move $p$ toward the north pole, then $M$ becomes $2\bar{M}$, a non-reduced scheme. Moreover the limiting special Lagrangian fibration will be the one above on $\mathbb{C}$ given by straight lines.

In general we should treat $\bar{M} \subset X_\Delta$ to have multiplicity $r$. This is because the zero sets of a family of sections of $K_{X_\Delta}^{-1} = O\left(r\bar{M}\right)$, which are Calabi-Yau manifolds, can converge to a non-reduced scheme $r\bar{M}$. The scheme structure on $\bar{M}$ does depend on the family.

# 7 Appendix: Deformations of SUSY B-cycles

We are going to study the deformation theory of B-cycles and structures of their moduli spaces. If we forget the deformed Hermitian-Yang-Mills equations, namely we consider deformations of a complex submanifold $C$ together with a holomorphic bundle $E$ over it, then they are parametrized infinitesimally by the cohomology group $Ext^1_{O_M}(\iota_* E, \iota_* E)$. To understand the deformation theory when the deformed Hermitian-Yang-Mills equation is included, we would need to have differential form representatives for classes in $Ext^1_{O_M}(\iota_* E, \iota_* E)$. We should see that

$$Ext^1_{O_M}(\iota_* E, \iota_* E) = H^1(C, \mathbb{E}) \times_{H^1(C,T_C)} H^0\left(C, N_{C/M}\right),$$

and we can represent elements on the right hand side by differential forms. First we consider only deformations of the deformed Hermitian-Yang-Mills equation while $C$ is kept fixed.

**Deformation of deformed Hermitian-Yang-Mills bundles**

To begin we consider infinitesimal deformations of B-cycles $(C, E)$ which fix the complex submanifold $C \subset M$. If we forget the Hermitian metric on $E$ and only deform the holomorphic structure of the vector bundle $E$, then they are parametrized by the cohomology group $H^1(C, End(E))$ infinitesimally. There are nontrivial [Th1] obstruction classes in $H^2(C, End(E))$ for these infinitesimal deformations to come from a honest deformation.

If we vary Hermitian-Yang-Mills connections on $E$, then it is well-known that they are parametrized by harmonic $(0,1)$ forms with valued in $ad(E)$. By Dolbeault-Hodge theorem this is isomorphic to $H^1(C, End(E))$. It is also not difficult to see that infinitesimal variations of deformed Hermitian-Yang-Mills bundles on $C$ are parametrized by $B \in \Omega^{0,1}(C, ad(E))$ satisfying

$$\begin{aligned} \bar{\partial} B &= 0, \\ \operatorname{Im} e^{i\theta}\left(\omega^{\mathbb{C}} + F\right)^{m-1} \wedge \partial B &= 0. \end{aligned}$$

In general we have the following definition ([LYZ], see also [L2]).

**Definition 3** *Let $E$ be a Hermitian bundle over an $m$ dimensional Kähler manifold $C$ with complexified Kähler form $\omega^{\mathbb{C}}$. A differential form $B \in \Omega^{0,q}(C, ad(E))$ is called a deformed $\bar{\partial}$-harmonic form if it satisfies*

$$\begin{aligned} \bar{\partial} B &= 0, \\ \operatorname{Im} e^{i\theta}\left(\omega^{\mathbb{C}} + F\right)^{m-q} \wedge \partial B &= 0. \end{aligned}$$

*Here $F$ is the curvature tensor of $E$.*

*We denote the space of their solutions as $QH^q(C, End(E))$.*

If the connection on $E$, the B-field on $C$ and the phase angle $\theta$ are all trivial, then a deformed $\bar{\partial}$-harmonic form is just an ordinary $\bar{\partial}$-harmonic form. It is useful to know if there is always a unique deformed $\bar{\partial}$-harmonic representative for each cohomology class in $H^q(C, End(E))$. One might need to require $\omega^{\mathbb{C}}+F$ to be sufficiently positive to ensure ellipticity of the equation.

**Deformation of B-cycles**

The situation becomes more interesting when we deform both the complex submanifold $C \subset M$ and a holomorphic vector bundle $E$ over $C$. The curvature tensor of $E$ determines a cohomology class in $H^{1,1}(C, End(E))$ which is called the Atiyah class of $E$. Using the isomorphism

$$H^{1,1}(C, End(E)) = Ext^1_{O_C}(T_C, End(E)),$$

we obtain an extension bundle $\mathbb{E}$,

$$0 \to End(E) \to \mathbb{E} \to T_C \to 0.$$

The cohomology group $H^1(C, \mathbb{E})$ is the space of infinitesimal deformations of the pair $(C, E)$ with $C$ a complex manifold and $E$ a holomorphic vector bundle over $C$. It fits into the long exact sequence

$$H^0(C, T_C) \to H^1(C, End(E)) \to H^1(C, \mathbb{E}) \to H^1(C, T_C),$$

which relates various deformation problems.

If we forget $E$ and look at abstract deformations of $C$ and deformations of $C$ inside $M$. Their relationships are explained by an exact sequence

$$H^0(C, T_M|_C) \to H^0\left(C, N_{C/M}\right) \to H^1(C, T_C),$$

which is induced from the short exact sequence

$$0 \to T_C \to T_M|_C \to N_{C/M} \to 0.$$

These cohomology groups (i) $H^0(C, T_M|_C)$, (ii) $H^0\left(C, N_{C/M}\right)$ and (iii) $H^1(C, T_C)$ parametrize infinitesimal deformations of (i) $C$ inside $M$ with fix complex structure on $C$; (ii) $C$ inside $M$ with various complex structures on $C$ and (iii) complex structures on $C$.

Now the infinitesimal deformations of a pair $(C, E)$ with $C$ a submanifold of $M$ and $E$ a holomorphic bundle over $C$ are parametrized by the Cartesian product of $H^1(C, \mathbb{E})$ and $H^0\left(C, N_{C/M}\right)$ over $H^1(C, T_C)$:

$$H^1(C, \mathbb{E}) \times_{H^1(C, T_C)} H^0\left(C, N_{C/M}\right)$$

It is pointed out by Thomas that this space is the same as $Ext^1(\iota_* E, \iota_* E)$ where $\iota : C \to M$ is the inclusion morphism.

The deRham representative of an element in this space is given by $B \in \Omega^{0,1}(C, End(E))$ and $v \in \Omega^0\left(N_{C/M}\right)$ satisfying (i) $\bar{\partial} v = 0$ and $\bar{\partial} B + \left(\bar{\partial} v\right) \lrcorner F = 0$ inside $\Omega^{0,2}(C, End(E))$ where $F$ is the curvature tensor of $E$.

Using Hodge decomposition we can choose harmonic representative in each class. Namely we can find unique pair $(B, v)$ in any fixed class in $H^1(\mathbb{E}) \times_{H^1(T_C)}$

$H^0\left(N_{C/M}\right)$ which satisfies $\bar{\partial}^* B = 0$ and $\bar{\partial} v = 0$. We can also rewrite these two equations as $\partial B \wedge \omega^{n-1} = 0$ and $\bar{\partial} v = 0$. Here $\partial$ denote the $(1,0)$ component of the covariant derivative for $End\,(E)$. Recall that there can be obstructions for such infinitesimal deformations to come from a honest family.

Finally we can write down the equations whose solutions parametrize infinitesimal variations of B-cycles $(C, E)$ , where the Hermitian metric on $E$ obeys the deformed Hermitian-Yang-Mills equations. They are a deformation of the above harmonic equations for $B \in \Omega^{0,1}(C, End\,(E))$ and $v \in \Omega^0\left(N_{C/M}\right)$:

$$\left\{ \begin{array}{r} \bar{\partial} B + \left(\bar{\partial} v\right) \lrcorner F = 0, \\ \left[\partial B \left(\omega^{\mathbb{C}} + F\right)^{m-1}\right]_{sym} = 0, \\ \bar{\partial} v = 0. \end{array} \right.$$

Remark: It is more natural to consider deformation of B-cycles inside $M$ together with the deformation of the complex structure on $M$. These infinitesimal deformations are parametrized by the Cartesian product,

$$H^1(C, \mathbb{E}) \times_{H^1(C, T_C)} H^1\left(T_M \to N_{C/M}\right),$$

if we ignore the deformed Hermitian-Yang-Mills equation.

**Moduli space of B-cycles and its correlation function**

Let us denote the moduli space of B-cycles on $M$ by ${}_B\mathcal{M}(M)$. As a moduli space of holomorphic objects, it carries a natural complex structure. It also have a natural pre-symplectic form ${}_B\omega$. At a given point $(C, E)$, we denote the curvature of $E$ as $F$. If $(v_1, B_1)$ and $(v_2, B_2)$ are two tangent vectors of ${}_B\mathcal{M}(M)$ at $(C, E)$, then their pairing with this pre-symplectic form is given by

$$\begin{aligned} &{}_B\omega\left((v_1, B_1), (v_2, B_2)\right) \\ &= \operatorname{Im} e^{i\theta} \int_C Tr\left[\left(\omega^{\mathbb{C}} + F\right)^{m-1} B_1 B_2\right]_{symm} + \int_C \omega\left(v_1, v_2\right) \omega^m. \end{aligned}$$

If we replace $\omega$ by a large multiple and set the phase angle to zero, then the dominating term does give a symplectic two form:

$$\int_C Tr B_1 B_2 \omega^{m-1} + \int_C \omega\left(v_1, v_2\right) \omega^m.$$

on the moduli space of usual Hermitian-Yang-Mills bundles over $C$.

Next we assume $M$ is a Calabi-Yau manifold, its holomorphic volume form $\Omega$ induces an $n$-form ${}_B\Omega$ on the moduli space ${}_B\mathcal{M}(M)$. In string theory terminology this is the correlation function of the theory.

Let us first describe ${}_B\Omega$ in explicit terms. If $(C, E)$ is a B-cycle on $M$ of dimension $m$. Let

$$(B_j, v_j) \in \Omega^{0,1}(C, End\,(E)) \times \Omega^0\left(N_{C/M}\right)$$

with $j=1,2,...,n$ represent tangent vectors of ${}_B\mathcal{M}(M)$ at $(C,E)$ as before. Then we have

$$\begin{aligned}&{}_B\Omega(C,E)\left((B_j,v_j)\right)_{j=1}^n\\&=\sum_{(j_1,\ldots,j_n)\in S_n}\int_C Tr\left[B_{j_1}\cdots B_{j_m}\right]_{sym}\iota_{v_{j_{m+1}}}\cdots\iota_{v_{j_n}}\Omega.\end{aligned}$$

We can also write down the correlation function in an intrinsic way: Consider the diagram

$$\begin{array}{ccc}C\times Map(C,M)\times\mathcal{A}_C(E) & \overset{ev}{\to} & M\\ \downarrow\pi & & \\ Map(C,M)\times\mathcal{A}_C(E) & & \end{array}$$

where $\mathcal{A}_C(E)$ is the space of Hermitian connections on $E$. Over $C\times\mathcal{A}_C(E)$ there is a universal connection and we denote its curvature tensor as $\mathbb{F}$. There is also an evaluation map $ev: C\times Map(C,M)\to M$. Then the correlation function ${}_B\Omega$ can be expressed as follows (see for example [L2]),

$${}_B\Omega=\int_C Tr\mathbb{F}^m\wedge ev^*\Omega.$$

This can be expressed simply as the composition,

$$\otimes^n Ext^1(\iota_*E,\iota_*E)\to Ext^n(\iota_*E,\iota_*E)\overset{Tr}{\to}H^n(M,O_M)\cong\mathbb{C},$$

provided that we can replace the deformed Hermitian-Yang-Mills equation by the usual Hermitian-Yang-Mills equation.

**Symplectic reduction**

On the configuration space $Map(C,M)\times\mathcal{A}_C(E)$ we can write down a similar pre-symplectic form in an intrinsic way:

$$\begin{aligned}{}_B\omega&=\operatorname{Im}e^{i\theta}\left[\int_C Tr\left(\omega^{\mathbb{C}}+\mathbb{F}\right)^{m+1}\right]^{(2)}+\left[\int_C ev^*\omega^{m+1}\right]^{(2)}\\&=\frac{1}{(m+1)!}\left[\int_C\operatorname{Im}Tre^{i\theta+\omega^{\mathbb{C}}+F+ev^*\omega}\right]^{(2)}.\end{aligned}$$

The group of gauge transformations of $E$ acts on $\mathcal{A}_C(E)$, and therefore on $Map(C,M)\times\mathcal{A}_C(E)$ preserving ${}_B\omega$. The complex submanifold of $Map(C,M)\times\mathcal{A}_C(E)$, consisting of complex submanifolds $C\subset M$ and holomorphic connections (i.e. $F^{2,0}=0$) on $E$, is invariant under this gauge group action.

The moment map equation is the deformed Hermitian-Yang-Mills equation and the symplectic quotient is simply the moduli space of B-cycles ${}_B\mathcal{M}(M)$.

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Email: leung@math.umn.edu