From Special Lagrangian to Hermitian-Yang-Mills via Fourier-Mukai Transform

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Abstract

We exhibit a transformation taking special Lagrangian submanifolds of a Calabi-Yau together with local systems to vector bundles over the mirror manifold with connections obeying deformed Hermitian-Yang-Mills equations. That is, the transformation relates supersymmetric A- and B-cycles. In this paper, we assume that the mirror pair are dual torus fibrations with flat tori and that the A-cycle is a section.

We also show that this transformation preserves the (holomorphic) Chern-Simons functional for all connections. Furthermore, on corresponding moduli spaces of supersymmetric cycles it identifies the graded tangent spaces and the holomorphic m-forms. In particular, we verify Vafa's mirror conjecture with bundles in this special case.

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1 Introduction

In this note we argue that a real version of the Fourier-Mukai transform would carry supersymmetric A-cycles to B-cycles. Roughly, special Lagrangian submanifolds (plus local systems) will be mapped to holomorphic submanifolds (plus bundles over them). The approach is similar to the one in [1], though our emphasis is geometric, as we focus on the differential equations defining D-branes.

Our goal is to provide the basis for a geometric functor relating the categories of D-branes on opposite mirror sides: the derived category of coherent sheaves on one hand, and Fukaya's category of Lagrangian submanifolds with local systems on the other. (In fact, we consider special Lagrangians as objects.[1]) At this point, providing a physical interpretation of the derived category is premature – even though recent discussions of the role of the brane-anti-brane tachyon offer glimpses – so we content ourselves with thinking of D-branes as vector bundles over holomorphic submanifolds, where "submanifold" may mean the entire space.

We assume, following [14], that the mirror manifold has a description as a dual torus fibration, blithely ignoring singular fibers for now. The procedure is now quite simple [1][2][4]: a point on a torus determines a line bundle on a dual torus. A section of a torus fibration then determines a family of line bundles on the mirror side. These can fit together to define a bundle. We will show how, in this idealized situation, the differential equations on the two sides are related under this transformation. We also show that the Chern-Simons action of an A-cycle equals the holomorphic Chern-Simons action of its transform, even off-shell. Vafa's version of mirror symmetry with bundles is then verified in this setting.

It will be interesting, though perhaps quite formidable, to generalize this procedure by relaxing some of our assumptions and extending the setting to include singular fibers and more general objects in the derived category.

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[1] One can't yet say which formulation of these categories will best fit the physics. The Donaldson-Uhlenbeck-Yau theorem relates stable bundles to solutions of the Hermitian-Yang-Mills equations. Analogously, one hopes that Lagrangians are equivalent to special Lagrangians up to Hamiltonian deformations. We are a long way away from proving such theorems, however. The recent preprint of R. Thomas investigates these two issues as moment map problems related by mirror symmetry.

2 Supersymmetric A- and B-cycles

The authors of [11] consider the supersymmetric p-brane action and determine the conditions for preserving supersymmetry (BPS).[2] They show that there are two kinds of supersymmetric cycles (C, L) on a Calabi-Yau threefold M where C is a (possibly singular and with multiplicity) submanifold of M and L is a complex line bundle over C together with a $U(1)$ connection D_A. Let us denote the Kähler form (resp. holomorphic volume form) on the Calabi-Yau threefold by ω (resp. Ω).

The type-A supersymmetric cycle is when C is a special Lagrangian submanifold of M and the curvature F_A of D_A vanishes,

$$F_A = 0,$$

namely D_A is a unitary flat connection. In the presence of a background B-field (an element of $H^2(M, \mathbf{R}/\mathbf{Z})$), F_A should be replaced by $F_A - B$, where B is understood to be pulled back to the submanifold. We take $B = 0$ in this paper. Recall that a Lagrangian submanifold C is called "special" if when restricting to C we have

$$\operatorname{Im}\Omega = \tan\theta \operatorname{Re}\Omega,$$

for some constant θ. Or equivalently, $\operatorname{Im} e^{i\theta}\Omega = 0$.

The type-B cycle is when C is a complex submanifold of M of dimension n and the curvature two form F_A of D_A satisfies following conditions:

$$\begin{aligned} F_A^{0,2} &= 0, \\ \operatorname{Im} e^{i\theta} (\omega + F_A)^n &= 0. \end{aligned}$$

The first equation says that the (0,1) component of the connection determines a holomorphic structure on L. The second equation is called the deformed Hermitian-Yang-Mills equation and it is equivalent to the following equation,

$$\operatorname{Im} (\omega + F_A)^n = \tan\theta \operatorname{Re} (\omega + F_A)^n.$$

For example when C is the whole Calabi-Yau manifold M of dimension three then the second equation says $F \wedge \omega^2/2 - F^3/6 = \tan\theta \left[\omega^3/6 - \left(F^2/2\right) \wedge \omega\right]$.

3 Fourier-Mukai Transform of A- and B-cycles

In this section we explain the Fourier-Mukai transform of supersymmetric cycles. The gist of the story is that, assuming mirror pairs are mirror torus fibrations,

[2] In this section, we follow what has become standard notation. Let us remark, however, that the IIA string theory, on which one normally considers the A-model, naturally has even-dimensional (hence type-B) branes, so-named because their compositions depend on the complex structure. IIB string theory has odd-dimensional (type-A) branes, whose composition depends only on the symplectic structure.

each point of a Lagrangian submanifold lies in some fiber – hence defines a bundle over the dual fiber. When done in families and with connections, we get a bundle with connection on the mirror, and the differential equations defining A-cycles map to those which define B-cycles on the mirror. Recall that the base of the fibration itself – the zero graph – should be dual to the six-brane with zero connection. Multi-sections are dual to higher-rank bundles, and are discussed in section 3.2. Other cases appear in 3.3.

We assume that the m dimensional Calabi-Yau mirror pair M and W have dual torus fibrations. To avoid the difficulties of singular fibers and unknown Calabi-Yau metrics, we will only consider a neighborhood of a smooth special Lagrangian torus and also assume the Kähler potential ϕ on M to be T^m-invariant (see for example p.20 of [7]). This is the semi-flat assumption of [14]. Notice that the Lagrangian fibrations on M and W are in fact special.

Therefore, let $\phi\left(x^j,y^j\right) = \phi\left(x^j\right)$. ($y$ is the coordinate for the fiber and x for the base B of the fibration on M. The holomorphic coordinates on M are $z^j = x^j + iy^j$'s.) As studied by Calabi, the Ricci tensor vanishes and $\Omega = dz^1 \wedge .... \wedge dz^m$ is covariant constant if and only if ϕ satisfies a real Monge-Ampère equation

$$\det \frac{\partial^2\phi}{\partial x^i \partial x^j} = const.$$

The Ricci-flat Kähler metric and form are

$$\begin{aligned} g &= \sum_{i,j} \frac{\partial^2\phi}{\partial x^i \partial x^j}\left(dx^i dx^j + dy^i dy^j\right), \\ \omega &= \frac{i}{2}\sum_{i,j} \frac{\partial^2\phi}{\partial x^i \partial x^j} dz^i \wedge d\overline{z}^j, \end{aligned}$$

(henceforth we sum over repeated indices). Notice that $\Omega \wedge \bar{\Omega}$ is a constant mulitple of ω^m and it is direct consequence of the real Monge-Ampère equation.

Also note from the form of the metric g that M is locally isometric to the tangent bundle of B with its metric induced from the metric $\sum_{i,j} \frac{\partial^2\phi}{\partial x^i \partial x^j} dx^i dx^j$ on B. If we use the metric on B to identify its tangent bundle with its cotangent bundle, then the above symplectic form ω is just the canonical symplectic form $dp \wedge dq$ on the cotangent bundle.

We can view the universal cover of M either as TB with the standard complex structure, or as T^*B with the standard symplectic structure. A solution of the real Monge-Ampère equation is used to determine the symplectic structure in the former case and to determine the metric structure, and therefore the complex structure in the latter case.

3.1 Transformation of a section

We will construct the transform for a special Lagrangian exhibited as a section of the fibration, i.e. a graph over the base.

Recall that a section of T^*B is Lagrangian with respect to the standard symplectic form if and only if it is a closed one form, and hence locally exact. Therefore (or by calculation), a graph $y(x)$ in M is Lagrangian with respect to ω if and only if $\frac{\partial}{\partial x^j}(y^l\phi_{lk}) = \frac{\partial}{\partial x^k}(\phi_{lj}y^l)$, where $\phi_{ij} = \frac{\partial^2\phi}{\partial x^i\partial x^j}$, from which we get

$$y^j = \phi^{jk}\frac{\partial f}{\partial x^k}$$

for some function f (locally), where ϕ^{jk} is the inverse matrix of ϕ_{jk}.

Now $dz^j = dx^j + i dy^j$ and on C we have $dy^j = \phi^{jl}\left(\frac{\partial^2 f}{\partial x^l\partial x^k} - \phi^{pq}\phi_{lkp}\frac{\partial f}{\partial x^q}\right)dx^k$. Therefore $dz^j = \left(\delta_{jk} + i\phi^{jl}\left(\frac{\partial^2 f}{\partial x^l\partial x^k} - \phi^{pq}\phi_{lkp}\frac{\partial f}{\partial x^q}\right)\right)dx^k$ over C. Notice that if we write $g = \phi_{jk}dx^j dx^k$ as the Riemannian metric on the base, then the Christoffel symbol for the Levi-Civita connection is $\Gamma^q_{lk} = \phi^{pq}\phi_{lkp}$. Therefore $Hess\,(f) = \left(\frac{\partial^2 f}{\partial x^l\partial x^k} - \phi^{pq}\phi_{lkp}\frac{\partial f}{\partial x^q}\right)dx^l dx^k$. Hence

$$\begin{aligned} dz^1\wedge ...\wedge dz^m|_C &= \det\left(I + ig^{-1}Hess\,(f)\right)dx^1\wedge ...\wedge dx^m \\ &= \det(g)^{-1}\det\left(g + iHess\,(f)\right)dx^1\wedge ...\wedge dx^m, \end{aligned}$$

so the special Lagrangian condition (with phase) $\mathrm{Im}\left(dz^1\wedge ....\wedge dz^m\right)|_C = \tan\theta\cdot Re\left(dz^1\wedge ....\wedge dz^m\right)|_C$ becomes

$$\mathrm{Im}\det\left(g + iHess\,(f)\right) = (\tan\theta)\,\mathrm{Re}\det\left(g + iHess\,(f)\right).$$

From these data, we want to construct a $U(1)$ connection over the mirror manifold W which satisfies the deformed Hermitian-Yang-Mills equation. The dual manifold W is constructed by replacing each torus fiber T in M by the dual torus $\widetilde{T} = Hom\left(T, S^1\right)$. If we write the dual coordinates to $y^1,...,y^m$ as $\widetilde{y}_1,...,\widetilde{y}_m$, then the dual Riemannian metric on W is obtained by taking the dual metric on each dual torus fiber $\widetilde{T}$:

$$\widetilde{g} = \sum_{i,j}\left(\phi_{ij}dx^i dx^j + \phi^{ij}d\widetilde{y}_i d\widetilde{y}_j\right).$$

We need to understand the complex structure and the symplectic structure on W (see for example [14] and [7]). First we rewrite $\widetilde{g}$ as follows,

$$\widetilde{g} = \sum_{i,j}\phi^{ij}\left(\left(\Sigma_k\phi_{ik}dx^k\right)\left(\Sigma_l\phi_{jl}dx^l\right) + d\widetilde{y}_i d\widetilde{y}_j\right).$$

Notice that $d\left(\Sigma_k\phi_{jk}dx^k\right) = 0$ because ϕ_{jkl} is symmetric with respect to interchanging the indexes. Therefore there exist functions $\widetilde{x}_j = \widetilde{x}_j(x)$'s such that $d\widetilde{x}_j = \Sigma_k\phi_{jk}dx^k$ locally – then $\frac{\partial\widetilde{x}_j}{\partial x^k} = \phi_{jk}$ – and we obtain

$$\widetilde{g} = \sum_{i,j}\phi^{ij}\left(d\widetilde{x}_i d\widetilde{x}_j + d\widetilde{y}_i d\widetilde{y}_j\right).$$

So we can use $\widetilde{z}_j = \widetilde{x}_j + i\widetilde{y}_j$'s as complex coordinates on W. It is easy to check that the corresponding symplectic form is given by

$$\widetilde{\omega} = \frac{i}{2}\sum_{i,j}\phi^{ij} d\widetilde{z}_i \wedge d\overline{\widetilde{z}}_j.$$

Moreover the covariant constant holomorphic m-form on W is given by

$$\widetilde{\Omega} = d\widetilde{z}_1 \wedge ... \wedge d\widetilde{z}_m.$$

Again, as a direct consequence of ϕ being a solution of the real Monge-Ampère equation, $\widetilde{\Omega} \wedge \overline{\widetilde{\Omega}}$ is a constant multiple of $\widetilde{\omega}^m$.

Remark 1 *The mirror manifold W can be interpreted as the moduli space of special Lagrangian tori together with flat $U(1)$ connections over them (see [14]). It is because the dual torus parametrizes isomorphism classes of flat $U(1)$ connections on the original torus. It can be checked directly that the L^2 metric, i.e. the Weil-Petersson metric, on this moduli space W coincides with our $\widetilde{g}$ above.*

In general, the relevent metric on the moduli space W is given by a two-point function computed via a path integral, wihch includes instanton contributions from holomorphic disks bounding the special Lagrangian torus fibers. However, for our local Calabi-Yau M such holomorphic disks do not exist. This is because M is homotopic to any one of its fibers; but any such holomorphic disk would define a non-trivial relative homology class. Therefore our metric $\widetilde{g}$ coincides with the physical metric on the moduli space W.

Remark 2 *We note the symmetry between g (resp. ω) and $\widetilde{g}$ (resp. $\widetilde{\omega}$). For one can write ϕ^{ij} as the second derivative of some function $\widetilde{\phi}$ with respect to the $\widetilde{x}_j$'s. Simply write $x^j = x^j(\widetilde{x})$, then $\frac{\partial x^j}{\partial \widetilde{x}_k} = \phi^{jk} = \frac{\partial x^k}{\partial \widetilde{x}_j}$ and therefore $x^j = \frac{\partial \Phi}{\partial \widetilde{x}_j}$ for some function, Φ, and it is easy to check that $\widetilde{\phi} = \Phi$.*

On each torus fiber, we have canonical isomorphisms $T = \mathrm{Hom}(\widetilde{T}, S^1) = \mathrm{Hom}(\pi_1(\widetilde{T}), S^1)$, therefore a point $y = (y^1, ..., y^m)$ in T defines a flat connection D_y on its dual $\widetilde{T}$. This is the real Fourier-Mukai transform. Explicitly, we have

$$\begin{aligned} g_y: \quad \widetilde{T} &\to i(\mathbb{R}/\mathbb{Z}) = S^1 \\ \widetilde{y} &\mapsto i\textstyle\sum_{j=1}^m y^j \widetilde{y}_j, \end{aligned}$$

and $D_y = d + A = d + idg_y = d + i\Sigma y^j d\widetilde{y}_j$.

In fact we get a torus family of one-forms, since y (hence A) has x- (or $\widetilde{x}$-) dependence. Namely, we obtain a $U(1)$ connection on W,

$$D_A = d + i\sum_j y^j d\widetilde{y}_j.$$

Its curvature two form is given by,

$$F_A = dA = \sum_{k,j} i \frac{\partial y^j}{\partial \widetilde{x}_k} d\widetilde{x}_k \wedge d\widetilde{y}_j .$$

In particular

$$F_A^{2,0} = \frac{1}{2} \sum_{j,k} \left(\frac{\partial y^k}{\partial \widetilde{x}_j} - \frac{\partial y^j}{\partial \widetilde{x}_k} \right) d\widetilde{z}_j \wedge d\widetilde{z}_k .$$

Therefore, that D_A is integrable, i.e. $F_A^{0,2} = 0$, is equivalent to the existence of $f = f(\widetilde{x})$ such that $y^j = \frac{\partial f}{\partial \widetilde{x}_j} = \phi^{jk} \frac{\partial f}{\partial x^j}$ because of $d\widetilde{x}_j = \Sigma_k \phi_{jk} dx^k$. Namely, the cycle $C \subset M$ must be Lagrangian. Now

$$\frac{\partial y^j}{\partial \widetilde{x}_k} = \frac{\partial^2 f}{\partial \widetilde{x}_j \partial \widetilde{x}_k} .$$

In terms of the x variable, this is precisely the Hessian of f, as discussed above. Therefore the cycle $C \subset M$ being special is equivalent to

$$\operatorname{Im} (\widetilde{\omega} + F_A)^m = (\tan \theta) \, Re \, (\widetilde{\omega} + F_A)^m .$$

For a general type-A supersymmetric cycle in M, we have a special Lagrangian C in M together with a flat $U(1)$ connection on it. Since as before, C is expressed as a section of $\pi : M \to B$ and is given by $y^j = \phi^{jk} \frac{\partial f}{\partial x^k}$, a flat $U(1)$ connection on C can be written in the form $d + ide = d + i\Sigma \frac{\partial e}{\partial x^k} dx^k$ for some function $e = e(x)$. Recall that the transformation of C alone is the connection $d + i\Sigma y^j d\widetilde{y}_j$ over W. When the flat connection on C is also taken into account, then the total transformation becomes

$$\begin{aligned} D_A &= d + i\Sigma y^j d\widetilde{y}_j + ide \\ &= d + i\Sigma \phi^{jk} \frac{\partial f}{\partial x^k} d\widetilde{y}_j + i\Sigma \frac{\partial e}{\partial \widetilde{x}^j} d\widetilde{x}_j . \end{aligned}$$

Here we have composed the function $e(x)$ with the coordinate transformation $x = x(\widetilde{x})$. Notice that the added term $\Sigma \frac{\partial e}{\partial \widetilde{x}^j} d\widetilde{x}_j$ is exact and therefore the curvature form of this new connection is the same as the old one. In particular D_A satisfies

$$\begin{aligned} F_A^{0,2} &= 0, \\ \operatorname{Im} e^{i\theta} (\omega + F)^m &= 0, \end{aligned}$$

so is a supersymmetric cycle of type-B in W. By the same reasoning, we can couple with C a flat connection on it of *any* rank and we would still obtain a non-Abelian connection D_A on W satisfying the above equations.

In conclusion, the transform of a type-A supersymmetric section in M is a type-B supersymmetric $2m$-cycle in W.

Remark 3 *The real Fourier-Mukai transform we discussed above exchanges the symplectic and complex aspects of the two theories.*

On the A-cycle side, Donaldson and Hitchin [7] introduce a symplectic form on the space of of maps $Map\,(C, M)$ *as follows. If* υ *is a fixed volume form on a three manifold* C*, then* $\int_C ev^*\omega \wedge \upsilon$ *is a symplectic form on* $Map\,(C, M)$ *and it equips with a Hamiltonian action by the group of volume preserving diffeomorphisms of* C*. The zero of the corresponding moment map is precisely the Lagrangian condition on* $f \in Map\,(C, M)$*.*

If one restricts to the infinite-dimensional complex submanifold of $Map\,(C, M)$ *consisting of those* f *which satisfy* $f^*\Omega = \upsilon$*, then the symplectic quotient is the moduli space of A-cycles.*

On the B-cycle side, we consider the pre-symplectic form $\mathrm{Im}\left[\int_W (\widetilde{\omega} + \mathbb{F})^m\right]^{[2]}$ *on the space of connections* $\mathcal{A}\,(W)$ *(see section 4.3 or compare [10]). This form is preserved by the group of gauge transformations and the corresponding moment map equation is the deformed Hermitian-Yang-Mills equations.*

If one restricts to the complex submanifold of $\mathcal{A}\,(W)$ *consisting of those connections which define a holomorphic structure on the bundle, then the symplectic quotient is the moduli space of B-cycles.*

Notice that the above real Fourier-Mukai transform exchanges the moment map condition on one side to the complex condition on the other side. Such exchanges of symplectic and complex aspects are typical in mirror symmetry. We expect this continues to hold true in general and not just for special Lagrangian sections in the semi-flat case.

Remark 4 *Unlike the Hermitian-Yang-Mills equation, solutions to the fully nonlinear deformed equations may not be elliptic. On the other hand, the special Lagrangian equation is always elliptic since its solutions are calibrated submanifolds. Nevertheless, deformed Hermitian-Yang-Mills connections obtained from the real Fourier-Mukai transformation as above are always elliptic.*

3.2 Transformation of a multi-section

When C is a multi-section of $\pi : M \to B$, the situation is more complicated. For one thing, holomorphic disks may bound C – a situation which cannot occur in the section case (see section 4.2). Here we propose to look at a line bundle on finite cover of W as the transform B-cycle. To begin we assume that C is smooth, $\pi : C \to B$ is a branched cover of degree r (as in algebraic geometry) and $\nabla_j \nabla_k \phi = \delta_{jk}$ for simplicity. Away from ramification locus, C determines r unitary connections on W locally, and each satisfies the above equation. One might be tempted to take the diagonal connection on their direct sum, so that this $U\,(r)$ connection satisfies a non-Abelian analogue of the equation. However, such a connection cannot be defined across ramification locus because of monodromy, which can interchange different summands of the diagonal connection.

In fact, as mentioned in [11], it is still open (even in physics) to find or derive a non-Abelian analogue of the deformed Hermitian-Yang-Mills equations via string theory (though there are some natural guesses). To remedy this problem, we would instead construct a $U(1)$ connection over a degree r cover of W. This finite cover $\hat{\pi}:\widehat{W}\to W$ is constructed via the following Cartesian product diagram

$$\begin{array}{ccc} \widehat{W}=C\times_B W & \xrightarrow{\hat{\pi}} & W \\ \downarrow & & \downarrow \\ C & \xrightarrow{\pi} & B. \end{array}$$

Notice that $\widehat{W}$ is a smooth manifold because C is smooth and $\pi: C\to B$ is a branched cover. Moreover, the construction as before determines a smooth unitary connection over $\widehat{W}$ satisfying the above equation (with ω replaced by $\hat{\pi}^*\omega$).

We remark that employing this construction is similar to the use of isogenies needed to define the categorical isomorphism which proves Kontsevich's conjecture in the case of the elliptic curve [13]. The point is that multi-sections transforming to higher-rank can be handled via single sections giving line bundles, by imposing functoriality after pushing forward under finite covers.

Even though this connection over $\widehat{W}$ satisfies $F^{0,2}=0$ in a suitable sense, $\widehat{W}$ is not a complex manifold.

3.3 Transformation for more general cycles

In the previous sections, we considered only sections or multi-sections on M and obtain holomorphic bundles on W which satisfied the deformed Hermitian-Yang-Mills equation. Notice that, for a Calabi-Yau threefold M, a multi-section of $\pi: M\to B$ can be characterized as special Lagrangian cycle C whose image under π is of dimension three – namely, the whole B. If the image has dimension zero, then the Lagrangian is a torus fiber plus bundle and its dual is the point (0-brane) it represents on the corresponding dual torus. This is the basis for the conjecture of [14]. Here we are going to look at the other cases, that is the dimension of the image of C under π is either (i) one or (ii) two. For simplicity we shall only look at the flat case, namely $M=T^6=B\times F$ where both B and F are flat three dimensional Lagrangian tori.

Case (i), when $\dim\pi(C)=1$. The restriction of π to C express C as the total space of an one-parameter family of surfaces. In fact we will see that this is a product family of an affine T^1 in B with an affine T^2 in F.

As before we denote the coordinates of B (resp. F) by x^1,x^2,x^3 (resp. y^1,y^2,y^3). Without loss of generality, we can assume that $\pi(C)$ is locally given by $x^2=f(x^1)$ and $x^3=g(x^1)$. Moreover the surface in C over any such point is determined by $y^3=h(x^1,y^1,y^2)$. In particular, C is parametrized by x^1,y^1 and y^2 locally.

The special condition $\operatorname{Im}\left(dz^1\wedge dz^2\wedge dz^3\right)|_C=0$ implies that h is independent of x^1. Namely the surface family C is indeed a product subfamily of

$M = F \times B$. Now the Lagrangian conditions read as follow:

$$\begin{aligned} 1+\frac{dg}{dx^1}\frac{\partial h}{\partial y^1} &= 0, \\ \frac{df}{dx^1}+\frac{dg}{dx^1}\frac{\partial h}{\partial y^2} &= 0. \end{aligned}$$

These imply that,

$$\begin{aligned} f = x^2 &= ax^1+\alpha \\ g = x^3 &= bx^1+\beta \\ h = y^3 &= -\frac{1}{b}y^1-\frac{a}{b}y^2+\frac{\gamma}{b}. \end{aligned}$$

By analyticity of special Lagrangians, these parametrizations hold true on the whole C. We can therefore express C as the product $C_B \times C_F$ with $C_F \subset F$ being a two torus and $C_B \subset B$ being a circle. Here

$$\begin{aligned} C_B &= \left\{ \left(x^1, x^2, x^3\right) = (1,a,b)\, x^1 + (0,\alpha,\beta) \right\}, \\ C_F &= \left\{ \left(y^1, y^2, y^3\right) : \left(y^1, y^2, y^3\right) \cdot (1,a,b) = \gamma \right\}. \end{aligned}$$

Since our primary interest is when C is a closed subspace of M, this implies that both a and b are rational numbers and C is a three torus which sits in $M = T^6$ as a totally geodesic flat torus. To describe a supersymmetric cycle, we also need a $U(1)$ flat connection D_A over C. If we parametrize C by coordinate functions x^1, y^2 and y^3 as above, then we have

$$D_A = d + i\left(\widetilde{\gamma} dx^1 + \widetilde{\alpha} dy^2 + \widetilde{\beta} dy^3\right),$$

for some real numbers $\widetilde{\alpha}, \widetilde{\beta}$ and $\widetilde{\gamma}$.

Now let us define the transformation of (C, L). First, the mirror of $M = B \times F$ equals $W = B \times \widetilde{F}$ where $\widetilde{F}$ is the dual three torus to F. A point $(x_1, x_2, x_3, \widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3) = (x, \widetilde{y}) \in W$ (note $x = \widetilde{x}$ here, by flatness of the metric) lies in the mirror $\left(\widetilde{C}, \widetilde{L}\right)$ of the SUSY cycle (C, L) if and only if the flat connection which is obtained by the restriction of D_A to $x \times C_F$ twisted by the one form $i \sum \widetilde{y}_j dy^j$ is in fact trivial. That is, $\widetilde{\alpha} dy^2 + \widetilde{\beta} dy^3 + \sum \widetilde{y}_j dy^j = 0$. Using the equation $\left(y^1, y^2, y^3\right) \cdot (1,a,b) = \gamma$, we obtain that

$$\begin{aligned} \widetilde{y}_2 &= a\widetilde{y}_1 - \widetilde{\alpha}, \\ \widetilde{y}_3 &= b\widetilde{y}_1 - \widetilde{\beta}. \end{aligned}$$

Or equivalently, $\widetilde{C} = C_B \times C_{\widetilde{F}}$ with $C_{\widetilde{F}} = \left\{ (\widetilde{y}_1, \widetilde{y}_2, \widetilde{y}_3) = (1,a,b)\, \widetilde{y}_1 - \left(0, \widetilde{\alpha}, \widetilde{\beta}\right) \right\}$ and $C_B = \{(x_1, x_2, x_3) = (1,a,b)\, x_1 + (0,\alpha,\beta)\}$. In particular $\widetilde{C}$ is a holomorphic curve in W. The last step would be to determine the $U(1)$ connection $D_{\widetilde{A}}$

on $\widetilde{C}$. By essentially the same argument as above and the fact that there is no transformation along the base direction, we obtain

$$D_{\widetilde{A}} = d + i\left(\widetilde{\gamma} dx_1 - \gamma d\widetilde{y}_1\right).$$

Now the deformed Hermitian-Yang-Mills equation $\operatorname{Im}\left(\omega + F_{\widetilde{A}}\right)^n = 0$ is equivalent to $F_{\widetilde{A}} = 0$, which is obviously true for the above connection $D_{\widetilde{A}}$ on $\widetilde{C}$.

Case (ii), when $\dim \pi(C) = 2$. The restriction of π to C express C as the total space of a two-parameter family of circles. As before, let us write the parametrizing surface S in B as $x^3 = f\left(x^1, x^2\right)$ and the one dimensional fiber over any point of it by $y^2 = g\left(x^1, x^2, y^1\right)$ and $y^3 = h\left(x^1, x^2, y^1\right)$. Namely C is parametrized by x^1, x^2 and y^1 locally.

Now the Lagrangian condition would imply that each fiber is an affine circle in $T^3 = F$, among other things. On the other hand, the special condition $\operatorname{Im}\left(dz^1 \wedge dz^2 \wedge dz^3\right)|_C = 0$ implies that the surface $S \subset T^3 = B$ satisfies a Monge-Ampère equation:

$$\det\left(\nabla^2 f\right) = 0.$$

We would like to perform a transformation on C which would produce a complex surface in W together with a holomorphic bundle on it with a Hermitian-Yang-Mills connection. Notice that the deformed Hermitian-Yang-Mills equation in complex dimension two is the same as the Hermitian-Yang-Mills equation. To transform C in this case is not as straight forward as before because the equation governing the family of affine circles is more complicated. However the above equation should imply that f is an affine function which would then simplify the situation a lot.

4 Correspondence of Moduli Spaces

Vafa [15] has argued that the topological open string theory describing strings ending on an A-cycle is equivalent to the topological closed-string model on a Calabi-Yau with a bundle.[3] Equating the effective string-field theories leads to the conjecture that the ordinary Chern-Simons theory on an A-cycle be equivalent to the holomorphic Chern-Simons theory on the transform B-cycle. Gopakumar and Vafa have verified equality of the partition functions for the dual resolutions of the conifold [5]. Further, all structures on the moduli spaces of branes must be equivalent.

In the previous section, we used the Fourier-Mukai transform in the semi-flat case to identify moduli spaces of A- and B-cycles as a set. In this section, we extend our analysis to the Chern-Simons functional for connections which are not necessarily flat or integrable. Then we study and relate various geometric objects on the moduli spaces related by the transform. In particular, we verify Vafa's conjecture in this case.

[3] In order to get a bundle, we consider only sections or multi-sections.

4.1 Chern-Simons functionals

In this section, we show the equivalence of the relevant Chern-Simons functionals for corresponding pairs of supersymmetric cycles in our semi-flat case. In fact, we will do this "off-shell," meaning that the equivalence holds even for connections which do not satisfy the flatness or integrability conditions, respectively. The argument is essentially the one given on pp. 4-5 of [6].

So, instead of flat connections, we consider a general $U(1)$ connection $d + A$ on the special Lagrangian section C in M, then the above transform will still produce a connection on W which might no longer be integrable. In real dimension three, flat connections of any rank can be characterized as those connections which are critical points of the Chern-Simons functional,

$$CS(A) = \int_C Tr\left(AdA + \frac{2}{3}A^3\right).$$

To be precise, one would need to impose boundary condition or growth condition for A because C is not a closed manifold.

There is also a complexified version of Chern-Simons for any holomorphic bundle E on a Calabi-Yau threefold W with holomorphic three form $\widetilde{\Omega}$ ([3] [16]). Namely, if $\mathcal{A}$ is a Hermitian connection on E which might not be integrable, then the holomorphic Chern-Simons functional is given by

$$CS_{hol}(\mathcal{A}) = \int_W Tr\widetilde{\Omega} \wedge \left(\mathcal{A}\bar{\partial}\mathcal{A} + \frac{2}{3}(\mathcal{A})^3\right).$$

Notice that $CS_{hol}(\mathcal{A})$ depends only on the $(0,1)$ component of $\mathcal{A}$. As in the real case, $\mathcal{A}$ is a critical point for the holomorphic Chern-Simons if and only if $F_{\mathcal{A}}^{0,2} = 0$, that is an integrable connection.

As argued in [15][16], the holomorphic Chern-Simons theory on W is conjectured to be *mirror* to the usual Chern-Simons theory on $C \subset M$, with instanton corrections given by holomorphic disks on M with boundary lying on C (as we will see in the next section, there are no such instantons in our setting). In fact, we can directly check that the Fourier-Mukai transform not only sends flat connections on C to integrable connections on W, but it preserves the Chern-Simons functional for an arbitrary connection on C which is not necessarily flat. Moreover, this holds true for connections of any rank over C.

We consider $C = \left\{y^j = \phi^{jk}\frac{\partial f}{\partial x^k}\right\} \subset M$ as in section 3.1, and now $d + A = d + ie_k(x)\,dx^k$ is an arbitrary rank r unitary connection on C. The real Fourier-Mukai transform of C alone (resp. C with the above connection) is the connection $\mathcal{A}_0 = d + i\phi^{jk}\frac{\partial f}{\partial x^k}d\widetilde{y}_j$ (resp. $\mathcal{A} = d + i\left(\phi^{jk}\frac{\partial f}{\partial x^k}d\widetilde{y}_j + e_k\phi^{jk}d\widetilde{x}_j\right)$) on W.

Recall that $(\mathcal{A}_0)^{0,1}$ determines a holomorphic structure on a bundle over W, which we use as the background $\bar{\partial}$ operator in defining the holomorphic Chern-Simons functional. That is, $CS_{hol}(\mathcal{A}) = CS_{hol}(\mathcal{A}, \mathcal{A}_0)$. In fact if we vary $\bar{\partial}$ continuously among holomorphic bundles, the holomorphic Chern-Simons functional remains the same.

Now,

$$CS_{hol}\left(\mathcal{A},\mathcal{A}_0\right)=\int_W Tr\widetilde{\Omega}\wedge\left(\mathcal{B}\left(\bar{\partial}-\frac{1}{2}\phi^{jk}\frac{\partial f}{\partial x^k}d\overline{\widetilde{z_j}}\right)\mathcal{B}+\frac{2}{3}\left(\mathcal{B}\right)^3\right),$$

where $\mathcal{B}=\left(\mathcal{A}-\mathcal{A}_0\right)^{0,1}=\frac{i}{2}e_k\phi^{jk}d\overline{\widetilde{z_j}}$. Now

$$\begin{aligned}\int_W Tr\widetilde{\Omega}\mathcal{B}\phi^{jk}\frac{\partial f}{\partial x^k}d\overline{\widetilde{z_j}}\mathcal{B} &= (const)\int_W Tr\widetilde{\Omega}\mathcal{B}\frac{\partial f}{\partial\widetilde{x}^j}d\overline{\widetilde{z_j}}\mathcal{B}\\ &= -(const)\int Tr\left(\varepsilon^2\right)df\,d\widetilde{y}_1d\widetilde{y}_2d\widetilde{y}_3\\ &= 0.\end{aligned}$$

Here $\varepsilon=\frac{i}{2}e_k\phi^{jk}d\widetilde{x}_k$ is a matrix-valued one form, and therefore $Tr\left(\varepsilon^2\right)=0$. Using the fact that $\bar{\partial}\mathcal{B}=\frac{i}{2}\frac{\partial}{\partial\widetilde{x}^l}\left(e_k\phi^{jk}\right)d\overline{\widetilde{z_l}}d\overline{\widetilde{z_j}}$, we have

$$\begin{aligned}CS_{hol}\left(\mathcal{A},\mathcal{A}_0\right) &= (const)\int_W Tr\widetilde{\Omega}\wedge\left(\mathcal{B}\bar{\partial}\mathcal{B}+\frac{2}{3}\left(\mathcal{B}\right)^3\right)\\ &= (const)\int_W Tr\left(\varepsilon d\varepsilon+\frac{2}{3}\varepsilon^3\right)d\widetilde{y}_1d\widetilde{y}_2d\widetilde{y}_3\\ &= (const)\int_x Tr\left(AdA+\frac{2}{3}A^3\right)\\ &= (const)\,CS\left(A\right).\quad\Box\end{aligned}$$

When the dimension of M is odd but bigger than three, the Fourier-Mukai transform still preserves the (holomorphic) Chern-Simons functional even though their critical points are no longer flat (or integrable) connection. Instead the Euler-Lagrange equation is $\left(F_A\right)^n=0$ (or $\left(F_{\mathcal{A}}^{0,2}\right)^n=0$) where $\dim_{\mathbb{C}}M=m=2n+1$.

4.2 Graded tangent spaces

Transforming A-cycles to B-cycles is only the first step in understanding mirror symmetry with branes [15]. The next step would be to analyze the correspondence between the moduli spaces of cycles (branes). In the next two sections, we identify the graded tangent spaces and the holomorphic m forms on the two moduli spaces of supersymmetric cycles. Generally, this would involve holomorphic disk instanton contributions for the A-cycles (analgously to the usual mirror symmetry A-model), but in our simplified setting we now show these are absent.[4]

Let D be a holomorphic disk whose boundary lies in the special Lagrangian section C. Since we are in the local case, C is homeomorphic to a ball and we can find a closed disk $D'\subset C$ with $\partial D=\partial D'$. Now using the assumption that

[4]The following argument is a variation of the one given on pp. 25-26 of [16].

C is a section and $\pi_2(T^m) = 0$, the closed surface $D \cup D'$ is contractible in M. Therefore $\int_{D\cup D'} \omega = 0$ by Stokes theorem. Now $\int_{D'} \omega = 0$ because D' lies inside a Lagrangian and $\int_D \omega > 0$ because it is the area of D. This is a contradiction. Hence there are no such holomorphic disks on M.

First we discuss the graded tangent spaces of the moduli of A- and B- cycles. After that we verify that the real Fourier-Mukai transform does preserve them in the semi-flat case.

For the A side, the tangent space to the moduli of special Lagrangians can be identified with the space of closed and co-closed one forms (we called such forms harmonic). This is proved by McLean [12]. We denote it by $H^1(C, \mathbb{R})$. If D_A is a flat $U(r)$ connection on a bundle E over C, then the tangent space of the moduli of such connections at D_A can be identified with the space of harmonic one forms with valued in $ad(E)$. We denote it by $H^1(C, ad(E))$. When $r = 1$ the spaces $H^1(C, ad(E))$ and $iH^1(C, \mathbb{R})$ are the same. For r bigger than one, it is expected that $H^1(C, ad(E))$ is the tangent space at the non-reduced point rC: If there is a family of special Lagrangians in M converging to C with multiplicity r, then it should determine a flat $U(r)$ connection on an open dense set in C. This connection would extend to the whole C if those special Lagrangians in the family are branched covers of C in T^*C. Then the tangent of this moduli space at rC should be $H^1(C, ad(E))$. It is useful to verify this statement.

The graded tangent spaces are defined to be $\oplus_k H^k(C, \mathbb{R}) \otimes \mathbb{C}$, or more generally $\oplus_k H^k(C, ad(E)) \otimes \mathbb{C}$, the space of harmonic k forms and those with coefficient in $ad(E)$.

On the B side, a cycle is a $U(r)$ connection $\mathcal{D}_A$ on a bundle $\mathcal{E}$ over W whose curvature $\mathcal{F}_A$ satisfies $\mathcal{F}_A^{0,2} = 0$ and $\operatorname{Im} e^{i\theta}(\widetilde{\omega} + \mathcal{F}_A)^m = 0$. If we replace the deformed Hermitian-Yang-Mills equation by the non-deformed one, then a tangent vector to this moduli space can be identified with an element $\mathcal{B}$ in $\Omega^{0,1}(W, ad(\mathcal{E}))$ satisfying $\bar{\partial}\mathcal{B} = 0$ and $\widetilde{\omega}^{m-1} \wedge \partial\mathcal{B} = 0$. The second equation is equivalent to $\bar{\partial}^*\mathcal{B} = 0$. That is $\mathcal{B}$ is a $\bar{\partial}$-harmonic form of type $(0,1)$ on W with valued in $ad(\mathcal{E})$. The space of such $\mathcal{B}$ equals the sheaf cohomology $H^1(W, End(E))$ by Dolbeault theorem, provided W is compact.

It is not difficult to see that a tangent vector $\mathcal{B}$ to the moduli of B-cycles is a *deformed $\bar{\partial}$-harmonic form* in the following sense:[5]

$$\begin{aligned} \bar{\partial}\mathcal{B} &= 0, \\ \operatorname{Im} e^{i\theta}(\widetilde{\omega} + \mathcal{F}_A)^{m-1} \wedge \partial\mathcal{B} &= 0. \end{aligned}$$

In general a differential form $\mathcal{B}$ of type $(0,q)$ is called a deformed $\bar{\partial}$-harmonic form (compare [10]) if it satisfies

$$\begin{aligned} \bar{\partial}\mathcal{B} &= 0, \\ \operatorname{Im} e^{i\theta}(\widetilde{\omega} + \mathcal{F}_A)^{m-q} \wedge \partial\mathcal{B} &= 0. \end{aligned}$$

We denote this space as $\widetilde{H}^q(W, End(\mathcal{E}))$. When the connection $\mathcal{D}_A$ and the phase angle θ are both trivial, a deformed $\bar{\partial}$-harmonic form is just an ordinary

[5] If the rank of E is bigger than one, then we need to symmetrize the product in the second equation, as done in [10].

$\bar{\partial}$-harmonic form. It is useful to know if there is always a unique deformed $\bar{\partial}$-harmonic representative for each coholomology class in $H^q(W, End(\mathcal{E}))$. One might want to require that $\widetilde{\omega}+\mathcal{F}_A$ is positive to ensure ellipticity of the equation.

As argued in [15], mirror symmetry with bundles leads to an identification between $H^q(C, ad(E))\otimes\mathbb{C}$ and $\widetilde{H}^q(W, End(\mathcal{E}))$ for each q.[6] This can be verified in our situation as follows. For simplicity we assume that the phase angle θ is zero and E is a line bundle.

First we need to define the transformation from a degree q form on $C\subset M$ to one on W. We need one transformation Φ corresponding to deformations of special Lagrangians and another Ψ corresponding to deformations of its flat unitary bundle.

$$\begin{array}{rcl}\Omega^q(C, ad(E))\otimes\mathbb{C} & \rightarrow & \Omega^{0,q}(W, End(\mathcal{E}))\\ B_1+iB_2 & \mapsto & \Phi(B_1)+i\Psi(B_2)\end{array}$$

If B is a q-form on the section $C\subset M$, in the coordinate system of x's we write $B=\Sigma b_{j_1\ldots j_q}(x)\,dx^{j_1}\cdots dx^{j_q}$. It suffices to define the transformations for $B=dx^j$, by naturality. The first (resp. second) transformation of dx^j is given by $\Phi(B)=\left(\phi^{jk}d\widetilde{y}_k\right)^{0,1}$ (resp. $\Psi(B)=\left(dx^j\right)^{0,1}=\left(\phi^{jk}d\widetilde{x}_k\right)^{0,1}$). When q equals one, these transformations are compatible with our identification of moduli spaces of A- and B-cycles.

Now let $B=\Sigma b_j(x)\,dx^j$ be any one-form on $C\subset M$. Then we have $\mathcal{B}=\Phi(B)=\left(\Sigma b_j\phi^{jk}d\widetilde{y}_k\right)^{0,1}=\frac{i}{2}\Sigma b_j(\widetilde{x})\,\phi^{jk}d\overline{\widetilde{z}_k}$. Therefore $\bar{\partial}\mathcal{B}=\frac{i}{2}\Sigma\frac{\partial}{\partial\widetilde{x}_l}\left(b_j(\widetilde{x})\,\phi^{jk}\right)d\overline{\widetilde{z}_l}d\overline{\widetilde{z}_k}$ and its vanishing is clearly equivalent to $dB=0$ under the coordinate change $\frac{\partial}{\partial\widetilde{x}_l}=\phi^{lk}\frac{\partial}{\partial x^k}$. It is also easy to see that this equivalence between $dB=0$ and $\bar{\partial}\mathcal{B}=0$ holds true for any degree q form too.

Our main task is to show that $d^*B=0$ if and only if $\mathrm{Im}\left(\widetilde{\omega}+\mathcal{F}_A\right)^{m-q}\wedge\partial\mathcal{B}=0$ for any degree q form B on $C\subset M$ and $\mathcal{B}=\Phi(B)$. Notice that, by type considerations, the latter condition is the same as $\mathrm{Im}\,d\left[\left(\widetilde{\omega}+\mathcal{F}_A\right)^{m-q}\mathcal{B}\right]=0$.

Let B be any degree q form on the special Lagrangian C. Using the symplectic form ω, we obtain a q-vector field v_B, i.e. a section of $\Lambda^q(T_M)$. Then using arguments as in [12] or [7], we have

$$*B=\pm\iota_{v_B}\,\mathrm{Im}\,\Omega.$$

Therefore $d^*B=0$ if and only if $\mathrm{Im}\,d\left(\iota_{v_B}\Omega\right)=0$ on C. If we write $B=\Sigma b_{j_1\ldots j_q}(x)\,dx^{j_1}\cdots dx^{j_q}$ then $v_B=\Sigma b_{j_1\ldots j_q}\phi^{j_1k_1}\cdots\phi^{j_qk_q}\frac{\partial}{\partial y^{k_1}}\cdots\frac{\partial}{\partial y^{k_q}}$. So on C, we have

$$\begin{array}{rcl}\iota_{v_B}\Omega & = & \iota_{v_B}\left(dx^1+idy^1\right)\wedge\cdots\wedge\left(dx^m+idy^m\right)\\ & = & \pm\Sigma b_{j_1\ldots j_q}\phi^{j_1k_1}\cdots\phi^{j_qk_q}i^q\prod\limits_{l\neq k_i}\left(dx^l+idy^l\right)\\ & = & \pm\Sigma b_{j_1\ldots j_q}\phi^{j_1k_1}\cdots\phi^{j_qk_q}i^q\prod\limits_{l\neq k_i}\left(dx^l+i\dfrac{\partial}{\partial x^p}\left(\phi^{lk}\dfrac{\partial f}{\partial x^k}\right)dx^p\right).\end{array}$$

[6] In [15], the author uses $H^k(W, End(\mathcal{E}))$ instead of $\widetilde{H}^k(W, End(\mathcal{E}))$.

Now we consider the corresponding $\mathcal{B} = \Phi(B)$ over W. Explicitly we have

$$\mathcal{B} = i^q \Sigma b_{j_1 \dots j_q} \phi^{j_1 k_1} \cdots \phi^{j_q k_q} d\overline{\widetilde{z}}_{k_1} \cdots d\overline{\widetilde{z}}_{k_q}.$$

Therefore we consider the form $(\widetilde{\omega} + \mathcal{F}_A)^{m-q} \mathcal{B}$ of type $(m-q, m)$ on W:

$$\begin{aligned} & (\widetilde{\omega} + \mathcal{F}_A)^{m-q} \mathcal{B} \\ = \quad & \left(\Sigma \left(\phi^{jk} + i \frac{\partial^2 f}{\partial \widetilde{x}_j \partial \widetilde{x}_k} \right) d\widetilde{z}_j \wedge d\overline{\widetilde{z}}_k \right)^{m-q} \left(i^q \Sigma b_{j_1 \dots j_q} \phi^{j_1 k_1} \cdots \phi^{j_q k_q} d\overline{\widetilde{z}}_{k_1} \cdots d\overline{\widetilde{z}}_{k_q} \right). \end{aligned}$$

After the coordinate transformation $\frac{\partial}{\partial \widetilde{x}_l} = \phi^{lk} \frac{\partial}{\partial x^k}$, it is now easy to see that $\operatorname{Im} d \left[(\widetilde{\omega} + \mathcal{F}_A)^{m-q} \mathcal{B} \right] = 0$ if and only if $\operatorname{Im} d (\iota_{v_B} \Omega) = 0$. That is, B is a co-closed form on C of degree q. So Φ carries a harmonic q-form on C to a deformed harmonic $(0,q)$-form on W. Now if E is a higher rank vector bundle over C, then the only changes we need in the proof are B and v_B now have valued in $ad(E)$ and we would replace the exterior differentiation by covariant differentiation and also we need to symmetrize the product in the $\bar{\partial}$-harmonic equation. The proof of the equivalence for Ψ is similar, and we omit it.

Therefore, we have proved that $B_1 + iB_2$ is a harmonic form of degree q over C if and only if $\Phi(B_1) + i\Psi(B_2)$ is a $\bar{\partial}$-harmonic form of degree $(0,q)$ over W. In particular $\Phi + i\Psi$ maps $H^q(C, ad(E)) \otimes \mathbb{C}$ to $\widetilde{H}^q(W, End(\mathcal{E}))$.

4.3 Holomorphic m-forms on moduli spaces

The moduli spaces of A-cycles and B-cycles on a Calabi-Yau m-fold have natural holomorphic m-forms. As explained in [15], these m-forms, which inclde holomorphic disk instanton corrections on the A-cycle side, can be identified with physical correlation functions derived from the Chern-Simons partition function. Under Vafa's version of the mirror conjecture with bundles, these partition functions and correlators should be the same for any mirror pair M and W, at least in dimension three. In this section we recall the definitions of the holomorphic m-forms and verify this equality our semi-flat case.

First we define a degree-m closed form on $Map(C, M)$ by $\int_C ev^* \omega^m$ where ω is the Kähler form on M and $C \times Map(C, M) \xrightarrow{ev} M$ is the evaluation map. For simplicity we will pretend C is a closed manifold, otherwise suitable boundary condition is required. If v is a normal vector field along a Lagrangian immersion $f \in Map(C, M)$, then v determines an one form η_v on C. At $f \in Map(C, M)$ we have $\int_C ev^* \omega^m (v_1, ..., v_m) = \int_C \eta_{v_1} \wedge ... \wedge \eta_{v_m}$...

Next we need to incorporate flat connections on C into the picture. We denote $\mathcal{A}(C)$ the affine space of connections on C. On $C \times \mathcal{A}(C)$ there is a naturally defined universal connection $\mathbb{D}$ and curvature $\mathbb{F}$ (see for example [10]). With respect to the decomposition of two forms on $C \times \mathcal{A}(C)$ as $\Omega^2(C) + \Omega^1(C) \otimes \Omega^1(\mathcal{A}) + \Omega^2(\mathcal{A})$, we write $\mathbb{F} = \mathbb{F}^{2,0} + \mathbb{F}^{1,1} + \mathbb{F}^{0,2}$. Then for $(x, A) \in C \times \mathcal{A}(C)$ and $u \in T_x C$, $B \in \Omega^1(C, End(E))$, we have $\mathbb{F}^{2,0}(x, A) = F_A$,

$\mathbb{F}^{1,1}(x,A)(u,B) = B(u)$ and $\mathbb{F}^{0,2} = 0$. We consider the following complex valued closed m form on $Map(C,M)\times\mathcal{A}(C)$,

$$_A\Omega = \int_C Tr\left(ev^*\omega + \mathbb{F}\right)^m .$$

Since the tangent spaces of the moduli of special Lagrangian and the moduli of flat $U(1)$ connections can both be identified with the space of harmonic one forms[7]. A tangent vector of the moduli space $_A\mathcal{M}(M)$ is a complex harmonic one form $\eta + i\mu$. Then $_A\Omega$ is given explicitly as follows

$$_A\Omega\left(\eta_1 + i\mu_1, ..., \eta_m + i\mu_m\right) = \int_C Tr\left(\eta_1 + i\mu_1\right)\wedge ... \wedge\left(\eta_m + i\mu_m\right).$$

On the W side we have universal connection and curvature on the space of connections $\mathcal{A}(W)$ as before and we have the following complex valued closed m form on $\mathcal{A}(W)$,

$$_B\Omega = \int_W \widetilde{\Omega}\wedge Tr\mathbb{F}^m.$$

As before, $_B\Omega$ descends to a closed m form on $_B\mathcal{M}(W)$, the moduli space of holomorphic bundles, or equivalently B-cycles, on W. It is conjectured by Vafa ([15]) that under mirror symmetry, these two forms $_A\Omega$ and $_B\Omega$ are equivalent after instanton correction by holomorphic disks.

In our case, where M is semi-flat and C is a section, there is no holomorphic disk. Also the real Fourier-Mukai transform gives mirror cycle. Now we can verify Vafa's conjecture in this situation. Namely $_A\Omega$ and $_B\Omega$ are preserved under the real Fourier-Mukai transform.

For a closed one form on C which represents an infinitesimal variation of a A-cycle in M, we can write it as $d\eta + id\mu$ for some function η and μ in x variables. Under the above real Fourier-Mukai transform, the corresponding infinitesimal variation of the mirror B-cycle is $\delta\mathcal{A} = i\left(\frac{\partial\eta}{\partial\widetilde{x}_j}d\widetilde{x}_j + \frac{\partial\mu}{\partial\widetilde{x}_j}d\widetilde{y}_j\right)$. Therefore its $(0,1)$ component is $(\delta\mathcal{A})^{0,1} = \frac{i}{2}\left(\frac{\partial\eta}{\partial\widetilde{x}_j} + i\frac{\partial\mu}{\partial\widetilde{x}_j}\right)\left(d\widetilde{x}_j - id\widetilde{y}_j\right)$. So

$$\begin{aligned}
_B\Omega\left(\delta\mathcal{A}_1, ..., \delta\mathcal{A}_m\right) &= \int_W \widetilde{\Omega}\wedge Tr\mathbb{F}^m\left(\delta\mathcal{A}_1, ..., \delta\mathcal{A}_m\right) \\
&= \int_W \widetilde{\Omega}\wedge\left[\delta\mathcal{A}_1\wedge\cdots\wedge\delta\mathcal{A}_m\right]_{sym} \\
&= (const)\int_W \Pi_j\left(d\eta_j + id\mu_j\right)d\widetilde{y_1}\cdots d\widetilde{y_m} \\
&= (const)'\int_C \Pi_j\left(d\eta_j + id\mu_j\right) \\
&= (const)'_A\,\Omega\left(d\eta_1 + id\mu_1, ..., d\eta_m + id\mu_m\right).
\end{aligned}$$

[7] If the rank is greater than one, these harmonic forms will take values in the corresponding local system.

Hence we are done. Note that the same argument also work for higher-rank flat unitary bundles over C.

5 B-cycles in M are A-cycles in T^*M

We now show that a B-cycle in M can also be treated as a special Lagrangian cycle in the cotangent bundle $X = T^*M \xrightarrow{\pi} M$. We include this observation for its possible relevance to the recovery of "classical" mirror symmetry from the version with branes, as outlined briefly in [8]. The reader is be warned that the $2n$-form that we use for the special condition on X may not be the most natural one.

Recall that cotangent bundle of M, or any manifold, carries a natural symplectic form $\vartheta = \Sigma dx^k \wedge du_k + \Sigma dy^k \wedge dv_k$ where x's and y's are local coordinates on M and u's and v's are the dual coordinates in T^*M. Moreover the conormal bundle of submanifold C in M is a Lagrangian submanifold with respect to this symplectic form on X. There are a couple other natural closed two-forms on X: (i) the pullback of Kähler form from M, namely $\pi^*\omega$ and (ii) the canonical holomorphic symplectic form ϑ_{hol} via the identification between T^*M and $(T^*M \otimes \mathbb{C})^{1,0}$. In term of local holomorphic coordinates $z^1, ..., z^n$ on M we have

$$\begin{aligned} \pi^*\omega &= i\Sigma g_{j\bar{k}} dz^j \wedge d\bar{z}^k, \\ \vartheta_{hol} &= \Sigma dz^j \wedge dw_j. \end{aligned}$$

Here $z^j = x^j + iy^j$ and $w_j = u_j + iv_j$. We define the $2n$ form Θ on X using a combination of $\pi^*\omega$ and ϑ_{hol}:

$$\Theta = (\pi^*\omega + \operatorname{Im} \vartheta_{hol})^n .$$

Notice that $\Theta\bar{\Theta} = \vartheta^{2n}$ and the restriction of Θ to the zero section is a constant multiple of the volume form on M...

A Lagrangian submanifold S in X is called special Lagrangian if the restriction of Θ to S satisfies $\operatorname{Im}\Theta = \tan\theta \operatorname{Re}\Theta$ for some phase angle θ. Equivalently $\operatorname{Im} e^{i\theta}\Theta$ vanishes on S.

Next we consider a Hermitian line bundle L over M. Let D_A be a Hermitian integrable connection on L, that is $F_A^{2,0} = 0$. With respect to a holomorphic trivialization of L, we can write $D_A = \partial + \bar{\partial} + \partial\phi$ locally for some real valued function $\phi(z, \bar{z})$. This determines a Lagrangian submanifold $S = \left\{ w_j = \frac{\partial\phi}{\partial z^j} : j = 1, ..., n \right\}$ in X with respect to ϑ. Notice that the definition of S depends on the holomorphic trivialization of L. The restriction ϑ_{hol} to S equals

$$\begin{aligned} \vartheta_{hol}|_S &= \Sigma dz^j \wedge d\left(\frac{\partial\phi}{\partial z^j}\right) \\ &= \Sigma dz^j \wedge \left(\frac{\partial^2\phi}{\partial z^j \partial z^k} dz^k + \frac{\partial^2\phi}{\partial z^j \partial \bar{z}^k} d\bar{z}^k\right) \\ &= \Sigma \frac{\partial^2\phi}{\partial z^j \partial \bar{z}^k} dz^j \wedge d\bar{z}^k. \end{aligned}$$

This form is pure imaginary because ϕ is a real valued function. Therefore the restriction of Θ to S equals

$$\Theta|_S = \left[\Sigma\left(ig_{j\bar{k}} + \frac{\partial^2\phi}{\partial z^j\partial\bar{z}^k}\right)dz^j \wedge d\bar{z}^k\right]^n = (\omega + F)^n\,.$$

Therefore S is a special Lagrangian in X if and only if (L, D_A) satisfies the deformed Hermitian-Yang-Mills equation on M.

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