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Symplectic Structures on Gauge Theory*

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Abstract: We study certain natural differential forms $\Omega^{[*]}$ and their $\mathcal{G}$ equivariant extensions on the space of connections. These forms are defined using the family local index theorem. When the base manifold is symplectic, they define a family of symplectic forms on the space of connections. We will explain their relationships with the Einstein metric and the stability of vector bundles. These forms also determine primary and secondary characteristic forms (and their higher level generalizations).

Contents

1. Introduction

In this paper, we study natural differential forms $\Omega^{[2k]}$ on the space of connections $\mathcal{A}$ on a vector bundle E over X,

$$\Omega^{[2k]}(A)(B_1, B_2, \ldots, B_{2k}) = \int_X \mathrm{Tr}[e^{\frac{i}{2\pi}F_A} B_1 B_2 \cdots B_{2k}]_{sym} \hat{A}(X).$$

They are $\mathcal{G}$ invariant closed differential forms on $\mathcal{A}$. They are introduced as the family local index for universal family of operators.

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We use them to define higher Chern-Simons forms of E:

$$ch(E; A_0, \ldots, A_l) = \Omega_l^{[*]} \circ L_l(A_0, \ldots, A_l),$$

and discuss their properties. When $l = 0$ and 1, they are just the ordinary Chern character and Chern-Simons forms of E. The usual properties of Chern-Simons forms are now generalized to

$$\begin{aligned}&(1)\ ch(E) \circ \partial_{\mathcal{A}} = d \circ ch(E),\\ &(2)\ ch(E; s \cdot \alpha) = (-1)^{|s|} ch(E; \alpha).\end{aligned}$$

When we restrict our attention to the case when X is a symplectic manifold (or a Kahler manifold) with integral symplectic form ω, $[\omega] = c_1(L)$. We study $\Omega^{[2]}$ on $\mathcal{A}(E \otimes L^k)$ for large k. Asymptotically, they define the symplectic form on $\mathcal{A}(E \otimes L^k)$. Using a connection D_L on L, we can identify all these $\mathcal{A}$'s with $\mathcal{A}(E)$ and get a one parameter family of symplectic forms Ω_k on $\mathcal{A}(E)$,

$$\Omega_k(D_A)(B_1, B_2) = \int_X \mathrm{Tr}\, [e^{\frac{i}{2\pi} F_A + k\omega I_E} B_1 B_2]_{sym} \hat{A}(X).$$

Their $\mathcal{G}$-moment maps are

$$\Phi_k(A) = [e^{\frac{i}{2\pi} F_A + k\omega I_E} \hat{A}(X)]^{(2n)}.$$

When we let the parameter k go to infinity, we will obtain the standard symplectic form Ω on $\mathcal{A}(E)$:

$$\Omega(A)(B_1, B_2) = \int_X \mathrm{Tr}\, B_1 B_2 \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

Notice that Ω is an affine constant form in the sense that it is independent of the point $A \in \mathcal{A}$. The moment map equation associated to Ω is

$$F_A \wedge \frac{\omega^{n-1}}{(n-1)!} = \mu_E \frac{\omega^n}{n!} I_E,$$

which is equivalent to the Hermitian Einstein equation (a semi-linear system of partial differential equations)

$$\bigwedge F_A = \mu_E I_E.$$

Nevertheless, the moment map equation for Φ_k is a fully non-linear system of equations (of Monge-Ampere type) which are closely related to stability of E as will be explained in Sect. 4.

Then we explain how to remove the assumption of integrality of ω in Sect. 5. Moreover, all $\mathcal{A}$'s will be canonically embedded inside a larger space $\tilde{\mathcal{A}}_{[E]}$, the space of generalized connections. Generalized connections satisfy a weaker constraint:

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} dg_{ij} + d\theta_{ij}.$$

However, they share many properties of an ordinary connection. The first Chern class defines a surjective affine homomorphism

$$c_1 : \tilde{\mathcal{A}}_{[E]} \to H^2(X, \mathbb{C}),$$

and $\Omega^{[2]}$ gives a relative symplectic form on it. We define the notion of extended moduli space and polarization which links to our previous picture on stability.

In Sect 6, we will generalize the moment map construction. We study equivariant extensions of all $\Omega^{[2k]}$ and their properties. They are $\Omega^{[2i,2j]} \in \Omega^{2i}(\mathcal{A},\ Symm^j(Lie\,\mathcal{G})^*)^{\mathcal{G}}$ given by

$$\Omega^{[2i,2j]}(D_A)(B_1,\ldots,B_{2i})(\phi_1,\ldots,\phi_j)$$
$$= \int_X \text{Tr}\ [e^{\frac{i}{2\pi}F_A} B_1\cdots B_{2i}\phi_1\cdots\phi_j]_{sym}\hat{A}(X)$$

In the last section, we will briefly explain how these higher $\Omega^{[2i,2j]}$ are related to family stability. We need to look at all connections on the universal bundle which induces the canonical relative universal connection $\mathbb{D}^{rel}$. Let $\mathbb{D} + S$ be such a connection, then its curvature is given by:

$$\mathbb{F}^{2,0}\ (x, A) = F_A \text{ at } x,$$
$$\mathbb{F}^{1,1}\ (x, A)(v, B) = B(v) + D_{A,v}(\mathbb{S}(A)(B)) \text{ at } x,$$
$$\mathbb{F}^{0,2} = d_{\mathcal{A}}\mathbb{S} + \mathbb{S}^2.$$

By twisted transgression of $\Omega^{[*]}$ (defined using $\mathbb{D}+S$), we get $Map(M,\mathcal{G})$-invariant closed forms on $Map(M,\mathcal{A})$, namely, $\Omega_M^{[*]}$. We will explain how these forms are related to stability for a family of bundles over X parametrized by M when both X and M are symplectic manifolds.

2. Higher Index of the Universal Family

In this section we consider the universal connection and the universal curvature. They induce canonical differential forms $\Omega^{[*]}$ on the space of connections which are important for our later discussions.

These forms are closely tied with the local index theorem for the universal family of certain operators. We shall discuss the Dirac operator and the $\overline{\partial}$ operator in detail.

Let X be a compact smooth manifold of dimension $2n$ and E be a rank r unitary vector bundle over X. We denote the space of connections on E by $\mathcal{A}$ and the space of automorphisms of E by $\mathcal{G}$, which is also called the group of Gauge transformations on E. The tangent space of $\mathcal{A}$ at a point $A \in \mathcal{A}$ is canonically isomorphic to the space of one forms on X with End(E) valued, that is,

$$T_A\mathcal{A} = \Omega^1(X, \text{End}(E)).$$

In particular, $\mathcal{A}$ is an affine space. (Later in this paper, we shall construct a natural affine extension of $\mathcal{A}$ by an one dimensional affine space.) $\mathcal{G}$ is an infinite dimensional Lie Group with Lie algebra being the space of sections of End(E) over X, that is,

$$Lie\,\mathcal{G} \ = \ \Omega^0(X, \text{End}\ \ E).$$

The center of $\mathcal{G}$ can be identified as the space of non-vanishing functions on X and hence the center of its Lie algebra can be identified as the space of all functions on X, $C(X)$, or the space of zero forms on X, $\Omega^0(X)$. By integrating over X, we can identify the dual of *Lie* $\mathcal{G}$ as the space of $2n$ forms on X with End(E) valued in the sense of distributions

$$(Lie\ \mathcal{G})^* \ = \ \Omega^{2n}(X, \text{End}(E))_{dist}.$$

It has a dense subspace $\Omega^{2n}(X, \mathrm{End}(E))$ consisting of smooth $2n$ forms which are enough in most purposes. The above identification follows from the fact that

$$\Omega^0(X, \mathrm{End}\ E) \otimes \Omega^{2n}(X, \mathrm{End}\ E)_{dist} \longrightarrow \mathbb{C},$$

$$\phi \otimes \alpha \longrightarrow \int_X \mathrm{Tr}\ \phi\alpha$$

is a perfect pairing.

By putting all connections together, one can form a universal connection as follow: Let $\mathbb{E} = \pi_1^* E$ be the universal bundle, where $\pi_1 : X \times \mathcal{A} \longrightarrow X$ is the projection to the first factor. Then we can patch up all connections on E to form a partial connection on $\mathbb{E}$ and since the bundle $\mathbb{E}$ is trivial along $\mathcal{A}$, we get a (universal) connection $\mathbb{D}$ on E.

To be more precise, let us describe $\mathbb{D}$ in terms of horizontal subspaces. A connection A on E is given by a $U(r)$ equivariant subbundle H of TE over E such that TE is isomorphic to $T_v E \bigoplus H$. Here $T_v E$ is the vertical tangent bundle of E with respect to $E \longrightarrow X$. Now, let $e \in \mathbb{E}$ be a point on the fiber of $\mathbb{E}$ over $(x, A) \in X \times \mathcal{A}$. Then $T_e \mathbb{E}$ is canonically isomorphic to $T_e E \bigoplus T_A \mathcal{A}$ and we can choose $H \bigoplus T_A \mathcal{A}$ as a horizontal bundle of $T\mathbb{E}$ (where H is the horizontal subbundle of TE defined by A). This horizontal subspace defines our universal connection $\mathbb{D}$.

The curvature $\mathbb{F}$ of $\mathbb{D}$ will be called the universal curvature, it is a End($\mathbb{E}$) valued two form on $X \times \mathcal{A}$. With respect to the natural decomposition of two forms on $X \times \mathcal{A}$:

$$\Omega^2(X \times \mathcal{A}) = \Omega^2(X) \bigoplus \Omega^1(X) \bigotimes \Omega^1(\mathcal{A}) \bigoplus \Omega^2(\mathcal{A}),$$

we decompose $\mathbb{F}$ into corresponding three components,

$$\mathbb{F} = \mathbb{F}^{2,0} + \mathbb{F}^{1,1} + \mathbb{F}^{0,2}.$$

They are given explicitly by

$$\begin{aligned} &\mathbb{F}^{2,0}\ (x, A) = F_A \text{ at } x, \\ &\mathbb{F}^{1,1}\ (x, A)(v, B) = B(v) \text{ at } x, \\ &\mathbb{F}^{0,2} = 0. \end{aligned}$$

where $(x, A) \in X \times \mathcal{A}$, $v \in T_x X$ and $B \in \Omega^1(X, \mathrm{End}(E))$.

Now, we want to use $\mathcal{A}$ to parametrize certain families of first order elliptic operators on X by coupling different connections with a fixed differential operator. The following three situations are of most interest to us.

2.1. Dirac operators. Let X be a spin manifold of even dimension $m = 2n$. For any Riemannian metric g on X, we get a Dirac operator D on X. By coupling D with a connection on E, $A \in \mathcal{A}$, we get a twisted Dirac operator D_A on X. Varying these connections, we then get a family of Dirac operators parametrized by $\mathcal{A}$.

The index of D_A (Index D_A = dim $Ker D_A$ − dim $Coker D_A$) can be computed by the Atiyah-Singer Index theorem:

$$\text{Index } D_A = \int_X ch(E)\hat{A}(X),$$

which can be expressed in terms of curvature of E via the Chern-Weil theory:

$$\text{Index } D_A = \int_X \text{Tr } e^{\frac{i}{2\pi}F_A}\hat{A}(X).$$

The virtual vector space (the formal difference of two vector spaces) $KerD_A - CokerD_A$ varies continuously with respect to A and forms a virtual vector bundle over $\mathcal{A}$ (despite the fact that $KerD_A$ or $CokerD_A$ might jump in dimensions separately). This is called the index bundle over $\mathcal{A}$, Ind ($\mathbb{D}$). We could formally regarded Ind ($\mathbb{D}$) as an element in the K group of $\mathcal{A}$, $K(\mathcal{A})$, even though $\mathcal{A}$ is of infinite dimension. This can be interpreted as Ind ($\mathbb{D}$) giving an element in $K(Y)$ for any compact subfamily $Y \subset \mathcal{A}$. Although Ind ($\mathbb{D}$) is only a virtual bundle, its determinant is an honest complex line bundle over $\mathcal{A}$, called the determinant line bundle for Dirac operators, Det ($\mathbb{D}$).

By the family index theorem of Atiyah and Singer, we have (for any compact subfamily $Y \subset \mathcal{A}$)

$$ch(\text{Ind } (\mathbb{D})) = \int_X \text{Tr } e^{\frac{i}{2\pi}\mathbb{F}}\hat{A}(X)$$

as cohomology classes.

Bismut and Freed [B+F] strengthens this result to the level of differential forms and give the local family index theorem. They introduce superconnection $\mathbb{A}_t$ on the infinite dimensional bundle $\pi_*\mathbb{E}$ and they proved the following equality on the level of differential forms:

$$\lim_{t\searrow 0} ch(\mathbb{A}_t) = \int_X \text{Tr } e^{\frac{i}{2\pi}\mathbb{F}}\hat{A}(X).$$

We decompose this differential form into a sum according to their different degrees $\Omega^{[0]} + \Omega^{[2]} + \cdots$, where $\Omega^{[2k]} \in \Omega^{2k}(\mathcal{A})$.

Proposition 1.

$$\Omega^{[2k]}(A)(B_1, B_2, \ldots, B_{2k}) = (\frac{i}{2\pi})^{2k}\int_X \text{Tr}[e^{\frac{i}{2\pi}F_A}B_1B_2\cdots B_{2k}]_{sym}\hat{A}(X),$$

where $A \in \mathcal{A}$ and $B_i \in T_A\mathcal{A} = \Omega^1(X, \text{End }(E))$.

The notation $[\cdots]_{sym}$ denotes the graded symmetric product of those elements inside.

Proof of Proposition.

$$\begin{aligned}
&\Omega^{[2k]}(A)(B_1, B_2, \ldots, B_{2k}) \\
&= \int_X \text{Tr } [e^{\frac{i}{2\pi}(\mathbb{F}^{2,0}+\mathbb{F}^{1,1})(x,A)}](B_1, B_2, \ldots, B_{2k})\hat{A}(X) \\
&= \int_X \text{Tr } [e^{\frac{i}{2\pi}F_A}(\frac{i}{2\pi})^{2k}\frac{(\mathbb{F}^{1,1})^{2k}}{(2k)!}(B_1, B_2, \ldots, B_{2k})]_{sym}\hat{A}(X) \\
&= (\frac{i}{2\pi})^{2k}\int_X \text{Tr } [e^{\frac{i}{2\pi}F_A}B_1B_2\cdots B_{2k}]_{sym}\hat{A}(X).
\end{aligned}$$

For the second equality, we used the fact that $\mathbb{F}^{2,0}$ has no effect on the B_i's and the last equality holds since $\mathbb{F}^{1,1}$ is the tautological element which can be described as below. Under the following relations:

$$\begin{aligned} & \Omega^1(X) \otimes \Omega^1(\mathcal{A}) \otimes \mathrm{End}(E) \\ \cong\ & Hom(T\mathcal{A}, \Omega^1(X, \mathrm{End}\,(E))) \\ \supset\ & Hom(T_A\mathcal{A}, \Omega^1(X, \mathrm{End}\,(E))) \\ =\ & Hom(\Omega^1(X, \mathrm{End}\,(E)), \Omega^1(X, \mathrm{End}\,(E))), \end{aligned}$$

$\mathbb{F}^{1,1}$ comes from the identity endomorphism of $\Omega^1(X, \mathrm{End}\,(E))$. Hence we have proved the proposition. □

Corollary 1. *$\Omega^{[2k]}$ is a $\mathcal{G}$ invariant closed form on $\mathcal{A}$ and such that $\Omega^{[2k]} = 0$ if $k > n$.*

Proof of Corollary. Both $\Omega^{[2k]} = 0$ for $k > n$ and the invariance of $\Omega^{[2k]}$ with respect to $\mathcal{G}$ action are clear from the above proposition. We will use the Bianchi identity and the simple fact that the antisymmetrization of a symmetric expression is always zero to show the closedness of $\Omega^{[2k]}$. Consider

$$\begin{aligned} d\Omega^{[2k]}(A)(B_0, \ldots, B_{2k}) &= \sum_{i=0}^{2k} (-1)^i B_i \left(\Omega^{[2k]}(A)(B_0, \ldots, \hat{B}_i, \ldots, B_{2k}) \right) \\ &+ \sum_{i<j} (-1)^{i+j} \Omega^{[2k]}(A)([B_i, B_j], B_0, \ldots, \hat{B}_i, \ldots, \hat{B}_j, \ldots, B_{2k}). \end{aligned}$$

Here, we have extended each $B_i \in T_A\mathcal{A}$ to a vector field on $\mathcal{A}$ by transporting it using the affine structure on $\mathcal{A}$. To put it another way, B_i is always the same element in $\Omega^1(X, \mathrm{End}\,(E))$ regardless of different $A \in \mathcal{A}$. Hence, we have $[B_i, B_j] \equiv 0$ for any i and j.

We shall show that $\sum_{i=0}^{2k} (-1)^i B_i \Omega^{[2k]}(A)(B_0, \ldots, \hat{B}_i, \ldots, B_{2k})$ is zero. In order to simplify the notation, let us assume temporary that $\hat{A}(X) = 1$. Then

$$\begin{aligned} & B_i \Omega^{[2k]}(A)(B_0, \ldots, \hat{B}_i, \ldots, B_{2k}) \\ &= (\frac{i}{2\pi})^{2n} (n-k) \int_X \mathrm{Tr}[F_A^{n-k-1} (D_A B_i) B_0 \cdot \cdot \hat{B}_i \cdot \cdot B_{2k}]_{sym}. \end{aligned}$$

because all B_i's are independent of the connection $A \in \mathcal{A}$. Hence,

$$\begin{aligned} & \sum_{i=0}^{2k} (-1)^i B_i \Omega^{[2k]}(A)(B_0, \ldots, \hat{B}_i, \ldots, B_{2k}) \\ &= (\frac{i}{2\pi})^{2n} (n-k) \int_X d\ \mathrm{Tr}[F_A^{n-k-1} B_0 \cdots B_{2k}]_{sym} \end{aligned}$$

Here, we have used the Bianchi identity $D_A F_A = 0$ and the fact that d Tr = Tr D_A. Now the vanishing of the above expression follows from the Stokes's theorem. Hence we have showed that $\Omega^{[2k]}$ is closed. An alternative way to prove the closedness is by observing that these forms come from the pushed forward of certain closed differential forms on $X \times \mathcal{A}$. □

2.2. $\overline{\partial}$-operator. The second case is when X is a compact complex manifold. Then (complexified) differential forms on X can be decomposed into holomorphic and anti-holomorphic components:

$$\Omega^l(X) \otimes \mathbb{C} = \sum_{i+j=l} \Omega^{i,j}(X).$$

As an End(E)-valued two form on X, the curvature form of a Hermitian connection can be decomposed accordingly: $F_A = F_A^{2,0} + F_A^{1,1} + F_A^{0,2}$. One should not confuse this decomposition of F_A with the previous decomposition of $\mathbb{F}$. If $F_A^{0,2} = 0$, then the $(0,1)$ part of D_A defines a differential complex on $\Omega^{0,*}(X, E)$. Let us denote $p \circ D_A$ by $\overline{\partial}_A$, where $p : \Omega^1(X, E) \otimes \mathbb{C} \longrightarrow \Omega^{0,1}(X, E)$ is the projection to the anti-holomorphic part. $\overline{\partial}_A$ gives a complex because $F_A^{0,2} = \overline{\partial}_A^2$.

Let $\mathcal{A}^{hol} = \{D_A \in \mathcal{A} | \ F_A^{0,2} = 0\}$, then $\mathcal{A}^{hol}$ parametrizes a family of elliptic complexes over X. Now, $\mathcal{A}$ is the space of connections on E, not necessary Hermitian. However, in order for it to parametrize elliptic complexes, we need to choose a compatible Hermitian metric on E (which in fact exists and is unique up to unitary gauge transformations) and a Hermitian metric on the base manifold X. Such a metric induces a Todd form on X representing the Todd class of X. As in the case of Dirac operators, we get a sequence of differential forms on $\mathcal{A}^{hol}$:

$$\int_X \mathrm{Tr} \ e^{\frac{i}{2\pi}\mathbb{F}} Td_X = \Omega^{[0]} + \Omega^{[2]} + \Omega^{[4]} + \cdots.$$

Here the only difference is the replacement of $\hat{A}(X)$ by the Todd form on X, Td_X.

Notice that the odd class and $\hat{A}$ class are closely related to each other: $Td = e^{c_1}\hat{A}$. If X is Spin Kahler manifold then this equality holds on the level of differential forms. In this case, the $\overline{\partial}$ operator on X is equal to the Dirac operator on X twisted by $K_X^{-\frac{1}{2}}$, where the canonical line bundle K_X is determined by the (almost) complex structure on X and its square root is given by the chosen spin structure on X. In this identification, Kahlerian is important, which ensures that the complex structure and the metric structure on X are compatible to each other.

Later on, we shall explain that asymptotic behaviors of $\Omega^{[2]}$ are closely related to stability in Mumford's Geometric Invariant theory [M+F]. The importance of higher $\Omega^{[2k]}$ will be briefly discussed in the last section.

3. Moment Map and Stable Bundles

In this section, we will explain the relationship between the form $\Omega^{[2]}$ we found earlier and stability properties of the bundle E. This relation is originally discovered in the author's thesis [Le1] and explained in [Le2]. Here, we shall give a global picture in this section and the next one. Then we shall also discuss higher $\Omega^{[2k]}$ later in this paper.

Now, we suppose X is a symplectic manifold with symplectic form ω. We also assume ω is integral which means $[\omega]$ represents an integer cohomology class in X, $[\omega] \in H^2(X, \mathbb{Z})$. The general case will be treated in the next session. $H^2(X, \mathbb{Z})$ classifies complex line bundles over X up to isomorphisms, that is, $[\omega] = c_1(L)$ for some L over X. In fact, we can find a Hermitian connection D_L on L such that the last equality holds on the level of forms, that is $\omega = \frac{i}{2\pi} F_L$, where F_L is the curvature form of a certain connection D_L on L.

We are interested in the behaviors of $\Omega^{[2]}$ on the space of connections on $E \bigotimes L^k$ for large k. In order to distinguish connections on different bundles, we write $\mathcal{A}(E \bigotimes L^k)$

for the space of connections on $E \bigotimes L^k$. However, by tensoring with D_L, we can identify different $\mathcal{A}(E \bigotimes L^k)$,

$$\mathcal{A}(E) \longrightarrow \mathcal{A}(E \otimes L^k),$$
$$D_A \mapsto D_A \otimes D_{L^k}.$$

Under this identification, the curvature $\frac{i}{2\pi} F_A$ of D_A will be sent to $\frac{i}{2\pi} F_A + k\omega I_E$. We shall denote the corresponding $\Omega_0 = \Omega^{[2]}$ by Ω_k. We therefore have

$$\Omega_k(D_A)(B_1, B_2) = \int_X \mathrm{Tr}\ [e^{\frac{i}{2\pi} F_A + k\omega I_E} B_1 B_2]_{sym} \hat{A}(X)$$

in the spin case (and similarly formula for other cases by replacing the $\hat{A}$ by suitable characteristic classes). Notice that, at a general point $D_A \in \mathcal{A}$, Ω_k may be degenerated. However, it will become non-degenerate if k is chosen large enough and therefore, in a suitable sense, Ω_k is asymptotically symplectic form on $\mathcal{A}$. (For the rest of this paper, a symplectic form may possobly degenerate but this deficiancy can usually be rescued by asymptotic non-degeneracy.)

Moment maps. For any k, Ω_k is always a $\mathcal{G}$ an invariant form and a moment map exists. That is, Ω_k can be extended to a $\mathcal{G}$ equivariant closed form on $\mathcal{A}$.

Lemma 1. *The moment map for the $\mathcal{G}$ action on $\mathcal{A}$ with symplectic form Ω_k is given by* $\Phi_k : \mathcal{A} \longrightarrow \Omega^{2n}(X, \mathrm{End}(E))$,

$$\Phi_k(A) = [e^{\frac{i}{2\pi} F_A + k\omega I_E} \hat{A}(X)]^{(2n)}.$$

Proof of Lemma. First, we need to show that Φ_k is a $\mathcal{G}$ equivariant map, but this is clear from the definition of Φ_k.

Next, we want to show that $\Omega_k + \Phi_k$ is a $\mathcal{G}$ equivariant form. That is, for any $\phi \in Lie\ \mathcal{G} = \Omega^0(X, \mathrm{End}(E))$,

$$\iota_\phi \Omega_k = d(\Phi_k(\phi))$$

as an ordinary one form on $\mathcal{A}$, or,

$$\Omega_k(A)(D_A\phi, B) = d(\int_X \mathrm{Tr}\ \ \Phi_k(A) \cdot \phi)(B)$$

for any $A \in \mathcal{A}$ and $B \in T_A\mathcal{A} = \Omega^1(X, \mathrm{End}(E))$,

$$\begin{aligned}
& d(\int_X \mathrm{Tr}\ \ \Phi_k(D_A)\phi)(B) \\
&= \frac{d}{dt}\Big|_{t=0} \int_X \mathrm{Tr}\ \ e^{\frac{i}{2\pi}(F_A + tD_A B + t^2 B^2) + k\omega I_E} \phi \hat{A}(X) \\
&= \frac{i}{2\pi} \int_X \mathrm{Tr}[e^{\frac{i}{2\pi} F_A + k\omega I_E} D_A B]_{sym} \phi \hat{A}(X) \\
&= \frac{i}{2\pi} \int_X \mathrm{Tr}[e^{\frac{i}{2\pi} F_A + k\omega I_E} D_A B \phi]_{sym} \hat{A}(X) \\
&= \frac{i}{2\pi} \int_X \mathrm{Tr}[e^{\frac{i}{2\pi} F_A + k\omega I_E} D_A \phi B]_{sym} \hat{A}(X) \\
&= \Omega_k(D_A)(D_A\phi, B)
\end{aligned}$$

Hence, $\Omega_k + \Phi_k$ is a $\mathcal{G}$ equivariant closed form and Φ_k is the moment map for the gauge group action on $\mathcal{A}$ with the symplectic structure given by Ω_k. $\square$

Remark . For each k, we obtain a $\mathcal{G}$ equivariant closed form on $\mathcal{A}$, namely $\Omega_k + \Phi_k$. It depends only on the choice of a connection D_L on L. Neverthless, the equivariant cohomology class it represents, $[\Omega_k + \Phi_k] \in H^2_{\mathcal{G}}(\mathcal{A})$, is independent of such a choice.

Proposition 2. $[\Omega_k + \Phi_k] \in H^2_{\mathcal{G}}(\mathcal{A})$ *is independent of the choice of the Hermitian connection* D_L *on* L.

Proof of Proposition. If D_L and D'_L are two connections on L, so they differ by a one form on X, say α. They induce $\Omega_k + \Phi_k$ and $\Omega'_k + \Phi'_k$ on $\mathcal{A}$. Their difference is a $\mathcal{G}$ equivariant exact form on $\mathcal{A}$ which can be written down explicitly (like the Chern-Simons construction). More precisely,

$$(\Omega'_k + \Phi'_k) - (\Omega_k + \Phi_k) = d^{\mathcal{G}} \Theta_{k, D_L, D'_L},$$

where

$$\Theta_{k, D_L, D'_L}(A)(B) = k \int_0^1 dt \int_X \alpha \ \mathrm{Tr}\ e^{\frac{i}{2\pi} F_A + k(\omega + t d\alpha) I_E} B \hat{A}(X).$$

Hence, the cohomology classes they represented are the same. $\square$

Large k limit of $\Omega_k + \Phi_k$. We now consider Ω_k and Φ_k for large k. If we expand Ω_k in powers of k, we have

$$\Omega_k(A)(B_1, B_2) = k^{n-1} \cdot \int_X \mathrm{Tr}\, B_1 B_2 \wedge \frac{\omega^{n-1}}{(n-1)!} + O(k^{n-2}).$$

The leading order term of it defines a (everywhere non-degenerate) symplectic form Ω on $\mathcal{A}$, moreover, it is a constant form on $\mathcal{A}$ in the sense that there is no dependence of A in its expression. The moment map defined by this constant symplectic form Ω is

$$\Phi(A) = F_A \wedge \frac{\omega^{n-1}}{(n-1)!}.$$

If we expand Φ_k for large k , we have

$$\Phi_k(A) = k^n \frac{\omega^n}{n!} I_E + k^{n-1}\Phi(A) + O(k^{n-2}).$$

The first term is a constant term and can be absorbed in the definition of Φ_k (since the moment map is uniquely determined only up to addition of any constant central element) and therefore, the leading (non-trivial) term for Φ_k becomes Φ for large k.

We now look at the symplectic quotient of $\mathcal{A}$ by Hamiltonian $\mathcal{G}$ actions with respect to different Ω_k. First choose a coadjoint orbit $\mathcal{O} \subset (Lie\mathcal{G})^*$, the inverse image $\Phi^{-1}(\mathcal{O})$ of $\mathcal{O}$ under the moment map Φ is $\mathcal{G}$ invariant. Then one can show that $\Phi^{-1}(\mathcal{O})/\mathcal{G}$ inherits a natural symplectic form (on the smooth points) from the original symplectic space. We can now apply this procedure to any of these Ω_k or Ω since they are all $\mathcal{G}$ invariant symplectic forms on $\mathcal{A}$.

The simplest choice of the coadjoint orbit would be 0, however, $\Phi^{-1}(0)$ may not be non-empty in general. We would choose those coadjoint orbits which consist of a single point only. The set of all these coadjoint orbits can be identified with the space of 2n forms on X (in the sense of distributions).

Hermitian Einstein metrics. We first consider Φ which depends on A is a (semi-)linear fashion. Let $\zeta \in \Omega^{2n}(X)$, then $D_A \in \Phi^{-1}(\zeta)$ if and only if

$$F_A \wedge \frac{\omega^{n-1}}{(n-1)!} = \zeta I_E.$$

By taking the trace and integrating it over X, we get a necessary condition for $\Phi^{-1}(\zeta)$ being non-empty which is

$$\int_X \zeta = \mu_E,$$

where $\mu_E = \frac{1}{rk(E)} < c_1(E) \cup \frac{c_1(L)^{n-1}}{(n-1)!}, [X] >$ is the slope of E with respect to the polarization L. For simplicity, we normalize the (symplectic) volume of X to be one. When we choose ζ to be a multuple of the symplectic volume form $\frac{\omega^n}{n!}$, then the equation defined by the connection being laid on the inverse image of ζ would be:

$$F_A \wedge \frac{\omega^{n-1}}{(n-1)!} = \mu_E \frac{\omega^n}{n!} I_E.$$

Using $\frac{\omega^n}{n!}$ and a metric on X, we can convert this equation into an equation of zero forms, it reads as:

$$\bigwedge F_A = \mu_E I_E,$$

where $\bigwedge$ is the adjoint to the multiplication operator $L = \omega \wedge (\,)$ (a metric on X is used to define the adjoint of an operator). In local coordinates, the equation is

$$g^{\alpha\overline{\beta}} F^i_{j\alpha\overline{\beta}} = \mu_E \delta^i_j,$$

where $\sqrt{-1} g_{\alpha\overline{\beta}} dz^\alpha \wedge dz^{\overline{\beta}}$ is the Kahler form ω on X and $(g^{\alpha\overline{\beta}})$ is the inverse matrix to $(g_{\alpha\overline{\beta}})$ provided X is Kahler and z's are local holomorphic coordinates on X. Suppose this is the case and E is an irreducible holomorphic vector bundle over it, then this equation is called the Hermitian Einstein equations or the Hermitian Yang Mills equations. It is called Einstein because $\bigwedge F_A$ is a Ricci curvature of E. (On a vector bundle, there is another kind of Ricci curvature on it, namely $\mathrm{Tr}_E F_A \in \Omega^2(X)$. This latter Ricci curature is a closed two form on X and the cohomology class it defined is the first Chern class of E.

If we consider only those connections whose (0,2) part of its curvature vanished, that is $A \in \mathcal{A}^{hol}$ as in the discussion of last section. Then we have

$$\int_X |F_A|^2 \frac{\omega^n}{n!} - \int_X |\bigwedge F_A|^2 \frac{\omega^n}{n!} = -8\pi^2 \int_X ch_2(E) \wedge \frac{\omega^{n-2}}{(n-2)!}.$$

Since X is Kahler, $\frac{\omega^n}{n!}$ is the volume form of X and the functional $\int_X |F_A|^2 \frac{\omega^n}{n!}$ is the standard Yang-Mills functional in Gauge theory. By the above equality, it is equivalent to the functional $\int_X |\bigwedge F_A|^2 \frac{\omega^n}{n!}$ up to a topological constant (when we restrict our attention to $\mathcal{A}^{hol}$). Now, the Euler-Lagrange equation for the functional $\int_X |\bigwedge F_A|^2 \frac{\omega^n}{n!}$ is $D_A(\bigwedge F_A) = 0$. It can be reduced to $\bigwedge F_A = \mu_E I_E$ if E is irrreducible. It is because the Euler-Lagrange equation implies that $\bigwedge F_A$ is a parallel endomorphism and therefore their eigenspaces are all holomorphic subbundles of E. When E is irreducible, E itself is the only non-trivial subbundle. So, the endomorphism $\bigwedge F_A$ has to be constant which is fixed by the topology of the bundle.

From another point of view, the Yang-Mills equation on E (that is the Euler-Lagrange equation of the functional $\int_X |F_A|^2 \frac{\omega^n}{n!}$) is $D_A^* F_A = 0$. When X is Kahler and $A \in \mathcal{A}^{hol}$, we have $D_A^* = [D_A, \bigwedge]$. So,

$$0 = D_A^* F = D_A(\bigwedge F_A) - \bigwedge(D_A F_A) = D_A(\bigwedge F_A)$$

by the Bianchi identity. Hence, the two equations are equivalent.

Stability. Existence of solutions to $\bigwedge F_A = \mu I_E$ on E can be rephased in algebraic geometric languages. By [N+S],[Do] and [U+Y], there exists a Hermitian Einstein metric on E if and only if E is a Mumford stable bundle. In such a case, the metric is also unique up to scaling by a global constant. (When E is not irreducible, then existence of the Hermitian Einstein metric on E is equivalent to E being a direct sum of irreducible Mumford stable bundles over X with the sum slope, we called such a bundle E a Mumford poly-stable bundle.)

Mumford stability is only the “linearization” of stability in studying moduli spaces of vector bundles using Mumford’s Geometric Invariant theory. In geometric invariant theory of vector bundles, Gieseker [Gi] found the correct notion of stability and stated them in geometric terms. In the thesis, the author found that stability is in fact equivalent to existence of solutions to moment map equations for Φ_k with k large (and control of the curvature of the solutions as k goes to infinity). To solve that (fully-nonlinear) equation, the author used a very involved singular perturbation method which identified the obstruction for perturbations is precisely the unstability [Le1, Le2]. The basic idea is when k goes to infinity, we expected the family of solutions will blow up (provided it exists). Different directions will blow up according to different rates. However, this information can be captured from algebraic geometry. Then we try to perturb the singular solution (whose existence can be proved by using a theorem of Uhlenbeck and Yau) to finite k; there will be numerous obstructions.

We can identify all these obstructions. In fact, obstructions vanish precisely when the bundle is Gieseker (poly-)stable. Instead of trying to solve the equations, we shall discuss the geometry underlying these Ω_k’s, Φ_k’s and their higher degree cousins.

First we want to set Φ_k equal to a constant multiple of $\frac{\omega^n}{n!} I_E$. The constant has to be $\frac{1}{rk(E)}\chi(X, E \otimes L^k)$, where χ is the index of the corresponding operator given by the Atiyah-Singer index theorem. Therefore, the equations defined by the moment map would become

$$[e^{\frac{i}{2\pi}F_A + k\omega I_E} Td(X)]^{(2n)} = \frac{1}{rk(E)}\chi(X, E \otimes L^k)\frac{\omega^n}{n!} I_E.$$

These equations describe the stability properties of the hlomorphic bundles. Unlike the Hermitian Einstein equations, this system of equations is fully nonlinear unless $n = 1$. Instead of $\bar{\partial}$ operator, we can use the Dirac operator and replace Td form by $\hat{A}$ forms.

4. Space of Generalized Connections

In this section, we shall explain a setting which is more suitable for studying the symplectic quotient of $\mathcal{A}$ (or $\mathcal{A}^{hol}$) by $\mathcal{G}$ with respect to $\Omega^{[2]}$. All the proofs here are elementary and sometimes we only sketch the reasonings. From the last section, we know that it is natural to put $\mathcal{A}(E \otimes L^k)$'s together for large k. Stability depends only on the 'direction' L and one should not distinguish E among $E \otimes L^k$'s. We are going to explain how to achieve these in a natural manner by allowing k to be any complex number. Moreover, in this approach, we do not need to assume that the symplectic form ω is an integral form. To do this, we need to define generalized connections on E.

Review of connections. Let us first recall the definition of connections in $\check{C}$ech languages. Let $\mathcal{U}$ be a good cover of X. That is, for any $U_i, U_j, \ldots, U_k \in \mathcal{U}$, $U_{i,j,\ldots,k} \equiv U_i \cap U_j \cap \cdots \cap U_k$ is a contractible set. Then the bundle E is determined by a collection of gluing functions: $\{g_{ij} : U_{ij} \to GL(r)\}$ which satisfies $g_{ij} = g_{ji}^{-1}$ and $g_{ij} g_{jk} g_{ki} = 1$. Two collections g and g' are equivalent if there exists $\{h_i : U_i \to GL(r)\}$ satisfying $g'_{ij} = h_i^{-1} g_{ij} h_j$.
A connection D_A is a collection of matrix-valued one form on $\mathcal{U}$, $\{A_i \in \Omega^1(U_i, gl(r))\}$. On U_{ij}, they satisfy

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij}.$$

Now, a generalized connection D_A is essentially the same thing except that we relax the last equality to

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij} + d\theta_{ij}$$

for some collection $\{\theta_{ij} : U_{ij} \to \mathbb{C}\}$ which satisfies $\theta_{ij} = -\theta_{ji}$ and $d(\theta_{ij} + \theta_{jk} + \theta_{ki}) = 0$. The definition of gauge equivalent for two generalized connections are the same as ordinary connections: $A'_i = g_i^{-1} A_i g_i + g_i^{-1} d g_i$ for some automorphism $\{g_i : U_i \to GL(r)\}$.

Let $F_i = dA_i + A_i \wedge A_i$, then one can prove that we still have $F_j = g_{ij}^{-1} F_i g_{ij}$. That is $F \in \Omega^2(X, \mathrm{End}\,(E))$. One can also verify that the notion of generalized connections is well-defined and independent of the choice of good cover $\mathcal{U}$ or the gluing functions $\{g_{ij}\}$ of E.

Generalized connections.

Definition 1. *A collection $D_A = \{A_i \in \Omega^1(U_i, gl(r))\}$ is called a generalized connection if*

$$A_j = g_{ij}^{-1} A_i g_{ij} + g_{ij}^{-1} d g_{ij} + d\theta_{ij}$$

for some collection $\{\theta_{ij}\}$ as before.

The curvature $F_A \in \Omega^2(X, \mathrm{End}\,(E))$ of D_A is defined to be

$$F_i = dA_i + A_i \wedge A_i$$

on U_i. $\tilde{\mathcal{A}}_{[E]}$ denotes the space of all generalized connections on E.

We collect some basic facts about $\tilde{\mathcal{A}}_{[E]}$ in the following lemma.

Lemma 2. *(1) $\mathcal{A}(E) \subset \tilde{\mathcal{A}}_{[E]}$.*
(2) $\tilde{\mathcal{A}}_{[E]} \equiv \tilde{\mathcal{A}}_{[E \otimes L]}$ for any line bundle L.

(3) $\tilde{\mathcal{A}}_{[E]}$ *has a natural affine structure such that* $\mathcal{A}(E)$ *is an affine subspace of it of codimension* $b_2(X)$*.*

In a sense, $\tilde{\mathcal{A}}_{[E]}$ should be regarded as $\bigcup_{L,k} \mathcal{A}(E \otimes L^k)$ where the L's forms a base of $H^2(X,\mathbb{Z})/Tor$ and $k \in \mathbb{C}$. Previously, in order to identify $\mathcal{A}(E)$ and $\mathcal{A}(E \otimes L)$, we have to pick a connection on L. But now, $\tilde{\mathcal{A}}_{[E]}$ and $\tilde{\mathcal{A}}_{[E\otimes L]}$ are always canonically isomorphic to each other. So it is natural to consider equivalent classes of vector bundles which are isomorphic to each other modulo tensoring with line bundles. We denote the equivalent class of E by $[E]$. That is $[E \otimes L] = [E]$ for any line bundle L.

Notice that $End([E])$ is a well-defined bundle on X and the group $\mathcal{G}$ of all gauge transformations of $[E]$ is also a well-defined notion for the equivalent class.

Let $c_1([E], D_A) = \frac{i}{2\pi} Tr\ F_A$ be the first Chern form of any generalized connection $D_A \in \tilde{\mathcal{A}}_{[E]}$. It is always a closed two form on X. However, its cohomology class is not determined by $[E]$. In fact, it induces a surjective affine homomorphism

$$c_1 : \tilde{\mathcal{A}}_{[E]} \to H^2(X, \mathbb{C})$$

and $\mathcal{G}$ acts on $\tilde{\mathcal{A}}_{[E]}$ preserving the fiber of c_1.

Now, $\Omega \equiv \Omega^{[2]}$ can be naturally extended to a $\mathcal{G}$ invariant relative closed two form on $\tilde{\mathcal{A}}_{[E]}$.

Relative forms. Let us first define the relative notion which will also be used in later sections. Suppose that $X \to P \xrightarrow{\pi} M$ is a fiber bundle over M. We have a canonical exact sequence of vector bundles over P:

$$0 \longrightarrow T_{vert}P \longrightarrow TP \longrightarrow \pi^* TM \longrightarrow 0,$$

where $T_{vert}P$ denote the vertical tangent bundle (or the relative tangent bundle). In fact, we can regard this exact sequence as the definition of $T_{vert}P$. Recall that a k-form is a section of $\bigwedge^k(TP)$ over P.

Definition 2. *A* **relative** k**-form** *is a section of* $\bigwedge^k(T_vP)$ *over* P*.*

A **relative connection** D_A^{rel} *on a bundle* E *over* P *is a first order relative differential operator*

$$D_A^{rel} : \Gamma(P, E) \longrightarrow \Gamma(P, T^*_{vert}P \otimes E),$$

that is,

$$D_A^{rel}(fs)(v) = v(f)s + f(D_A^{rel}s)(v)$$

for any $f \in \mathcal{C}^\infty(P)$*,* $s \in \Gamma(P, E)$ *and* $v \in T_{vert}P$*.*

The **curvature** *of* D_A^{rel} *is* $F_A^{rel} = (D_A^{rel})^2$*.*

An ordinary k-form (resp. connection of E) on P induces a relative k-form (resp. relative connection of E). Notice that the curvature of a relative connection is a relative two form on P with $End(E)$-valued.

If we are given a splitting of TP as a direct sum of $T_{vert}P$ and π^*TM then a relative form can be extended to a form on P by assigning zero to π^*TM. For example, when P has a Riemannian metric or P is a product of X and M, then $TP \equiv T_{vert}P \oplus \pi^*TM$.

Extended moduli space. Now, Ω is a $\mathcal{G}$ invariant relative closed two form on $\tilde{\mathcal{A}}_{[E]}$. (The theory of deRham model of equivariant cohomology can be extended to the relative case.) Then Ω admits a $\mathcal{G}$-equivariant closed extension $\Omega + \Phi$. Just as before, for any $D_A \in \tilde{\mathcal{A}}_{[E]}$ and $\phi \in \Omega^0(X, End([E]))$, Φ is given by

$$\Phi(D_A)(\phi) = \int_X Tr\, e^{\frac{i}{2\pi}F_A}\phi\, \hat{A}(X).$$

For any non-vanishing top form μ on X, $\Phi^{-1}(\mathbb{C}\cdot\mu I_E)$ would be preserved by $\mathcal{G}$.

Definition 3. *$\tilde{\mathcal{M}}_\mu := \Phi^{-1}(\mathbb{C}\cdot\mu I_E)/\mathcal{G}$ is called the extended moduli space.*

Now, c_1 descends to give a map $c_1 : \tilde{\mathcal{M}}_\mu \to H^2(X, \mathbb{C})$.

To study the stability problem, we have to introduce polarization to specify a particular direction in $\tilde{\mathcal{A}}_{[E]}$.

Definition 4. *A polarization $\mathcal{P}$ is a line field in $\tilde{\mathcal{A}}_{[E]}$ which is invariant under affine translations. $\mathcal{P}$ is called regular if it is transverse to the fiber of c_1. Two regular polarizations are equivalent to each other if they induce the same line field in $H^2(X, \mathbb{C})$.*

It is easy to see that given a line bundle L and a connection D_L on it. They induce a regular polarization in $\tilde{\mathcal{A}}_{[E]}$. Different choices of connections always give equivalent polarizations.

Remark . The word polarizatioon we used here comes from algebraic geometry which means a choice of an ample line bundle up to numerical equivalency. This should not be confused with the notion of polarizations used in geometric quantization. We fix a line bundle L on X and choose a connection D_L on L. Let $\omega = \frac{i}{2\pi}F_L$ be its curvature. Then the top form $\mu = \frac{\omega^n}{n!}$ is non-vanishing if and only if ω is a symplectic form. We are interested in the ends of $\mathcal{M}_\mu$. It being non-empty near the $+\infty$ ends is related to stability properties of E as explained in the last section. However, now the picture we have here is more geometric. All the symplectic quotients $\mathcal{A}(E\otimes L^k)//\mathcal{G}$'s ($k \in \mathbb{Z}$) are embedded inside a bigger continuous family which is the symplectic quotient of the space of generalized connections. Moreover, "$\mathcal{M}_\mu|_{c_1=+\infty}$" should be regarded as the moduli space of Hermitian Einstein connections on E. Loosely speaking, asymptotically near $c_1 = +\infty$, $\Omega^{[2]}$ defines a Poisson structure on $\tilde{\mathcal{A}}_{[E]}$ and restricting to each fiber on c_1, it is a symplectic form.

5. Higer Level Chern–Simons Forms

In this section, we shall show that these $\Omega*$ deteremine Chern character and Chern–Simons forms of E. t the sma time, we define higher level generalzytions of ern–Simons forms. In order to simplify the diskussion below, we shall assume that the characteristic form of X is one. The author would like to thank the referee who pointed ot that this section is closely related to earlier work of Gefand and Wang. In fact, the author found out later that this is also related to certain previous work of BOtt.

Inside $\mathcal{A}$, there is a canonical affine foliation such that any two connections D_A and D'_A are in the dame leaf if and only if $D'_A = D_A + \alpha I_E$ for some one form α on X. Interm of an (integrable) subbundle of A, it is given by $\Omega^1(X) \subset \left(\Omega^1 \operatorname{End}(E)\right) = T_A\mathcal{A}$.

We restrict $\Omega^{[*]}$ to a differential form along the foliation (see relative forms in Sect. 3). We shall still denote this relative differential form an $(\mathcal{A})$ by $\Omega^{[*]}$. We are going to see that $\Omega^{[*]}$ is equivalent to the operation of taking the Chern character form of any connection.

For exaple, $\omega^{[2k]} \subset \Gamma\left(\mathcal{A}, \bigwedge 2kT^*\mathcal{A}\right)$ now gives a homomorphism:

$$\mathcal{A} \times \Omega^{2k} \hookrightarrow \gamma(\mathcal{A}, \bigwedge^{2k}(TcalA)) \xrightarrow{\Omega^{[2k]}} \mathbb{C}$$

By abuse of languages, we also denote this homomorphism by $\Omega^{[2k]}$. The we have

$$\Omega^{[2k]}(D_A, \gamma) = \int_x \mathrm{Tr}\, e^{\frac{i}{2\pi}F_A}\gamma.$$

If we regard $\Omega^{[2k]}$ as a map $\mathcal{A} \to \Omega^{2n-2k}(X)_{dist}$, the the image of D_A is just given by the $n-k^{\text{th}}$ Chern character form of D_A. Therefore, taking the Chern character form of any connection is equivalent to the restriction of the differential form $\Omega^{[*]}$ to the above foliation of $\mathcal{A}$.

To recover secondary (or higher level generalized) characteristic forms on E, we look at the chain complex of $\mathcal{A}$, $S_*(\mathcal{A})$ and $\Omega^{[*]}$ defines a chain homomorphism

$$\Omega^{[*]} : \mathcal{S}(\mathcal{A}) \to \Omega^{[*]}(X).$$

Recall that an element in $\mathcal{S}_l(\mathcal{A})$ (a singular chain) is a finite linear combination of singular l-simplexes σ's on $\mathcal{A}$ with complex coefficient. Where a singular l-simplex is a smooth map from the standard l-simplex Δ_l to $\mathcal{A}$, $\sigma : \Delta_l \to \mathcal{A}$. There is a boundary homomorphism $\partial : \mathcal{S}_{\updownarrow}(\mathcal{A}) \to \mathcal{S}_{l-1}(\mathcal{A})$ with $\partial^2 = 0$ and its homology is called the singular homology of $\mathcal{A}$, $H^*_{sin}(\mathcal{A})$. For completeness, let us recall the notations:

$$\Delta_L = \left\{ \sum_{j=0}^{l} t_j P_j \,|sum^l_{j=j} t_j = 1 \right\}.$$

where P_0 is the origin of $\mathbb{R}^\infty$ and P_i in the i^{th} standard basis vector in $\mathbb{R}^\infty$. To define ∂, it is sufficient to know its effect on a singular l-simplex $\sigma : \Delta_l \to \mathcal{A}$. We have:

$$\partial\sigma = \sum_{i=0}^{l} (-1)^i \sigma \circ \partial^i_l,$$

where $\partial^i_l : \delta_{l-i} \to \Delta_l$ is te i^{th} face map given by

$$\partial^i_l \left(\sum^{\prime} j = 0 t_j P_j \right) = \sum_{j=0}^{i=1} + \sum_{j=i+1}^{l} t_{j-1} P_j.$$

Now we will imitate the previous constructions to obtain characteristic forms for any singular chain in $\mathcal{A}$,

$$\omega^{[*]}_l : \mathcal{S}_l \to \Omega^*(X)_{dist}.$$

We no longer nedd to assume that X is even dimensional. It is enough for us to construct it for a single singular l-simplex $\sigma : \Delta_l \to \mathcal{A}$ ansd extend it by linearity.

For any such σ, we get a connection $\mathcal{A}_\sigma$ on $\pi_1^* E \to X \times \Delta_l$ by pulling back the universal connection $\mathbb{A}$ over $\mathbb{E} \to X \times A$. We denote its curvature by F_σ.

Definition 5. *We define* $\Omega_l^{[*]} : \mathcal{S}_l(A) \to \Omega^*(X)_{dist}$ *by*

$$\Omega_l^{[*]}(\sigma)(\gamma) = \int_{X \times \Delta_L} \mathrm{Tr}\, e^{\frac{1}{2}F_\alpha}\gamma.$$

Lemma 3. $\Omega_*^{[*]}$ *commute with the (co-)boundary operators of both complexes. That is,*

$$d \circ \Omega_l^{[*]} = \Omega_{l-1}^{[*]} \circ \partial.$$

This lemma follows from the Stokes's theorem. Next, we shall use $\Omega_l^{[*]}$ define secondary characteristic forms (Chern–Simons forms) and their higher level generalizations. We first define a map L which is similar to the homotopy operator $K : \mathcal{S} : *(\mathcal{A}) \to \mathcal{S}' * (\mathcal{A})$ defined by one construction. (since $[K, \partial] = 1$ or -1, this was used to prove that the singular homology of an affine space is trivial.) Now,

$$L_l : \prod^{l+1} \mathcal{A} \to \mathcal{S}_L(\mathcal{A}).$$

Roughly speaking, L_l is given by the complex polyhedron spanned by $l + 1$ points in $\mathcal{A}$. Notice that both K and L use strongly the affine structure on $\mathcal{A}$. Let us write $\sigma = L_l(D_{A_0}, \ldots, D_{A_l})$, then L is defined by as

$$\sigma\left(\sum_{i=0}^{l} t_j P_j\right) = \sum_{j=o}^{l} t_j D_{Aj}.$$

Here, $\sum_{j=o}^{l} t_j D_{Aj}$ is a well-defined element in $\mathcal{A}$ because of $\sum_{j=0}^{l} t_j = 1$.

We can extend L_l by linearity to the free Abelian group $\mathcal{A}_l$ generated by elements in $\prod^{l+1} \mathcal{A}$, that is,

$$L_l : \mathcal{A} := \mathbb{Z}\left[\prod^{l+1} \mathcal{A}\right] \to \mathcal{S}_l(\mathcal{A})$$

On $\mathcal{A}_*$, there is also a boundary operator $\partial_{\mathcal{A}} : \mathcal{A}_L \to \mathcal{A}_{l-1}$ defined by

$$\partial_{\mathcal{A}}(A_0, \ldots, A : l) = \sum_{i=0}^{l}(-1)^i(A_0, \ldots, \hat{A}_i, \ldots, A_l)$$

on any generator $(A_0, \ldots, A_l)$ in $\mathcal{A}_L$. The proof of the following lemma is straightforward.

Lemma 4. *(1)* $\partial_{\mathcal{A}}^2 ==$*,*
(2)$L \circ \partial_{\mathcal{A}} = \partial \circ L$*.*

Now we can define a higher Chern–Simons form as follow:

Definition 6. *The higher Chern–Simons is the linear homomorphism*

$$ch(E) = \Omega_*^{[*]} \circ L_* : \mathcal{A}_* \to \Omega^*(X)_{dist}$$

such that on any generator $(A_0, \ldots, A_l)$ *in* $\mathcal{A}_l$*, it is given by*

$$ch(E; A_0, \ldots, A : l) = \Omega_L^{[*]} circ L_L(A_0, \ldots, A_l).$$

In fact, $ch(E; A_0, \ldots, A_l)$ i always a smooth diffrential form on X. Noticce that $ch(E; A_=)$ is just the ordinary Chern character form of A_0 and $ch(E; A_0, A_1)$ is the ordinary Chern–Simons form of the pair A_0 and A_1.

Proposition 3. $ch(E) \circ \partial_{\mathcal{A}} = d \circ ch(E)$.

This proposition follows from the previous two lemmas. When $l = 0$, this proposition implies that the ordinary Chern character form is always a closed differential form. When $L = 1$, the proposition says that

$$ch/E; A_1) - ch(E; A_0) = d?, ch(E; , A_0, A_1)$$

, which is the most important property of Chern–Simons forms as the image under d of a canonical diffferential form (by the proposition for $l = 2$:

$$ch(E; A_1, A_2) - ch(E; A_0, A_2) + ch(E; A_0, A_1) = d\, ch(E; A_0, A_1, A_2).$$

Using Stokes' theorem we have

$$\int_X ch(E; A_1, A_2) - \int_X ch(E; A_0, A_2) + \int_X ch(E; A_0, A_1) = 0,$$

which is useful when we use $\int_X ch(E; A_1, A_2)$ as an action functional. For instance, the Chern–Simons theory over a three manifold uses such a functional and has important impacts on knot theory and three dimensional topology in recent years.

These $ch(E)$'s have another symmetry property. The symmetry group of l+1 elements $\mathcal{S}_{l+1}$, acts on $\mathcal{A}_l$ and we have the following proposition.

Proposition 4. *For any $s \in S_{l+1}$ and $\alpha \in \mathcal{A}_l$, we have*

$$ch(E; s \cdot \alpha) = (-1)^{|s|} ch(E; \alpha).$$

The proof of this proposition is left to the readers. When we look at the case for $l = 1$, we get

$$ch(E; A_0, A_1) = -ch(E; A_1, A_0),$$

which is a well-known property for Chern–Simons forms.

Hence, we have finished constructing higher Chern–Simons forms of E using $\omega^{[*]}$'s.

6. Equivariant Extensions of $\Omega^{[*]}$

In the last section, we analyze $\Omega^{[2]}(= \Omega)$ and its $\mathcal{G}$-moment map Φ. We showed that how they are related to the Hermitian Einstein metric and stability when X is a symplectic manifold. In this section, we first generalize the moment map construction to all $\Omega^{[*]}$'s and study their equivariant extensions $\Omega^{[*,*]}$.

Let us first review some basic materials about equivariant cohomology. (See, for example [A+B] [M+Q] for details.) We shall decribe the Cartan model here. When M admits a G (compact connected Lie Group) action, then the equivariant cohomology of M is defined as

$$H^*_G(M) = H^*(EG \times_G M),$$

where EG is the Universal space for G. It can be computed as the cohomology of the following differential complex on $(\Omega^*(M) \otimes Symm^*(Lie\, G)^*)^G$. Choose any base ϕ_i's of $Lie\ G$ and let ϕ^i's be its dual base. For any $\alpha \otimes \psi \in \Omega^*(M) \otimes Symm^*(Lie\ G)^*$, we define the differential D to be

$$D(\alpha \otimes \psi) = d\alpha \otimes \psi - \sum_i \iota_{\phi_i} \alpha \otimes (\phi_i \psi).$$

It is not difficult to check that D^2 vanishes on G invariant elements in $\Omega^*(M) \otimes Symm^*(Lie\ \ G)^*$. Therefore, it defines a differential complex on $(\Omega^*(M) \otimes Symm^*(Lie\ \ G)^*)^G$ whose cohomology can be proved to be $H^*_G(M)$. Although, in our situation, the group $\mathcal{G}$ is not a compact finite dimensional Lie Group, we shall still consider the same complex and call it the equivariant complex. Elements that are D-closed (-exact) are called equivariant closed (exact) forms.

Equivariant extensions of $\Omega^{[*]}$. Now, we want to extend each $\Omega^{[2k]}$ as a $\mathcal{G}$ equivariant closed form on $\mathcal{A}$ for any k. That is, we should have $\sum_{i+j=k} \Omega^{[2i,2j]}$, where

$$\Omega^{[2i,2j]} \in \Omega^{2i}(\mathcal{A},\ Symm^j(Lie\ \mathcal{G})^*)^{\mathcal{G}},$$

such that $\Omega^{[2k,0]} = \Omega^{[2k]}$ and for any $\phi \in Lie\ \mathcal{G} = \Omega^0(X, \text{End}\,(E))$, we have

$$\iota_\phi \Omega^{[2i,2j]} = d(\Omega^{[2i-2,2j+2]}(\phi)).$$

Theorem 1.

$$\Omega^{[2i,2j]}(D_A)(B_1, \ldots, B_{2i})(\phi_1, \ldots, \phi_j)$$
$$= \int_X Tr\ [e^{\frac{i}{2\pi}F_A} B_1 \cdots B_{2i} \phi_1 \cdots \phi_j]_{sym} \hat{A}(X),$$

where $D_A \in \mathcal{A}$, $B \in T_A\mathcal{A} = \Omega^1(X, \text{End}\,(E))$, $\phi \in Lie\ \mathcal{G} = \Omega^0(X, \text{End}\,(E))$.

Proof of Theorem. Before proving the theorem, let us first explain some notations. $\Omega^{[2i,2j]}$ is a $2i$-form on $\mathcal{A}$ valued in the j^{th} symmetric power of the dual of Lie algebra of $\mathcal{G}$ and it is also $\mathcal{G}$-invariant. At a point $D_A \in \mathcal{A}$ and $2i$ tangent vector B's at D_A, $\Omega^{[2i,2j]}$ gives an element in $Symm^j(Lie\ \mathcal{G})^*$. Evaluating it on the j element of $Lie\mathcal{G}$, then it gives us a number which we claimed to be given by the above formula.

First, $\mathcal{G}$ invariancy of $\Omega^{[2i,2j]}$ is clear from the formula. $\Omega^{[2k,0]} = \Omega^{[2k]}$ can also be seen directly. It remains to verify that

$$\sum_{l=0}^{j} \iota_{\phi_l} \Omega^{[2i,2j]}(\phi_0, \ldots, \hat{\phi}_l, \ldots, \phi_j) = d(\Omega^{[2i-2,2j+2]}(\phi_0, \ldots, \phi_j)),$$

where each $\phi_l \in Lie\ \mathcal{G} = \Omega^0(X, \text{End}\,(E))$. Now, we take any $B_1, \ldots, B_{2i-1} \in T_A\mathcal{A} = \Omega^1(X, \text{End}\,(E))$ and extend it to vector fields over all of $\mathcal{A}$ by parallel transport. Then $[B_i, B_j]$ is always a zero vector field on $\mathcal{A}$. (Here, [,] means the bracket of two vector fields on the space $\mathcal{A}$.) Therefore,

$$d(\Omega^{[2i-2,2j+2]}(\phi_0, \ldots, \phi_j))(B_1, \ldots, B_{2i-1})$$
$$= \sum_{l=1}^{2i-1} (-1)^{l+1} B_l(\Omega^{[2i-2,2j+2]}(\phi_0, \ldots, \phi_j)(B_1, \ldots, \hat{B}_l, \ldots, B_{2i-1}))$$
$$= \sum_{l=1}^{2i-1} (-1)^{l+1} \int_X Tr[e^{\frac{i}{2\pi}F_A}(D_A B_l) B_1 \cdot \cdot \hat{B}_l \cdot \cdot B_{2i-1} \phi_0 \cdot \cdot \phi_j]_{sym} \hat{A}(X)$$

On the other hand,

$$\begin{aligned}&\iota_{\phi_l}\Omega^{[2i,2j]}(\phi_0,\ldots,\hat{\phi}_l,\ldots,\phi_j)(B_1,\ldots,B_{2i-1})\\ &= \int_X Tr[e^{\frac{i}{2\pi}F_A}(D_A\phi_l)B_1\cdot\cdot B_{2i-1}\phi_0\cdot\cdot\hat{\phi}_l\cdot\cdot\phi_j]_{sym}\hat{A}(X)\end{aligned}$$

Hence,

$$\begin{aligned}&\Big(\sum_{l=0}^{j}\iota_{\phi_l}\Omega^{[2i,2j]}(\phi_0,\ldots,\hat{\phi}_l,\ldots,\phi_j)-d(\Omega^{[2i-2,2j+2]}(\phi_0,\ldots,\phi_j))\Big)(B_1,\ldots,B_{2i-1})\\ &= \int_X Tr[e^{\frac{i}{2\pi}F_A}D_A(B_1\cdot\cdot B_{2i-1}\phi_0\cdot\cdot\phi_j)]_{sym}\hat{A}(X)\\ &= \int_X d\,Tr[e^{\frac{i}{2\pi}F_A}B_1\cdot\cdot B_{2i-1}\phi_0\cdot\cdot\phi_j]_{sym}\hat{A}(X)\\ &= 0\end{aligned}$$

by the Stokes's theorem. So we have proved the theorem. □

In particular, when we look at $\Omega^{[2]}=\Omega_0$, we recover the moment map Φ_0 as $\Omega^{[0,2]}$. The extension of the total $\Omega^{[*]}$ as $\mathcal{G}$ equivariant closed form on $\mathcal{A}$ would be $\sum_{i+j\leq n}\Omega^{[2i,2j]}$ since the highest degree form in $\Omega^{[*]}$ is of degree $2n$. We can relax the restriction $i+j\leq n$ and obtain an equivariant closed form of arbitrart large total degree, we shall denote it by $\Omega^{[*,*]}$.

$\Omega^{[*,*]}$ *as equivariant pushforwards.* From previous section, we know that $\Omega^{[*]}$ occurred as the Chern character of the virtual bundle by the family local theorem. Recall that

$$\Omega^{[*]}=\int_X Tr\ e^{\frac{i}{2\pi}\mathbb{F}}\hat{A}(X).$$

and weextended $\Omega^{[*]}$ as a $\mathcal{G}$ equivariant closed form on $\mathcal{A}$. Now, we are going to extend the results in section two. So we want to extend $Tr\ e^{\frac{i}{2\pi}\mathbb{F}}\hat{A}(X)$ to a $\mathcal{G}$ equivariant closed form on $X\times\mathcal{A}$ such that we get $\Omega^{[*,*]}$ by integrating it over X.

Let us first consider an example where we want to find an element in $\Omega^{2n}(X\times\mathcal{A},(Lie\ \mathcal{G})^*)^{\mathcal{G}}$ such that it gives us ϕ by integrating it over X. This is clear that this is given by the $2n$-form $Tr[e^{\frac{i}{2\pi}F_A}\phi]^{(2n)}(x)$ at a point $(x,D_A)\in X\times\mathcal{A}$ when evaluating it on $\phi\in Lie\ \mathcal{G}=\Omega^0(X,\mathrm{End}\,(E))$. To generalize it to other $\Omega^{[*,*]}$'s, we introduce Δ and such that $Tr\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\hat{A}(X)$ would be our $\mathcal{G}$ equivariant form on $X\times\mathcal{A}$.

Definition 7. *Let* $\Delta\in\Omega^0(X\times\mathcal{A},End(\mathbb{E}))(Lie\ \mathcal{G})^*$ *be given by*

$$\Delta(x,D_A)(\phi)=\phi_x,$$

where $(x,D_A)\in X\times\mathcal{A}$ *and* $\phi\in Lie\ \mathcal{G}=\Omega^0(X,\mathrm{End}\,(E))$.

By evaluating Δ on ϕ at (x,D_A), we should arrive at an element in $(End\mathbb{E})(x,A)=EndE_x$, and Δ is defined such that this element is just given by ϕ at x. Therefore, Δ is an tautological element and independent of the variable $D_A\in\mathcal{A}$. Put it another way, Δ is the pullback from the identity element in

$$\Omega^0(X,\mathrm{End}\,(E))(Lie\ \mathcal{G})^*\equiv Map(\Omega^0(X,\mathrm{End}\,(E)),\Omega^0(X,\mathrm{End}\,(E))).$$

Now we consider $Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\hat{A}(X)$. In the exponential sums, we multiple forms on $X \times \mathcal{A}$ by exterior product and elements in $(Lie\ \mathcal{G})^*$ by symmetric product. Therefore, we have

$$Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\hat{A}(X) \in \Omega^*(X \times \mathcal{A}, End(\mathbb{E}))(Symm^*(Lie\ \mathcal{G})^*).$$

Theorem 2. *$Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\hat{A}(X)$ is a $\mathcal{G}$ equivariant closed form on $X \times \mathcal{A}$.*

Proof of Theorem. To prove the theorem, we can forget the $\hat{A}(X)$ term. It is rather straight forward to check that $Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}$ is $\mathcal{G}$ equivariant. To prove that it is equivariantly closed, we choose any $\phi \in Lie\ \mathcal{G} = \Omega^0(X, \mathrm{End}\,(E))$. Then, by Bianchi identity $\mathbb{D}\mathbb{F} = 0$, we have

$$d(Tr\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}(\phi)) = Tr\ [e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\mathbb{D}(\Delta\phi)]_{sym},$$

where $\mathbb{D}$ is the universal connection on $\mathbb{E}$.

On the other hand, we have

$$\iota_\phi Tr\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta} = Tr\ [e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\iota_\phi\mathbb{F}]_{sym}.$$

Therefore, in order to prove that $Tr\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}$ is equivariantly closed, it is enough to check that

$$\iota_\phi\mathbb{F} = \mathbb{D}(\Delta\phi).$$

At a point $(x, D_A) \in X \times \mathcal{A}$, ϕ induces the vector field $(0, D_A\phi) \in T_{(x,D_A)}(X \times \mathcal{A}) = T_xX \times T_{D_A}\mathcal{A} = T_xX \times \Omega^1(X, \mathrm{End}\,(E))$. If we choose any other vector $(v, B) \in T_{(x,D_A)}(X \times \mathcal{A})$, then

$$\begin{aligned}
&\iota_\phi\mathbb{F}(x, D_A)(v, B)\\
&= \mathbb{F}(x, D_A)((v, B), (0, D_A\phi))\\
&= (\mathbb{F}^{2,0} + \mathbb{F}^{1,1})(x, D_A)((v, B), (0, D_A\phi))\\
&= F_A(x)(v, 0) + (D_A\phi)(x, v)\\
&= (D_A\phi)(x, v).
\end{aligned}$$

For $\mathbb{D}(\Delta\phi)$, we have

$$\begin{aligned}
&\mathbb{D}(\Delta\phi)(x, A)(v, B)\\
&= (D_{A,x}\phi)(x)\\
&= (D_A\phi)(x, v).
\end{aligned}$$

Therefore, we have $\iota_\phi\mathbb{F} = \mathbb{D}(\Delta\phi)$ and $Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}$ is a $\mathcal{G}$ equivariantly closed form on $\mathcal{A}$. $\square$

Next, we shall show that the pushforward of this form gives $\Omega^{[*,*]}$ on $\mathcal{A}$ which generalizes $\int_X Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}}\hat{A}(X) = \Omega^{[*]}$.

Theorem 3.

$$\int_X Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\hat{A}(X) = \Omega^{[*,*]}.$$

Proof of Theorem. For simplicity, we shall omit some factors of the power of $\frac{i}{2\pi}$ in the calculation below. We consider the component of $\int_X Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}$ in $\Omega^{2i}(\mathcal{A}, Symm^j(Lie\ \mathcal{G})^*)$. We evaluate it on 2i tangent vectors B's on $\mathcal{A}$ and j elements ϕ's in $Lie\mathcal{G}$ at a point $A \in \mathcal{A}$,

$$
\begin{aligned}
&\Big(\int_X Tr_{\mathbb{E}}\ e^{\frac{i}{2\pi}\mathbb{F}+\Delta}\hat{A}(X)\Big)(D_A)(B_1,\ldots,B_{2i})(\phi_1,\ldots,\phi_j)\\
&= \int_X Tr\ e^{(\frac{i}{2\pi}(\mathbb{F}^{2,0}+\mathbb{F}^{1,1})+\Delta)(x,D_A)}(B_1,\ldots,B_{2i})(\phi_1,\ldots,\phi_j)\hat{A}(X)\\
&= \int_X Tr\ [e^{\frac{i}{2\pi}F_A}\frac{(\mathbb{F}^{1,1})^{2k}}{(2k)!}(B_1,\ldots,B_{2i})\frac{\Delta^{2j}(x,D_A)}{(2j)!}(\phi_1,\ldots,\phi_j)]_{sym}\hat{A}(X)\\
&= \int_X Tr\ [e^{\frac{i}{2\pi}F_A}B_1\cdot\cdot B_{2i}\phi_1\cdot\cdot\phi_j]_{sym}\hat{A}(X)\\
&= \Omega^{[2i,2j]}(D_A)(B_1,\ldots,B_{2i})(\phi_1,\ldots,\phi_j).
\end{aligned}
$$

Hence the theorem. $\square$

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