Bando Futaki Invariants and Kähler Einstein metric

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ABSTRACT. We show that on a Kähler Einstein manifold, existence of almost Kähler Einstein metrics if and only if Futaki Bando invariants are all zero. We also discuss their relationships with the supertrace of heat kernel.

CONTENTS

1. INTRODUCTION

Narasiham and Seshadri, Donaldson, Uhlenbeck and Yau proved that an irreducible holomorphic vector bundle E over a Kähler manifold X is Mumford stable if and only if there exists a Hermitian Einstein metric. In order to uncover stability from geometric invariant theory point of view, we need to study Gieseker stability (Mumford stability is only a linearized version of Gieseker stability in some suitable sense.)

In [Leung], the author discovered the following equation

$$\left[e^{k\omega+F}\ Td_X(\omega)\right]^{(n,n)} = \chi_k \tfrac{\omega^n}{n!} I_E$$

which explains Gieseker stability in term of differential geometry. Solutions to the above equations are called almost Hermitian Einstein metrics.

For Kähler Einstein manifold, we shall study a similar equation:

$$\boxed{\left[e^{k\omega}Td_X(\omega)\right]^{(n,n)}=\chi\left(X,O_X(L^k)\right)\tfrac{\omega^n}{n!}}$$

As we shall see in later section that this expression is related to the super-trace of heat kernel that appeared in the Atiyah-Singer local index theorem.

The main result of this paper is to prove that the existence of solution to this equation for large value of k is equivalent to vanishing of certain Bando Futaki invariants.

More generally, one can look at equation of the following form: For any characteristic class $A[X] = A_0[X] + A_2[X] + ... + A_{2n}[X]$ with $A_l[X] \in H^{2l}(X,\mathbb{R})$ of X

satisfying (i) $A_0[X] \neq 0$ and (ii) $A_1[X] = Cc_1(M)$ for some nonzero constant C. We consider the following equation:

$$\left[e^{k\omega}A(X,\omega)\right]^{(n,n)} = C_k \frac{\omega^n}{n!}$$

where $A(X,\omega)$ being the characteristic form of X representing $A[X]$ defined using ω via the Chern Weil theory and C_k is a suitable constant. Examples of such $A[X]$ include the Todd class Td_X and the $\widehat{A}-$ class of X, $\widehat{A}[X]$. Our methods in this paper can be used to deal with any equation of this form.

For simplicity, we choose $A[X]$ to be the full Chern character class of X, $ch(X)$. Then we have $\left[e^{k\omega}ch(X,\omega)\right]^{(n,n)} = Tr\left(k\omega + \frac{i}{2\pi}F\right)^n/n!$ and we call the corresponding equation the almost Kähler Einstein equation:

$$\boxed{\mathrm{Tr}\left(k\omega + \frac{i}{2\pi}F\right)^n = C_k \frac{\omega^n}{n!}}$$

or equivalently

$$\left(\omega + \frac{i\eta}{2\pi}F\right)^n = \chi_\eta I_{T_X} \frac{\omega^n}{n!}$$

where $\eta = \frac{1}{k}$.

The goal is to understand its relation with stability of the manifold X itself. However, in this case, there are new obstructions to the existence of solutions which comes up even when X carries a Kähler Einstein metric. We identify those obstructions precisely as Bando Futaki invariants. Recently, Tian has some important progress on this problem of relating manifold stability and existence of canonical metric.

To begin with, we suppose that X is a compact Kähler manifold of dimension n whose first Chern class has a definite sign. We shall assume the Kähler class is proportional to the first Chern class. The Kähler Einstein equation of X is as follow:

$$Rc_{\alpha\bar{\beta}} = Cg_{\alpha\bar{\beta}}$$

where $Rc_{\alpha\bar{\beta}}$ is the Ricci curvature form and the constant C (after the metric being normalized) is $1, 0$ or -1 according to the sign of $c_1(X)$. We can also rewrite this equation in an equivalent form:

$$\frac{i}{2\pi}F \wedge \frac{\omega^{n-1}}{(n-1)!} = C\frac{\omega^n}{n!} I_{T_X}$$

where $F \in \Omega^{1,1}(X, End(T_X))$ is the full curvature tensor of X with respect to the Kähler form ω. Motivated from [Leung]. we look at the following perturbed equation:

$$\left(\omega + \frac{i\eta}{2\pi}F\right)^n = \chi_\eta I_{T_X} \frac{\omega^n}{n!}$$

As η goes to zero, then we will recover the Kähler Einstein equation from these equations.

We would be interested in elliptic solutions of these equations for small positive η. Here, χ_η is a topological constant given by

$$\chi_\eta = \frac{1}{n \cdot vol\,(X)} \sum_{j=0}^{n} \int_X \eta^j ch_j\,(X) \wedge \tfrac{\Omega^{n-j}}{(n-j)!}$$

where $ch_j\,(X)$ is the j^{th} Chern character of X and Ω is the Kähler class.

Because of Kählerian, to solve the Kähler Einstein equation, we only need to solve the trace of the equation. To be precise, it is the equation of constant scalar curvature:

$$Tr\ F \wedge \omega^{n-1} = C\omega^n$$

Similarly, we will be interested to the trace of the perturbed equation, that is

$$Tr\left(\omega + \tfrac{i\eta}{2\pi}F\right)^n \quad = \chi_\eta \tfrac{\omega^n}{(n-1)!}$$

Solutions to this equation will be called *almost Kähler Einstein metrics.*

This is a fully non-linear *fourth* order equation in the Kähler potential. To solve it in general would be very difficult. In this paper, we shall show that the existence of solutions to these equations is closely related to the vanishing of Bando-Futaki invariants.

2. BANDO FUTAKI INVARIANTS

In [Futaki], Futaki introduced an invariant on any Kähler manifold with positive first Chern class (that is Fano manifolds). This invariant is an obstruction for the existence of Kähler Einstein metric. By viewing the Einstein condition as harmonicity of the first Chern form, Bando [Bando] generalized Futaki invariant to obstruction for harmonicity of higher Chern forms of X.

Let ω be a Kähler form in the (given) Kähler class Ω. Let $ch_j\,(\omega) = Tr\ \left(\frac{i}{2\pi}F\right)^j$ denotes the j^{th} Chern character form of X, where F is the curvature tensor of X with respect to the Kähler metric defined by ω. We denote the harmonic part of $ch_j\,(\omega)$ by $Hch_j\,(\omega)$ as in the Hodge decomposition. Since both $ch_j\,(\omega)$ and $Hch_j\,(\omega)$ are in the same cohomology class, there exists a (real) form $b_j\,(\omega)$ such that

$$ch_j\,(\omega) - Hch_j\,(\omega) = \tfrac{i}{2\pi}\partial\bar{\partial} b_j\,(\omega)$$

Such a $b_j\,(\omega)$ is unique up to addition of a $\partial\bar{\partial}$-closed form.

Definition 1 [Bando-Futaki Invariants]**.** $f_j : H^0\,(X,\ T_X) \to \mathbb{C}$

$$f_j\,(V) = \int_X \mathcal{L}_V b_j\,(\omega) \wedge \omega^{n-j+1}$$

Each f_j is a well-defined Lie algebra homomorphism independent of the particular choice of the Kähler metric ω in the Kähler class Ω. Moreover, if $ch_j(\omega)$ is harmonic for some Kähler form $\omega \in \Omega$, then f_j will be identically zero.

In general, it is rather rare for higher Chern character forms to be harmonic because the corresponding equations are overdetermined. However, these invariants are useful for our almost Kähler Einstein equation as indicated in our theorem below:

Theorem 1. *Let X be a compact Kähler Einstein manifold. If $c_1(X)$ is non-positive, then*

$$Tr\ \left(\omega I_{T_X} + \eta \tfrac{i}{2\pi} F\right)^n \ = \chi_\eta \tfrac{\omega^n}{(n-1)!}$$

have elliptic solutions for all sufficiently small positive η.

If $c_1(X)$ is positive, then it has elliptic solutions for all sufficiently small positive η if and only if all Bando-Futaki invariants f_j 's are zero homomorphisms.

3. Proof of the theorem

Since X admits Kähler Einstein, which is the same as the existence of solution of our equation for $\eta = 0$ after rewriting the equation from

$$Tr\ \left(\omega + \tfrac{i\eta}{2\pi} F\right)^n \ = \chi_\eta \tfrac{\omega^n}{(n-1)!}$$

to

$$Tr\ \left(F \wedge \tfrac{\omega^{n-1}}{(n-1)!} + \eta\ F^2 \wedge \tfrac{\omega^{n-2}}{(n-2)!}\ + ...\right)\ = \left(\tfrac{\chi_\eta - 1}{\eta}\right) \tfrac{\omega^n}{(n-1)!}$$

First, we suppose that the first Chern class of X is non-positive. In this case, by Yau's solution [Yau] to Calabi conjecture, there exists a unique Kähler Einstein metric in Ω. A simple application of implicit function theorem on suitable Banach space will show that the above almost Kähler Einstein equation can be solved for sufficiently small positive η.

Fix any background metric $\omega_0 \in \Omega$. For simplicity, we assume that ω_0 is a Kähler Einstein metric. Let $\mathfrak{B}_+^{k+2,\alpha}$ be the Banach manifold of all $C^{k+2,\alpha}$ functions φ on X such that $\int_X \varphi = 0$ and $\omega_0 + i\partial\bar{\partial}\varphi$ is a positive (1,1) form. Also, let $\mathfrak{B}^{k-2,\alpha}$ be the Banach space of all $C^{k-2,\alpha}$ functions φ on X with $\int_X \varphi = 0$. We consider a Fréchet continuously differentiable map $\Phi_\eta : \mathfrak{B}_+^{k+2,\alpha} \to \mathfrak{B}^{k-2,\alpha}$ defined by

$$\Phi_\eta(\varphi) \tfrac{\omega^n}{n!} = \frac{1}{\eta}\left[Tr\ \left(\omega I_{T_X} + \eta \tfrac{i}{2\pi} F\right)^n \ - \chi_\eta \tfrac{\omega^n}{(n-1)!}\right]$$

where $\omega = \omega_0 + i\partial\bar{\partial}\varphi$ and F is the curvature of X defined by ω. Then, $\Phi_{\eta=0}(0) = 0$. The differential of $\Phi_{\eta=0}$ at $\varphi = 0$ is given by $d\Phi_{\eta=0}(0) : \mathfrak{B}^{k+2,\alpha} \to \mathfrak{B}^{k-2,\alpha}$

$$d\Phi_{\eta=0}(0)(\psi) = \triangle(-\triangle\psi + C\psi)$$

where $\triangle$ is the Laplacian operator.

Since C is either 0 or -1, $d\Phi_{\eta=0}(0)$ is invertible. Therefore, by implicit function theorem [G-T], $d\Phi_\eta(\varphi) = 0$ can be solved for small η.

From now on, we assume $c_1(X)$ is positive and $\Omega = c_1(X)$.

Second, we assume that our equation $Tr\left(\omega I_{T_X} + \eta\frac{i}{2\pi}F\right)^n = \chi_\eta \frac{\omega^n}{(n-1)!}$ have solutions for all sufficiently small η and we shall proves that Futaki-Bando invariants all vanish in this case. The proof of this part uses Bourguognon's arguments for Bando's results.

Lemma 2. *Under the above situation, we have*

$$\sum_{j=0}^{n} \eta^j f_j(V) = \int_X \psi_V(\omega)\ Tr\left(\omega I_{T_X} + \eta\tfrac{i}{2\pi}F\right)^n$$

where $\psi_V(\omega)$ is the unique function on X determined by $\int_X \psi_V(\omega)\ \frac{\omega^n}{n!} = 0$ and $\mathcal{L}_V\omega = \frac{i}{2\pi}\partial\bar{\partial}\psi_V(\omega)$.

From our assumption, $Tr\left(\omega I_{T_X} + \eta\frac{i}{2\pi}F\right)^n = \chi_\eta \frac{\omega^n}{(n-1)!}$, we have

$$\begin{aligned} & \int_X \psi_V(\omega)\, Tr\ \left(\omega I_{T_X} + \eta\tfrac{i}{2\pi}F\right)^n \\ = & \quad \chi_\eta \int_X \psi_V(\omega)\ \tfrac{\omega^n}{(n-1)!} \\ = & \quad 0 \end{aligned}$$

from the definition of $\psi_V(\omega)$. So $\sum_{j=0}^{n} \eta^j f_j(V) = 0$ for all small η. It implies that $f_j(V) = 0$ for all j and for all holomorphic vector field $V \in H^0(X, T_X)$.

Third, we are going to prove the converse. Suppose f_j 's are all zero and ω_0 is a Kähler Einstein metric of X. As in the first part of the proof, we have

$$d\Phi_{\eta=0}(0)(\psi) = \triangle(-\triangle\psi + \psi)$$

However, $d\Phi_{\eta=0}(0)$ is only invertible on the orthogonal complement to the eigenspace of $\triangle$ with eigenvalue one, which we call $\mathfrak{B}_-$. That is

$$\Pi \circ d\Phi_{\eta=0}(0) : \mathfrak{B}_-^{k+2,\alpha} \to \mathfrak{B}_-^{k-2,\alpha}$$

is an invertible linear operator. Here $\Pi : \mathfrak{B} \to \mathfrak{B}_-$ is the orthogonal projection operator. We apply the implicit function theorem on $\mathfrak{B}_-$,

$$\Pi\left(Tr\ \left(\omega I_{T_X} + \eta\tfrac{i}{2\pi}F\right)^n / \omega^n\right) = \frac{\chi_\eta}{(n-1)!}$$

can then be solved for all small η.

Let ω_η be an analytic path (in η) of Kähler metrics which solve the above equation and let

$$\varphi_\eta = Tr\ \left(\omega_\eta I_{T_X} + \eta \tfrac{i}{2\pi} F_\eta\right)^n / \omega^n - \tfrac{\chi_\eta}{(n-1)!}$$

Suppose that $\frac{\partial \varphi_\eta}{\partial \eta}|_{\eta=0} \neq 0$, then

$$0 = \sum_{j=0}^{n} \eta^j f_j\left(V\right) = \int_X \psi_V\left(\omega\right) \varphi_\eta \omega_\eta^n$$

by our assumption on vanishing of Bando-Futaki invariants. This implies that

$$0 = \int_X \psi_V\left(\omega_0\right) \left(\tfrac{\partial \varphi_\eta}{\partial \eta}|_{\eta=0}\right) \omega_0^n$$

Lemma 3. *Under the above situation, we have*

$$-\triangle\,\psi_V\left(\omega_0\right) + \psi_V\left(\omega_0\right) = 0$$

and $V \mapsto \psi_V\left(\omega_0\right)$ *is an isomorphism from* $H^0\left(X, T_X\right)$ *to the eigenspace* $H_{\lambda=1}$*of* $\triangle$ *with eigenvalue one.*

The previous equation shows that $\frac{\partial \varphi_\eta}{\partial \eta}|_{\eta=0}$ is perpendicular to every element in $H_{\lambda=1}$. One the other hand, $\frac{\partial \varphi_\eta}{\partial \eta}|_{\eta=0} \in H_{\lambda=1}$. Therefore, $\frac{\partial \varphi_\eta}{\partial \eta}|_{\eta=0}$ is the zero element which contradicts our assumption.

In a similar fashion, we obtain that $\frac{\partial^r \varphi_\eta}{\partial \eta^r}|_{\eta=0} = 0$ for all $r > 0$. This implies that $\varphi_\eta \equiv \varphi_0 = 0$. That is,

$$Tr\ \left(\omega I_{T_X} + \eta \tfrac{i}{2\pi} F\right)^n = \chi_\eta \tfrac{\omega^n}{(n-1)!}$$

can be solved for all sufficiently small positive η. Hence, we have our theorem.

Remark 1. *Most of the materials presented here can be generalized to Kähler metric with constant scalar curvature easily.*

4. RELATION TO THE SUPERTRACE OF HEAT KERNEL

Now we come back and look at equations corresponding to $A\left[X\right] = Td_X$:

$$\boxed{\left[e^{k\omega} Td_X\left(\omega\right)\right]^{(n.n)} = \chi\left(X, O_X(L^k)\right) \tfrac{\omega^n}{n!}}$$

where $Td_X\left(\omega\right)$ is the Todd form of X written in terms of the Kähler form ω via Chern Weil theory and $\chi\left(X, O_X(L^k)\right)$ is the Hilbert polynomial of X with respect certain polarization L.

As in the almost Kähler Einstein equation case, the limiting equation as k goes to infinity becomes the Kähler Einstein equation $Rc_{\alpha\bar{\beta}} = Cg_{\alpha\bar{\beta}}$ since the first Todd class is represented by the Ricci form of ω. Moreover, over a Kähler Einstein manifold X, existence of solution to this equation for large value of k is equivalent to vanishing of certain Bando Futaki invariants (as readers can easily verify this using methods from last section).

Let us now recall the heat kernel of the operator $\bar{\partial}$ on $\Omega^{0,*}\left(X, L^k\right)$ and the local Atiyah-Singer index theorem. The Dirac operator for Kähler manifolds can be identified as $D = \sqrt{2}\left(\bar{\partial} + \bar{\partial}^*\right)$ acting on $\Omega^{0,*}\left(X, L^k\right)$. Let $k_t(x,x)$ denote the restriction of heat kernel of D^2 to the diagonal.. Then by the local Atiyah Singer index theorem (see [BGV, BGV] for more details), the limit to the supertrace of k_t exists as t goes to zero. More precisely one has the formula:

$$\lim_{t\to 0^+} Tr_s k_t(x,x)\,|dx| = \left[e^{k\omega} Td_X(\omega)\right]^{(n.n)}$$

Combining with our earlier discussions, we know that for Kähler Einstein manifold with vanishing Bando Futaki invariants, we can find Kähler metric (depending on k) on it such that for large enough k, the supertrace of heat kernel of Dirac operator on L^k will converge to a constant multiple of its volume form as t goes to zero.

More generally, we can look at the twisted Dirac operator D on $\Omega^{p,*}\left(X, L^k\right)$. We obtain the following theorem:

Theorem 4. *Let M be a compact Kähler Einstein manifold with polarization L. We denote the twisted Dirac operator acting on $\Omega^{p,*}\left(X, L^k\right)$ by D.*

Suppose that Bando Futaki invariants of M all vanish. Then for k sufficiently large, there exists a canonical Kähler metric ω on M such that the limit of supertrace of the heat kernel of D as t goes to zero will exist and equal to a constant multiple of the volume form. That is

$$\lim_{t\to 0^+} Tr_s k_t(x,x)\,|dx| = C\frac{\omega^n}{n!}$$

for some constant C depending on Todd class of M only.

The proof of the theorem are similar to the untwisted case. First we need to choose $A[X]$ to be $ch\left(\Lambda^p T_X^*\right) \wedge Td_X$. In the perturbation scheme that we used in earlier sections, we started with a Kähler Einstein metric on X. By applying the method we used in previous sections, we know that as long as the Bando Futaki invariants of M vanish, then we can find ω to solve the equation

$$\left[ch\left(\Lambda^p T_X^*, \omega\right)\ Td_X(\omega)\right]^{(n,n)} = \chi_k \frac{\omega^n}{n!}$$

for large enough value of k. Then we can apply the local Atiyah Singer index theorem and concludes our results.

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