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Seiberg Witten invariants and uniformizations

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1 Introduction

Recently, Seiberg and Witten introduced new coupled equations on any compact smooth four dimensional oriented manifold X. In [W], Witten studied the space of solutions of these equations and defined differentiable invariants for X when $b_2^+(X) \geqq 2$. He also identifies these invariants when X is a Kähler manifold and proves a vanishing theorem for manifolds admitting Riemannian metric with positive scalar curvature. In [L], LeBrun studied these equations and generalized the (four dimensional) Miyaoka-Yau inequality to Einstein manifolds with non-vanishing Seiberg Witten invariants. He gives a new characterization of Einstein four manifolds which can be uniformized by the complex two dimensional ball $B^2 = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 < 1\}$.

In this paper, we shall use the Seiberg Witten equations to study uniformization problems. In the first part, we assume that X is a Kähler surface and examine when X can be uniformized by the product of two unit disks, $D^1 \times D^1 = \{(z, w) \in \mathbb{C} | |z| < 1, \ |w| < 1\}$. We shall study the Seiberg Witten equations for the reversed orientation of X. Instead of changing the complex orientation, we will keep the orientation of X and change the equations to

$$\begin{cases} F_A^- = \zeta\bar{\zeta} \\ \not{D}_A \zeta = 0 \end{cases},$$

where D_A is a connection on a line bundle L and ζ is a section on S_L^-. We call these equations the negative Seiberg Witten equations. In section two, we will show that the invariant defined by these equations is non-trivial (in fact, the invariant is one) for surfaces which can be uniformized by product of two unit disks.

Then we study the negative Seiberg Witten invariants for algebraic surfaces with Kähler-Einstein metric of negative type. (By Yau's theorem, such a metric exists and is unique on any algebraic surface whose canonical line bundle is positive/ample). This in turn gives a characterization of surfaces which can be uniformized by the product of two disks. We show in section three,

Theorem 1. *If X is an algebraic surface with ample canonical line bundle and its negative Seiberg-Witten invariant is non-zero, then X has non-negative index, $\sigma(X) \geqq 0$.*

Moreover, the index of X is zero if and only if X is uniformized by the product of two unit discs.

Algebraic manifolds which can be uniformized by product of unit disks were investigated by Simpson [S] in the compact case and Yau [Y2] in the noncompact case. In those characterizations, we usually assume that the tangent bundle of X splits holomorphically into a direct sum of holomorphic line bundles. Instead of a holomorphic splitting, we assume the non-triviality of the negative Seiberg Witten invariants. This then implies the following (weaker version of) a result of Jost and Yau [J+Y]:

Corollary. *Let X be an algebraic surface with ample canonical line bundle. If X is diffeomorphic to an algebraic surface which can be uniformized by the product of two disks. Then X can be uniformized by the product of two disks. (B. Wong brought to my attention the conjugate complex structures)*

Remark. When both X and its complex conjugate admit complex structures, then by classification theory, their canonical line bundles must be numerically effective. If moreover, they are ample, then from the above result, we know that the universal cover of X must be the product of two unit disks.

The studies of the negative Seiberg Witten invariants are in some sense parallel to LeBrun's paper [L] where his main interest are manifolds which can be uniformized by the complex two ball.

In the second part of this paper, we shall study non-unitary $U(2,1)$ connections and their relation to Seiberg Witten equations. First, a Kähler Einstein metric of negative type is in particular a Hermitian Einstein metric on the tangent bundle TX. In fact, it induces a Hermitian Einstein Higgs structure on $TX \oplus \mathcal{O}_X$ which can be regarded as a $U(2,1)$ Einstein metric on $TX \oplus \mathcal{O}_X$. These ideas were introduced by Hitchin [Hi] on Riemann surfaces and generalized by Simpson [S] to higher dimensional X to study uniformization problems.

In our discussion here, X is always of four real dimensions. Going from TX to $TX \oplus \mathcal{O}_X$ gives us a bundle interpretation of the Chern number inequality

$$c_1^2(X) \leqq 3c_2(X)$$

on a Kähler Einstein manifold.

When X is a real four dimensional spin Einstein manifold, the bundle of negative spinors S_- is anti-self-dual with respect to the induced metric [AHS]. When X is also a Kähler manifold, then we have the identification $TX = S_-^{\mathbb{C}}$ or $TX = S_- \oplus K_X^{-1/2}$ when X is spin. We shall imitate the Kähler case and study $U(2,1)$ connection on $S_-^{\mathbb{C}} \oplus \mathcal{O}_X$. We will show that the perturbed anti-self-dual equation on $S_-^{\mathbb{C}} \oplus \mathcal{O}_X$ decouples into two sets of equations. One is the Einstein equation on X and the second one is precisely the Seiberg Witten equations.

From these, we can give a geometric interpretation of LeBrun's version [L] of the Miyaoka-Yau inequality for Einstein four-manifolds with non-trivial Seiberg-Witten invariants.

Theorem 2. (1) *If X is a four manifold with Riemannian metric g. Suppose that for some almost complex structure on X, there is a solution to the Seiberg Witten equation, (D_A, φ). Then $\mathbb{D}_{(A,\varphi)} = \begin{pmatrix} D_A & \frac{1}{\sqrt{6}}\bar{\varphi} \\ \frac{1}{\sqrt{6}}\bar{\varphi}^* & d \end{pmatrix}$ is a $U(2,1)$ connection on $E = S_-^{\mathbb{C}} \oplus \mathcal{O}_X$ with self-dual part of its curvature equals*

$$\mathbb{F}^+_{(A,\varphi)} = \begin{pmatrix} Rc^o - \frac{1}{3}F_A^+ \cdot I & 0 \\ 0 & -\frac{1}{3}F_A^+ \end{pmatrix}.$$

(2) *When X is an Einstein metric, then $\mathbb{D}_{(A,\varphi)}$ is a perturbed Anti-Self-Dual $U(2,1)$ connection on E. We have the Chern number inequality*

$$c_1^2(E) \leqq 3c_2(E)$$

which is equivalent to

$$3\sigma(X) \leqq \chi(X).$$

(3) *Moreover, equality holds if and only if the above $U(2,1)$ connection is projectively flat and X is a Kähler manifold which can be uniformized by $\mathbb{C}^2$ or the complex two ball B^2.*

In the last section, we study dimensional reduction for perturbed Anti-Self-Dual $U(2,1)$ equations. We show that Ω^2_- invariancy will decouple the equation into two sets of equations: Seiberg-Witten equations and Einstein equations. More precisely,

Theorem 3. *Let $\mathbb{D}$ be any $U(2,1)$ perturbed Anti-Self-Dual connection on $E = S_L^- \otimes L^{-1} \oplus \mathcal{O}_X$. Then $\mathbb{D}$ is Ω^2_- invariant if and only if the Riemannian metric is Einstein and the Seiberg-Witten equations are satisfied. That is,*

$$\begin{cases} Rc^o = 0 \\ F_A^+ = 6\varphi\bar{\varphi} \\ \not{D}\varphi = 0 \end{cases}$$

2 Seiberg-Witten invariants

The Seiberg-Witten invariants are introduced and studied by Witten [W]. Let L be a complex line bundle over (X, g) with $2\chi(X) + 3\sigma(X) = c_1(L)^2$ and $S_L^\pm$ be the corresponding complex spinor bundles over X with spin^c structure induced by L. Notice that the numerical condition on L implies that there exists an almost complex structure on X such that $c_1(L)$ equals to its first Chern class of X. (Locally, $S_L^\pm$ is isomorphic to $S_\pm \otimes L^{1/2}$. Even though $S_\pm$ and $L^{1/2}$ do not make sense separately, their tensor products do. See [L+M] for clarifications).

The Seiberg Witten equations are

$$\begin{cases} F_A^+ = \varphi\bar{\varphi} \\ \not{D}_A\varphi = 0 \end{cases},$$

where D_A is a connection on L and φ is a section on $S_L^{\pm}$. A solution for these equations is a pair (D_A, φ). When $2\chi(X) + 3\sigma(X) = c_1(L)^2$, then the expected dimension of the moduli space of solutions of the Seiberg Witten equations up to gauge transformations will be zero. (Here $\chi(X)$ and $\sigma(X)$ are the Euler characteristic and index of X). From now on, we shall only consider those cases where the formal dimension of the moduli space of solutions is zero.

When $b_2^+(X) \geqq 2$ or ($b_2^+(X) = 1$ and $2\chi(X) + 3\sigma(X) > 0$), then for a generic choice of metric g on X, the number of solutions (up to gauge) counted with sign will be a differentiable invariant for X. That is, it is independent of choices of g. This is called the Seiberg Witten invariant and it is denoted by $SW_L(X)$. For more details, readers can refer to following papers: [W], [K+M] and [L].

If we reverse the orientation of X, then the corresponding invariant will be called the negative Seiberg Witten invariant, $SW_L^-(X)$. Put another way, we can keep the original orientation but change the equations to

$$\begin{cases} F_A^- = \zeta\bar{\zeta} \\ \not{D}_A\zeta = 0 \end{cases},$$

where D_A is still a connection on L but ζ is a section on S_L^-.

It is observed by Witten that for a metric with positive scalar curvature $R > 0$, there is no solution to the Seiberg Witten equation. However, the story is very different for a metric with negative scalar curvature. In [L], LeBrun showed that for Kähler metrics with constant negative scalar curvature, $SW_K(X) = 1$ where K is the canonical line bundle for X.

Instead, we are going to study the negative Seiberg Witten invariant on Kähler manifold X. Suppose that X is covered by product of two unit discs $D^1 \times D^1$. Then (possibly, up to double covering) the holomorphic tangent bundle TX of X has a holomorphic splitting into a direct sum of two holomorphic line subbundles, $TX = L_1 \oplus L_2$. Let $L = L_1^{-1} \otimes L_2$, then we have

$$S_L^- = \mathcal{O}_X \oplus L,$$

where $\mathcal{O}_X$ is the trivial line bundle on X. For simplicity, if TX does not split holomorphically, we automatically replace X by its two-fold cover where its tangent bundle splits holomorphically.

Lemma. $2\chi(X) - 3\sigma(X) = -c_1(L)^2$.

Proof of Lemma.

$$\begin{aligned} &-c_1(L)^2 \\ =&-(-c_1(L_1) + c_1(L_2))^2 \\ =&-(c_1(L_1) + c_1(L_2))^2 + 4c_1(L_1) \cdot c_1(L_2) \\ =&-c_1(L_1 \otimes L_2)^2 + 4c_1(L_1) \cdot c_1(L_2)\,. \end{aligned}$$

Since $TX = L_1 \oplus L_2$, by Whitney's formula, we have

$$\begin{aligned} & -c_1(L)^2 \\ = & -c_1(X)^2 + 4c_2(X) \\ = & \ 2c_2(X) - (c_1(X)^2 - 2c_2(X)) \\ = & \ 2\chi(X) - 3\sigma(X)\,. \end{aligned}$$

Hence the lemma.

We want to solve equations

$$\begin{cases} F_A^- = \zeta\bar{\zeta} \\ \not{D}_A\zeta = 0 \end{cases},$$

where D_A is a connection on L and ζ is a section on $S_L^- = \mathcal{O}_X \oplus L$. By the Atiyah-Singer index theorem and the above lemma, the expected dimension of the moduli space of solutions is zero.

In fact, if we choose g to be the unique Kähler Einstein metric on X, then the Levi-Civita connection on TX induces a connection D_A on L. Let $\zeta = (1,0) \in \mathcal{O}_X \oplus L$, then (D_A, ζ) gives an irreducible solution to the negative Seiberg Witten equations. One can also show that this solution is transverse.

We are going to show that this is in fact the only solution (up to gauge transformations).

Let ω be the Kähler form associated to the Kähler Einstein metric on X. Then $\omega = \omega_1 + \omega_2$ where both ω_1 and ω_2 are covariantly constant and $\omega_i = -c_1(L_i)$. By using Weitzenböck formula for the Dirac operators

$$\not{D}_A^*\not{D}_A\zeta = \nabla_A^*\nabla_A\zeta + \frac{R}{4}\zeta + F_A^- \cdot \zeta$$

we have [W] a C^0 norm estimate for any solution (D_A, ζ) to the negative Seiberg-Witten equations:

$$|\zeta|^2 \leqq -R\,.$$

Here R is the scalar curvature which is -1 in our case and therefore $|\zeta|^2 \leqq 1$. Using the equation $F_A^- = \zeta\bar{\zeta}$, we have

$$\begin{aligned} \int_X |F_A^-|^2 \leqq \int_X |Rc_X|^2 &= \int_X |\omega_1 + \omega_2|^2 \\ &= \int_X |\omega_1 - \omega_2|^2\,. \end{aligned}$$

The last inequality holds since ω_1 and ω_2 are perpendicular forms (pointwisely). Now $\omega_1 - \omega_2$ is a parallel form, (in particular, it is harmonic) and the cohomology class it represented is $-c_1(L)$. Moreover, $\omega_1 - \omega_2$ is an anti-self-dual form, we have

$$\int_X |\omega_1 - \omega_2|^2 \leqq -2\pi^2 c_1(L)^2 + \frac{1}{2}\int_X |F_A|^2 = \int_X |F_A^-|^2$$

by the Chern-Weil formula.

Therefore, all inequality signs must be equal signs. By $|\zeta|^2 \equiv 1$ and the Weitzenböck formula, we have $\nabla_A \zeta = 0$. We also have $\frac{i}{2\pi} F_A = \omega_2 - \omega_2$. These implies $\zeta = (1,0) \in \mathcal{O}_X \oplus L$ and D_A is induced from the Levi-Civita connection from the Kähler Einstein metric. Hence, the solution to the negative Seiberg Witten equation is unique (up to gauge transformation). One can also show that this solution is transverse as in [L]. We therefore have the following theorem.

Theorem. *If X is an algebraic surface which can be uniformized by product of two discs. Then for the line bundle L as before, its negative Seiberg Witten invariant is* ± 1.

We should notice that the above arguments are very similar to LeBrun [L] where he studied the self-dual Seiberg Witten invariant and proved that

Theorem (LeBrun). *Let X be a Kähler surface with constant negative scalar curvature, then $SW_K(X) = 1$.*

3 Uniformization

In [Y], Yau proved that any algebraic surface X with ample canonical class admits a unique Kähler-Einstein metric whose scalar curvature is negative. A direct consequence of this is a Chern number inequality

$$c_1^2(X) \leqq 3c_2(X) .$$

Moreover, when the equality sign holds, X can be uniformized by the complex two dimensional ball $B^2 = SU(2,1)/U(2)$. In [L], LeBrun generalized this result to Einstein four manifold with non-zero Seiberg Witten invariant, $SW_K(X) \neq 0$.

In complex two dimension, there are only two Hermitian symmetric domains. One is $B^2 = SU(2,1)/U(2)$ and the other one is the product of two unit discs, $D^1 \times D^1 = SU(1,1) \times SU(1,1)/U(1) \times U(1)$. We will give a characterization of algebraic surfaces which can be uniformized by $D^1 \times D^1$ using negative Seiberg Witten invariants. The idea is for algebraic surfaces with ample canonical bundle which support solutions to the negative Seiberg Witten equation, then certain Chern number inequality must hold. Moreover, this Chern number inequality can be achieved only for those that can be uniformized by the product of two unit disks.

Notice that, by the Hirzebruch proportionality principle, any such surface has zero index, $\sigma(X) = 0$. Since $3\sigma(X) = c_1^2(X) - 2c_2(X)$, we have

$$c_1^2(X) = 2c_2(X) .$$

Theorem 1. *If X is an algebraic surface with ample canonical line bundle and its negative Seiberg Witten invariant is non-zero, then X has non-negative* *index, $\sigma(X) \geqq 0$, or equivalently,*

$$c_1^2(X) \geqq 2c_2(X)\,.$$

Moreover, the index of X is zero if and only if X can be uniformized by product of two unit discs $D^1 \times D^1$.

Proof of Theorem 1. By [Y], X admits an unique Kähler Einstein metric ω, $Rc_X = -\omega$. Suppose that L is a complex line bundle on X for which $SW_L^-(X) \neq 0$, then there exists solution (D_A, ζ) to the negative Seiberg Witten equations. As before, we have $|\zeta|^2 \leqq -R = 1$ by using the Weitzenböck formula. Using the equation, we get $|F_A^-|^2 \leqq R^2/8$. Because we assume the formal dimension of the moduli space of solutions to the Seiberg Witten equations is zero, we obtain $2\chi(X) - 3\sigma(X) + c_1(L)^2 = 0$ by the Atiyah Singer index theorem.

We therefore have

$$\begin{aligned} &2\chi(X) - 3\sigma(X) \\ &= -\,c_1(L)^2 \\ &= \frac{1}{4\pi^2}\int(|F_A^-|^2 - |F_A^+|^2) \\ &\leqq \frac{1}{4\pi^2}\int|F_A^-|^2 \\ &\leqq \frac{1}{32\pi^2}\int R^3 \\ &= c_1^2(X)\,. \end{aligned}$$

The last equality follows from X being Kähler Einstein. However, $c_1^2(X) = 3\sigma(X) + 2\chi(X)$, we get

$$\sigma(X) \geqq 0\,,$$

which is equivalent to

$$c_1^2(X) \geqq 2c_2(X)\,.$$

Now we suppose $\sigma(X) = 0$. We then have $F_A^+ = 0$ and $|\zeta|^2 = 1$. By the same argument as in the last section, we have $\nabla_A\zeta = 0$ and $\nabla_A F_A^- = 0$.

Using ζ, we can split S_L^- into a direct sum of two holomorphic subbundles for which one of them is a trivial holomorphic line bundle $\mathcal{O}_X$. This in turn decomposes TX into $L_1 \oplus L_2$ and L can be identified as $L^{-1} \otimes L_2$. ζ is the constant one section of $\mathcal{O}_X$ inside $S_L^- = \mathcal{O}_X \oplus L$.

Since F_A is the curvature for $L_1^{-1} \otimes L_2$ and Rc_X is the curvature for $L_1 \otimes L_2$, we get $\omega_1 = \frac{1}{2}(Rc_X - F_A)$ and $\omega_2 = \frac{1}{2}(Rc_X + F_A)$ curvatures for line bundles L_1 and L_2 which are both parallel because $Rc_X = -\omega$ and $\nabla_A F_A = 0$. So, $\omega = \omega_1 + \omega_2$. By a standard holonomy argument, the universal cover of X is the product of two unit discs, $D^1 \times D^1$ which proved theorem one.

Remark. Simpson [S] and Yau [Y2] gave certain characterizations of algebraic manifolds which can be uniformized by the product of unit discs. In those characterizations, TX is usually assumed to split into direct sum of line bundles holomorphically.

Because both the index and the negative Seiberg Witten invariants, are differentiable invariants, we obtain the following result immediately.

Corollary. *Let X be an algebraic surface with ample canonical line bundle. If X is diffeomorphic to an algebraic surface which can be uniformized by the product of two disks, then X itself can be uniformized by the product of two disks.*

In fact, a stronger result is proved by Jost and Yau [J+Y] by using a harmonic mapping method where instead of diffeomorphic, they only need to assume homotopic equivalency.

4 Anti-self-dual $U(2,1)$ connections

In this section, we shall discuss the relationship between solutions of Seiberg Witten equations and $U(2,1)$ anti-self-dual connections.

We first explain the case when X carries a Kähler Einstein metric with negative scalar curvature. By Yau's theorem, such a metric always exists whenever the canonical line bundle K_X of X is ample. From the existence of Kähler Einstein metric on X, one has a Chern number inequality: $c_1^2(X) \leqq 3c_2(X)$.

Since X is Kähler Einstein, in particular, its tangent bundle TX carries a Hermitian Einstein metric, $\Lambda F = \mu I$. By Narashima-Seshadra, Donaldson, Uhlenbeck-Yau, it is equivalent to say that TX is a poly-stable bundle over X. Recall that a holomorphic bundle E over X is called a (Mumford) stable bundle if for any non-trivial torsion-free subsheaf $\mathscr{S}$ of E, we have $\mu(\mathscr{S}) < \mu(E)$ where $\mu(\mathscr{S})$, $\mu(E)$ are slopes of corresponding torsion-free sheaves:

$$\mu(E) = \frac{1}{\operatorname{rank}(E)} \cdot \int_X c_1(E) \cdot \omega ,$$

where ω is the Kähler form on X. And a holomorphic bundle is called poly-stable if it is the direct sum of stable bundles of the same slope. For a poly-stable/Hermitian-Einstein bundle of rank r, we have a Chern number inequality

$$c_1^2(E) \leqq \frac{2r}{r-1} c_2(E) .$$

When we let $E = TX$, we have,

$$c_1^2(X) \leqq 4c_2(X)$$

which is weaker than $c_1^2(X) \leqq 3c_2(X)$.

Nevertheless, we are saved by the fact that the Hermitian Einstein connection on TX is torsion-free in our situation. From this, we have a Hermitian-Einstein-Higgs bundle $TX \oplus \mathcal{O}_X$ as follows: On the trivial factor $\mathcal{O}_X$, we put the minus of the standard metric and the trivial connection d on it. On TX, we use the Kähler Einstein metric ω.

We construct a $U(2,1)$ connection on $TX \oplus \mathcal{O}_X$ by

$$D_A = \begin{pmatrix} \nabla^{L.C.} & \varphi \\ \varphi^* & d \end{pmatrix}$$

where $\nabla^{L.C.}$ is the Levi-Civita connection of ω and the second fundamental form $\varphi \in \Omega^1(\mathrm{Hom}(\mathcal{O}_X, TX)) = \Omega^1(TX)$ is the tautological one, $\varphi = \sum dz^i \otimes \frac{\partial}{\partial z^i}$ locally. The curvature of D_A is

$$F_A = \begin{pmatrix} Rm + \varphi\varphi^* & \nabla_A\varphi \\ \nabla_A\varphi^* & +\varphi^*\varphi \end{pmatrix},$$

where Rm is the curvature tensor for ω. Next, we are going to look at the endomorphism

$$\begin{aligned} \Lambda F_A &= \frac{F_A \wedge \omega^{n-1}}{\omega^n} \cdot n \\ &= \begin{pmatrix} \Lambda Rm + \Lambda\varphi\varphi^* & 0 \\ 0 & \Lambda\varphi^*\varphi \end{pmatrix}. \end{aligned}$$

It is easy to see that $\Lambda\varphi\varphi^* = I_{TX}$ and $\Lambda\varphi^*\varphi = 2$. Moreover, ΛRm can be identified with the usual Ricci curvature because of Kähler identity. But ω is Kähler Einstein, $Rc = -\omega$; we have $\Lambda Rm = -I_{TX}$. Therefore, we have $\Lambda F_A = -2I_{TX\oplus\mathcal{O}_X}$.

Notice that Hermitian-Einstein-Higgs bundles are in one to one correspondence to Higgs poly-stable holomorphic bundles ([H], [S]) which generalized the previous stated correspondence between Hermitian Einstein and poly-stable bundles. In our situation, the bundle is $E = TX \oplus \mathcal{O}_X$ and the Higgs field is

$$\Phi = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix} \in \Omega^1(\mathrm{End}\, E),$$

where clearly satisfies $\Phi \wedge \Phi = 0 \in \Omega^2(\mathrm{End}\, E)$ and $\bar{\partial}\Phi = 0$. (See [H], [S] for more details). Recall that a pair (E, Φ) is called Higgs stable if for any nontrivial torsion-free subsheaf $\mathcal{S}$ of E which is Φ-invariant (i.e. $\Phi(\mathcal{S}) \subseteq \Omega^1(\mathcal{S})$), we have

$$\mu(\mathcal{S}) < \mu(E).$$

For Hermitian-Einstein-Higgs/Higgs-poly-stable bundle, we still have Chern number inequality

$$c_1^2(E) \leqq \frac{2r}{r-1} c_2(E).$$

Looking at our situation where $E = TX \oplus \mathcal{O}_X$, we therefore obtain

$$\begin{aligned} c_1^2(X) &= c_1^2(E) \\ &\leqq \frac{2 \cdot 3}{3-1} c_2(E) \\ &= 3c_2(X). \end{aligned}$$

The strong form of Chern number inequality. From this approach, one can also get uniformation by the unit ball B^2 when the equality sign holds.

Now, let (X,g) be a Riemannian four manifold with an almost complex structure J and $K_X = \Lambda^2 T^*X$ be the canonical line bundle associated to J. Suppose that there exists a connection D_A on K_X and a section φ on $S_+^{\mathbb{C}} = S_+ \otimes K_X^{-1/2}$ which satisfies the Seiberg Witten equations

$$\begin{cases} F_A^+ = \varphi\bar{\varphi} \\ \not{D}_A\varphi = 0\,. \end{cases}$$

Using φ, we shall construct a connection on $E = S_-^{\mathbb{C}} \oplus \mathcal{O}_X = S_- \otimes K_X^{-1/2} \oplus \mathcal{O}_X$. Via Clifford multiplication, there is a homomorphism $\Gamma(S_+^{\mathbb{C}}) \to \Omega^1(S_-^{\mathbb{C}}) = \Omega^1(\mathrm{Hom}(\mathcal{O}_X, S_-^{\mathbb{C}}))$. We call the image of φ under this homomorphism by $\tilde{\varphi}$. On the other hand, from the Riemannian metric g on X, we have a Levi-Civita connection of T_X and it induces a connection on S_+. Combining it with D_A on K_X, we obtain a connection on $S_+^{\mathbb{C}} = S_+ \otimes K_X^{-1/2}$ which we call it D_A again. Because of φ is a harmonic spinor, we have the following lemma.

Lemma. $P_-(D_A\tilde{\varphi}) = 0$.

Here P_- is the orthogonal projection of a two form to its anti-self-dual part.

Proof of Lemma. We shall consider the following diagram: (For simplicity, we have not included the twisting $K_X^{-1/2}$ here).

$$\begin{array}{ccccc} S_+ & \xrightarrow{D} & \Omega^1(S_+) & \xrightarrow{\mathrm{cl}} & S_- \\ {\scriptstyle \mathrm{cl}}\downarrow & & {\scriptstyle \mathrm{cl}}\downarrow & & \downarrow \\ \Omega^1(S_-) & \xrightarrow{D} & \Omega^2(S_-) & \xrightarrow{P_-} & \Omega^2_-(S_-) \end{array}$$

The composite of the two maps on the upper row is the Dirac operator $\not{D}_A = \mathrm{cl} \circ D : \Gamma(S_+) \to \Gamma(S_-)$. The Clifford multiplication follows by the two maps on the bottom row on φ is our element $P_-(D\tilde{\varphi})$.

The homomorphism in the right column $S_- \to \Omega^2_-(S_-)$ is defined as follows: There is a canonical isomorphism of Ω^2_- with trace-free endomorphisms of S_-; that is, $\Omega^2_- = su(S_-)$. Therefore, by projecting to the trace-free part, $u(S_-) \to su(S_-)$, we get the homomorphism $S_- \to \Omega^2_-(S_-)$ in the right column of the above diagram. This is an injective homomorphism. In fact, we can obtain this homomorphism directly from representation theory of $sl_2(V)$. As a representation of $sl_2(V)$, $sl_2(V) \otimes V$ splits into two irreducible components, one is S^3V and the other is V itself which corresponds to our factor S_- inside $\Omega^2(S_-)$.

Now, the lemma will follow from the commutativity of the above diagram.

For the box on the left, the commutativity is clear from the compatibility of the connection D and the metric g on X. For the box on the right, we shall first complexify each term. Notice that $\Omega^1 \otimes \mathbb{C} = S_+ \otimes S_-$ and $S_\pm$ are both $SU(2)$-bundles. In particular, $S_\pm^* = S_\pm$. Moreover, $\Omega^2 \otimes \mathbb{C} = sl(S_+) \oplus sl(S_-)$.

Now $\Omega^1(S_+)_{\mathbb{C}} \to \Omega^2_-(S_-)_{\mathbb{C}}$ is given by first taking trace of S_+ in $S_+ \otimes (S_+ \otimes S_-)$ and then mapping S_- to $\Omega^2_-(S_-)_{\mathbb{C}}$ via the homomorphism we have above. This concludes the proof of the lemma.

Nevertheless, it is also interesting to see the other projection $\Omega^1(S_+)_{\mathbb{C}} \to \Omega^2_+(S_-)_{\mathbb{C}}$. This one is induced from the projection of an endomorphism of S_+ to its trace-free part:

$$\begin{array}{ccc} \Omega^1(S_+)_{\mathbb{C}} & \longrightarrow & \Omega^2_+(S_-)_{\mathbb{C}} \\ \Big\Vert & & \Big\Vert \\ S_+ \otimes (S_+ \otimes S_-) & \longrightarrow & sl(S_+) \otimes S_- \, . \end{array}$$

Next, we shall compare $\varphi\bar{\varphi}$ with $\tilde{\varphi}\tilde{\varphi}^*$ and $\tilde{\varphi}^*\tilde{\varphi}$. Since $\tilde{\varphi}$ is in $\Omega^1(S_-)$, we can define its adjoint $\tilde{\varphi}^*$ in $\Omega^1(\mathrm{Hom}(S_-, \mathcal{O}_X))$. By taking wedge product, we have $\tilde{\varphi}\tilde{\varphi}^*$ in $\Omega^2(\mathrm{End}(S_-))$ and $\tilde{\varphi}^*\tilde{\varphi}$ in Ω^2. By projecting to the self-dual two form part, we

$$P_+(\tilde{\varphi}\tilde{\varphi}^*) = \varphi\bar{\varphi}I \, ,$$

$$P_+(\tilde{\varphi}^*\tilde{\varphi}) = -2\varphi\bar{\varphi} \, .$$

To prove these two equalities, one can trace all identifications via representations of Spin(4).

Now we define a connection $\mathbb{D}_{(A,\varphi)}$ on $E = S_-^{\mathbb{C}} \oplus \mathcal{O}_X$ by

$$\mathbb{D}_{(A,\varphi)} = \begin{pmatrix} D_A & \frac{1}{\sqrt{6}}\tilde{\varphi} \\ \frac{1}{\sqrt{6}}\tilde{\varphi}^* & d \end{pmatrix} .$$

Notice that $\mathbb{D}_{(A,\varphi)}$ is *not* a Hermitian connection on E with respect to the direct sum Hermitian metric $h_{S_-^{\mathbb{C}}} + h_{\mathrm{trivial}}$ on E. Here $h_{S_-^{\mathbb{C}}}$ is the induced metric on $S_-^{\mathbb{C}}$ from g and h_{trivial} is the trivial product metric on $\mathcal{O}_X$.

Instead, we are interested in the Hermitian form $h_E = h_{S_-^{\mathbb{C}}} - h_{\mathrm{trivial}}$, which is a semi-Hermitian metric of signature (2,1). $\mathbb{D}_{(A,\varphi)}$ preserves this Hermitian form h_E. Therefore, we call $\mathbb{D}_{(A,\varphi)}$ an $U(2,1)$ connection.

By coupling with h_E, we can decompose $\Omega^2(X, \mathrm{End}\, E)$ into self-dual and anti-self-dual parts as follows: Let $* : \Omega^2(X) \to \Omega^2(X)$ be the Hodge star operator on X defined using g. Let $\tau : \mathrm{End}(E) \to \mathrm{End}(E)$ be the parity operator on $\mathrm{End}(E)$. In essence, τ equals identity on diagonal block endomorphisms and equals minus identity on off-diagonal block endomorphisms. The best way to rephrase these are in terms of the 'super' (or $\mathbb{Z}_2$ graded) language.

By composing $*$ and τ, we get

$$*_E : \Omega^2(X, \mathrm{End}\, E) \to \Omega^2(X, \mathrm{End}\, E)$$

with $*_E^2 = \mathrm{Id}$. In terms of the block decomposition of matrix of two forms, we have,

$$*_E \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} *A & -*B \\ -*C & *D \end{pmatrix} .$$

Hence, $\Omega^2(X, \mathrm{End}\, E)$ is decomposed into direct sum of self-dual and anti-self-dual components.

We shall denote the self-dual component of an element $\mathbb{F} \in \Omega^2(X, \mathrm{End}\, E)$ by $\mathbb{F}^+$. By a simple calculation, we have the following result.

Proposition. *If $\mathbb{D}_{(A,\varphi)}$ is as before with (A, φ) a solution to the Seiberg-Witten equation. Let $\mathbb{F}_{(A,\varphi)}$ be the curvature of $\mathbb{D}_{(A,\varphi)}$ on $E = S_-^{\mathbb{C}} \oplus \mathcal{O}_X$, then*

$$\mathbb{F}^+_{(A,\varphi)} = \begin{pmatrix} Rc^o - \frac{1}{3}F_A^+ \cdot I & 0 \\ 0 & -\frac{1}{3}F_A^+ \end{pmatrix} .$$

Proof of Proposition. The curvature of $\mathbb{D}_{(A,\varphi)}$ is

$$\mathbb{F}_{(A,\varphi)} = \begin{pmatrix} F_{S_-^{\mathbb{C}}} + \frac{1}{6}\tilde{\varphi}\tilde{\varphi}^* & \frac{1}{\sqrt{6}} D_A\tilde{\varphi} \\ \frac{1}{\sqrt{6}} D\tilde{\varphi}^* & \frac{1}{6}\tilde{\varphi}^*\tilde{\varphi} \end{pmatrix} .$$

When we project $\mathbb{F}$ to its self-dual part, the off-diagonal term is $P_-(D_A\tilde{\varphi})$ which becomes zero by the above lemma.

Now, $F_{S_-^{\mathbb{C}}} = F_{S_-} - \frac{1}{2}F_A \cdot I_{S_-}$ since $S_-^{\mathbb{C}} = S_- \otimes K_X^{-1/2}$ and F_A is the curvature of D_A on K_X. We have

$$\begin{aligned} F^+_{S_-^{\mathbb{C}}} &= F^+_{S_-} - \tfrac{1}{2}F_A^+ \cdot I_{S_-} \\ &= Rc^o - \tfrac{1}{2}\varphi\bar{\varphi} I_{S_-} . \end{aligned}$$

It is because $F^+_{S_-} = Rc^o$ for the connection on S_- which is induced from the Levi-Civita connection on X [AHS]. Here Rc^o denotes the trace-free part of the Ricci curvature of g. We have also used the Seiberg Witten equation $F_A^+ = \varphi\bar{\varphi}$ in the above equality.

Using equalities $P_+(\tilde{\varphi}\tilde{\varphi}^*) = \varphi\bar{\varphi} I$ and $P_+(\tilde{\varphi}^*\tilde{\varphi}) = -2\varphi\bar{\varphi}$, we get

$$\begin{aligned} F^+_{S_-^{\mathbb{C}}} + \tfrac{1}{6}(\tilde{\varphi}\tilde{\varphi}^*)^+ &= Rc^o - \tfrac{1}{2}\varphi\bar{\varphi} I + \tfrac{1}{6}\varphi\bar{\varphi} I \\ &= Rc^o - \tfrac{1}{3}\varphi\bar{\varphi} I \end{aligned}$$

and

$$\tfrac{1}{6}(\tilde{\varphi}^*\tilde{\varphi})^+ = \tfrac{1}{6} \times (-2\varphi\bar{\varphi}) = -\tfrac{1}{3}\varphi\bar{\varphi} .$$

Therefore, we have

$$\begin{aligned} \mathbb{F}^+_{(A,\varphi)} &= \begin{pmatrix} Rc^o - \frac{1}{3}\varphi\bar{\varphi} \cdot I & 0 \\ 0 & -\frac{1}{3}\varphi\bar{\varphi} \end{pmatrix} \\ &= \begin{pmatrix} Rc^o - \frac{1}{3}F_A^+ \cdot I & 0 \\ 0 & -\frac{1}{3}F_A^+ \end{pmatrix} . \end{aligned}$$

Corollary. *If, moreover, g is an Einstein metric, then $\mathbb{D}_{(A,\varphi)}$ is a perturbed Anti-Self-Dual $U(2,1)$ connection on $E = S_-^{\mathbb{C}} \oplus \mathcal{O}_X$.*

Recall that an anti-self-dual connection means a connection whose curvature is anti-self-dual and a perturbed anti-self-dual connection means a connection such that the self-dual part of its curvature is central, that is of the form αI_E for some two form α on X. In holomorphic category, a perturbed anti-self-dual Hermitian connection is related to a Hermitian Einstein metric on the bundle.

We can generalized Chern number inequality for perturbed Anti-Self-Dual $U(2,1)$ connection. In general, we have,

Proposition. *Let E be any rank r complex vector bundle over X with a $U(p,q)$ Hermitian form on E. Let $\mathbb{D}$ be an $U(p,q)$ connection on E such that it curvature $\mathbb{F}$ satisfies*

$$\mathbb{F}^+ = \alpha I_E$$

for some two form α, that is, $\mathbb{D}$ is a perturbed anti-self-dual connection on E.

If some multiple of α is the self-dual part of certain integral two form, then we have

$$c_1^2(E) \leqq \frac{2r}{r-1} c_2(E) .$$

Moreover, equality sign holds if and only if $\mathbb{D}$ is a projectively flat $U(p,q)$ connection on E.

Combining these results, we obtain

Theorem 2. *If X is a four manifold with Riemannian metric g. Suppose that for some almost complex structure on X, there is a solution to the Seiberg Witten equation, (D_A, φ). Then $\mathbb{D}_{(A,\varphi)} = \begin{pmatrix} D_A & \frac{1}{\sqrt{6}}\tilde{\varphi} \\ \frac{1}{\sqrt{6}}\tilde{\varphi}^* & d \end{pmatrix}$ is a $U(2,1)$ connection on $E = S_-^{\mathbb{C}} \oplus \mathcal{O}_X$ with self-dual part of its curvature equals*

$$\mathbb{F}^+_{(A,\varphi)} = \begin{pmatrix} Rc^o - \frac{1}{3}F_A^+ \cdot I & 0 \\ 0 & -\frac{1}{3}F_A^+ \end{pmatrix} .$$

When X is an Einstein metric, then $\mathbb{D}_{(A,\varphi)}$ is a perturbed Anti-Self-Dual $U(2,1)$ connection on E. We have Chern number inequality

$$c_1^2(E) \leqq 3c_2(E)$$

which is equivalent to

$$3\sigma(X) \leqq \chi(X) .$$

Moreover, equality holds if and only if the above $U(2,1)$ connection is projectively flat and X is a Kähler manifold which can be uniformized by $\mathbb{C}^2$ or the complex two ball B^2.

In general, it would be interesting to know when an Einstein metric is actually a Kähler Einstein metric. We shall give one such criterion in terms of the self-dual Weyl tensor.

Proposition. *Let X be a four manifold with non-trivial Seiberg Witten invariant. If X carries an Einstein metric g, then we have*

$$\frac{3}{4\pi^2}\int_X |W_+|^2 \leqq 2\chi(X) + 3\sigma(X).$$

Moreover, the equality sign hold if and only if g is a Kähler Einstein metric.

Proof of Proposition. From the existence of solution to the Seiberg Witten equation with respect to some line bundle L, one applies the Weitzenböck formula and get $|\varphi|^2 \leqq R^2$. We have

$$\begin{aligned}
&2\chi(X) + 3\sigma(X)\\
&= c_1(L)^2\\
&= \frac{1}{4\pi^2}\int (|F_A^+|^2 - |F_A^-|^2)\\
&\leqq \frac{1}{4\pi^2}\int |F_A^+|^2\\
&= \frac{1}{32\pi^2}\int |\varphi|^4\\
&\leqq \frac{1}{32\pi^2}\int R^2
\end{aligned}$$

However, when g is an Einstein metric, we have

$$2\chi(X) + 3\sigma(X) = \frac{1}{2\pi^2}\int \left(|W_+|^2 + \frac{1}{48}R^2\right).$$

Putting these together, we obtain

$$\frac{3}{4\pi^2}\int |W_+|^2 \leqq 2\chi(X) + 3\sigma(X).$$

When the equality sign holds, we have $F_A^- \equiv 0$, $D_A\varphi \equiv 0$ and $|\varphi|^2 \equiv R^2$. In particular, F_A is a parallel self-dual two form which will reduce the holonomy group of g from $SO(4)$ to $U(2)$. Hence, we have a Kähler Einstein metric g. The converse is easy by observing that W_+ is proportional to the scalar curvature for any Kähler metric and $2\chi(X) + 3\sigma(X) = c_1^2(X)$. Hence we have the proposition.

5 Dimension reduction

In this section, we shall study Anti-Self-Dual equation for $U(2,1)$ connections. By imposing certain symmetry constraints, this equation will be decoupled into two independent sets of equations. One of them gives the Seiberg Witten equations. The other one is the Einstein equation for the Riemannian metric.

We let E be the bundle $S_L^- \otimes L^{-1} \oplus \mathcal{O}_X$ with a hermitian form of signature (2.1) on it (as in the last section). Let $\mathscr{A}(E)$ be the space of $U(2,1)$

connections on E. An element in $\mathscr{A}(E)$ can be decomposed as

$$\mathbb{D} = \begin{pmatrix} D_{S_L^- \otimes L^{-1}} & \tilde{\varphi} \\ \tilde{\varphi}^* & d + \alpha \end{pmatrix}$$

where α is an ordinary one form on X and $\tilde{\varphi} \in \Omega^1(S_L^- \otimes L^{-1})$. We can write a connection on $S_L^- \otimes L^{-1}$ as the tensor product of a $SU(2)$ connection on S_- and a $U(1)$ connection on L. On S_-, we have a canonical connection induced from the Levi-Civita connection on X. Therefore, any other connection on S_- is of the form $\nabla^{L.C.} + B$ for some $su(S_-)$ valued one form on X. We can regard B as an element in $\Omega^1 \otimes \Omega^2_-$ by our previous identification, $\Omega^2_- = su(S_-)$.

Moreover, we have a Lie algebra bundle structure on Ω^2_- via this identification. Using this and the wedge product on one forms, we get $[B, B]$ in $\Omega^2 \otimes \Omega^2_-$. Ω^2_- also acts on Ω^1 via interior multiplication. If $\eta \in \Omega^2_-$ and $\tilde{v} \in \Omega^1$, then

$$\eta \cdot \tilde{v} = \iota_v(\eta)$$

where v is a vector dual to $\tilde{v}$ using the metric g and ιv is the interior multiplication.

Now the $U(2,1)$ perturbed Anti-Self-Dual equations for $\mathbb{D}$ can be written as

$$\begin{cases} Rc^o + (dB + \frac{1}{2}[B,B])^+ - \frac{1}{2}F_A^+ \cdot I + (\tilde{\varphi}\tilde{\varphi}^*)^+ = d^+\alpha \cdot I + (\tilde{\varphi}^*\tilde{\varphi})^+ \cdot I \\ P_-(D\tilde{\varphi}) = 0 \end{cases}$$

where D_A is a connection on L.

If we assume that $\tilde{\varphi}$ is Ω^2_--invariant, then the first equation will be decoupled and the second equation can also be simplified. Here Ω^2_- acts on $\tilde{\varphi}$ via the action of Ω^2_- on Ω^1 and S_-.

For simplicity, we shall assume α is zero since, by gauge transformation, we can absorb $d\alpha$ inside F_A.

Proposition. *If $\tilde{\varphi}$ is Ω^2_--invariant, then the above $U(2,1)$ perturbed Anti-Self-Dual equations are equivalent to the Seiberg-Witten equations and* $Rc^o = -(dB + \frac{1}{2}[B,B])^+$.

Proof of Proposition. $\tilde{\varphi}$ being Ω^2-invariant will imply that $(\tilde{\varphi}\tilde{\varphi}^*)^+$ lies in the center (or the trace part) of $\mathrm{End}(S_L^-)$. Therefore, the first equation will be decoupled into two parts: one involving the trace part and the other involving the trace-free part which is $Rc^o + (dB + \frac{1}{2}[B,B])^+ = 0$ via Clifford multiplication. From the discussion in last section, we have

$$(\tilde{\varphi}\tilde{\varphi}^*)^+ = \varphi\bar{\varphi}I$$

and $(\tilde{\varphi}^*\tilde{\varphi})^+ = -2\varphi\bar{\varphi}$.

Therefore, the trace part of the first equation becomes

$$-\tfrac{1}{2}F_A^+ + \varphi\bar{\varphi} = -2\varphi\bar{\varphi}$$

or

$$F_A^+ = 6\varphi\bar{\varphi}\,.$$

Again, from the lemma in the last section, we know that $P_-(D\tilde{\varphi}) = 0$ is equivalent to the Dirac equation $\not{D}\,\varphi = 0$ where $\tilde{\varphi}$ is Ω^2_--invariant.

Hence, we have our proposition.

In addition, if we assume both B and $\tilde{\varphi}$ are Ω^2-invariant, then the above equations will be equivalent to the Seiberg-Witten equations for (D_A, φ) and Einstein equation for the Riemannanian metric g.

Notice that B is in $\Omega^1 \otimes \Omega^2_-$ and Ω^2_- acts on both Ω^1 and Ω^2_-. After complexification, we have $\Omega^1 \otimes \Omega^2_- = S_+ \otimes S_- \otimes sl(S_-)$. As a representation of $sl(S_-) = \Omega^2_-$, we have $S_+ \otimes S_- \otimes sl(S_-) \cong S_+ \otimes (S_- \oplus S^3 S_-)$, which contains no trivial summand. Therefore, if B is $sl(S_-)$-invariant element, it must be zero. Putting these together, we have

Theorem 3. *Let $\mathbb{D}$ be any $U(2,1)$ perturbed Anti-Self-Dual connection on $E = S_L^- \otimes L^{-1} \oplus \mathcal{O}_X$. Then $\mathbb{D}$ is Ω^2_--invariant if and only if the Riemannian metric is Einstein and the Seiberg-Witten equations are satisfied. That is,*

$$\begin{cases} Rc^o = 0 \\ F_A^+ = 6\varphi\bar{\varphi} \\ \not{D}\,\varphi = 0 \end{cases}$$

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