# A PRODUCT FORMULA FOR LOG GROMOV-WITTEN INVARIANTS 

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#### Abstract

The purpose of this short article is to prove a product formula relating the $\log$ Gromov-Witten invariants of $V \times W$ with those of $V$ and $W$ in the case the $\log$ structure on $V$ is trivial.


## 0. Introduction

Product formulas in the literature of Gromov-Witten (GW) theory started with [15] for the genus zero Gromov-Witten invariants. It was soon generalized in [8] to (absolute) GW invariants in any genus. There is also an orbifold version in [7]. In this paper, we extend the product formula to the setting of relative $G W$ theory or more generally $\log G W$ theory.

The absolute Gromov-Witten theory studies the intersection theory on the moduli stacks of stable maps $\overline{\mathcal{M}}_{g, n}(X, \beta)$ from an n-pointed curve with arithmetic genus $g$ to a fixed nonsingular projective variety $X$ with a fixed degree $\beta$ in the Mori cone $N E(X)$ of effective curves in $X$

$$
f:\left(C, x_{1}, x_{2}, \ldots, x_{n}\right) \rightarrow X, \text { such that } f_{*}([C])=\beta .
$$

For notational convenience, we denote such a class by $[f]$. Intuitively, GW invariants count the numbers of curves passing through $n$ fixed cycles $\alpha_{1}, \ldots, \alpha_{n}$ in $X$ with the above given conditions. To put it on a mathematically sound setting, one defines the invariants as intersection numbers on $\overline{\mathcal{M}}_{g, n}(X, \beta)$ in the following way. By the functorial properties of the moduli stacks, there are the evaluation morphisms

$$
e v_{i}: \overline{\mathcal{M}}_{g, n}(X, \beta) \rightarrow X, i=1, \ldots, n,
$$

and stabilization morphism

$$
\pi: \overline{\mathcal{M}}_{g, n}(X, \beta) \rightarrow \overline{\mathcal{M}}_{g, n}
$$

where $\overline{\mathcal{M}}_{g, n}$ is the moduli stack of stable genus $g$, $n$-pointed curves and

$$
e v_{i}([f])=f\left(x_{i}\right) \in X, \quad \pi([f])=\left[\left(\bar{C}, \bar{x}_{1}, \bar{x}_{2}, \ldots, \bar{x}_{n}\right)\right] \in \overline{\mathcal{M}}_{g, n}
$$

where $\bar{C}$ is the stabilization of the source curve $C$. The GW invariants can be defined as

$$
\int_{\left[\overline{\mathcal{M}}_{g, n}(X, \beta)\right]^{v i r}} \pi^{*}(\gamma) \prod_{i} e v_{i}^{*}\left(\alpha_{i}\right)
$$

where $\left[\overline{\mathcal{M}}_{g, n}(X, \beta)\right]^{v i r}$ is the virtual fundamental class. One can rephrase these invariants in terms of the cohomological field theory

$$
R_{g, n, \beta}^{X}: H^{*}(X)^{\otimes n} \rightarrow H^{*}\left(\overline{\mathcal{M}}_{g, n}\right)
$$

[^0]via
$$
\int_{\left[\overline{\mathcal{M}}_{g, n}(X, \beta)\right]^{v i r}} \pi^{*}(\gamma) \prod_{i} e v_{i}^{*}\left(\alpha_{i}\right)=\int_{\overline{\mathcal{M}}_{g, n}} \gamma \cdot R_{g, n, \beta}^{X}\left(\prod_{i} \alpha_{i}\right)
$$

The details can be found in [8] and references therein.
Intuitively, the relative invariants as defined in [17] and [18] can be considered as refined counting. Let $(X, D)$ be a pair consisting of a nonsingular projective variety $X$ and a smooth divisor $D$ in $X$. If the curve $C$ does not lie in $D$, it intersects with $D$ at $\rho$ points with multiplicities $\mu_{1}, \ldots, \mu_{\rho}$ such that

$$
\sum_{j} \mu_{j}=\int_{\beta}[D] .
$$

The refined counting is to fix the profile $\left(\mu_{1}, \ldots, \mu_{\rho}\right)$ and constraint the $\rho$ points to lie in chosen cycles $\left\{\delta_{j}\right\}$ in $D$. Similarly, one can define the relative invariants as intersection numbers on relative moduli stacks as above. See [18] and [17] for details. There is also a similar reformulation in terms of cohomological field theory. See Section 4 .

The divisor $D$ in $X$ gives rise to a divisorial $\log$ structure on $X$. Recently relative invariants have been generalize to the setting of $\log$ GW ([2, 9, 13]). See Section 1 for a brief summary of log geometry and $\log$ GW theory.

In a sense, the study of Gromov-Witten theory is the study of the virtual fundamental classes. The product formula is a statement that GW invariants of $V$ and $W$ determine the invariants of $V \times W$. See Equation (3). It can be written in the form of certain functorial properties of virtual classes. In [8], K. Behrend proves a product formula for absolute GW invariants by first establishing a corresponding functorial property of virtual fundamental classes. In this note, we approach the product formula in the relative and log settings in a similar way. The main results are Theorem 2.2 and Corollary 4.1, which expresses the log/relative invariants of $X \times(Y, D)$ by invariants of $X$ and of $(Y, D)$. The logarithmic approach to relative GW invariants of Abramovich-Chen and Gross-Siebert ( $[2,9,13 \mid)$ avoids the expanded degenerations of Li ([18]) and allows us to adapt Behrend's original proof, but it also presents new technical difficulties. We were not able to prove the product formula in the general log setting and we have to assume the log structure on one of the factors to be trivial. For the general case, see Section 3 for some initial attempts and speculations.

This product formula could be useful in the study of Gromov-Witten theory, even when no explicit product or $\log$ geometry is involved in the statement. For example, it plays a role in proving the crepant transformation conjecture for ordinary flops with non-split vector bundles in [16]. There the degeneration technique is extensively employed and the product formula is applied to treat fiber integrals which naturally occur in the degeneration process.

## 1. Preliminaries

1.1. Log geometry. We work over the base log scheme Spec $\mathbb{C}$ with the trivial log structure. We refer to [14, Sections 1-4] for general background on log structures on schemes, and [22, Section 5] for log stacks. For the reader's convenience, we recall the basic properties we need about Olsson's Log stack and saturated morphisms.
1.1.1. $\mathcal{T}^{\text {or }}{ }_{X}$. Denote by $\operatorname{LogSt}$ the category of fine saturated (fs) log algebraic stacks, and $\mathbf{S t}$ the category of algebraic stacks. 1

For a fs $\log$ stack $X, \mathcal{T}$ or $_{X}$ is introduced in [22, Remark 5.26] to parametrize all fs log schemes over $X$. It follows from the definition of $\mathcal{T}^{\text {or }}{ }_{X}$ that a map $S \rightarrow X$ in LogSt factors as $S \rightarrow \mathcal{T}^{\text {or }}{ }_{X} \rightarrow X$. We can view $X$ as an open substack of $\mathcal{T}^{\operatorname{or}}{ }_{X}$ parametrizing strict maps $S \rightarrow X$. Note that $\mathcal{T}$ or ${ }_{X}$ has a natural fs $\log$ structure so that the factorization $S \rightarrow \mathcal{T}$ or $_{X}$ above is strict, and $\mathcal{T}$ or ${ }_{X} \rightarrow X$ is log étale.

It is easy to check when $X$ is $\log$ smooth, it is open dense in $\mathcal{T}^{\text {or }} \mathrm{X}$. In particular if $X$ is $\log$ smooth and irreducible, $\mathcal{T}^{\operatorname{tr}}{ }_{X}$ is irreducible.

Remark 1.1. Let Sch be the category of schemes over C, Grpd the category of (small) groupoids. Recall a stack is a functor Sch $\rightarrow$ Grpd that satisfies certain 'sheaf' condition. A $\log$ stack is a stack with a $\log$ struture, and a $\log$ structure can be understood as a map from the stack to $\mathcal{L o g}_{C}$.

It is more natural to describe a fs log moduli stack $X$ by its functor of points than specifying its underlying stack and log structure. As the Yoneda embedding for the $(2,1)$ category $\operatorname{LogSt}$ is fully faithful, an object $X$ in $\operatorname{LogSt}$ is uniquely determined by:

$$
\operatorname{hom}_{\operatorname{LogSt}}(-, X): \text { LogSch } \rightarrow \text { Grpd. }
$$

Here the stack condition for $X$ allows us to restrict hom $_{\text {LogSt }}(-, X)$ from $\operatorname{LogSt}$ to its full subcategory $\mathbf{L o g S c h}$ of fs $\log$ schemes.

Given a $\log$ moduli functor $\mathbb{X}:$ LogSch $\rightarrow$ Grpd such that it is isomorphic to $\operatorname{hom}_{\text {LogSt }}(-, X)$ for some fs $\log$ stack $X$, one might recover $X$ by considering the stack $\mathcal{T}^{\text {or }}{ }_{X}:$ Sch $\rightarrow$ Grpd which corresponds to $\mathcal{T}^{\text {or }}{ }_{X}$, then the underlying stack $\underline{X}$ of $X$ is the substack of $\mathcal{T}$ or ${ }_{X}$ satisfying a minimal condition, and the $\log$ structure of $X$ is the restriction of the natural $\log$ structure on $\mathcal{T}_{\text {or }}^{X}$. See [11] for details.

### 1.1.2. Saturated morphisms.

Definition 1.2. Let $P, Q$ be saturated monoids, a map $P \rightarrow Q$ is saturated, if it is integral and the push out of

is saturated when $R$ is saturated.
Definition 1.3. Let $\left(X, M_{X}\right),\left(Y, M_{Y}\right)$ be fs log schemes, a map $f:\left(X, M_{X}\right) \rightarrow\left(Y, M_{Y}\right)$ is called saturated, if for any $x \in X, y=f(x)$, the induced map between characteristics $\bar{M}_{Y, \bar{y}} \rightarrow \bar{M}_{X, \bar{x}}$ is saturated.
Remark 1.4. Given the above definition, a saturated morphism between fs log stacks can be defined locally with respect to the lisse-étale topology.

It follows from the definition that saturated morphisms satisfy the following properties:

- They are stable under composition and base change in LogSt.

[^1]- For a cartesian square

in $\log \mathbf{S t}$, when $f$ is saturated, the underlying diagram of stacks

is cartesian in $\mathbf{S t}$.


### 1.2. Log Gromov-Witten theory.

1.2.1. Log curves. Let $\overline{\mathcal{M}}_{g, n}$ (resp. $\mathfrak{M}_{g, n}$ ) be the algebraic stack of stable (resp. prestable) curves of genus $g$ with $n$ marked points. The $\log$ structure of $\overline{\mathcal{M}}_{g, n}$ is the divisorial $\log$ structure associated with its boundary consisting of singular curves. The log structure of $\mathfrak{M}_{g, n}$ is determined from the smooth chart $\sqcup_{m} \overline{\mathcal{M}}_{g, n+m} \rightarrow \mathfrak{M}_{g, n}$, where the map from $\overline{\mathcal{M}}_{g, n+m} \rightarrow \mathfrak{M}_{g, n}$ is forgetting the extra m marked points without stabilizing. We view $\overline{\mathcal{M}}_{g, n}$ and $\mathfrak{M}_{g, n}$ as log stacks from now on using the same notation.

Note that the forgetful map $\overline{\mathcal{M}}_{g, n+1} \rightarrow \overline{\mathcal{M}}_{g, n}$ is saturated, as it can be identified as the universal $\log$ curve. This implies the stabilization map $\mathfrak{M}_{g, n} \rightarrow \overline{\mathcal{M}}_{g, n}$ is saturated.
1.2.2. Log stable maps. For a projective log smooth scheme $V$, let $\mathscr{M}_{g, n}(V)$ be the fs $\log$ stack parameterizing stable $\log$ maps from $\log$ curves of genus $g$ with $n$ marked points (see [2, 4, 9, 13] for more details). For a fs $\log$ scheme $S$, a log map from $S$ to $\mathscr{M}_{g, n}(V)$ corresponds to a stable log map:


We note that the underlying map of the diagram is a usual stable map.
A $\log$ map $V \rightarrow W$ induces a stabilization map $\mathscr{M}_{g, n}(V) \rightarrow \mathscr{M}_{g, n}(W)$. Given $S \rightarrow \mathscr{M}_{g, n}(V)$ which corresponds to a diagram (1), the composition $S \rightarrow \mathscr{M}_{g, n}(V) \rightarrow$ $\mathscr{M}_{g, n}(W)$ corresponds to

where the underlying map of $\bar{C} \rightarrow W$ is the stabilization of the underlying map $C \rightarrow V \rightarrow W$, and the $\log$ structure on $\bar{C}$ is the push forward of the $\log$ structure on $C$ with respect to the partial stabilization $C \rightarrow \bar{C}$. (See [5, appendix B].)
1.2.3. Perfect obstruction theories. We have a natural $\log \operatorname{map} \mathscr{M}_{g, n}(V) \rightarrow \mathfrak{M}_{g, n}$ forgetting the map to $V$. As $\log$ deformations for $\mathscr{M}_{g, n}(V) \rightarrow \mathfrak{M}_{g, n}$ is the same as deformations for the underlying map of $\mathscr{M}_{g, n}(V) \rightarrow \mathcal{T}^{\text {or }_{\mathfrak{M}_{g, n}}, \text { a (relative) perfect }}$ obstruction theory for the $\log \operatorname{map} \mathscr{M}_{g, n}(V) \rightarrow \mathfrak{M}_{g, n}$ is by definition a perfect obstruction theory for the underlying map of $\mathscr{M}_{g, n}(V) \rightarrow \mathcal{T}^{\left(\mathcal{M}_{\mathfrak{M}_{g, n}}\right.}$.

A perfect obstruction theory for $\mathscr{M}_{g, n}(V) \rightarrow \mathfrak{M}_{g, n}$ is defined in [13, Section 5], tangent space and obstruction space at $[f: C \rightarrow V] \in \mathscr{M}_{g, n}(V)$ are given by $H^{0}\left(f^{*} T_{V}\right)$ and $H^{1}\left(f^{*} T_{V}\right)$ respectively, where $T_{V}$ is the log tangent bundle of V.

If we have a factorization $\mathscr{M}_{g, n}(V) \rightarrow \mathfrak{N} \rightarrow \mathfrak{M}_{g, n}$, and $\mathfrak{N} \rightarrow \mathfrak{M}_{g, n}$ is log étale, a perfect obstruction theory for $\mathscr{M}_{g, n}(V) \rightarrow \mathfrak{M}_{g, n}$ induces a perfect obstruction theory for $\mathscr{M}_{g, n}(V) \rightarrow \mathfrak{N}$ since $\mathcal{T}^{\text {or }} \mathfrak{N} \rightarrow \mathcal{T}_{\text {or }_{\mathfrak{M}_{g, n}}}$ is étale.

## 2. Product formula in terms of virtual classes

2.1. Setup. We start by introducing relevant commutative diagrams, which are the log enhancement of those in [8].

We define a fs $\log$ stack $\mathfrak{D}$ by its functor of points. Given any fs log scheme $S$, a $\operatorname{map} S \rightarrow \mathfrak{D}$ corresponds to the data consisting of $n$-pointed prestable log curves $C, C^{\prime}, C^{\prime \prime}$ of genus $g$ over $S$, together with partial stabilizations $p^{\prime}: C \rightarrow C^{\prime}$ and $p^{\prime \prime}: C \rightarrow C^{\prime \prime}$, such that no component of $C$ is contracted by both $p^{\prime}, p^{\prime \prime}$. Note that as $\log$ curves over $S$, the $\log$ structure of $C^{\prime}$ resp. $C^{\prime \prime}$ is given by the pushforward of the $\log$ structure of $C$ along $p^{\prime}$ resp. $p^{\prime \prime}$.

Define $e: \mathfrak{D} \rightarrow \mathfrak{M}_{g, n}$ as the forgetful morphism taking $\left(p^{\prime}, p^{\prime \prime}\right)$ to $C$. It is proved in [8, Lemma 4] that the underlying map of $e$ is étale. As $e$ is strict, it is log étale.

We claim that the following commutative diagram is cartesian in $\mathbf{L o g S t}$,


Here the top horizontal arrow is determined by stabilization maps $\mathscr{M}_{g, n}(V \times W) \rightarrow$ $\mathscr{M}_{g, n}(V), \mathscr{M}_{g, n}(V \times W) \rightarrow \mathscr{M}_{g, n}(W)$ induced from the projections $\mathrm{pr}_{1}: V \times W \rightarrow$ $V, \mathrm{pr}_{2}: V \times W \rightarrow W$. The left vertical arrow is defined by retaining the curve together with its partial stabilizations with respect to $\mathrm{pr}_{1}, \mathrm{pr}_{2}$.

The reason for the diagram being cartesian is essentially the same as that in [8, Proposition 5]. For a fs log scheme S, a commutative diagram

corresponds to stable $\log$ maps $C^{\prime} \rightarrow V, C^{\prime \prime} \rightarrow W$ over $S$, together with stabilization between log curves $C \rightarrow C^{\prime}, C \rightarrow C^{\prime \prime}$. This recovers a stable log map $C \rightarrow C^{\prime} \times C^{\prime \prime} \rightarrow V \times W$. Indeed, $\mathfrak{D}$ is constructed to make (2) cartesian.

[^2]We then extend the above cartesian diagram in LogSt to


Here all squares are constructed by taking fiber product. If $V$ and $W$ have trivial $\log$ structures, this reduces to Diagram (2) of [8].

Definition 2.1. We say that the product formula (of virtual fundamental classes) holds for $V$ and $W$ if

$$
\begin{equation*}
h_{*}\left(\left[\mathscr{M}_{g, n}(V \times W)\right]^{\mathrm{vir}}\right)=\Delta^{!}\left(\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}\right) \tag{3}
\end{equation*}
$$

In [8], Behrend showed that the product formula holds for $V$ and $W$ when $V$ and $W$ are smooth projective schemes, or $\log$ smooth schemes with trivial log structures, our goal of this paper is to extend his result to the case when $W$ has nontrivial log structure.
2.2. Product formula in $\log$ GW theory. When $W$ has nontrivial $\log$ structure, we factor

$$
a: \mathscr{M}_{g, n}(V) \times \mathscr{M}_{g, n}(W) \rightarrow \mathfrak{M}_{g, n} \times \mathfrak{M}_{g, n}
$$

into

$$
\mathscr{M}_{g, n}(V) \times \mathscr{M}_{g, n}(W) \xrightarrow{a^{\prime}} \mathfrak{M}_{g, n} \times \mathcal{T}_{\text {or }_{\mathfrak{M}_{g, n}}} \rightarrow \mathfrak{M}_{g, n} \times \mathfrak{M}_{g, n}
$$

We then have the following commutative diagram in which all squares are cartesian in LogSt,


Main Lemma. (I) The underlying square diagrams in $\mathbf{S t}$ are all cartesian.
(II) The relative perfect obstruction theories for $a^{\prime}$ and $c^{\prime}$ are compatible.
(III) $\mathfrak{D}^{\prime}, \mathfrak{P}^{\prime}$ are irreducible, and $l^{\prime}$ is of degree 1.
(IV) $\phi^{\prime}$ is a l.c.i. and is compatible with $\Delta$ in the sense that the cotangent complex $\mathcal{L}_{\Delta}$ pulls back to $\mathcal{L}_{\phi^{\prime}}$.

Theorem 2.2. Using the notations in (4), we have

$$
h_{*}\left(\left[\mathscr{M}_{g, n}(V \times W)\right]^{\mathrm{vir}}\right)=\Delta^{!}\left(\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}\right)
$$

in the $\log G W$ setting, assuming the trivial log structure on $V$.
Proof. We have virtual pullbacks $c^{\prime!}, a^{\prime!}, \phi^{\prime!}$, and $\Delta^{!}$. (See [19, Section 3.1])
For the upper left square of diagram (4), we have

$$
h_{*}\left(\left[\mathscr{M}_{g, n}(V \times W)\right]^{\mathrm{vir}}\right)=a^{\prime!}\left[\mathfrak{P}^{\prime}\right]
$$

by (II), (III), and Costello's pushforward theorem [10, Theorem 5.0.1].
For the upper right square of diagram (4), note that $\phi^{\prime!}\left(\left[\mathfrak{M}_{g, n}\right] \times\left[\mathcal{T}_{\left.\left.\text {or }_{\mathfrak{M}_{g, n}}\right]\right)}\right.\right.$ equals $\left[\mathfrak{P}^{\prime}\right]$, and $a^{\prime!}\left(\left[\mathfrak{M}_{g, n}\right] \times\left[\mathcal{T}^{\prime}\right.\right.$ r $\left.\left._{\mathfrak{M}_{g, n}}\right]\right)$ is $\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}$. By [19, Theorem 4.3], we know that $a^{\prime!} \phi^{\prime!}=\phi^{\prime!} a^{\prime!}$, so

$$
a^{\prime!}\left[\mathfrak{P}^{\prime}\right]=\phi^{\prime!}\left(\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}\right)
$$

Now (IV) gives

$$
\left.\phi^{\prime!}\left(\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}\right]\right)=\Delta^{!}\left(\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}\right)
$$

and we can complete the proof by combining these equations.

### 2.3. Proof of Main Lemma.

(I). As $a^{\prime}$ is strict, the first row of squares are cartesian. Since $\mathfrak{M}_{g, n} \rightarrow \overline{\mathcal{M}}_{g, n}$ is saturated, the bottom square is cartesian.

For the second row, to prove the two squares are cartesian is the same as showing the squares in the following diagram are cartesian in St.


We claim that $l \circ \mathrm{pr}_{2} \circ \phi$ and $\mathrm{pr}_{2} \circ \phi$ are both saturated. $l \circ \mathrm{pr}_{2} \circ \phi$ is saturated because a partial stabilization map locally is given by forgetting marked points. (See the proof of [8, Proposition 3]. What we need is that Diagram (4) there to be commutative.) $\mathrm{pr}_{2} \circ \phi$ is saturated as it is the base change by $\mathfrak{M}_{g . n} \rightarrow \overline{\mathcal{M}}_{g, n}$


Thus, both squares in (5) are cartesian.
(II). We remark that as $\mathcal{T}^{\operatorname{or}} \mathfrak{M}_{g, n} \rightarrow \mathfrak{M}_{g, n}$ is $\log$ étale, $\mathfrak{D}^{\prime}$ is log étale over $\mathfrak{D}$. Thus the relative perfect obstruction theory for the log map $\mathscr{M}_{g, n}(V \times W) \rightarrow \mathfrak{M}_{g, n}$ can be viewed as a relative perfect obstruction theory for the underlying map of $c^{\prime}$. Compatibility check is the same as in [8, Propsition 6].
(III). Start with Diagram (5). The open embedding $\mathfrak{M}_{g, n} \rightarrow \mathcal{T}_{\text {or }_{\mathfrak{M}_{g, n}}}$ induces

$\mathfrak{M}_{g, n}$ being $\log$ smooth, it is open and dense in $\mathcal{T}_{\text {or }}^{\mathfrak{M}_{g, n}}$. Note that both $\mathrm{pr}_{2} \circ \phi, l \circ$ $\mathrm{pr}_{2} \circ \phi$ are flat surjective, we conclude $\mathfrak{D}$ (resp. $\mathfrak{P}$ ) are open dense substacks of $\mathfrak{D}^{\prime}$ (resp. $\mathfrak{P}^{\prime}$ ). (III) then follows from properties of $\mathfrak{D}, \mathfrak{P}$ and $l$. (See [8, Proposition 3]).


$$
\mathfrak{M}_{g, n} \times \mathcal{T}_{\operatorname{or}_{M_{g, n}}} \rightarrow \overline{\mathcal{M}}_{g, n} \times \mathcal{T}_{\operatorname{or}_{\mathfrak{M}_{g, n}}} \rightarrow \overline{\mathcal{M}}_{g, n} \times \mathcal{T}_{\operatorname{or}_{\mathcal{M}_{g, n}}} \rightarrow \overline{\mathcal{M}}_{g, n} \times \overline{\mathcal{M}}_{g, n}
$$

First arrow is saturated and flat, and second arrow strict and smooth.
For the third arrow, note that

is cartesian in $\mathbf{L o g S t}$ as well as in $\mathbf{S t}$, and horizontal arrows are compatible l.c.i. maps. It is then easy to see $\phi^{\prime}$ and $\Delta$ are compatible.
2.4. Product formula for families. As we need the product formula for equivariant invariants in [16], we will adapt the arguments above to families.

Let $X \rightarrow S$ and $Y \rightarrow T$ be two families of log smooth projective varieties over $\log$ smooth and irreducible bases. We would like to relate GW invariants of the family $X \times Y \rightarrow S \times T$ to those of $X \rightarrow S$ and $Y \rightarrow T$.

Denote by $\mathscr{M}_{g, n}(X / S), \mathscr{M}_{g, n}(Y / T), \mathscr{M}_{g, n}(X \times Y / S \times T)$ log stacks of stable log maps from genus $g$ curves with $n$ marked points to these families.

Remark 2.3. Let $X \rightarrow S$ be a family of $\log$ smooth projective varieties over a $\log$ stack $S$. As a cartesian digram

in $\mathbf{L o g} \mathbf{S t}$ induces a cartesian diagram

in LogSt. It follows that $\mathscr{M}_{g, n}(X / S)$ is an algebraic stack locally of finite type over $S$ by [13, Theorem 0.1].

It is easy to see we have a cartesian diagram in LogSt

and we consider the diagram

as before.
Theorem 2.4. Let $X \rightarrow S$ be a family of smooth projective varieties over a smooth, pure dimensional stack $S$, and $Y \rightarrow T$ a family of log smooth projective varieties over a stack $T$ which is log smooth and irreducible.

Then we have

$$
h_{*}\left(\left[\mathscr{M}_{g, n}(X \times Y / S \times T)\right]^{\mathrm{vir}}\right)=\Delta^{!}\left(\left[\mathscr{M}_{g, n}(X / S)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(Y / T)\right]^{\mathrm{vir}}\right)
$$

Proof. Consider the cartesian diagram in LogSt,


As $T$ is $\log$ smooth, $\mathcal{T}_{\operatorname{or}_{\mathfrak{M}}^{g, n}} \times T \rightarrow \mathcal{T}_{\text {or }_{\mathfrak{M}_{g, n}}}$ is smooth, the perfect obstruction theory for $\mathscr{M}_{g, n}(Y / T) \rightarrow \mathfrak{M}_{g, n} \times T$ then induces a perfect obstruction theory for the composition

$$
\mathscr{M}_{g, n}(Y / T) \rightarrow \mathfrak{M}_{g, n} \times T \rightarrow \mathfrak{M}_{g, n}
$$

by the arguments in [12, Appendix B]. Similarly, we have a perfect obstruction theory for

$$
\mathscr{M}_{g, n}(X / S) \rightarrow \mathfrak{M}_{g, n} \times S \rightarrow \mathfrak{M}_{g, n}
$$

and

$$
\mathscr{M}_{g, n}(X \times Y / S \times T) \rightarrow \mathfrak{D} \times(S \times T) \rightarrow \mathfrak{D}
$$

Now we are in the same situation as in Section 2.2, identical arguments finish the proof. Note that (II) follows from standard properties of cotangent complexes of algebraic stacks.

## 3. Remarks for the general case

3.1. When $V$ has nontrivial $\log$ structure, we might consider factoring $a$ by

In this case, the underlying diagram of

is no longer cartesian in St. If it is true that $\mathscr{M}_{g, n}(V) \rightarrow \mathfrak{M}_{g, n}$ is saturated, we can replace $\mathcal{T}^{\text {or }} \mathfrak{M}_{g_{, n}}$ by its open substack parameterizing saturated maps and the argument we used can be adapted to this more general setting.

However, the map $\mathscr{M}_{g, n}(V) \rightarrow \mathfrak{M}_{g, n}$ is not even integral in general, and further understanding concerning the log structure of $\mathscr{M}_{g, n}(V)$ seems necessary.
3.2. To prove the product formula holds for $V$ and $W$, we can assume $V$ and $W$ are smooth and log smooth. This is achieved by using desingularizations of $\log$ smooth schemes and invariance of virtual classes under certain log modifications.

By [20, Theorem 5.10], for any log smooth variety $X$, there exists a log blow up $\pi: Y \rightarrow X$ such that $Y$ is smooth, $\log$ smooth and $\pi$ is birational. In particular, $\pi$ is proper, birational, and $\log$ étale.
Lemma 3.1. Let $\Phi: \widetilde{V} \rightarrow V$ and $\Psi: \widetilde{W} \rightarrow W$ be proper, birational, log étale maps between $\log$ smooth projective varities. If the product formula holds for $\widetilde{V}$ and $\widetilde{W}$, then it holds for $V$ and $W$.
Proof. Consider the diagram

where

$$
\begin{aligned}
& \mathscr{M}(\Phi): \mathscr{M}_{g, n}(\widetilde{V}) \rightarrow \mathscr{M}(V), \\
& \mathscr{M}(\Psi): \mathscr{M}_{g, n}(\widetilde{W}) \rightarrow \mathscr{M}(W),
\end{aligned}
$$

and

$$
\mathscr{M}(\Psi \times \Psi): \mathscr{M}_{g, n}(\widetilde{V} \times \widetilde{W}) \rightarrow \mathscr{M}(V \times W)
$$

are maps between moduli stacks induced by $\Phi, \Psi$ and $\Phi \times \Psi$ respectively.
As $\Phi$ is proper, birational, log étale, by [6, Theorem 1.1.1],

$$
\mathscr{M}(\Phi)_{*}\left[\mathscr{M}_{g, n}(\widetilde{V})\right]^{\mathrm{vir}}=\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}}
$$

Similarly, we have

$$
\mathscr{M}(\Psi)_{*}\left[\mathscr{M}_{g, n}(\widetilde{W})\right]^{\mathrm{vir}}=\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}
$$

and

$$
\mathscr{M}(\Psi \times \Psi)_{*}\left[\mathscr{M}_{g, n}(\widetilde{V} \times \widetilde{W})\right]^{\mathrm{vir}}=[\mathscr{M}(V \times W)]^{\mathrm{vir}}
$$

Now pushforwarding the relation

$$
\widetilde{h}_{*}\left[\mathscr{M}_{g, n}(\widetilde{V} \times \widetilde{W})\right]^{\mathrm{vir}}=\Delta^{!}\left(\left[\mathscr{M}_{g, n}(\widetilde{V})\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(\widetilde{W})\right]^{\mathrm{vir}}\right)
$$

along $\widetilde{P} \rightarrow P$ gives

$$
h_{*}\left(\left[\mathscr{M}_{g, n}(V \times W)\right]^{\mathrm{vir}}\right)=\Delta^{!}\left(\left[\mathscr{M}_{g, n}(V)\right]^{\mathrm{vir}} \times\left[\mathscr{M}_{g, n}(W)\right]^{\mathrm{vir}}\right)
$$

Remark 3.2. Let $V$ be a smooth projective variety with log structure coming from a simple normal crossing divisor $\cup D_{i}$. Motivated by the results proved in [1, 5], we expect that genus zero $\log \mathrm{GW}$ invariants of $V$ are related to orbifold GW invariants of root stacks $V\left(\sqrt[r]{D_{i}}\right)$. If this naive expectation is valid, then the product formula for orbifolds proved in [7] would imply the product formula of log GW invariants in genus zero.

## 4. Applications to relative Gromov-Witten invariants

We apply this to the relative GW invariants as defined by A. Li and Y. Ruan [17], and J. Li [18].

Let $X$ and $Y$ be nonsingular projective varieties, and $D$ a a smooth divisor in $Y$. We further assume $H^{1}(Y)=0$, so a curve class of $X \times Y$ is of the form $\left(\beta_{X}, \beta_{Y}\right)$ where $\beta_{X}$ (resp. $\beta_{Y}$ ) is a curve class of $X$ (resp. $Y$ ).

Let $\overline{\mathcal{M}}_{\Gamma_{Y}}(Y, D)$ be the relative moduli stack. Here $\Gamma_{Y}=\left(g, n, \beta_{Y}, \rho, \mu\right)$ encodes the discrete data: $g$ for the genus, $n+\rho$ for the number of marked points, $\beta_{Y}$ the curve class, $\mu=\left(\mu_{1}, \ldots \mu_{\rho}\right)$ an ordered partition of $\int_{\beta_{Y}}[D]$.

We have evaluation maps

$$
e v_{Y}: \overline{\mathcal{M}}_{\Gamma_{Y}}(Y, D) \rightarrow Y^{n}, \quad e v_{D}: \overline{\mathcal{M}}_{\Gamma_{Y}}(Y, D) \rightarrow D^{\rho}
$$

and the stabilization map

$$
\pi: \overline{\mathcal{M}}_{\Gamma_{Y}}(Y, D) \rightarrow \overline{\mathcal{M}}_{g, n+\rho}
$$

Relative GW invariants can be viewed as the Gromov-Witten transformation

$$
R_{\Gamma_{Y}}: H^{*}(Y)^{\otimes n} \otimes H^{*}(D)^{\otimes \rho} \rightarrow H^{*}\left(\overline{\mathcal{M}}_{g, n+\rho}\right)
$$

defined as

$$
\operatorname{PD}\left(\pi_{*}\left(e v_{Y}^{*}(\alpha) e v_{D}^{*}(\delta) \cap\left[\overline{\mathcal{M}}_{\Gamma_{Y}}(Y, D)\right]^{v i r}\right)\right)
$$

where PD stands for the Poincaré duality.

Let $\Gamma_{X \times Y}=\left(g, n,\left(\beta_{X}, \beta_{Y}\right), \rho, \mu\right)$. Similarly we have

$$
R_{\Gamma_{X \times Y}}: H^{*}(X \times Y)^{\otimes n} \otimes H^{*}(X \times D)^{\otimes \rho} \rightarrow H^{*}\left(\overline{\mathcal{M}}_{g, n+\rho}\right)
$$

Let $\Gamma_{X}=\left(g, n+\rho, \beta_{X}\right)$. The map

$$
R_{\Gamma_{X}}: H^{*}(X)^{\otimes(n+\rho)} \rightarrow H^{*}\left(\overline{\mathcal{M}}_{g, n+\rho}\right)
$$

is the Gromov-Witten correspondence $R_{g, n+\rho, \beta_{X^{\prime}}}^{X}$ defining a cohomological field theory.

## Corollary 4.1.

$$
\begin{aligned}
& R_{\Gamma_{X \times Y}}\left(\left(\alpha_{1} \otimes \alpha_{1}^{\prime}\right) \otimes \ldots \otimes\left(\left(\alpha_{n} \otimes \alpha_{n}^{\prime}\right)\right) ;\left(\alpha_{n+1} \otimes \delta_{1}\right) \otimes \ldots \otimes\left(\left(\alpha_{n+\rho} \otimes \delta_{\rho}\right)\right)\right. \\
= & R_{\Gamma_{X}}\left(\alpha_{1} \otimes \ldots \otimes \alpha_{n+\rho}\right) R_{\Gamma_{Y}}\left(\alpha_{1}^{\prime} \otimes \ldots \otimes \alpha_{n}^{\prime} ; \delta_{1} \otimes \ldots \otimes \delta_{\rho}\right)
\end{aligned}
$$

where $\alpha_{i} \in H^{*}(X), \alpha_{i}^{\prime} \in H^{*}(Y)$ and $\delta_{j} \in H^{*}(D)$.
Proof. This follows directly from Theorem 2.2 and the comparison result between relative and $\log$ GW invariants in [5, Theorem 1.1, Section 2.3].

Acknowledgements. We wish to thank Q. Chen, W. D. Gillam, and S. Marcus for helpful discussions, and the referee for valuable comments. This research is supported in part by the NSF.

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[^0]:    2010 Mathematics Subject Classification. Primary 14N35; Secondary 14A20, 14C17.
    Key words and phrases. log Gromov-Witten invariants, product formula.

[^1]:    ${ }^{1}$ In a higher categorical sense. In other words, $\mathbf{L o g S t}$ and $\mathbf{S t}$ are 2-categories, or $(2,1)$ categories.

[^2]:    ${ }^{2}$ Note that $\mathscr{M}_{g, n}(V)(S) \rightarrow \mathfrak{M}_{g, n}(S)$ and $\mathscr{M}_{g, n}(W)(S) \rightarrow \mathfrak{M}_{g, n}(S)$ are iso-fibrations of groupoids.

