

Virtual Fundamental Classes of Zero Loci

David A. Cox

Department of Mathematics and Computer Science
Amherst College
Amherst, MA 01002-5000
dac@cs.amherst.edu

Sheldon Katz

Department of Mathematics
Oklahoma State University
Stillwater, OK 74078-0613
katz@math.okstate.edu

Yuan-Pin Lee

Department of Mathematics
University of California at Los Angeles
Los Angeles, CA 90095-1555
yplee@math.ucla.edu

1 Introduction and a conjecture

Let X be a smooth projective variety over \mathbb{C} . For each nonnegative integer n and homology class $\beta \in H_2(X, \mathbb{Z})$, we are interested in n -pointed genus 0 stable maps to X of class β . The concept of a stable map was introduced in [Kontsevich]. We will for the most part follow the notation of [CK]. In particular, such a stable map is

$$f : (C, p_1, \dots, p_n) \rightarrow X,$$

where $p_a(C) = 0$ and $f_*[C] = \beta$. Families of stable maps can be described either by a coarse moduli space or by a stack. Here, we will use the language of stacks, and we will denote this stack by $\overline{M}_{0,n}(X, \beta)$. For more about stacks, see [DM, BEFFGK]. Roughly speaking, $\overline{M}_{0,n}(X, \beta)$ is a functor

$$\overline{M}_{0,n}(X, \beta) : (\mathbb{C}\text{-Schemes}) \longrightarrow (\text{Sets})$$

which associates to any scheme S over \mathbb{C} the set of all isomorphism classes of families of n -pointed genus 0 stable maps to X over S in the class β . (As with many moduli problems, $\overline{M}_{0,n}(X, \beta)$ is actually a groupoid rather than a functor.)

This is explained in the appendix to [Vistoli].) One can show that $\overline{M}_{0,n}(X, \beta)$ is a Deligne-Mumford stack [BM]. The associated coarse moduli space was shown to exist in [Alexeev] and was explicitly constructed in [FP].

It is well known that the stack $\overline{M}_{0,n}(X, \beta)$ can have pathological behavior. For instance, even its dimension does not remain constant under deformation of complex structure of X . The problem is due to obstructions, which arise in a very general setting and can behave erratically.

Specifically, the tangent space to $\overline{M}_{0,n}(X, \beta)$ at $f : (C, p_1, \dots, p_n) \rightarrow X$ is

$$\mathrm{Ext}_C^1(f^*\Omega_X^1 \rightarrow \Omega_C^1(\sum_{i=1}^n p_i), \mathcal{O}_C),$$

while the obstructions lie in

$$\mathrm{Ext}_C^2(f^*\Omega_X^1 \rightarrow \Omega_C^1(\sum_{i=1}^n p_i), \mathcal{O}_C).$$

The dimensions of these Ext^1 s and Ext^2 s can vary as the complex structure of X varies, but the difference $\dim(\mathrm{Ext}^1) - \dim(\mathrm{Ext}^2)$ remains constant, equal to

$$d = \int_{\beta} c_1(X) + n + \dim X - 3.$$

This quantity is called the *virtual dimension* of $\overline{M}_{0,n}(X, \beta)$.

These considerations lead to natural notions of an obstruction theory, and an associated virtual fundamental class [LT, BF]

$$[\overline{M}_{0,n}(X, \beta)]^{\mathrm{virt}} \in A_d(\overline{M}_{0,n}(X, \beta)).$$

The virtual fundamental class is invariant under deformations of the complex structure of X in a precise sense. For example, if X is a Calabi-Yau threefold and $n = 0$, then the virtual dimension is 0, and we have the Gromov-Witten invariant

$$N_{\beta} = \deg [\overline{M}_{0,0}(X, \beta)]^{\mathrm{virt}} \in \mathbb{Q}.$$

This number is independent of the complex structure of X , and is closely related to the “number of rational curves” on X in the homology class β . One approach to making sense of the “number of curves” is to introduce the notion of an *instanton number* n_{β} defined recursively by

$$(1) \quad N_{\beta} = \sum_{k|\beta} \frac{n_{\beta/k}}{k^3}.$$

The individual terms in (1) morally account for the contribution to N_{β} of degree k covers of $n_{\beta/k}$ distinct embedded curves with homology class β/k . The n_{β} are conjectured to be integers. However, even if this integrality were proven, these numbers are not quite the same as the elusive “number of rational curves”.¹ See [CK, Section 7.4.4] for further discussion.

¹The n_{β} and their generalizations to higher genus have a direct meaning in physics [GV]. If these ideas could be transformed into rigorous mathematics, integrality would follow immediately. Some evidence is presented in [BKL, BP, KKV].

We are now ready to formulate our conjecture. For each $k = 1, \dots, n$ we have the usual evaluation maps $e_k : \overline{M}_{0,n}(X, \beta) \rightarrow X$, which on geometric points take $f : (C, p_1, \dots, p_n) \rightarrow X$ to $f(p_k)$. Furthermore, the universal stable curve over $\overline{M}_{0,n}(X, \beta)$ is the map $\pi_{n+1} : \overline{M}_{0,n+1}(X, \beta) \rightarrow \overline{M}_{0,n}(X, \beta)$ which ignores the last marked point and contracts any components which have become unstable. In Section 3, we will also need to use the n tautological sections s_j for $j = 1, \dots, n$, where s_j takes a pointed stable map to the point p_j , identifying the fiber of the universal curve over $f : (C, p_1, \dots, p_n) \rightarrow X$ with C . Further details of this situation are given in [CK, Section 10.1.1].

Now let V be a convex vector bundle on X , so that $H^1(f^*V) = 0$ for all genus 0 stable maps f to X . We then put

$$\mathcal{V}_{\beta,n} = (\pi_{n+1})_* e_{n+1}^* V,$$

which is a vector bundle on $\overline{M}_{0,n}(X, \beta)$ by the convexity of V . Let $i : Y \hookrightarrow X$ be the inclusion of the zero locus of a regular section of V . For any homology class $\gamma \in H_2(Y, \mathbb{Z})$, this induces a natural inclusion $j_\gamma : \overline{M}_{0,n}(Y, \gamma) \hookrightarrow \overline{M}_{0,n}(X, i_*\gamma)$.

Conjecture 1.1. *For any $\beta \in H_2(X, \mathbb{Z})$, we have*

$$(2) \quad \sum_{i_*\gamma=\beta} (j_\gamma)_* [\overline{M}_{0,n}(Y, \gamma)]^{\text{virt}} = c_{\text{top}}(\mathcal{V}_{\beta,n}) \cap [\overline{M}_{0,n}(X, \beta)]^{\text{virt}}.$$

Here and throughout the rest of the paper, we use the notation $c_{\text{top}}(E)$ to denote the top chern operator of the vector bundle E .

As a special case, this conjecture explains Kontsevich's original formulation of the Gromov-Witten invariants of the quintic threefold. Here, $X = \mathbb{P}^4$, $V = \mathcal{O}(5)$, and Y is a quintic threefold. Homology classes on Y and on \mathbb{P}^4 can be identified with the degree d . Putting $n = 0$, Conjecture 1.1 reads

$$(j_d)_* [\overline{M}_{0,0}(Y, d)]^{\text{virt}} = c_{\text{top}}(\mathcal{V}_{d,0}) \cap [\overline{M}_{0,0}(\mathbb{P}^4, d)] = \text{Euler}(\mathcal{V}_{d,0}),$$

since $[\overline{M}_{0,0}(\mathbb{P}^4, d)]^{\text{virt}}$ is the usual fundamental class by the convexity of \mathbb{P}^4 . We then have

$$(3) \quad N_d = \deg [\overline{M}_{0,0}(Y, d)]^{\text{virt}} = \int_{\overline{M}_{0,0}(\mathbb{P}^4, d)} \text{Euler}(\mathcal{V}_{d,0}),$$

agreeing with Kontsevich's definition [Kontsevich]. A proof of the conjecture in this case can be found in [CK, Example 7.1.5.1].

The intuition behind Conjecture 1.1 is easy to explain. If Y is the zero locus of a regular section s of V , then (roughly speaking), we should have the following:

- The fiber of $\mathcal{V}_{d,0}$ over $f : C \rightarrow X$ is $H^0(f^*(V))$. Thus $f \mapsto f^*(s)$ is a section of $\mathcal{V}_{d,0}$ over $\overline{M}_{0,n}(X, \beta)$.
- This section vanishes when f factors through $i : Y \hookrightarrow X$. Thus the zero locus of this section is $\coprod_{i_*\gamma=\beta} \overline{M}_{0,n}(Y, \gamma)$.

- At the level of fundamental classes, the zero locus of a regular section of a vector bundle is given by evaluation of its top chern operator on the fundamental class.

In a few cases (including the quintic three discussed above), this naive reasoning can be made rigorous. In the general case, the presence of the virtual fundamental class makes the situation more complicated, though one expects the conjecture to follow easily once the general properties of the virtual fundamental class are fully understood. We will give a special case of how this works in the next section.

We should also mention that Conjecture 1.1 generalizes the conjecture stated in [CK, (11.81)]. As will we see in Section 3, this has implications for the Gromov-Witten invariants of certain nef complete intersections in smooth toric varieties.

2 Proof in a special case

The basic idea of this section is that Conjecture 1.1 is true when V is an *embedding vector bundle* on X . This is defined as follows. Let $G(r, k)$ be the Grassmannian of r dimensional quotients of \mathbb{C}^k .

Definition 2.1. *A vector bundle V of rank r on X is an **embedding bundle** if V is generated by global sections and the map $X \rightarrow G(r, k)$, $k = h^0(X, V)$, induced by the exact sequence*

$$H^0(X, V) \otimes \mathcal{O}_X \rightarrow V \rightarrow 0$$

is an embedding.

In particular, when V is an embedding bundle, it is the restriction of the universal quotient bundle \mathcal{Q} on $G(r, k)$. Of course, when $r = 1$, “embedding” is equivalent to “very ample”. However, when $r > 1$, “embedding” differs from the notion of “positive” or “ample” vector bundle defined in [Griffiths] or [Hartshorne].

We can now prove our conjecture in the case of an embedding bundle.

Proposition 2.2. *Conjecture 1.1 is true when the vector bundle V on X is embedding and $Y \subset X$ is the zero locus of a generic section of V .*

Proof. We will use an argument in [Gathmann] together with standard facts about refined Gysin maps from [Fulton]. The rough idea of the proof is that the conjecture is obviously true for a Grassmannian $G(r, k)$, and then the results of [Gathmann] and [Fulton] will show that it remains true when we pull back from $G(r, k)$ to X .

To see how this works in detail, we first recall some notation. Suppose we

have a cartesian diagram of Deligne-Mumford stacks:

$$(4) \quad \begin{array}{ccc} M & \xrightarrow{j} & M_2 \\ \downarrow & & \downarrow f \\ M_1 & \xrightarrow{i} & S \end{array}$$

where $M_1 \rightarrow S$ is a regular embedding. In this situation, we have the refined Gysin map $i^! : A_*(M_2) \rightarrow A_*(M)$ defined in [Fulton, Chapter 6], where we use [Vistoli] to extend from schemes to stacks.

Now assume that there is also a vector bundle \mathcal{V} on S such that $i : M_1 \subset S$ is the inclusion of the zero locus of a regular section of \mathcal{V} . By [Fulton, Ex. 6.3.4], it follows that

$$(5) \quad j_* i^! (\gamma) = c_{\text{top}}(f^* \mathcal{V}) \cap \gamma$$

for all $\gamma \in A_*(M_2)$, where j and f are from (4).

To apply this to our situation, suppose that V is an embedding bundle on X and $Y \subset X$ is the zero locus of a section s of V . By Definition 2.1, we can assume that X is embedded in the Grassmannian $G(r, k)$. Furthermore, s induces a section $s_{\mathcal{Q}}$ of the universal quotient bundle \mathcal{Q} on $G(r, k)$ via the tautological quotient mapping

$$H^0(X, V) \otimes \mathcal{O}_{G(r, k)} \rightarrow \mathcal{Q}.$$

Let $G \subset G(r, k)$ be the zero locus of $s_{\mathcal{Q}}$. It follows from this description that $s_{\mathcal{Q}}$ is a regular section of \mathcal{Q} . This description also implies that $G \simeq G(r, H^0(X, V)/\mathbb{C} \cdot s) \simeq G(r, k - 1)$ and that $Y = X \cap G$.

Now fix $\beta \in H_2(X)$ and suppose that β maps to $d \in H_2(G(r, k)) \simeq H_2(G) \simeq \mathbb{Z}$. Then we have a cartesian diagram:

$$(6) \quad \begin{array}{ccc} \coprod_{i_* \gamma = \beta} \overline{M}_{0, n}(Y, \gamma) & \xrightarrow{j} & \overline{M}_{0, n}(X, \beta) \\ \downarrow & & \downarrow f \\ \overline{M}_{0, n}(G, d) & \xrightarrow{i} & \overline{M}_{0, n}(G(r, k), d), \end{array}$$

where i and f are the natural inclusions.

Since $G(r, k)$ and $G \simeq G(r, k - 1)$ are homogeneous spaces, it follows that $\overline{M}_{0, n}(G, d) \rightarrow \overline{M}_{0, n}(G(r, k), d)$ is a regular embedding of smooth stacks. Applying the above construction, we get the class

$$i^!([\overline{M}_{0, n}(X, \beta)]^{\text{virt}}) \in \sum_{i_* \gamma = \beta} A_*(\overline{M}_{0, n}(Y, \gamma)).$$

Using the argument of [Gathmann, Lemma 4.2], we see that

$$(7) \quad i^!([\overline{M}_{0, n}(X, \beta)]^{\text{virt}}) = \sum_{i_* \gamma = \beta} [\overline{M}_{0, n}(Y, \gamma)]^{\text{virt}}.$$

Gathmann's lemma uses \mathbb{P}^N and $H \simeq \mathbb{P}^{N-1}$ rather than $G(r, k)$ and $G \simeq G(r, k-1)$, but the proof still applies since $G(r, k)$ and G are convex. Also, while the statement of Lemma 4.2 in [Gathmann] is different from (7), the final sentence of his proof shows that (7) follows from his argument.

In Section 1, we discussed how V induces the bundle $\mathcal{V}_{\beta, n}$ on $\overline{M}_{0, n}(X, \beta)$. In a similar way, the universal quotient bundle \mathcal{Q} induces a bundle $\mathcal{V}_{d, n} = (\pi_{n+1})_* \mathcal{E}_{n+1}^* \mathcal{Q}$ on $\overline{M}_{0, n}(G(r, k), d)$. The map $i : \overline{M}_{0, n}(G, d) \rightarrow \overline{M}_{0, n}(G(r, k), d)$ is the zero locus of a section of $\mathcal{V}_{d, n}$ by [Pandharipande, Sect. 2.1]. Using (5), it follows that

$$j_* i^!(\gamma) = c_{\text{top}}(f^* \mathcal{V}_{d, n}) \cap \gamma$$

for all $\gamma \in A_*(\overline{M}_{0, n}(X, \beta))$, where j and f are from (6). Since $f^* \mathcal{V}_{d, n} = \mathcal{V}_{\beta, n}$, we obtain

$$j_* i^!([\overline{M}_{0, n}(X, \beta)]^{\text{virt}}) = c_{\text{top}}(\mathcal{V}_{\beta, n}) \cap [\overline{M}_{0, n}(X, \beta)]^{\text{virt}}.$$

Combining this with (7), the proposition follows. \square

Remark 2.3. The obvious embedding

$$G(r_1, k_1) \times G(r_2, k_2) \hookrightarrow G(r_1 + r_2, k_1 + k_2)$$

shows that a direct sum of embedding bundles is embedding. In particular, a direct sum of very ample line bundles is an embedding bundle. Thus Conjecture 1.1 holds when Y is a generic complete intersection of very ample hypersurfaces in X .

In some versions of the mirror theorem, the ambient space is a smooth toric variety. Here, a line bundle is very ample if and only if it is ample, so that Conjecture 1.1 holds when Y is a generic complete intersection of ample hypersurfaces in a smooth toric variety X .

In Remark 4.5 of [Gathmann], Gathmann says that he expects that his results should hold under the weaker hypothesis that V is generated by global sections. (Gathmann only considers line bundles, but if his argument extends to line bundles generated by global sections, then it should also work for vector bundles generated by global sections.)

For a smooth (or simplicial) toric variety X , one can easily show that a line bundle is convex if and only if it is generated by global sections (this follows from Reid's description [Reid, Prop. 1.6] of the Mori cone of X). Thus, if Gathmann's result can be extended to vector bundles generated by global sections, then it would follow that Conjecture 1.1 would hold when the vector bundle V is a direct sum of convex line bundles on a smooth toric variety X .

3 Relation to mirror theorems

This section will discuss the relationship between Conjecture 1.1 and various approaches to the mirror theorem.

This basic idea is that computing Gromov–Witten invariants of hypersurfaces $Y \subset X$ (or more generally, the zero locus of a section of a vector bundle V on X) is a two-stage process:

- First, one must relate the Gromov–Witten invariants of Y to invariants defined using the ambient space X and the vector bundle V . Conjecture 1.1 shows how to do this.
- Second, one must compute the invariants defined using X and V . This is what the various *mirror theorems* in the literature do.

We will briefly discuss three approaches to the mirror theorem and describe the role of equation (2) from Conjecture 1.1 in each of these (related) approaches.

Givental’s approach [Givental1] was largely based on the fact that the quantum cohomology of X can be described by a quantum \mathcal{D} -module generated by a single formal function, which we denote by J_X . We recall the definition, following [CK, Chapters 10 and 11].

Let $\{T_0, \dots, T_m\}$ be a basis for $H^*(X)$ with $T_0 = 1$, and $\{T_1, \dots, T_r\}$ a basis for $H^2(X)$. Let $\{T^a\}$ be the dual basis under the intersection pairing. Introduce variables t_i and put $\delta = \sum_{i=1}^r t_i T_i$.

We recall the definition of the gravitational correlators. The universal curve $\overline{M}_{0,2}(X, \beta) \rightarrow \overline{M}_{0,1}(X, \beta)$ has the section s_1 as described in Section 1. We put $\mathcal{L} = s_1^*(\omega)$, where ω is the relative dualizing sheaf of $\overline{M}_{0,2}(X, \beta) \rightarrow \overline{M}_{0,1}(X, \beta)$. Then the 1-point genus 0 gravitational correlators are defined by the equation

$$(8) \quad \langle \tau_n T_a \rangle_{0, \beta} = \int_{\overline{M}_{0,1}(X, \beta)} c_1(\mathcal{L})^n \cup e_1^*(T_a) \cap [\overline{M}_{0,1}(X, \beta)]^{\text{virt}}.$$

Let q^β be formal symbols satisfying $q^\beta \cdot q^{\beta'} = q^{\beta+\beta'}$ and let \hbar be a formal parameter. Then we define the formal function J_X by the equation

$$(9) \quad J_X = e^{(t_0+\delta)/\hbar} \left(1 + \sum_{\beta \neq 0} \sum_{a=0}^m \sum_{n=0}^{\infty} \hbar^{-(n+2)} \langle \tau_n T_a \rangle_{0, \beta} T^a q^\beta \right).$$

The goal is to relate J_Y to a variant of J_X which takes the bundle V into account. We denote the kernel of the natural “evaluation map” $\mathcal{V}_{\beta,1} \rightarrow e_1^*V$ by $\mathcal{V}'_{\beta,1}$. We then put

$$(10) \quad J_{\mathcal{V}} = e^{(t_0+\delta)/\hbar} \text{Euler}(\mathcal{V}) \left(1 + \sum_{\beta \neq 0} q^\beta e_{1!} \left(\frac{\text{Euler}(\mathcal{V}'_{\beta,1})}{\hbar(\hbar-c)} \right) \right),$$

where the expression $1/(\hbar-c)$ is understood to be expanded as

$$\frac{1}{\hbar-c} = \sum_{n=0}^{\infty} c^n \hbar^{-(n+1)} = \sum_{n=0}^{\infty} c_1(\mathcal{L})^n \hbar^{-(n+1)}.$$

It was shown in [CK] Section 11.2 that when $X = \mathbb{P}^n$, Conjecture 1.1 implies that

$$(11) \quad i_* J_Y = J_{\mathcal{V}},$$

relating the Gromov-Witten invariants on Y to invariants defined using \mathbb{P}^n and V , as desired.

Givental’s approach to the mirror theorem computes $J_{\mathcal{V}}$ explicitly in terms of hypergeometric functions. This is described in [Givental2] (see the references therein) and Theorem 11.2.16 of [CK], and is extended by the Quantum Hyperplane Section Principle of [Kim]. Once one has $J_{\mathcal{V}}$, one needs a formula such as (11) in order to compute the Gromov-Witten invariants of Y .

For convex bundles on projective space, a result analogous to (11) is proved in [CKYZ], with an extension to direct sums of convex/concave line bundles on projective space in [Elezi]. Hence (11) and its analogues provide a bridge between Gromov-Witten invariants on Y and hypergeometric functions. Once Conjecture 1.1 is proved, we expect that similar formulas should hold whenever Y is the zero locus of a regular section of a convex vector bundle V on X . This will be useful when generalizing the results of [CKYZ, Elezi] to other ambient spaces.

Another approach to the mirror conjecture is the “Mirror Principle” proposed by Lian, Liu and Yau [LLY1, LLY2, LLY3, LLY4]. In the third of these papers, the basic object of interest is the integral

$$(12) \quad K_{\beta} = \int_{[\overline{M}_{0,0}(X,\beta)]^{\text{virt}}} b(\mathcal{V}_{\beta,0}),$$

where:

- X is a projective manifold and $\beta \in H_2(X)$.
- V is the concavex vector bundle on X . (A *concavex bundle* is a direct sum of a convex bundle and a concave bundle.)
- When V is convex, $\mathcal{V}_{\beta,0}$ is the vector bundle $\pi_* e_1^* V$ on $\overline{M}_{0,0}(X,\beta)$ as defined in Section 1. When V is concave, $\mathcal{V}_{\beta,0} = R^1 \pi_* e_1^* V$.
- b is a multiplicative characteristic class, such as the Euler class.

The paper [LLY3] discusses the properties of the generating function which has the K_{β} as coefficients. Explicit formulas for this generating function are given when X is a balloon manifold and b is either the Euler class or the Chern polynomial [LLY3, section 4].

It follows that there are many situations where the K_{β} can be computed. This raises the question of interpreting these numbers in terms of Gromov-Witten invariants. Let us explain how this works when V is a convex bundle on X and $Y \subset X$ is the zero locus of a regular section of V .

In this situation, suppose that the multiplicative characteristic class b is the Euler class and assume also that Conjecture 1.1 holds for V and $Y \subset X$. Then

the integral K_β defined in (12) is a sum of Gromov-Witten invariants of Y as follows:

$$K_\beta = \int_{[\overline{M}_{0,0}(X,\beta)]^{\text{virt}}} \text{Euler}(\mathcal{V}_{\beta,0}) = \sum_{i_*\gamma=\beta} \int_{[\overline{M}_{0,0}(Y,\gamma)]^{\text{virt}}} 1 = \sum_{i_*\gamma=\beta} \langle I_{0,0,\gamma} \rangle,$$

where we are using the notation of [CK, Chapter 7] for the Gromov-Witten invariants $\langle I_{0,0,\gamma} \rangle$ of Y . Notice how this generalizes (3).

Remark 3.1. In the literature, one finds two algebro-geometric definitions of the virtual fundamental class, one due to Behrend and Fantechi [BF] and the other due to Li and Tian [LT]. In Sections 1 and 2, we used the Behrend-Fantechi definition of virtual fundamental class. In particular, the argument of Gathmann used in the proof of Theorem 2.2 uses the definition of $[\overline{M}_{0,n}(X,\beta)]^{\text{virt}}$ given in [BF]. Lian-Liu-Yau, on the other hand, use the Li-Tian virtual fundamental class. However, they never use the explicit construction of Li and Tian. What is needed in their papers are the functorial properties of virtual fundamental classes, which work for Behrend-Fantechi classes as well.

It is expected that the virtual fundamental classes defined by Behrend-Fantechi and Li-Tian are equal, but to the best of our knowledge, the details of this argument have not been written down.

There is another proof of the mirror conjecture in [Bertram] for the case of complete intersections in \mathbb{P}^r . His approach also uses the J -function of the zero locus Y of a section of V in X via Conjecture 1.1, and relates the J -function to hypergeometric functions. Let $X = \mathbb{P}^r$ and $V = \mathcal{O}_{\mathbb{P}^r}(l)$ for simplicity. Recall that there is a *birational* morphism

$$\varphi : \overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1)) \rightarrow N_d$$

where $N_d = \mathbb{P}^{(r+1)d+r}$ ([CK] Section 11.1.2). In this approach one first observes that there is a \mathbb{C}^* -action on $\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$ induced from the \mathbb{C}^* -action on the second factor \mathbb{P}^1 and that there are fixed point components which can be identified with $\overline{M}_{0,1}(\mathbb{P}^r, d)$. Let

$$i : \overline{M}_{0,1}(\mathbb{P}^r, d) \hookrightarrow \overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$$

be the inclusion of such a component. Consider the vector bundle $\mathcal{V}_d = \pi_{1*}e_1^*(V)$ on $\overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$, where

$$\pi_1 : \overline{M}_{0,1}(\mathbb{P}^r \times \mathbb{P}^1, (d,1)) \rightarrow \overline{M}_{0,0}(\mathbb{P}^r \times \mathbb{P}^1, (d,1))$$

is the universal curve and

$$e_1 : \overline{M}_{0,1}(\mathbb{P}^r \times \mathbb{P}^1, (d,1)) \rightarrow \mathbb{P}^r$$

is the evaluation morphism. It is easy to see that

$$c_{\text{top}}(\mathcal{V}_{d,1}) = i^*(c_{\text{top}}(\mathcal{V}_d)).$$

The key point is that on the complement of a boundary divisor, one has an equivariant isomorphism

$$\mathcal{V}_d \simeq H^0(\mathbb{P}^1, \mathcal{O}(dl)) \otimes \varphi^*(\mathcal{O}_{N_d}(l)),$$

which then gives

$$c_{\text{top}}(\mathcal{V}_d) = \varphi^* H_d + \text{boundary terms},$$

where $H_d = c_{\text{top}}(H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(dl)) \otimes \mathcal{O}_{N_d}(l))$ is an explicit class on the projective space N_d and the boundary terms are supported on the exceptional divisors of φ . It is then shown that the class H_d gives the hypergeometric series and that the boundary terms are responsible for the mirror transformations, which are certain changes of variables (see [CK, Chapter 11]).

One novelty of this approach is that, unlike the previous two approaches, it does not use the torus action on \mathbb{P}^r . It therefore opens the way to prove mirror theorems for other projective varieties X . This was recently realized in [Lee], where equation (2) (in the special case proved in Proposition 2.2) again serves as a starting point of the proof.

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