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A Lawson-type exponential integrator for the Korteweg–de Vries equation

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We propose an explicit numerical method for the periodic Korteweg–de Vries equation. Our method is based on a Lawson-type exponential integrator for time integration and the Rusanov scheme for Burgers' nonlinearity. We prove first-order convergence in both space and time under a mild Courant–Friedrichs–Lewy condition $\tau = O(h)$, where τ and h represent the time step and mesh size for solutions in the Sobolev space $H^3((-\pi,\pi))$, respectively. Numerical examples illustrating our convergence result are given.

Keywords: exponential integrators; Lawson methods; Korteweg–de Vries equation; error estimates; Rusanov scheme.

1. Introduction

Consider the Korteweg–de Vries (KdV) equation

$$\begin{aligned} &u_t + u_{xxx} + uu_x = 0, \quad x \in \Omega = (-\pi,\pi), \quad t > 0, \\ &u(x,0) = u_0(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{1.1}$$

where we impose periodic boundary conditions for practical implementation. The KdV equation is a generic model for the study of weakly nonlinear long waves. It describes the propagation of shallow water waves in a channel (Korteweg & de Vries, 1895) and is widely applied in science and engineering, such as in plasma physics where it gives rise to ion acoustic solitons (Das & Sarma, 1998), and in geophysical fluid dynamics where it describes long waves in shallow seas and deep oceans (Ostrovsky & Stepanyants, 1989; Osborne, 1995). The KdV equation is also relevant for studying the interaction between nonlinearity and dispersion.

For the well-posedness of the periodic KdV equation we refer to Bourgain (1993), Colliander *et al.* (2003) and Gubinelli (2012). It was shown in Colliander *et al.* (2003) that the equation is globally well-posed for initial data in $H^s(\Omega)$ with $s \geq -1/2$. For its numerical solution various methods have been proposed and analyzed in the literature, such as finite difference methods (FDM) (Vliegenthart, 1971; Taha & Ablowitz, 1984; Helal & Mehanna, 2007; Holden *et al.*, 2014), finite element methods (Winther, 1980; Arnold & Winther, 1982; Aksan & Özdeş, 2006; Dutta *et al.*, 2015), Fourier spectral methods (Chan & Kerkhoven, 1985; Maday & Quarteroni, 1988; Rashid *et al.*, 2004; Rashid, 2006, 2007), splitting methods (Holden *et al.*, 1999, 2011; Klein, 2008) and Petrov–Galerkin methods for the KdV equation with nonperiodic boundary condition (Ma & Sun, 2000, 2001; Shen, 2003). Numerical methods for the Kadomtsev–Petviashvili equation, which is a two-dimensional generalization of the KdV equation, were considered in Einkemmer & Ostermann (2018) and Klein & Roidot (2011).

For FDM, linear stability has been analyzed in Goda (1975), Taha & Ablowitz (1984) and Vliegenthart (1971). The explicit leap-frog scheme in Taha & Ablowitz (1984) and the Lax–Friedrichs scheme in Vliegenthart (1971) require both the rather severe stability condition $\tau = O(h^3)$, where τ and h represent the discretization parameters in time and space, respectively. To weaken the stability restriction some implicit FDM were proposed in Goda (1975) and Taha & Ablowitz (1984). Recently the Lax–Friedrichs scheme with an implicit dispersion was proved to converge uniformly to the solution of the KdV equation for initial data in H^3 under the stability condition $\tau = O(h^{3/2})$ for both the decaying case on the full line and the periodic case (Holden *et al.*, 2014). However, no convergence rate was obtained. Very recently, for the θ-right winded FDM, which applies the Rusanov scheme for the hyperbolic flux term and a 4-point θ-scheme for the dispersive term, first-order convergence in space was proved under a hyperbolic Courant–Friedrichs–Lewy (CFL) condition $\tau = O(h)$ for $\theta \geq \frac{1}{2}$ and under an Airy CFL condition $\tau = O(h^3)$ for $\theta < \frac{1}{2}$, for solutions in $H^6(\mathbb{R})$ (Courtès *et al.*, 2017).

On the other hand, the numerical approximation by Fourier spectral/pseudospectral methods has been studied by many authors (Ma & Guo, 1986; Maday & Quarteroni, 1988). Maday & Quarteroni (1988) showed that for solutions in H^r the error of the Fourier spectral method is of order $O(h^{r-1})$ in the L^2 norm, while the error of the pseudospectral method is of order $O(h^{r-2})$ in the H^1 norm. The corresponding L^2 estimate for the Fourier pseudospectral method was established in Ma & Guo (1986) with the aid of artificial viscosity, to avoid the nonlinear instability caused by the aliasing error. More specifically, first-order convergence in time was shown in Ma & Guo (1986) for the fully discrete pseudospectral method under the stability condition $\tau = O(h^3)$ for explicit and $\tau = O(h^2)$ for implicit discretization of the nonlinear term. For the rigorous analysis of splitting methods we refer to Holden *et al.* (2011, 2013).

Nowadays, exponential time integration methods are widely applied for parabolic and hyperbolic problems (Hochbruck & Ostermann, 2010; Bao *et al.*, 2013; Hofmanová & Schratz, 2017). In particular, a distinguished exponential-type integrator was derived for the KdV equation by Hofmanová & Schratz (2017) using a ‘twisting’ technique. For this integrator first-order convergence in time was proved without any CFL condition required. However, the success of this scheme strongly depends on the particular form of the equation. The resulting key relation $k_1^3 + k_2^3 - (k_1 + k_2)^3 = -3(k_1 + k_2)k_1k_2$ in Fourier space allows one to integrate the stiff part involving ∂_x^3 exactly without loss of regularity. Such an integrator, however, can hardly be extended to more general equations, e.g., the fifth-order KdV equation, without additional regularity assumptions. Furthermore, the spatial error was not considered in Hofmanová & Schratz (2017).

In the present paper we propose a Fourier pseudospectral method based on a classical Lawson-type exponential integrator, which integrates the linear part exactly, and the Rusanov scheme for Burgers’ nonlinearity with an added artificial viscosity. The method is explicit, implemented with fast Fourier transform (FFT) and efficient in practical computation. First-order convergence in both space and time is shown under a mild CFL condition $\tau = O(h)$. Moreover, the method can be easily extended to other dispersive equations with Burgers’ nonlinearity.

The rest of this paper is organized as follows. In Section 2 we present the necessary notation, the numerical scheme and the main convergence result. Section 3 is devoted to the details of the error analysis. Numerical results are reported in Section 4 to illustrate our error bounds.

Throughout the paper C represents a generic constant, which is independent of the discretization parameters and the exact solution u.

2. The Fourier pseudospectral method

We make use of standard Sobolev spaces and denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and inner product in $L^2(\Omega)$, respectively. For $m \in \mathbb{N}$ we denote by $H_{\mathrm{p}}^m(\Omega)$ the H^m functions on the one-dimensional torus $\Omega = (-\pi,\pi)$. In particular, these functions have derivatives up to order $m-1$ that are all 2π-periodic. The space is equipped with the standard norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$.

Let $\tau = \Delta t > 0$ be the time step size and denote the temporal grid points by $t_k := k\tau$ for $k = 0, 1, 2, \ldots$ Given a mesh size $h := 2\pi/(2N+1)$ with N being a positive integer let

$$x_j := -\pi + jh, \qquad j = 0, 1, \ldots, 2N,$$

be the spatial grid points in $[-\pi,\pi)$. Denote

$$X_N := \mathrm{span}\left\{e^{ikx} : |k| \le N\right\}, \qquad \widetilde{X}_N := \left\{v = \sum_{k=-N}^{N} v_k\, e^{ikx} \in \mathbb{R}\right\} \subseteq X_N,$$

$$Y_N := \left\{v = (v_0, v_1, \ldots, v_{2N}) \in \mathbb{C}^{2N+1}\right\}, \qquad \widetilde{Y}_N = Y_N \cap \mathbb{R}^{2N+1}.$$

For any $u, v \in C(\Omega)$, define the following discrete inner product and norm by

$$\langle u, v\rangle_N = h \sum_{j=0}^{2N} u(x_j)\overline{v(x_j)}, \qquad |||u|||_N = \langle u, u\rangle_N^{1/2}.$$

For a periodic function $v(x)$ and a vector $v \in Y_N$, let $P_N : L^2(\Omega) \to X_N$ be the standard orthogonal projection operator and $I_N : C(\Omega) \to X_N$ or $I_N : Y_N \to X_N$ be the interpolation operator (Shen *et al.*, 2011), i.e.,

$$(P_N v, \varphi) = (v, \varphi), \qquad \text{for all} \quad \varphi \in X_N;$$
$$(I_N v)(x_j) = v(x_j), \qquad \text{or} \quad (I_N v)(x_j) = v_j, \quad j = 0, \ldots, 2N.$$

More specifically $P_N v$ and $I_N v$ can be written as

$$(P_N v)(x) = \sum_{l=-N}^{N} \widehat{v}_l\, e^{ilx}, \qquad (I_N v)(x) = \sum_{l=-N}^{N} \widetilde{v}_l\, e^{ilx},$$

where $\widehat{v}_l$ and $\widetilde{v}_l$ are the Fourier and discrete Fourier coefficients, respectively, defined as

$$\widehat{v}_l = \frac{1}{2\pi}\int_0^{2\pi} v(x)\, e^{-ilx}\,\mathrm{d}x, \qquad \widetilde{v}_l = \frac{1}{2N+1}\sum_{j=0}^{2N} v_j\, e^{-ilx_j}, \quad l = -N, \ldots, N.$$

It was proved in Shen *et al.* (2011) that for any $u, v \in C(\Omega)$,

$$\langle u, v\rangle_N = (I_N u, I_N v), \qquad |||u|||_N = \|I_N u\|. \tag{2.1}$$

The semidiscrete pseudospectral method for (1.1) consists in finding u_N in $\widetilde{X}_N$ such that

$$\begin{aligned} &\partial_t u_N(x,t) + \partial_x^3 u_N(x,t) + \frac{1}{2} I_N\big((u_N^2)_x\big)(x,t) = 0, \quad x \in \Omega = (-\pi,\pi), \quad t > 0, \\ &u_N(x,0) = I_N(u_0)(x), \quad x \in \overline{\Omega}. \end{aligned} \tag{2.2}$$

Thus, by Duhamel's formula, we have

$$u_N(t_n+\tau) = e^{-\tau\partial_x^3} u_N(t_n) - \frac{1}{2}\int_0^\tau e^{-(\tau-s)\partial_x^3} I_N\big((u_N^2)_x(t_n+s)\big)\,\mathrm{d}s.$$

By applying the approximation $u_N(t_n+s) \approx u_N(t_n)$ and the first-order Lawson method (Lawson, 1967; Hochbruck & Ostermann, 2005) we get a first-order approximation as

$$u_N(t_n+\tau) \approx e^{-\tau\partial_x^3} u_N(t_n) - \frac{\tau}{2} e^{-\tau\partial_x^3} I_N\big((u_N^2)_x(t_n)\big). \tag{2.3}$$

To ensure the stability we apply the Rusanov scheme (see, e.g., Trangenstein, 2009; Courtès *et al.*, 2017) for Burgers' nonlinearity, which consists of a centered hyperbolic flux and an added artificial viscosity. The scheme then reads as

$$u_N^{n+1} = e^{-\tau\partial_x^3} u_N^n - \frac{\tau}{2} e^{-\tau\partial_x^3} I_N \delta_x^0\big((u_N^n)^2\big) + \frac{c\tau h}{2} e^{-\tau\partial_x^3}\delta_x^2 u_N^n, \quad n \geq 0; \quad u_N^0 = I_N(u_0), \tag{2.4}$$

where the constant c is the so-called Rusanov coefficient, which has to satisfy a certain condition (cf. (3.30)). Moreover, we have used the notation

$$\delta_x^0 v(x) = \frac{v(x+h) - v(x-h)}{2h}, \qquad \delta_x^2 v(x) = \frac{v(x+h) - 2v(x) + v(x-h)}{h^2},$$

where $v(x) = v(x \pm 2\pi)$. Similarly, for a vector $v \in Y_N$, define the standard finite difference operators as

$$\delta_x^0 v_j = \frac{v_{j+1} - v_{j-1}}{2h}, \quad \delta_x^2 v_j = \frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}, \quad \delta_x^+ v_j = \frac{v_{j+1} - v_j}{h}, \quad j = 0, 1, \ldots, 2N,$$

with $v_{j\pm(2N+1)} = v_j$ when necessary.

We are now in the position to present the main result of the paper.

THEOREM 2.1 Assume that the solution of (1.1) satisfies $u \in C(0,T;H^3_{\mathrm{p}}(\Omega))$ and let $c_0 > 0$ be given by condition (3.30). Then, for $c > c_0$, there exists $h_0 > 0$ such that for all $h \le h_0$ and $\tau \le h/c$, the error of scheme (2.4) satisfies

$$\left\| u^n_N - u(t_n) \right\| \le M(\tau + h), \qquad n\tau \le T. \tag{2.5}$$

Here both of the constants M and h_0 depend on T, c and $\|u\|_{L^\infty(0,T;H^3_{\mathrm{p}}(\Omega))}$ (cf. (3.32) and (3.31)).

3. Error estimate

The purpose of this section is to prove Theorem 2.1.

3.1 *Some lemmas*

We recall three lemmas from the literature and then prove an additional lemma. All these lemmas are used in the proof of Theorem 2.1.

LEMMA 3.1 (Shen *et al.*, 2011). For any $u \in H^m_{\mathrm{p}}(\Omega)$ and $0 \le \mu \le m$

$$\|P_N u - u\|_\mu \le C h^{m-\mu} |u|_m, \quad \|P_N u\|_m \le C\|u\|_m. \tag{3.1}$$

In addition, if $m > \frac{1}{2}$, then

$$\|I_N u - u\|_\mu \le C h^{m-\mu} |u|_m, \quad \|I_N u\|_m \le C\|u\|_m. \tag{3.2}$$

LEMMA 3.2 (Nikolski's inequality; Shen *et al.*, 2011). For any $u \in X_N$ and $1 \le p \le q \le \infty$

$$\|u\|_{L^q} \le \left(\frac{Np_0 + 1}{2\pi} \right)^{\frac{1}{p} - \frac{1}{q}} \|u\|_{L^p},$$

where p_0 is the smallest even integer $\ge p$. In particular, for $p = 2$ and $q = \infty$, we have

$$\|u\|_{L^\infty} \le h^{-1/2} \|u\|. \tag{3.3}$$

LEMMA 3.3 (Bernstein's inequality; Shen *et al.*, 2011). For any $u \in X_N$ and $0 \le \mu \le m$

$$\|u\|_m \le C h^{\mu - m} \|u\|_\mu. \tag{3.4}$$

LEMMA 3.4 For $a = (a_0, a_1, \ldots, a_{2N})$, $b = (b_0, b_1, \ldots, b_{2N}) \in \widetilde{Y}_N$, we have

$$\langle a, \delta_x^2 b \rangle_N = -\langle \delta_x^+ a, \delta_x^+ b \rangle_N, \tag{3.5}$$

and

$$\sum_{j=0}^{2N} \left(\delta_x^2 a_j\right)^2 = \frac{4}{h^2} \sum_{j=0}^{2N} \left[\left(\delta_x^+ a_j\right)^2 - \left(\delta_x^0 a_j\right)^2\right], \tag{3.6}$$

$$\sum_{j=0}^{2N} a_j a_{j+1} \delta_x^+ a_j = -\frac{h^2}{3} \sum_{j=0}^{2N} \left(\delta_x^+ a_j\right)^3, \qquad \sum_{j=0}^{2N} a_{j-1} a_{j+1} \delta_x^0 a_j = -\frac{4h^2}{3} \sum_{j=0}^{2N} \left(\delta_x^0 a_j\right)^3, \tag{3.7}$$

$$\sum_{j=0}^{2N} \delta_x^2 a_j \delta_x^0 (ab)_j = -\frac{1}{h^2} \sum_{j=0}^{2N} a_j a_{j+1} \delta_x^+ b_j + \frac{1}{h^2} \sum_{j=0}^{2N} a_{j-1} a_{j+1} \delta_x^0 b_j. \tag{3.8}$$

Proof. The identity (3.5) is the discrete version of the integration by parts formula:

$$\begin{aligned}\langle a, \delta_x^2 b\rangle_N &= \frac{1}{h} \sum_{j=0}^{2N} a_j (b_{j+1} - 2b_j + b_{j-1}) = \frac{1}{h} \sum_{j=0}^{2N} a_j (b_{j+1} - b_j) - \frac{1}{h} \sum_{j=0}^{2N} a_j (b_j - b_{j-1}) \\ &= \sum_{j=0}^{2N} a_j \delta_x^+ b_j - \sum_{j=0}^{2N} a_{j+1} \delta_x^+ b_j = -h \sum_{j=0}^{2N} \left(\delta_x^+ a_j\right)\left(\delta_x^+ b_j\right) = -\langle \delta_x^+ a, \delta_x^+ b\rangle_N.\end{aligned}$$

The equalities (3.6)–(3.8) were established in Courtès *et al.* (2017) for infinite sequences. By applying the same arguments we can get (3.6)–(3.8) for periodic sequences here. We refer to Courtès *et al.* (2017) for details. □

3.2 *Local error analysis*

We introduce the local truncation error ξ^{n+1} as defect

$$\xi^{n+1} = u(t_{n+1}) - e^{-\tau \partial_x^3} u(t_n) + \frac{\tau}{2} e^{-\tau \partial_x^3} \left[\delta_x^0 (u(t_n)^2) - ch\delta_x^2 u(t_n)\right], \quad n \geq 0. \tag{3.9}$$

The local error can be bounded as follows.

LEMMA 3.5 For $u \in C(0, T; H_{\mathrm{p}}^3(\Omega))$ we have

$$\|\xi^{n+1}\| \leq M_3 \tau^2 + M_2 \tau h,$$

where M_3 and M_2 depend on $\|u\|_{L^\infty(0,T;H^3(\Omega))}$ and $\|u\|_{L^\infty(0,T;H^2(\Omega))}$, respectively.

Proof. We first recall

$$u(t_{n+1}) = e^{-\tau \partial_x^3} u(t_n) - \frac{1}{2} \int_0^\tau e^{-(\tau - s)\partial_x^3} (u^2)_x (t_n + s)\, \mathrm{d}s$$

and that $e^{t\partial_x^3}$ is a linear isometry for all $t \in \mathbb{R}$. This yields that

$$\begin{aligned}
\|\xi^{n+1}\| &\le \frac{1}{2}\Big\| \int_0^\tau \Big[e^{-(\tau-s)\partial_x^3}(u^2)_x(t_n+s) - e^{-\tau\partial_x^3}(u^2)_x(t_n) \Big]\,\mathrm{d}s \Big\| \\
&\quad + \frac{\tau}{2}\Big\| e^{-\tau\partial_x^3}\big[(u^2)_x(t_n) - \delta_x^0(u(t_n)^2)\big] \Big\| + \frac{c\tau h}{2}\Big\| e^{-\tau\partial_x^3}\delta_x^2 u(t_n) \Big\| \\
&= \Big\| \int_0^\tau (\tau-s)\partial_s\big[e^{-(\tau-s)\partial_x^3}(uu_x)(t_n+s)\big]\,\mathrm{d}s \Big\| + \frac{\tau}{2}\Big\|(u^2)_x(t_n) - \delta_x^0(u(t_n)^2)\Big\| + \frac{c\tau h}{2}\Big\|\delta_x^2 u(t_n)\Big\| \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

For the first part we get

$$\begin{aligned}
I_1 &= \Big\| \int_0^\tau (\tau-s)\, e^{-(\tau-s)\partial_x^3}\Big[\partial_x^3(uu_x) + \partial_s(uu_x)\Big](t_n+s)\,\mathrm{d}s \Big\| \\
&= \Big\| \int_0^\tau (\tau-s)\, e^{-(\tau-s)\partial_x^3}\Big[3u_x\partial_x^3 u + 3\big(\partial_x^2 u\big)^2 - u^2\partial_x^2 u - 2uu_x^2\Big](t_n+s)\,\mathrm{d}s \Big\| \\
&\le C\tau^2 \sup_{0\le t\le T} \|u\|_3 \Big[\|u_x\|_{L^\infty} + \|\partial_x^2 u\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|u\|_{L^\infty}\|u_x\|_{L^\infty}\Big] \\
&\le C\tau^2 \sup_{0\le t\le T} \big(\|u\|_3^2 + \|u\|_3^3\big),
\end{aligned}$$

where we employed equation (1.1) and the Sobolev imbedding theorem $H^3(\Omega) \hookrightarrow W^{2,\infty}(\Omega)$. Further, using Taylor expansion and Hölder's inequality, we have

$$\begin{aligned}
I_2 &= \frac{\tau}{4h}\Big\| \int_0^h (h-y)\Big[\partial_x^2(u^2)(\cdot+y,t_n) - \partial_x^2(u^2)(\cdot-y,t_n)\Big]\,\mathrm{d}y \Big\| \\
&\le \frac{\tau h^{1/2}}{4}\Big(\int_{-\pi}^{\pi}\int_0^h \Big[\partial_x^2(u^2)(\cdot+y,t_n) - \partial_x^2(u^2)(\cdot-y,t_n)\Big]^2\,\mathrm{d}y\,\mathrm{d}x \Big)^{1/2} \\
&\le \tau h\|\partial_x^2(u^2)(t_n)\| \le 2\tau h\big(\|u_x(t_n)\|_{L^\infty}\|u(t_n)\|_1 + \|u(t_n)\|_{L^\infty}\|u(t_n)\|_2\big) \\
&\le C\tau h\|u(t_n)\|_2^2 \le C\tau h \sup_{0\le t\le T}\|u\|_2^2.
\end{aligned}$$

A similar calculation shows that

$$\begin{aligned}
I_3 &= \frac{c\tau}{2h}\Big\| \int_0^h (h-y)\Big[\partial_x^2 u(\cdot+y,t_n) + \partial_x^2 u(\cdot-y,t_n)\Big]\,\mathrm{d}y \Big\| \\
&\le c\tau h^{1/2}\Big(\int_{-\pi}^{\pi}\int_0^h \Big[\partial_x^2 u(\cdot+y,t_n) + \partial_x^2 u(\cdot-y,t_n)\Big]^2\,\mathrm{d}y\,\mathrm{d}x \Big)^{1/2} \\
&\le 2c\tau h \sup_{0\le t\le T}\|u\|_2,
\end{aligned}$$

which completes the proof. □

3.3 *Proof of Theorem* 2.1

Proof. Denote $\omega_N^n = P_N(u(t_n))$ and $\eta^n = u_N^n - \omega_N^n \in \widetilde{X}_N$. In view of (3.1) and the triangle inequality it is sufficient to show

$$\|\eta^n\| \le M(\tau + h), \tag{3.10}$$

where M is independent of τ and h for $0 \le n\tau \le T$.

The proof is given by induction. For $n = 0$ it is obvious by using Lemma 3.1

$$\|\eta^0\| = \|I_N(u_0) - P_N(u_0)\| \le Ch\|u_0\|_1. \tag{3.11}$$

Suppose the claim is true for $n = 0, 1, \ldots, k$. We prove that $\|\eta^{k+1}\| \le M(\tau + h)$. Subtracting (2.4) from the projection of (3.9) in X_N and noticing that the operator P_N commutes with $e^{-\tau\partial_x^3}$, we get for $n = 0, 1, \ldots, k$,

$$\begin{aligned}
\eta^{n+1} &= e^{-\tau\partial_x^3}\eta^n - \frac{\tau}{2}e^{-\tau\partial_x^3}\left[I_N\delta_x^0\big((u_N^n)^2\big) - P_N\delta_x^0(u(t_n)^2)\right] + \frac{c\tau h}{2}e^{-\tau\partial_x^3}\delta_x^2\eta^n - P_N(\xi^{n+1}) \\
&= e^{-\tau\partial_x^3}\left[\eta^n + \frac{c\tau h}{2}\delta_x^2\eta^n - \frac{\tau}{2}I_N\delta_x^0\left(\left(u_N^n\right)^2 - \left(\omega_N^n\right)^2\right) + \zeta^{n+1}\right],
\end{aligned} \tag{3.12}$$

where

$$\zeta^{n+1} = \frac{\tau}{2}\left[P_N\delta_x^0(u(t_n)^2) - I_N\delta_x^0\big((\omega_N^n)^2\big)\right] - e^{\tau\partial_x^3}P_N(\xi^{n+1}).$$

It follows from Lemma 3.1, (2.1) and Lemma 3.5 that

$$\begin{aligned}
\|\zeta^{n+1}\| &\le \frac{\tau}{2}\left\|P_N\delta_x^0(u(t_n)^2) - I_N\delta_x^0(u(t_n)^2)\right\| + \frac{\tau}{2}\left\|I_N\delta_x^0\big(u(t_n)^2 - (\omega_N^n)^2\big)\right\| + \|P_N(\xi^{n+1})\| \\
&\le C\tau h\|\delta_x^0(u(t_n)^2)\|_1 + \frac{\tau}{2}\left\|\left|\delta_x^0\left(u(t_n)^2 - \left(\omega_N^n\right)^2\right)\right\|\right|_N + M_3(\tau^2 + \tau h) \\
&\le C\tau h\|u(t_n)^2\|_2 + \frac{\tau}{2}\left\|u(t_n)^2 - \left(\omega_N^n\right)^2\right\|_1 + M_3(\tau^2 + \tau h) \\
&\le C\tau h\|u(t_n)\|_2^2 + C\tau\|u(t_n) + \omega_N^n\|_1\|u(t_n) - \omega_N^n\|_1 + M_3(\tau^2 + \tau h) \\
&\le C\tau h\|u(t_n)\|_2^2 + C\tau h\|u(t_n)\|_1\|u(t_n)\|_2 + M_3(\tau^2 + \tau h) \le M_3(\tau^2 + \tau h),
\end{aligned} \tag{3.13}$$

where M_3 depends on $\|u\|_{L^\infty(0,T;H^3_{\mathrm{p}}(\Omega))}$. Here for the third inequality we used the properties

$$\|\delta_x^0 v\|^2 = \frac{1}{4h^2}\int_{-\pi}^{\pi}(v(x+h)-v(x-h))^2\,\mathrm{d}x = \frac{1}{4h^2}\int_{-\pi}^{\pi}\left(\int_{-h}^{h} v'(x+y)\,\mathrm{d}y\right)^2\mathrm{d}x$$
$$\le \frac{1}{2h}\int_{-\pi}^{\pi}\int_{-h}^{h}\left(v'(x+y)\right)^2\mathrm{d}y\,\mathrm{d}x = |v|_1^2,$$
$$|||\delta_x^0 v|||_N^2 = \frac{1}{4h}\sum_{j=0}^{2N}\left(v(x_j+h)-v(x_j-h)\right)^2 = \frac{1}{4h}\sum_{j=0}^{2N}\left(\int_{-h}^{h} v'(x_j+y)\,\mathrm{d}y\right)^2$$
$$\le \frac{1}{2}\sum_{j=0}^{2N}\int_{-h}^{h}\left(v'(x_j+y)\right)^2\mathrm{d}y = |v|_1^2$$

and the well-known bilinear estimate $\|fg\|_1 \le C\|f\|_1\|g\|_1$. For simplicity of notation we denote $u_j^n = u_N^n(x_j)$, $\omega_j^n = \omega_N^n(x_j)$, $\eta_j^n = \eta^n(x_j)$ and $\zeta_j^{n+1} = \zeta^{n+1}(x_j)$. Recall that $u_j^n, \omega_j^n, \eta_j^n, \zeta_j^{n+1} \in \mathbb{R}$ by definition. Applying (3.12), Young's inequality, (2.1) and (3.5) we obtain

$$\begin{aligned}
\|\eta^{n+1}\|^2 &= \left\|\eta^n + \frac{c\tau h}{2}\delta_x^2\eta^n - \frac{\tau}{2}I_N\delta_x^0\left(\left(u_N^n\right)^2 - \left(\omega_N^n\right)^2\right) + \zeta^{n+1}\right\|^2 \\
&= \left\|\eta^n + \zeta^{n+1}\right\|^2 + \frac{\tau^2}{4}\left\|ch\delta_x^2\eta^n - I_N\delta_x^0((\eta^n)^2) - 2I_N\delta_x^0\left(\eta^n\omega_N^n\right)\right\|^2 \\
&\quad + \tau\left(\eta^n + \zeta^{n+1}, ch\delta_x^2\eta^n - I_N\delta_x^0((\eta^n)^2) - 2I_N\delta_x^0\left(\eta^n\omega_N^n\right)\right) \\
&\le (1+\tau)\|\eta^n\|^2 + \left(1+\frac{1}{\tau}\right)\|\zeta^{n+1}\|^2 + \frac{\tau^2}{4}|||ch\delta_x^2\eta^n - \delta_x^0((\eta^n)^2) - 2\delta_x^0\left(\eta^n\omega_N^n\right)|||_N^2 \\
&\quad + \tau\left\langle\eta^n + \zeta^{n+1}, ch\delta_x^2\eta^n - \delta_x^0((\eta^n)^2) - 2\delta_x^0(\eta^n\omega_N^n)\right\rangle_N \\
&= (1+\tau)\|\eta^n\|^2 + \left(1+\frac{1}{\tau}\right)\|\zeta^{n+1}\|^2 + \frac{c^2\tau^2h^2}{4}|||\delta_x^2\eta^n|||_N^2 + \frac{\tau^2}{4}|||\delta_x^0((\eta^n)^2)|||_N^2 \\
&\quad + \tau^2|||\delta_x^0\left(\eta^n\omega_N^n\right)|||_N^2 - \frac{c\tau^2h}{2}\left\langle\delta_x^2\eta^n, \delta_x^0((\eta^n)^2)\right\rangle_N - c\tau^2h\left\langle\delta_x^2\eta^n, \delta_x^0\left(\eta^n\omega_N^n\right)\right\rangle_N \\
&\quad + \tau^2\left\langle\delta_x^0((\eta^n)^2), \delta_x^0\left(\eta^n\omega_N^n\right)\right\rangle_N - c\tau h|||\delta_x^+\eta^n|||_N^2 + c\tau h\left\langle\zeta^{n+1}, \delta_x^2\eta^n\right\rangle_N - \tau\left\langle\eta^n, \delta_x^0((\eta^n)^2)\right\rangle_N \\
&\quad - \tau\left\langle\zeta^{n+1}, \delta_x^0((\eta^n)^2)\right\rangle_N - 2\tau\left\langle\eta^n, \delta_x^0\left(\eta^n\omega_N^n\right)\right\rangle_N - 2\tau\left\langle\zeta^{n+1}, \delta_x^0\left(\eta^n\omega_N^n\right)\right\rangle_N.
\end{aligned} \tag{3.14}$$

Next we estimate the terms in (3.14) separately by using similar arguments as in Courtès *et al.* (2017). By definition and (3.6) we have

$$\frac{c^2\tau^2h^2}{4}|||\delta_x^2\eta^n|||_N^2 = \frac{c^2\tau^2h^3}{4}\sum_{j=0}^{2N}\left(\delta_x^2\eta_j^n\right)^2 = c^2\tau^2\left(|||\delta_x^+\eta^n|||_N^2 - |||\delta_x^0\eta^n|||_N^2\right). \tag{3.15}$$

Moreover, it follows from (3.3), by induction $\|\eta^n\| \le M(\tau+h)$ and the assumption $\tau \le h/c$ that

$$\|\eta^n\|_{L^\infty} \le h^{-1/2}\|\eta^n\| \le Mh^{-1/2}(\tau+h) \le M(1+1/c)h^{1/2} \le c, \tag{3.16}$$

whenever

$$h \le h_1 = M^{-2}c^4(1+c)^{-2}. \tag{3.17}$$

Thus, when $h \le h_1$ we have

$$\frac{\tau^2}{4}|||\delta_x^0((\eta^n)^2)|||_N^2 = \frac{\tau^2h}{4}\sum_{j=0}^{2N}\left(\delta_x^0\eta_j^n\right)^2\left(\eta_{j+1}^n+\eta_{j-1}^n\right)^2 \le c\tau^2\|\eta^n\|_{L^\infty}|||\delta_x^0\eta^n|||_N^2. \tag{3.18}$$

In view of the Sobolev inequality and (3.1) we have

$$\|\omega_N^n\|_{L^\infty} \le C\|\omega_N^n\|_1 \le C\|u(t_n)\|_1 \le M_1, \quad \|\partial_x\omega_N^n\|_{L^\infty} \le C\|\omega_N^n\|_2 \le C\|u(t_n)\|_2 \le M_2, \tag{3.19}$$

where M_1 and M_2 depend on $\|u\|_{L^\infty(0,T;H_\mathrm{p}^1(\Omega))}$ and $\|u\|_{L^\infty(0,T;H_\mathrm{p}^2(\Omega))}$, respectively. This yields

$$\begin{aligned}
\tau^2|||\delta_x^0(\eta^n\omega_N^n)|||_N^2 &= \tau^2h\sum_{j=0}^{2N}\left(\eta_{j+1}^n\delta_x^0\omega_j^n + \omega_{j-1}^n\delta_x^0\eta_j^n\right)^2 \\
&\le 2\tau^2h\sum_{j=0}^{2N}\left(\left(\eta_{j+1}^n\right)^2\left(\delta_x^0\omega_j^n\right)^2 + \left(\omega_{j-1}^n\right)^2\left(\delta_x^0\eta_j^n\right)^2\right) \\
&\le 2\tau^2\|\partial_x\omega_N^n\|_{L^\infty}^2\|\eta^n\|^2 + 2\tau^2\|\omega_N^n\|_{L^\infty}^2|||\delta_x^0\eta^n|||_N^2 \\
&\le 2\tau^2\left(M_2^2\|\eta^n\|^2 + M_1^2|||\delta_x^0\eta^n|||_N^2\right).
\end{aligned} \tag{3.20}$$

Applying (3.7) and (3.16) we obtain

$$\begin{aligned}
-\frac{c\tau^2 h}{2}\left\langle \delta_x^2\eta^n, \delta_x^0((\eta^n)^2)\right\rangle_N &= \frac{c\tau^2}{2}\sum_{j=0}^{2N}\eta_j^n\eta_{j+1}^n\delta_x^+\eta_j^n - \frac{c\tau^2}{2}\sum_{j=0}^{2N}\eta_{j-1}^n\eta_{j+1}^n\delta_x^0\eta_j^n \\
&= -\frac{c\tau^2h^2}{6}\sum_{j=0}^{2N}\left(\delta_x^+\eta_j^n\right)^3 + \frac{2c\tau^2h^2}{3}\sum_{j=0}^{2N}\left(\delta_x^0\eta_j^n\right)^3 \\
&\le -\frac{c\tau^2h^2}{6}\sum_{j=0}^{2N}\left(\delta_x^+\eta_j^n\right)^3 + \frac{2c\tau^2}{3}\|\eta^n\|_{L^\infty}|||\delta_x^0\eta^n|||_N^2. \qquad (3.21)
\end{aligned}$$

Similarly, using (3.8), (3.19) and the assumption $c\tau \le h$ yields

$$\begin{aligned}
-c\tau^2 h\left\langle \delta_x^2\eta^n, \delta_x^0\left(\eta^n\omega_N^n\right)\right\rangle_N &= c\tau^2\sum_{j=0}^{2N}\eta_j^n\eta_{j+1}^n\delta_x^+\omega_j^n - c\tau^2\sum_{j=0}^{2N}\eta_{j-1}^n\eta_{j+1}^n\delta_x^0\omega_j^n \\
&\le 2\tau\|\partial_x\omega_N^n\|_{L^\infty}\|\eta^n\|^2 \le 2M_2\tau\|\eta^n\|^2. \qquad (3.22)
\end{aligned}$$

Some tedious calculations give

$$\begin{aligned}
&\tau^2\left\langle \delta_x^0((\eta^n)^2), \delta_x^0\left(\eta^n\omega_N^n\right)\right\rangle_N = \tau^2 h\sum_{j=0}^{2N}\left(\delta_x^0\eta_j^n\right)\left(\eta_{j+1}^n + \eta_{j-1}^n\right)\left(\omega_{j+1}^n\delta_x^0\eta_j^n + \eta_{j-1}^n\delta_x^0\omega_j^n\right) \\
&= \tau^2 h\sum_{j=0}^{2N}\left(\delta_x^0\eta_j^n\right)^2\omega_{j+1}^n\left(\eta_{j+1}^n + \eta_{j-1}^n\right) + \tau^2 h\sum_{j=0}^{2N}\eta_{j-1}^n\left(\eta_{j+1}^n + \eta_{j-1}^n\right)\left(\delta_x^0\eta_j^n\right)\left(\delta_x^0\omega_j^n\right) \\
&\le 2\tau^2\|\omega_N^n\|_{L^\infty}\|\eta^n\|_{L^\infty}|||\delta_x^0\eta^n|||_N^2 + \frac{\tau^2}{h}\|\eta^n\|_{L^\infty}\|\partial_x\omega_N^n\|_{L^\infty}h\sum_{j=0}^{2N}|\eta_{j-1}^n|\left(|\eta_{j+1}^n| + |\eta_{j-1}^n|\right) \\
&\le 2\tau^2\|\omega_N^n\|_{L^\infty}\|\eta^n\|_{L^\infty}|||\delta_x^0\eta^n|||_N^2 + 2\tau\frac{\tau}{h}\|\eta^n\|_{L^\infty}\|\partial_x\omega_N^n\|_{L^\infty}\|\eta^n\|^2 \\
&\le 2\tau^2M_1\|\eta^n\|_{L^\infty}|||\delta_x^0\eta^n|||_N^2 + 2M_2\tau\|\eta^n\|^2. \qquad (3.23)
\end{aligned}$$

Applying Young's inequality we have

$$\begin{aligned}
c\tau h\left\langle \zeta^{n+1}, \delta_x^2\eta^n\right\rangle_N &= c\tau\sum_{j=0}^{2N}\zeta_j^{n+1}\left(\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n\right) \\
&\le 2c\sum_{j=0}^{2N}(\zeta_j^{n+1})^2 + \frac{c\tau^2}{8}\sum_{j=0}^{2N}\left(\eta_{j+1}^n - 2\eta_j^n + \eta_{j-1}^n\right)^2 \\
&\le 2c\sum_{j=0}^{2N}(\zeta_j^{n+1})^2 + 2c\tau^2\sum_{j=0}^{2N}(\eta_j^n)^2 \le \frac{2}{\tau}\|\zeta^{n+1}\|^2 + 2\tau\|\eta^n\|^2. \qquad (3.24)
\end{aligned}$$

Furthermore, a straightforward calculation yields that

$$-\tau\langle\eta^n,\delta_x^0((\eta^n)^2)\rangle_N = -\frac{\tau}{2}\sum_{j=0}^{2N}\eta_j^n\left(\left(\eta_{j+1}^n\right)^2-\left(\eta_{j-1}^n\right)^2\right) = -\frac{\tau}{2}\sum_{j=0}^{2N}\left[\eta_j^n\left(\eta_{j+1}^n\right)^2-\eta_{j+1}^n\left(\eta_j^n\right)^2\right]$$
$$= \frac{\tau}{6}\sum_{j=0}^{2N}\left[\left(\eta_{j+1}^n\right)^3-3\eta_j^n\left(\eta_{j+1}^n\right)^2+3\eta_{j+1}^n\left(\eta_j^n\right)^2-\left(\eta_j^n\right)^3\right] = \frac{\tau h^3}{6}\sum_{j=0}^{2N}\left(\delta_x^+\eta_j^n\right)^3, \tag{3.25}$$

$$-2\tau\langle\eta^n,\delta_x^0(\eta^n\omega_N^n)\rangle_N = -\tau h\sum_{j=0}^{2N}\eta_j^n\eta_{j+1}^n\delta_x^+\omega_j^n \le \tau\|\partial_x\omega_N^n\|_{L^\infty}\|\eta^n\|^2 \le M_2\tau\|\eta^n\|^2. \tag{3.26}$$

Similarly, one derives that

$$-\tau\left\langle\zeta^{n+1},\delta_x^0((\eta^n)^2)\right\rangle_N - 2\tau\left\langle\zeta^{n+1},\delta_x^0\left(\eta^n\omega_N^n\right)\right\rangle_N$$
$$= -\frac{\tau}{2}\sum_{j=0}^{2N}\zeta_j^{n+1}\left[\left(\eta_{j+1}^n\right)^2-\left(\eta_{j-1}^n\right)^2+2\eta_{j+1}^n\omega_{j+1}^n-2\eta_{j-1}^n\omega_{j-1}^n\right]$$
$$\le 2\sum_{j=0}^{2N}(\zeta_j^{n+1})^2+\frac{\tau^2}{4}\|\eta^n\|_{L^\infty}^2\sum_{j=0}^{2N}\left(\eta_{j+1}^n-\eta_{j-1}^n\right)^2+\frac{\tau^2}{4}\sum_{j=0}^{2N}\left(\eta_{j+1}^n\omega_{j+1}^n-\eta_{j-1}^n\omega_{j-1}^n\right)^2$$
$$\le \frac{\tau}{h}\left[\frac{2}{\tau}\|\zeta^{n+1}\|^2+\tau\left(\|\eta^n\|_{L^\infty}^2+\|\omega_N^n\|_{L^\infty}^2\right)\|\eta^n\|^2\right] \le \frac{1}{c}\left[\frac{2}{\tau}\|\zeta^{n+1}\|^2+\tau(c^2+M_1^2)\|\eta^n\|^2\right]. \tag{3.27}$$

Combining (3.14) and (3.15)–(3.27), we obtain that

$$\|\eta^{n+1}\|^2 \le (1+A\tau)\|\eta^n\|^2+\left(1+\frac{3}{\tau}+\frac{2}{c\tau}\right)\|\zeta^{n+1}\|^2+B\tau^2\|\delta_x^0\eta^n\|^2$$
$$+\tau h\sum_{j=0}^{2N}(h-c\tau)\left(\frac{h}{6}\delta_x^+\eta_j^n-c\right)\left(\delta_x^+\eta_j^n\right)^2, \tag{3.28}$$

where

$$A = 3+c+5M_2+2\tau M_2^2+M_1^2/c,$$
$$B = 2M_1^2-c^2+\|\eta^n\|_{L^\infty}\left(2M_1+5c/3\right).$$

Applying (3.16) we get

$$B \le 2M_1^2-c^2+2M(1+1/c)\left(M_1+c\right)h^{1/2},$$

which implies that $B \le 0$ whenever

$$c > c_0 = \sqrt{2}M_1, \tag{3.29}$$

and

$$h \le h_0 = \left(\frac{c^2 - 2M_1^2}{2M(1+1/c)(M_1+c)} \right)^2, \tag{3.30}$$

where M_1 is given by (3.20) depending on $\|u\|_{L^\infty(0,T;H^1_{\mathrm{p}}(\Omega))}$. It is easily observed that $h_0 \le h_1$. In view of (3.17) we have

$$\frac{h}{6}\delta_x^+ \eta_j^n - c \le \frac{1}{3}\|\eta^n\|_{L^\infty} - c \le -\frac{2c}{3} < 0, \quad \text{if} \quad h \le h_0.$$

This together with the CFL condition $c\tau \le h$ and (3.29) yields that for $n = 0, \dots, k$,

$$\begin{aligned} \|\eta^{n+1}\|^2 &\le (1+\tau C(M_2,c))\|\eta^n\|^2 + \frac{C(c)}{\tau}\|\zeta^{n+1}\|^2 \\ &\le (1+\tau C(M_2,c))\|\eta^n\|^2 + \tau C(M_3,c)(\tau+h)^2, \end{aligned}$$

where $C(c,d)$ indicates that C depends on c and d. Hence,

$$\begin{aligned} \|\eta^{k+1}\|^2 &\le e^{\tau C(M_2,c)}\|\eta^k\|^2 + \tau C(M_3,c)(\tau+h)^2 \\ &\le e^{2\tau C(M_2,c)}\|\eta^{k-1}\|^2 + \tau C(M_3,c)(\tau+h)^2\big(1 + e^{\tau C(M_2,c)}\big) \le \dots \\ &\le e^{(k+1)\tau C(M_2,c)}\|\eta^0\|^2 + \tau C(M_3,c)(\tau+h)^2\big(1 + e^{\tau C(M_2,c)} + \dots + e^{k\tau C(M_2,c)}\big) \\ &\le e^{(k+1)\tau C(M_2,c)}\left[\|\eta^0\|^2 + \frac{C(M_3,c)}{C(M_2,c)}(\tau+h)^2 \right] \\ &\le e^{TC(M_2,c)} C(M_3,c)(\tau+h)^2, \end{aligned}$$

which gives the error (3.10) for $n = k+1$ by setting

$$M = C(T, M_3, c) = e^{TC(M_2,c)/2} C^{1/2}(M_3,c). \tag{3.31}$$

This concludes the proof. □

4. Numerical experiments

In this section we present some numerical experiments to illustrate our analytic convergence rate given in Theorem 2.1. In practical computation the interpolation I_N is implemented via FFT, which is very efficient.

Example 1. The well-known solitary-wave solution of the KdV equation (1.1) is given by

$$u(x,t) = 12\lambda\, \mathrm{sech}^2(\sqrt{\lambda}(x - 4\lambda t - a)), \quad a \in \mathbb{R}, \quad \lambda > 0. \tag{4.1}$$

It represents a single bump moving to the right with speed 4λ. Here we choose $\lambda = 1/4$ and the torus $\Omega = (-30, 30)$, which is large enough such that the periodic boundary conditions do not introduce significant errors, i.e., the soliton is far enough away from the boundary for the considered time interval.

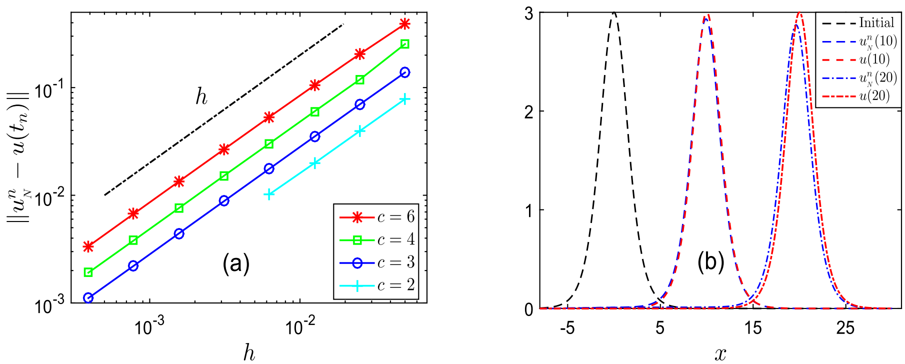

FIG. 1. Numerical simulation for the solitary-wave solution (4.1). (a) The error of the first-order scheme (2.4) at $T = 2$ for various choices of h and c. The time step size τ satisfies $\tau = h/c$. The broken line has slope one. (b) The numerical solution at $T = 10, 20$ was obtained by the scheme (2.4) with $h = 1/200$ and $\tau = h/4$.

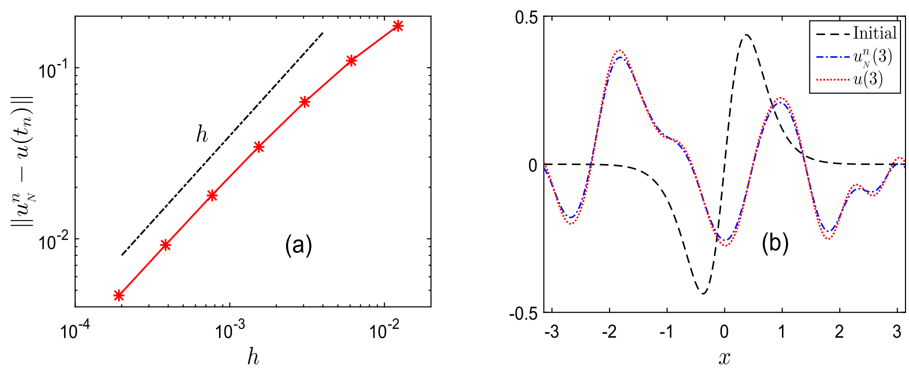

FIG. 2. Numerical simulation for the initial value (4.2). (a) The error of the first-order scheme (2.4) at $T = 3$ for various choices of h with $c = 3$ and $\tau = h/\pi$. The broken line has slope one. (b) The numerical solution at $T = 3$ was computed with the scheme (2.4) using $h = \pi/2^{11}$ and $\tau = h/\pi$.

Figure 1(a) displays the discretization errors for the scheme (2.4) at $T = 2$ for various choices of h and c with $\tau = h/c$. The results for $\tau = dh$ with $d \leq 1/c$ are similar, which are omitted here for brevity. It can be clearly observed that the scheme (2.4) converges linearly in space under the condition $\tau \leq h/c$. Moreover, the error decreases as c gets smaller, which is reasonable due to the fact that c is the coefficient of the added artificial viscosity. The constraint of $c \geq c_0$ is verified by the fact that the numerical solution blows up when $h \leq 1/320$ for $c = 2$. On the other hand, the solution also explodes when $\tau = dh$ with $d > 1/c$, which shows the CFL condition $\tau \leq h/c$ in Theorem 2.1 is sharp. Figure 1(b) illustrates the time evolution of the solitary wave and the corresponding first-order approximate solutions for fixed $h = 1/200$ and $\tau = h/4$.

Example 2. The initial data of the KdV equation (1.1) is now chosen as

$$u_0(x) = 3\,\mathrm{sech}^2(2x)\sin(x), \qquad x \in [-\pi, \pi]. \tag{4.2}$$

The initial data and the numerical solution for $T = 3$ with $c = 3$, $h = \pi/2^{11}$ and $\tau = h/\pi$ are displayed in Fig. 2 (b), where the reference solution is obtained by the second-order exponential integrator of Hofmanová & Schratz (2017) with $\tau = 10^{-6}$ and $h = \pi/2^{15}$. The error of the scheme (2.4) with $c = 3$ and $\tau = h/\pi$ is shown in Fig. 2 (a). The graph clearly shows first-order convergence of the scheme (2.4).

References

Aksan, E. & Özdeş, A. (2006) Numerical solution of Korteweg–de Vries equation by Galerkin B-spline finite element method. *Appl. Math. Comput.*, **175**, 1256–1265.

Arnold, D. N. & Winther, R. (1982) A superconvergent finite element method for the Korteweg–de Vries equation. *Math. Comp.*, **38**, 23–36.

Bao, W., Dong, X. & Zhao, X. (2013) An exponential wave integrator sine pseudospectral method for the Klein–Gordon–Zakharov system. *SIAM J. Sci. Comput.*, **35**, A2903–A2927.

Bourgain, J. (1993) Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations. *Geom. Funct. Anal.*, **3**, 209–262.

Chan, T. & Kerkhoven, T. (1985) Fourier methods with extended stability intervals for the Korteweg–de Vries equation. *SIAM J. Numer. Anal.*, **22**, 441–454.

Colliander, J., Keel, M., Staffilani, G., Takaoka, H. & Tao, T. (2003) Sharp global well-posedness for kdv and modified kdv on $\mathbb{R}$ and $\mathbb{T}$. *J. Amer. Math. Soc.*, **16**, 705–749.

Courtès, C., Lagoutière, F. & Rousset, F. (2018) Numerical analysis with error estimates for the Korteweg–de Vries equation. *MA J. Numer. Anal.*, **00**, 1–58.

Das, G. & Sarma, J. (1998) A new mathematical approach for finding the solitary waves in dusty plasma. *Phys. Plasmas*, **5**, 3918–3923.

Dutta, R., Koley, U. & Risebro, N. H. (2015) Convergence of a higher order scheme for the Korteweg–de Vries equation. *SIAM J. Numer. Anal.*, **53**, 1963–1983.

Einkemmer, L. & Ostermann, A. (2018) A split step Fourier/discontinuous Galerkin scheme for the Kadomtsev–Petviashvili equation. *Appl. Math. Comput.*, **334**, 311–325.

Goda, K. (1975) On stability of some finite difference schemes for the Korteweg–de Vries equation. *J. Physical Soc. Japan*, **39**, 229–236.

Gubinelli, M. (2012) Rough solutions for the periodic Korteweg–de Vries equation. *Comm. Pure Appl. Anal.*, **11**, 709–733.

Helal, M. & Mehanna, M. (2007) A comparative study between two different methods for solving the general Korteweg–de Vries equation (GKdV). *Chaos Solitons Fractals*, **33**, 725–739.

Hochbruck, M. & Ostermann, A. (2005) Explicit exponential Runge–Kutta methods for semilinear parabolic problems. *SIAM J. Numer. Anal.*, **43**, 1069–1090.

Hochbruck, M. & Ostermann, A. (2010) Exponential integrators. *Acta Numer.*, **19**, 209–286.

Hofmanová, M. & Schratz, K. (2017) An exponential-type integrator for the KdV equation. *Numer. Math.*, **136**, 1117–1137.

Holden, H., Karlsen, K. H. & Risebro, N. H. (1999) Operator splitting methods for generalized Korteweg–de Vries equations. *J. Comput. Phys.*, **153**, 203–222.

Holden, H., Karlsen, K., Risebro, N. & Tao, T. (2011) Operator splitting for the KdV equation. *Math. Comp.*, **80**, 821–846.

Holden, H., Koley, U. & Risebro, N. (2014) Convergence of a fully discrete finite difference scheme for the Korteweg–de Vries equation. *IMA J. Numer. Anal.*, **35**, 1047–1077.

Holden, H., Lubich, C. & Risebro, N. (2013) Operator splitting for partial differential equations with burgers nonlinearity. *Math. Comp.*, **82**, 173–185.

Klein, C. (2008) Fourth order time-stepping for low dispersion Korteweg–de Vries and nonlinear Schrödinger equation. *Electron. Trans. Numer. Anal.*, **29**, 116–135.

Klein, C. & Roidot, K. (2011) Fourth order time-stepping for Kadomtsev–Petviashvili and Davey–Stewartson equations. *SIAM J. Sci. Comput.*, **33**, 3333–3356.

Korteweg, D. & de Vries, G. (1895) On the change of form of long waves advancing in a rectangular channel, and a new type of long stationary wave. *Philos. Mag.*, **39**, 422–443.

Lawson, J. D. (1967) Generalized Runge–Kutta processes for stable systems with large Lipschitz constants. *SIAM J. Numer. Anal.*, **4**, 372–380.

Ma, H. & Guo, B. (1986) The Fourier pseudospectral method with a restrain operator for the Korteweg–de Vries equation. *J. Comput. Phys.*, **65**, 120–137.

Ma, H. & Sun, W. (2000) A Legendre–Petrov–Galerkin and Chebyshev collocation method for third-order differential equations. *SIAM J. Numer. Anal.*, **38**, 1425–1438.

Ma, H. & Sun, W. (2001) Optimal error estimates of the Legendre–Petrov–Galerkin method for the Korteweg–de Vries equation. *SIAM J. Numer. Anal.*, **39**, 1380–1394.

Maday, Y. & Quarteroni, A. (1988) Error analysis for spectral approximation of the Korteweg–de Vries equation. *ESAIM Math. Model. Numer. Anal.*, **22**, 499–529.

Osborne, A. (1995) The inverse scattering transform: tools for the nonlinear Fourier analysis and filtering of ocean surface waves. *Chaos Solitons Fractals*, **5**, 2623–2637.

Ostrovsky, L. & Stepanyants, Y. A. (1989) Do internal solitons exist in the ocean? *Rev. Geophys.*, **27**, 293–310.

Rashid, A. (2006) Convergence analysis of three-level Fourier pseudospectral method for Korteweg–de Vries burgers equation. *Comput. Math. Appl.*, **52**, 769–778.

Rashid, A. (2007) Numerical solution of Korteweg–de Vries equation by the Fourier pseudospectral method. *Bull. Belg. Math. Soc.*, **14**, 709–721.

Rashid, A., Mahmood, T. & Mustafa, G. (2004) An explicit pseudospectral scheme for Korteweg–de Vries burgers equation. *Inter. J. Pure Appl. Math.*, **16**, 439–449.

Shen, J. (2003) A new dual-Petrov–Galerkin method for third and higher odd-order differential equations: application to the KdV equation. *SIAM J. Numer. Anal.*, **41**, 1595–1619.

Shen, J., Tang, T. & Wang, L.-L. (2011) *Spectral Methods: Algorithms, Analysis and Applications*. Berlin, Heidelberg: Springer.

Taha, T. R. & Ablowitz, M. I. (1984) Analytical and numerical aspects of certain nonlinear evolution equations. III. Numerical, Korteweg–de Vries equation. *J. Comput. Phys.*, **55**, 231–253.

Trangenstein, J. A. (2009) *Numerical Solution of Hyperbolic Partial Differential Equations*. Cambridge: Cambridge University Press.

Vliegenthart, A. (1971) On finite–difference methods for the Korteweg–de Vries equation. *J. Engrg. Math.*, **5**, 137–155.

Winther, R. (1980) A conservative finite element method for the Korteweg–de Vries equation. *Math. Comp.*, **34**, 23–43.