Asymptotic Analysis of Pulse Dynamics in Mode-Locked Lasers

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Solitons of the power-energy saturation (PES) equation are studied using adiabatic perturbation theory. In the anomalous regime individual soliton pulses are found to be well approximated by solutions of the classical nonlinear Schrödinger (NLS) equation with the key parameters of the soliton changing slowly as they evolve. Evolution equations are found for the pulse amplitude(s), velocity(ies), position(s), and phase(s) using integral relations derived from the PES equation. The results from the integral relations are shown to agree with multi-scale perturbation theory. It is shown that the single soliton case exhibits mode-locking behavior for a wide range of parameters, while the higher states form effective bound states. Using the fact that there is weak overlap between tails of interacting solitons, evolution equations are derived for the relative amplitudes, velocities, positions, and phase differences. Comparisons of interacting soliton behavior between the PES equation and the classical NLS equation are also exhibited.

1. Introduction

Mode-locking in laser systems is used to produce ultrashort pulses for a variety of applications in fields ranging from communications [1], to optical clock

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technology [2], to particle accelerators [3]. The study of mode-locked (ML) lasers is long standing with its origins dating back more then four decades [4, 1]; however, only in recent years have researchers begun to probe their complicated dynamics. Mode-locking may be achieved passively through the use of the so-called Kerr-lens mechanism which allows for strong nonlinear focusing via the $\chi^{(3)}$ contribution. Models for the passive mode-locking of lasers should take into account the effects of chromatic dispersion, self-phase modulation, saturable gain and filtering and intensity discrimination.

Here the power-energy saturation (PES) equation is considered. This is a variant of the classical nonlinear Schrödinger (NLS) equation used to model the effects of dispersion and self-phase modulation. The effects of gain, loss and filtering are taken to be perturbations to the NLS (not necessarily small!). The gain and filtering terms are saturated with energy and the loss term is saturated with power. Other models used to describe this propagation include Ginzburg-Landau (GL) type equations [5, 6] and the "master-equation" [7, 8]. The PES equation is written in nondimensional form as

$$i\psi_z + \frac{d_0}{2}\psi_{tt} + |\psi|^2\,\psi = \frac{ig}{1+E/E_{sat}}\psi + \frac{i\tau}{1+E/E_{sat}}\psi_{tt} - \frac{il}{1+P/P_{sat}}\psi \tag{1}$$

where $E = \int_{-\infty}^{\infty}|\psi|^2 dt$ is the pulse energy, E_{sat} is the normalized saturation energy, $P = |\psi|^2$ is the normalized pulse power, and P_{sat} is the saturation power. The dispersion is in the anomalous regime: $d_0 > 0$; g, τ and l are positive real constants. Here we take $d_0 = 1$, $E_{sat} = 1$, $P_{sat} = 1$, $\tau = l = 0.1$ and vary g. Typical physical parameters can be found in [9]. This equation has been shown numerically to exhibit mode-locking behavior for a large range of parameters with solutions that are found to be well approximated by NLS solitons [9]. Mode-locking associated with the other models is much more limited. As with the master-equation, the PES model has gain and filtering saturated by energy. Unlike the master-equation in which cubic nonlinearity models additional loss–gain, the PES equation has loss modeled by the power saturation term, which reflects the "iris" effect in the laser media (e.g., Ti:Sapphire crystal) where a portion of the field power is transmitted.[1] For small P/P_{sat} the PES equation reduces to the master-equation by taking the first-order Taylor series expansion of the loss term. The model can then be further reduced by taking pulse energy to be constant, in which case the "master-equation" reduces to a GL type system.

[1]Mode-locking requires higher gain in the cavity for optical pulses than for a continuous wave beam. This lensing effect is more pronounced for higher intensities. Pulsed operation, which results in higher peak intensities, may be enforced by carefully placing an intracavity aperture and low intensity operation is prohibited through loss. Another method of creating pulsed operation is to have a fast saturable absorber in the cavity; the saturable absorber creates fractional loss in its ML operation [10].

2. Single soliton evolution

We begin by studying the effects of the right-hand side of the PES equation as perturbations on a soliton solution and use the following form for the four parameter soliton family of solutions to the NLS

$$\psi = ue^{i\phi}. \tag{2a}$$

$$\xi = \int_0^z V dz + t_0, \quad \theta = t - \xi, \quad \sigma = \int_0^z \left[\mu + \frac{V^2}{2}\right] dz + \sigma_0, \quad \phi = V\theta + \sigma \tag{2b}$$

with

$$u = \eta \operatorname{sech}(\eta\theta) \tag{3}$$

where μ, V, t_0, σ_0 are arbitrary constants defining the height/width, speed, temporal shift and phase shift of the soliton, respectively. We also note that η is directly related to μ by $\mu = \eta^2/2$.

The gain, filtering, and loss terms are considered perturbations to the NLS

$$i\psi_z + \frac{1}{2}\psi_{tt} + |\psi|^2 \psi = F[\psi] \tag{4}$$

where F is assumed small. Remarkably, the contribution of the individual terms in F may not be small, but the resulting term always is, thus, providing only the mode-locking mechanism [9]. We account for this perturbation in our solution by letting the four free parameters of the soliton solution to vary slowly in z. Rather than introducing a new slow space variable related to a small term $O(\epsilon)$ and then factoring an ϵ out of F, we directly consider η, V, t_0, σ_0 as functions of z. The assumption that they vary *adiabatically*, i.e., slowly, in z is due to the smallness of F.

The analysis uses integrated conservation laws modified by the right-hand side terms and integral identities derived from Equation (4) given by

$$\frac{d}{dz}\int |\psi|^2 = 2\,\mathrm{Im}\int F[\psi]\psi^* \tag{5a}$$

$$\frac{d}{dz}\mathrm{Im}\int \psi\psi_t^* = 2\mathrm{Re}\int F[\psi]\psi_t^* \tag{5b}$$

$$\frac{d}{dz}\int t\,|\psi|^2 = -d_0\,\mathrm{Im}\int \psi\psi_t^* + 2\,\mathrm{Im}\int tF[\psi]\psi^* \tag{5c}$$

$$\mathrm{Im}\int \psi_z\psi_\mu^* = -\frac{d_0}{2}\,\mathrm{Re}\int \psi_t\psi_{t\mu}^* + \mathrm{Re}\int |\psi|^2\,\psi\psi_\mu^* - \mathrm{Re}\int F[\psi]\psi_\mu^* \tag{5d}$$

where all integrals are taken over $-\infty < t < \infty$.

The virtue of using conservation laws rather then multi-scale perturbation theory is that it allows one to readily obtain a convenient system of ordinary differential equations for η, V, t_0, σ_0 without formulating detailed perturbation theory and secularity conditions. Substituting the soliton solution into the above integral relations and allowing the functions η, V, t_0, σ_0 to vary slowly in z satisfies the following list of conservation laws

$$\frac{dE}{dz} = \int_{-\infty}^{\infty} u(Fe^{-i\phi} - F^* e^{i\phi})d\theta \tag{6a}$$

$$E\frac{dV}{dz} = \int_{-\infty}^{\infty} u_\theta(Fe^{-i\phi} + F^* e^{i\phi})d\theta \tag{6b}$$

$$E\frac{dt_0}{dz} = \int_{-\infty}^{\infty} u\theta(Fe^{-i\phi} - F^* e^{i\phi})d\theta \tag{6c}$$

$$E_\mu V\frac{dt_0}{dz} + \frac{d\sigma_0}{dz} = \int_{-\infty}^{\infty} u_\mu(Fe^{-i\phi} + F^* e^{i\phi})d\theta. \tag{6d}$$

The above equations are in fact the same secularity equations found from the method of multi-scale perturbation theory [11, 12], and are valid for the generalized NLS with a nonlinear term of the form $N(|\psi|^2)\psi$–see Appendix.

Substituting Equation (3) into (6) yields the following set of differential equations governing η, V, t_0, σ_0, respectively

$$\frac{d\eta}{dz} = g\left(\frac{2\eta}{2\eta+1}\right) - \tau\frac{1}{2\eta+1}\left(\frac{2}{3}\eta^3 + 2V^2\eta\right) + l\left(2\eta\frac{1}{a-b}\log\left(\frac{a}{b}\right)\right) \tag{7a}$$

$$\frac{dV}{dz} = -\tau V\left(\frac{4}{3}\frac{\eta^2}{2\eta+1}\right) \tag{7b}$$

$$\frac{dt_0}{dz} = 0 \tag{7c}$$

$$\frac{d\sigma_0}{dz} = 0 \tag{7d}$$

where a and b are the roots of the polynomial $x^2 + 2(1 + 2\eta^2)x + 1 = 0$. Note, a and b can be chosen to be either root of the quadratic equation because they are interchangeable in the equation. This is a convenient way to write the contribution from the loss. Indeed, recall that from Equations (6) the integral corresponding to the loss is

$$2l\int_{-\infty}^{\infty}\frac{|u|^2}{1+|u|^2}d\theta = 2l\int_{-\infty}^{\infty}\frac{\eta^2\text{sech}^2(\eta\theta)}{1+\eta^2\text{sech}^2(\eta\theta)}d\theta = 2\eta l\int_{-1}^{1}\frac{dy}{(1+\eta^2)-\eta^2 y^2}$$

where $y = \tanh(\eta\theta)$. The last integral can be easily evaluated using partial fractions resulting in the loss term in the first of Equations (7). Also, t_0 and σ_0 are constant and the first two equations are independent of t_0, σ_0.

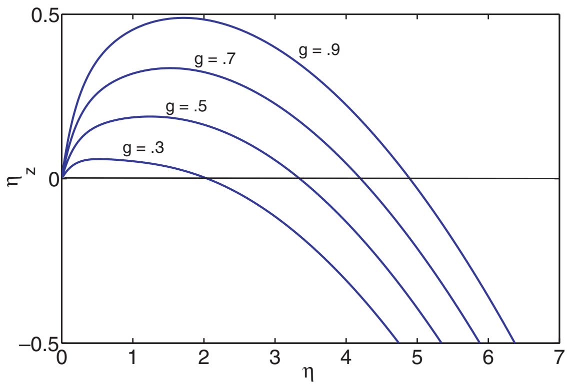

Figure 1. Phase portrait of the amplitude equation for several values of g illustrating the single stable equilibrium.

Consider first the case $V(z) = 0$; in fact this turns out to be a stable equilibrium for any $\eta > 0$ ($\eta < 0$ is only a phase shift from its positive counterpart). This results in the following equation for η

$$\frac{d\eta}{dz} = g\left(\frac{2\eta}{2\eta+1}\right) - \tau\frac{1}{2\eta+1}\left(\frac{2}{3}\eta^3\right) + l\left(2\eta\frac{1}{a-b}\log\left(\frac{a}{b}\right)\right).$$

Clearly, η has at least one equilibrium at 0. To classify its stability we consider the first derivative of $\frac{d\eta}{dz}$ with respect to η evaluated at $\eta = 0$.

$$\frac{d}{d\eta}\left(\frac{d\eta}{dz}\right) = g\left(\frac{2}{(2\eta+1)^2}\right) - \tau\eta^2\left(\frac{2}{3}\frac{4\eta+1}{(2\eta+1)^2}\right)$$
$$-2l\left[\log\left(\frac{a}{b}\right)\frac{1}{b-a} + \eta\left(\frac{b}{a}\right)\left(\frac{a}{b}\right)_\eta + \eta\log\left(\frac{a}{b}\right)\frac{a_\eta - b_\eta}{(b-a)^2}\right] \Rightarrow$$
$$\Rightarrow \frac{d}{d\eta}\left(\frac{d\eta}{dz}\right)_{\eta=0} = 2\left[g - l\right].$$

When $g > l$, $\eta = 0$ is an unstable equilibrium while for $g < l$, $\eta = 0$ is stable. The phase portrait for various values of $g < l$ indicates that $\frac{d\eta}{dz}$ is a monotonic decreasing function in η and so $\eta \to 0$ as $z \to \infty$ for any initial condition $\eta > 0$. Physically this corresponds to loss overtaking gain and the pulse decaying. For $g > l$, $\frac{d\eta}{dz}$ is initially positive and as η increases the filtering term, $\tau\frac{1}{2\eta+1}(\frac{2}{3}\eta^3)$ becomes dominant and $\frac{d\eta}{dz}$ takes on negative values. Thus, there must exist another equilibrium. Plotting the phase portrait for various values of g for $g > l$ shows there is a single stable equilibrium for $\eta > 0$, as shown in Figure 1.

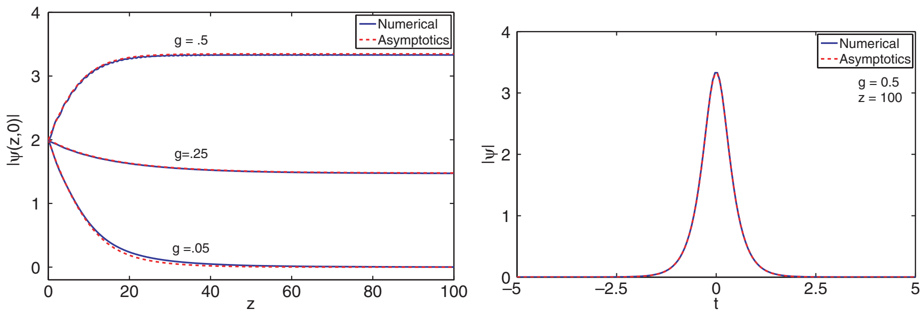

Figure 2. Comparison of asymptotic results to numerical simulations for the peak values of the soliton η (left) and for the resulting profile (right).

As η approaches its stable equilibrium, say $\eta_* \neq 0$, and V approaches its stable equilibrium, say $V_* = 0$, the system of Equations (7) tend to

$$\left.\frac{d\eta}{dz}\right|_{\eta=\eta^*} = \left.\frac{dV}{dz}\right|_{V=V^*} = 0$$

which means the solition tends to a particular NLS soliton of speed zero and height/width determined by η_*, thus suggesting the mode-locking capabilities of Equation (1). In fact, η_* is an attractor that any arbitrary initial condition will eventually converge to, even when it is far way from its soliton solution [9, 13].

This agrees with the results found numerically for the PES equation. An example of the final state found numerically compared to what is predicted is given in Figure 2(right). Numerically ML solutions are observed for $g > l = 0.1$; for $g < l = 0.1$ initial pulses decay to zero which agrees with the turn over point $g = l$ found above regarding the stability of the zero solution for η. For soliton initial conditions the evolution of the height matches very well (see Figure 2(left)).

3. Soliton interactions and bound states

Let us begin with the unperturbed NLS equation; i.e., $F = 0$. We only consider the case that two solitons are widely separated. Then to leading order approximation, the solution is just the superposition of two solitons, i.e., $\psi = \psi_1 + \psi_2$. Take ψ_1 to be the soliton on the left and ψ_2 to be the soliton on the right. Let's consider the small effect (due to the wide separation) of soliton 2 on soltion 1. This means that ψ_2 is locally small relative to ψ_1 and we can expand the nonlinear term around ψ_1. Thus the evolution of ψ when ψ_1 is the dominant term is well approximated by

$$i\psi_{1z} + \frac{1}{2}\psi_{1tt} + |\psi_1|^2\,\psi_1 = -2\,|\psi_1|^2\,\psi_2 - \psi_1^2\;\psi_2^* = G[u_1, u_2] \qquad (8)$$

and similarly an equation for when ψ_2 dominates is found to satisfy

$$i\psi_{2z} + \frac{1}{2}\psi_{2tt} + |\psi_2|^2\,\psi_2 = -2\,|\psi_2|^2\,\psi_1 - \psi_2^2\;\psi_1^* = G[u_2, u_1]. \qquad (9)$$

Notice that if ψ_1 and ψ_2 satisfy this system of coupled PDEs then their sum, ψ, satisfies NLS. This is not a requirement for the method to work, but an added feature of having a cubic nonlinearity. See [14] for a more general perturbative treatment of such interactions.

We can now use the same procedure from the one soliton case to derive equations for the evolution of the eight parameters which define our two solitons. Substituting the perturbations in Equations (8)–(9) for F in the integral relations (5) we arrive at the following set of differential equations

$$\frac{d\eta_k}{dz} = (-1)^k \int_{-\infty}^{\infty} u_k^3 u_{3-k}\sin(\Delta\phi)d\theta \qquad (10a)$$

$$\eta_k\frac{dV_k}{dz} = -\int_{-\infty}^{\infty} 3u_{k\theta}u_k^2 u_{3-k}\cos(\Delta\phi)d\theta \qquad (10b)$$

$$\eta_k\frac{dt_{0k}}{dz} = (-1)^k \int_{-\infty}^{\infty} \theta u_k^3 u_{3-k}\sin(\Delta\phi)d\theta \qquad (10c)$$

$$V_k\frac{dt_{0k}}{dz} + \frac{d\sigma_{0k}}{dz} = -\int_{-\infty}^{\infty} 3u_{k\eta}u_k^2 u_{3-k}\cos(\Delta\phi)d\theta \qquad (10d)$$

where $\Delta\phi = \phi_2 - \phi_1$ for $k = 1, 2$. Further simplifications are needed to get more explicit results. We assume that $\Delta V = V_2 - V_1$ is small and then from the definition $\Delta\phi \approx -\bar{V}\Delta\xi + \Delta\sigma$ which, importantly, is independent of θ so those terms may be taken outside the integral. We also assume $\Delta\eta = \eta_2 - \eta_1$ is small and approximate η_1 and η_2 as $\bar{\eta}$. Hereafter, the bar denotes the mean over the two solitons and Δ the difference of the right minus the left. We finally approximate $u_{3-k} \approx 2\bar{\eta}e^{(-1)^k\bar{\eta}\theta_{3-k}} \approx 2\bar{\eta}e^{(-1)^k\bar{\eta}\theta_k}e^{-\bar{\eta}\Delta\xi}$ using the tail which interacts with u_k. Then, Equations (10) may now be reduced to

$$\frac{d\eta_k}{dz} = (-1)^k \int_{-\infty}^{\infty} u_k^3 2\bar{\eta}e^{(-1)^k\bar{\eta}\theta}\sin(\Delta\phi)e^{-\bar{\eta}\Delta\xi}d\theta$$

$$\eta_k\frac{dV_k}{dz} = -\int_{-\infty}^{\infty} 3u_{k\theta}u_k^2 2\bar{\eta}e^{(-1)^k\bar{\eta}\theta}\cos(\Delta\phi)e^{-\bar{\eta}\Delta\xi}d\theta$$

$$\eta_k\frac{dt_{0k}}{dz} = (-1)^k \int_{-\infty}^{\infty} \theta u_k^3 2\bar{\eta}e^{(-1)^k\bar{\eta}\theta}\sin(\Delta\phi)e^{-\bar{\eta}\Delta\xi}d\theta$$

$$V_k\frac{dt_{0k}}{dz} + \frac{d\sigma_{0k}}{dz} = -\int_{-\infty}^{\infty} 3u_{k\eta}u_k^2 2\bar{\eta}e^{(-1)^k\bar{\eta}\theta}\cos(\Delta\phi)e^{-\bar{\eta}\Delta\xi}d\theta$$

and because we know the form of u_k, the integrals can now be evaluated to give

$$\frac{d\eta_k}{dz} = (-1)^k 4\bar{\eta}^3 \sin(\Delta\phi) e^{-\bar{\eta}\Delta\xi} \tag{11a}$$

$$\frac{dV_k}{dz} = (-1)^{k+1} 4\bar{\eta}^3 \cos(\Delta\phi) e^{-\bar{\eta}\Delta\xi} \tag{11b}$$

$$\frac{dt_0}{dz} = 2\bar{\eta}^2 \sin(\Delta\phi) e^{-\bar{\eta}\Delta\xi} \tag{11c}$$

$$\frac{d\sigma_0}{dz} = -6\bar{\eta}^2 \cos(\Delta\phi) e^{-\bar{\eta}\Delta\xi} - V_k 2\bar{\eta}^2 \sin(\Delta\phi) e^{-\bar{\eta}\Delta\xi}. \tag{11d}$$

Since these equations are in terms of $\Delta\phi$ and $\Delta\xi$ it is useful to derive evolution equations directly for these variables as well. First we combine Equations (11) (sums and differences) to get the following stationary variables

$$\frac{d\bar{\eta}}{dz} = 0, \quad \frac{d\bar{V}}{dz} = 0, \quad \frac{d\Delta t_0}{dz} = 0, \quad \frac{d\Delta\sigma_0}{dz} = 0.$$

And now we may use the definitions of $\Delta\phi = -\bar{V}\Delta\xi + \Delta\sigma$ and $\Delta\xi = \int_0^z (\Delta V) dz - \Delta t_0$ and take the derivative to arrive at

$$\begin{aligned}
\frac{d\Delta\xi}{dz} &= \Delta V + \Delta t_{0z} = \Delta V \\
\frac{d\Delta\phi}{dz} &= -\bar{V}_z \Delta\xi - \bar{V}\Delta\xi_z + \Delta\sigma_z \\
&= -\bar{V}\Delta V + (\bar{\eta}\Delta\eta + \bar{V}\Delta V) + \Delta\sigma_{0z} = \bar{\eta}\Delta\eta.
\end{aligned}$$

Now consider the combined effect of the perturbation associated with soliton interactions and the perturbation from the PES equation

$$\begin{aligned}
i\psi_{1z} + \frac{1}{2}\psi_{1tt} + |\psi_1|^2 \psi_1 &= G[u_1, u_2] + F[u_1] \\
i\psi_{2z} + \frac{1}{2}\psi_{2tt} + |\psi_2|^2 \psi_2 &= G[u_2, u_1] + F[u_2].
\end{aligned}$$

The effect of interaction in the F terms is a higher-order term and omitted for our first-order approximation. This method has also been successfully applied to the case of unperturbed generalized NLS [15]. Because the integral relations used to derive evolution equations for the soliton parameters are linear in the perturbation we may find a set of equations for the evolution of two weakly interacting solitons in the PES equation by taking the superposition

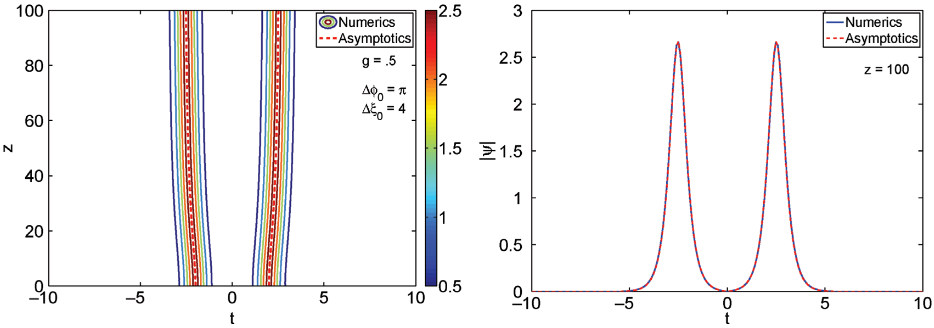

Figure 3. Comparison of the asymptotic results to the contour plots found numerically for $\Delta\phi_0 = \pi$ (left) and comparison of the final profile (right).

of our previous results. For this, it is convenient to define the following functions

$$S_1(\eta, V, \bar{\eta}) := g\left(\frac{2\eta}{E_0+1}\right) - \tau\frac{1}{E_0+1}\left(\frac{2}{3}\eta^3 + 2V^2\eta\right) + l\left(2\eta\frac{1}{a-b}\log\left(\frac{a}{b}\right)\right)$$

$$S_2(\eta, V, \bar{\eta}) := -\tau V\left(\frac{4}{3}\frac{\eta^2}{E_0+1}\right).$$

Here the energy is computed to be $E_0 = 4\bar{\eta}$. Because the energy saturation is a nonlocal effect this adds yet another interaction between the solitons. The final set of evolution equations governing the weak interaction is

$$\begin{aligned}
\frac{d\eta_k}{dz} &= S_1(\eta_k, V_k) + (-1)^k 4\bar{\eta}^3 \sin(\Delta\phi) e^{-\bar{\eta}\Delta\xi} \\
\frac{dV_k}{dz} &= S_2(\eta_k, V_k) + (-1)^{k+1} 4\bar{\eta}^3 \cos(\Delta\phi) e^{-\bar{\eta}\Delta\xi} \\
\frac{d\Delta t_0}{dz} &= 0 \\
\frac{d\Delta\sigma_0}{dz} &= 0 \\
\frac{d\Delta\xi}{dz} &= \Delta V \\
\frac{d\Delta\phi}{dz} &= \bar{\eta}\Delta\eta.
\end{aligned}$$

For two solitons whose respective peaks differ in phase by $\Delta\phi$ we find that for $0 \le \Delta\phi < \pi/2$ the solitons are attracted to each other, while for $\pi/2 < \Delta\phi \le \pi$ they are repelled. In Figures 3 and 4 we show some comparisons to numerical results. Here initial conditions are NLS solitons of height $\eta_{10} = \eta_{20} = 2.67$ and peak separation $\Delta\xi_0 = 4$ with an initial phase difference $\Delta\phi_0$. In certain cases

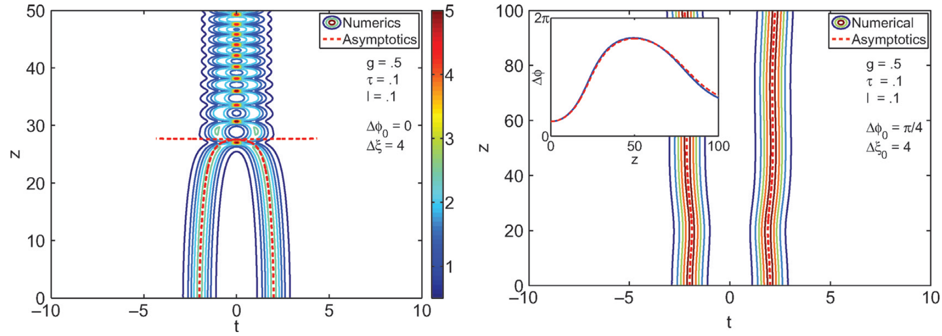

Figure 4. Comparison of the asymptotic results to the contour plots found numerically for $\Delta\phi_0 = 0$ (left) and for $\Delta\phi_0 = \pi/4$ (right).

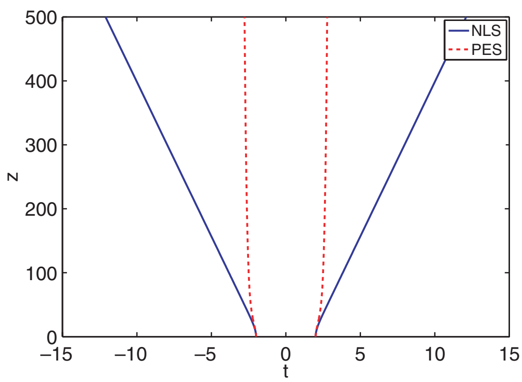

Figure 5. Comparison of the interaction behavior for the classical NLS equation and the PES equation, for the initial conditions used in Figure 3.

the two solitons collide. Prior to collision, perturbation theory gives excellent agreement with direct numerical simulation as in Figure 4 (left). Near and after collision the effects can no longer be modeled by a weak interaction. In these cases the perturbation equations experience blow up.

Comparing the results for the classical NLS to what is found for the PES equation when the interactions result in the solitons eventually moving away from each other out to infinity we find the rate at which the solitons move apart is an order of magnitude different, as indicated in Figure 5. Indeed, direct numerical evaluation indicates that the departure rate of the PES equation is even slower than that predicted from perturbation theory. Effectively we have *bound states* in the PES equation. This also agrees with experimental observations [16]. The difference between the equations can be explained by the strong damping effect the filtering has on the velocity of the solitons. This is shown analytically later.

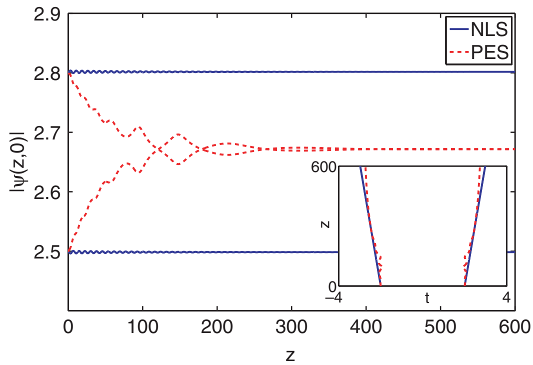

Figure 6. Comparison of classical NLS equation and the PES equation for the heights and centers of two solitons. Here $\Delta\phi_0 = \pi/4$ and $\Delta\xi_0 = 4$.

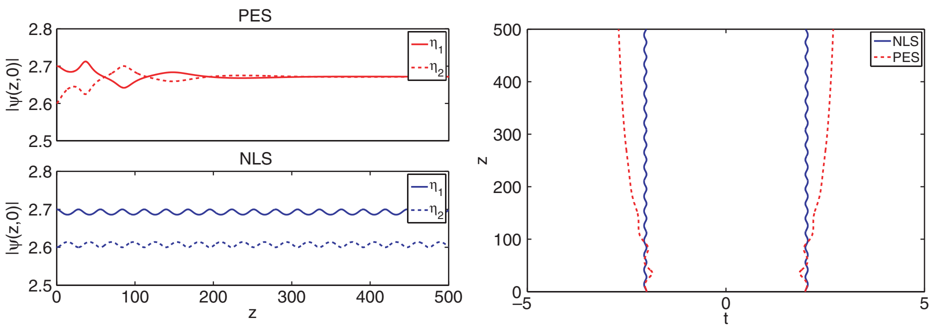

Figure 7. An example of the oscillating *bound states* found for classical the NLS, but not apparent for the PES equation. Here $\Delta\phi_0 = \pi$ and $\Delta\xi_0 = 4$.

Similar to the one soliton case, the heights of the two solitons in the PES equation both tend toward a fixed height η^* defined by $S_1(\eta^*, 0, \eta^*) = 0$. As η_1 and η_2 tend to equilibrium tends to π. In contrast, for the classical NLS, and for $\eta_1 \neq \eta_2$, has (small decaying) oscillations in the heights as illustrated in Figure 6; generally the distance between solitons increases linearly while logarithmically for the PES equation.

In some special cases in the classical NLS equation the distance between solitons only oscillates. This is found for initial conditions $\eta_1 \neq \eta_2$ and $\Delta\phi_0 = 0$ or π. For these oscillating *bound states* the oscillations in the heights do not decay as opposed to the case of seperating solitons; an example is given in Figure 7.

Using the fact that $\eta_1 = \eta_2 \to \eta^*$ and $\Delta\phi \to \pi$ as $z \to \infty$ we can derive the asymptotic behavior of $\Delta\xi$. First, we take the difference of $\frac{dV_2}{dz}$ and $\frac{dV_1}{dz}$ and the derivative of $\frac{d\Delta\xi}{dz}$ giving us

$$\frac{d\Delta V}{dz} = -\tau\left(\frac{4}{3}\frac{\eta^{*2}}{4\eta^{*}+1}\right)\Delta V + 8\eta^{*3}e^{-\eta^{*}\Delta\xi}$$
$$\frac{d^2}{dz^2}\Delta\xi = \frac{d\Delta V}{dz}$$

which may now be combined to form a second-order differential equation for $\Delta\xi$

$$\frac{d^2}{dz^2}\Delta\xi = -\tau\left(\frac{4}{3}\frac{\eta^{*2}}{4\eta^{*}+1}\right)\frac{d}{dz}\Delta\xi + 8\eta^{*3}e^{-\eta^{*}\Delta\xi}.$$

Since the evolution of $\Delta\xi$ is seen from numerical simulations to be evolving slowly in z we expect that $\frac{d^2}{dz^2}\Delta\xi \ll \frac{d}{dz}\Delta\xi$ and so $\frac{d^2}{dz^2}\Delta\xi$ is dropped as being a higher-order term leaving

$$\tau\left(\frac{4}{3}\frac{\eta^{*2}}{4\eta^{*}+1}\right)\frac{d}{dz}\Delta\xi = 8\eta^{*3}e^{-\eta^{*}\Delta\xi}$$

which we solve to get the long term behavior of $\Delta\xi$

$$\Delta\xi \to \frac{1}{\eta^{*}}\ln\left[\frac{\eta^{*}(1+4\eta^{*})}{\tau}6z\right].$$

This logarithmic growth in the soliton separation differs significantly from the linear separation growth in the classical NLS. In fact, after an initial interaction period, during which there is transient motion the soliton centers tend essentially to a fixed distance; the motion is hardly noticeable numerically. We call this an *effective* bound state because the small variations would be too small and take too long to seen experimentally. For example, when the initial $\Delta\xi$ is taken to be nine times the full-width at half-maximum the change in separation over a thousand units in z is approximately 0.1.

4. Conclusions

We presented an analytic theory based on asymptotic analysis to describe soliton propagation and soliton interactions in ML lasers. The power-saturation equation was used. This is a generalization of the classical NLS, appropriately modified to model gain, loss and filtering in the laser. It has been shown, that solitons of the PES system are essentially attractors that even arbitrary initial condition will tend to under evolution. The mode-locking capabilities of the model distinguishes it from others, such as the master-equation and other GL systems. Effective multi-soliton bound states are found. Future research includes the study of the PES system in the normal regime [17] and for dispersion managed systems.

Acknowledgments

This research was partially supported by U.S. Air Force Office of Scientific Research, under Grant FA4955-06-1-0237 and by the National Science Foundation, under Grant DMS-0505352.

Appendix

Here we show the equivalence of using conservation laws and multiple-scale perturbation theory for the perturbed generalized NLS equation

$$i\psi_z + \frac{d_0}{2}\psi_{tt} + N[|\psi|^2]\psi = F[\psi] = \epsilon\hat{F}[\psi]. \quad \text{(A.1)}$$

In both cases we assume a solution of the form (2) where u satisfies

$$-\mu u + \frac{1}{2}u_{\theta\theta} + N[u^2]u = 0$$

and μ, V, t_0, σ_0 are all functions of $Z = \epsilon z$. The first three conservation laws (first three integral identities from the set of Equations (5)) remain the same replacing F with $\epsilon\hat{F}$, while the forth (5d) is replaced by

$$\text{Im}\int \psi_z\psi_\mu^* = -\frac{d_0}{2}\text{Re}\int \psi_t\psi_{t\mu}^* + \text{Re}\int N[|\psi|^2]\psi\psi_\mu^* - \text{Re}\int \epsilon\hat{F}[\psi]\psi_\mu^*.$$

In either case Equations (6) will result by substituting Equations (2) into the integral identities. For the multiple-scales we begin by substituting Equations (2) into Equations (A.1) to arrive at

$$iu_z - \mu u + \frac{1}{2}u_{\theta\theta} + N(u^2)u = G$$

where

$$G = \epsilon\hat{F}[\psi]e^{-i\phi} - \epsilon[iu_\mu\mu_Z - -iu_\theta t_{0Z}] - \epsilon[Vt_{0Z} - V_Z\theta + \sigma_{0Z}]u.$$

Expanding u

$$u = u_0 + \epsilon u_1 + O(\epsilon^2)$$

we see that the leading order is satisfied immediately since u_0 satisfies Equation (A.2) and at order ϵ the equation for u_1 can be written as

$$iU_z + LU = \tilde{G} \quad \text{(A.2)}$$

where

$$U = \begin{pmatrix} \mathrm{Re}(u_1) \\ -i\,\mathrm{Im}(u_1) \end{pmatrix}, \quad L = \begin{pmatrix} 0 & L_0 \\ L_1 & 0 \end{pmatrix}, \quad \tilde{G} = \begin{pmatrix} -i\,\mathrm{Im}(G) \\ -\mathrm{Re}(G) \end{pmatrix}$$

and

$$L_0 = -\frac{1}{2}\partial_{\theta\theta} + \mu - N(u^2)$$
$$L_1 = -\frac{1}{2}\partial_{\theta\theta} + \mu - N(u^2) - 2u^2 N'(u^2).$$

The operator L has two pairs of solutions (eigenfunctions and generalized eigenfunctions) corresponding to the zero eigenvalue:

$$V_1 = \begin{pmatrix} u_\theta \\ 0 \end{pmatrix}, \quad \hat{V}_1 = \begin{pmatrix} 0 \\ -\theta u/2 \end{pmatrix}$$

and

$$V_2 = \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad \hat{V}_2 = \begin{pmatrix} -u_\mu \\ 0 \end{pmatrix}.$$

For the solution U of Equation (A.2) to be nonsecular for large z, the inhomogeneous term $\hat{G}$ must be orthogonal to the eigenfunction and generalized eigenfunctions given earlier; i.e.

$$\langle H, V_k \rangle = \langle H, \hat{V}_k \rangle = 0, \; k = 1, 2 \tag{A.3}$$

where the inner product is defined by

$$\langle A, B \rangle = \int_{-\infty}^{\infty} A^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} B d\theta.$$

Evaluating the four integrals defined by (A.3), the evolution equations for μ, V, t_0, σ_0 are found to be

$$\frac{dE}{dZ} = \int_{-\infty}^{\infty} u(\hat{F}e^{-i\phi} - \hat{F}^* e^{i\phi}) d\theta$$

$$E\frac{dV}{dZ} = \int_{-\infty}^{\infty} u_\theta(\hat{F}e^{-i\phi} + \hat{F}^* e^{i\phi}) d\theta$$

$$E\frac{dt_0}{dZ} = \int_{-\infty}^{\infty} u\theta(\hat{F}e^{-i\phi} - \hat{F}^* e^{i\phi}) d\theta$$

$$E_\mu V\frac{dt_0}{dZ} + \frac{d\sigma_0}{dz} = \int_{-\infty}^{\infty} u_\mu(\hat{F}e^{-i\phi} + \hat{F}^* e^{i\phi}) d\theta$$

which are exactly Equations (5) after the simple change of variables $Z = \epsilon z$ and $\hat{F} = F/\epsilon$.

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(Received December 10, 2008)