Differential Graded Algebras Over Some Reductive Group

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Abstract

In this paper, we study the general properties of commutative differential graded algebras in the category of representations over a reductive algebraic group with an injective central cocharacter. Besides describing the derived category of differential graded modules over such an algebra, we also provide a criterion for the existence of a t-structure on the derived category together with a characterization of the coordinate ring of the Tannakian fundamental group of its heart.

1 Introduction

In [12], Kriz and May develop a general theory about Adams graded commutative differential algebra (cdga) over $\mathbb{Q}$. More precisely, given an Adams cdga A, they consider the bounded derived category $\mathcal{D}_A^f$ of Adams graded dg A-modules and show a lot of formal properties of $\mathcal{D}_A^f$. Assume that A is cohomologically connected, which says that negative cohomological degree parts of A vanish and zero-th cohomological degree part of A is isomorphic to $\mathbb{Q}$. Then they show that $\mathcal{D}_A^f$ has a t-structure. Furthermore, they can use the reduced bar construction (Section 9) to describe the heart $\mathcal{H}_A^f$ of $\mathcal{D}_A^f$. We recall that the reduced bar construction $\bar{B}(A)$ is a differential graded Hopf algebra. Taking the first cohomological part will get a Hopf algebra $H^0(\bar{B}(A))$, which corresponds to a pro-affine group scheme G_A over $\mathbb{Q}$ with $\mathbb{G}_m$-action. Then $\mathcal{H}_A^f$ is equivalent to the category of graded representations of G_A in finite dimensional $\mathbb{Q}$-vector spaces.

Let us mention one application of the theory of Adams cdgas. Given any field k, Kriz and May take A to be Bloch's cycle complex $\mathcal{N}_k$. When $\mathcal{N}_k$ is cohomologically connected (depending on k), $\mathcal{H}^f_{\mathcal{N}_k}$ will exist and coincide with an earlier construction of mixed Tate motives by Bloch and Kriz [3]. For the definition of Bloch's cycle complex $\mathcal{N}_k$, we refer to [13]. Later Spitzweck [15] defines an equivalence

$$\theta_k : \mathcal{D}^f_{\mathcal{N}_k} \to DMT(k, \mathbb{Q})$$

for any field k, where $DMT(k, \mathbb{Q})$ is the full rigid tensor subcategory of Veovodsky's triangulated category of geometric motives generated by Tate objects. The precise definition of $DMT(k, \mathbb{Q})$ can be found in [13].

If we view Adams cdgas as cdgas in the category of representations over $\mathbb{G}_m$, the above example provides us a motivation to study motives by the general theory of cdgas over a reductive group. Using the theory of cdgas over GL_2, we generalize Spitzweck' equivalence to the case of motives for an elliptic curve without complex multiplication in [5].

We now state our main result together with the outline of this paper. Assume G is a reductive group with an injective central cocharacter and A is cdga over G (Definition 2.4). Like the case of Adams graded cdgas, we can also define the derived category of dg A-modules, denoted by $\mathcal{D}_A^G$ and study their properties, which is the content of Section 2 - Section 7. Furthermore if we assume that A is cohomologically connected, the full subcategory $\mathcal{D}_A^{G,f}$ of $\mathcal{D}_A^G$ consisting of compact objects works well (Section 5). Then we have:

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Theorem 1.1. *(Theorem 8.3 and Theorem 11.2) Suppose A is cohomologically connected. There exists a non-degenerate t-structure on $\mathcal{D}_A^{G,f}$ with heart $\mathcal{H}_A^{G,f}$. Furthermore,*

- *There is a functor: $\rho : D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}$.*
- *The functor ρ constructed above is an equivalence of triangulated categories if and only if A is 1-minimal.*

The proof of the first part of the above theorem is included in Section 8. In order to complete the proof of the second part, we need to use the reduced bar construction (Section 9) to give other descriptions about $\mathcal{H}_A^f$ (Section 10). Then we finish the whole proof in Section 11. If $G = \mathbb{G}_m$, the above theorem coincides with the works of Kriz and May.

The theory of cdgas over a reductive group is closely related with the weighted completion (see Definition 12.3). We explain their relation in Section 12. Another observation is that $\mathcal{H}_A^f$ belongs to a special kind of Tannakian categories (Definition 13.4), whose Tannakian fundamental groups are a semi-product of a prounipotent algebraic group and a reductive group. In the last section of the paper, we give a precise description about the coordinate ring of the Tannakian fundamental groups of these Tannkain categories by framed objects [2].

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2 Basic definitions

Convention 2.1. Let G be a reductive algebraic group over $\mathbb{Q}$ and $w : \mathbb{G}_m \to G$ be a central cocharacter – that is, the image of w is contained in the center of G. We assume that w is injective. Using the map w, we can define the weight of representations of G. (See Definition 2.2). Fix a finite dimensional faithful representation $\mathbf{F}$ of G with positive weights.

Definition 2.2. Let V be a rational G-representation. For any $r \in \mathbb{Z}$, define the weight r part of V to be a sub representation of V:

$$V\langle r| = \{x \in V | w(\lambda) \cdot x = \lambda^r x \quad \text{for any} \quad \lambda \in \mathbb{G}_m(k)\}.$$

A rational G-representation V is called pure of weight r if $V\langle r| = V$.

Convention 2.3. The Adams degree for a pure weight r representation W of G over $\mathbb{Q}$ is defined to be $-r$. Given a complex of linear G-representations A^*, the Adams degree r part of A^* is denoted by $A^*|r\rangle$. We call the category of linear G-representations over $\mathbb{Q}$ simply as the category of G-representations.

Definition 2.4. A cdga $(A^*, d, \cdot)$ over G consists of a complex (A^*, d) in the category of G-representations, where $d = \oplus_n d^n : A^n \to A^{n+1}$ is a homomorphism between G-representations, satisfying:

- there exists a homomorphism of complexes of G-representations: $\cdot : A^* \otimes A^* \to A^*$, which is unital, graded commutative and associative.
- $d^{n+m}(a \cdot b) = d^n a \cdot b + (-1)^n a \cdot d^m b$, where $a \in A^n, b \in A^m$.

For the existence of such $\mathbf{F}$, we refer to Corollary 2.5 in [7].

- the Adams grading gives a decomposition of A^* into subcomplexes $A^* = \oplus_{r\in\mathbb{Z}} A^*|r\rangle$ and $\mathbb{Q}$ (the trivial G-representation) is a direct summand of $A^*|0\rangle$.

A^* is called Adams connected if the Adams decomposition satisfies $A^* = \oplus_{r\geq 0} A^*|r\rangle$ and $A^*|0\rangle = \mathbb{Q}$. Furthermore, A^* is called connected (resp. cohomologically connected) if $A^n = 0$ for $n < 0$ and $A^0 = \mathbb{Q}$ (resp. $H^n(A^*) = 0$ for $n < 0$ and $H^0(A^*) = \mathbb{Q}$).

For $x \in A^n|r\rangle$, we call n the cohomological degree of x, denoted by $n = deg(x)$, and r the Adams degree of x, denoted by $r = |x|$.

Definition 2.5. Let A be a cdga over G. A dg A-module (M^*, d) over G consists of a complex M^* of G-representations with the differential d, together with a map $A^* \otimes M^* \to M^*, a \otimes m \to a \cdot m$, which makes M^* into a A^*-module, and satisfies the Leibniz rule

$$d(a \cdot m) = da \cdot m + (-1)^{dega} a \cdot dm; a \in A^*, m \in M^*.$$

Remark 2.6. By definition, there exists a decomposition of M^* into subcomplexes $M^* = \oplus_s M^*|s\rangle$ satisfying $A^*|r\rangle \cdot M^*|s\rangle \subset M^*|r+s\rangle$, which is called the Adams decomposition of M^*.

Definition 2.7. Let M and N be two dg A-modules. A morphism f between M and N is a morphism between the underlying complexes of G-representations of M and N such that $a \cdot f(m) = f(a \cdot m)$ for any $a \in A$ and $m \in M$.

Example 2.8. Let $A[n]$ denote the A^*-module which is A^{m+n} in degree m, with a natural action of A^* by multiplication. Given A^* a cdga over G, we let $A\langle r\rangle[n]$ be A^*-module which is $\bigoplus_{t\in\mathbb{Z}} A^{m+n}|t\rangle \otimes \mathbf{F}^{\otimes r}|s-t\rangle$ in bi-degree (m, s), with the action given by multiplication. More generally, given any G-representation W, $A[n] \otimes W$, with $\bigoplus_{t\in\mathbb{Z}} A^{m+n}|t\rangle \otimes W|s-t\rangle$ in degree (m, s), is also a dg A-module over G. When W is a rational representation of G, $A[n] \otimes W$ is called the generalized sphere A-modules for any $n \in \mathbb{Z}$.

Definition 2.9. A dg A-module M is a cell module if

1. There is an isomorphism of A-modules in the category of G-representations:

$$\oplus_{j\in J} A[-n_j] \otimes V_j \to M,$$

   where all the V_j are rational representations of G and all n_j are integers.

2. There is a filtration on the index set J:

$$J_{-1} = \emptyset \subset J_0 \subset J_1 \cdots \subset J$$

   such that $J = \bigcup_{n=0}^{\infty} J_n$ and for $j \in J_n$,

$$db_j = \sum_{i\in J_{n-1}} a_{ij} b_i,$$

   where b_j is in the cohomological degree n_j part of the complex $Hom_G(V_j, M)$ and a_{ij} is in A.

A finite cell module is a cell module with finite index set J.

Remark 2.10. Given M a cell module, using the condition 1 and 2, one can construct a filtration of sub cell modules M_n, where M_n is isomorphic to $\oplus_{j\in J_n} A[-n_j] \otimes V_j$ as complexes of G-representations. $\{M_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is called the sequential filtration of M.

In other words, a cell A-module M can be considered as the union of an expanding sequence of sub A-modules M_n such that $M_0 = 0$ and M_{n+1} is the cofiber of a map $\phi_n : F_n \to M_n$, where all the F_n are generalized sphere A-modules.

Definition 2.11. A cell module is called effective if all the generalized sphere modules appearing in the definition are of the form $A[-n] \otimes V_j$ with V_j a direct summand of $\mathbf{F}^{\otimes i_j}$ for some $i_j \in \mathbb{Z}_{\geq 0}$.

We denote the category of dg A-modules over G by $\mathcal{M}_A^G$, the category of cell A-modules by $\mathcal{CM}_A^G$ and the category of finite cell modules by $\mathcal{CM}_A^{G,f}$. Furthermore, we denote the category of effective cell A-modules by $\mathcal{CM}_A^{G,eff}$ and the category of finite effective cell modules by $\mathcal{CM}_A^{G,eff,f}$. Finally we denote the full subcategory of cell A-modules of Tate-type by $\mathcal{CM}_A^{\mathbb{G}_m}$.

3 The derived category of dg modules

Let A be a cdga over G and let M and N be dg A-modules. Let $\mathcal{H}om_A(M,N)$ be the dg A-module over G with $\mathcal{H}om_A(M,N)^n$ consisting of linear maps $f : M \to N$ with $f(M^a) \subset N^{a+n}$ and with the differential d defined by $df(m) = d(f(m)) - (-1)^n f(dm)$ for $f \in \mathcal{H}om_A(M,N)^n$. Let $Hom_A(M,N)$ be the dg $\mathbb{Q}$-module over G with $Hom_A(M,N)^n$ consisting of linear maps $f : M \to N$ with $f(M^a) \subset N^{a+n}$, $f(am) = (-1)^{np} af(m)$ for $a \in A^p$ and $m \in M^a$, and with the differential d defined by $df(m) = d(f(m)) - (-1)^n f(dm)$ for $f \in Hom_A(M,N)^n$.

Definition 3.1. For $f : M \to N$ a morphism of dg A-modules, we let $Cone(f)$ be the dg A-module with:

$$Cone(f)^n(r) = N^n(r) \oplus M^{n+1}(r)$$

and the differential is given by $d(n,m) = (dn + f(m), -dm)$.

Given M a dg A-module, we let $M[1]$ denote a dg A-module such that $M[1]^n = M^{n+1}$ with the differential $-d$, where d is the differential of M. Then we have the following sequence:

$$M \xrightarrow{f} N \xrightarrow{i} Cone(f) \to M[1],$$

which is called a cone sequence.

Definition 3.2. We let $\mathcal{K}_A^G$ denote the homotopy category of the category of dg A-modules over G. The objects are the same as $\mathcal{M}_A^G$ and $Hom_{\mathcal{K}_A^G}(M,N) = Hom_G(\mathbb{Q}, H^0(Hom_A(M,N)))$.

The derived category $\mathcal{D}_A^G$ of dg A-modules over G is the localization of $\mathcal{K}_A^G$ with respect to quasi-isomorphisms between dg A-modules, which are defined as morphisms $M \to N$ being quasi-isomorphic on the underlying complexes of $\mathbb{Q}$-vector spaces.

Given $M, N \in \mathcal{M}_A^G$ (resp. $\mathcal{CM}_A^G$), we can define their direct sum to be the direct sum $M \oplus N$ of the chain complexes of GL_2-representations which is equipped with a natural A-module structure (resp. cell A-module structure). Furthermore, the infinite direct sum exists in both $\mathcal{M}_A^G$ and $\mathcal{CM}_A^G$.

Lemma 3.3. *The infinite direct sums defined above is the categorical sum in* $\mathcal{K}_A^G$.

Convention 3.4. Let I be the complex

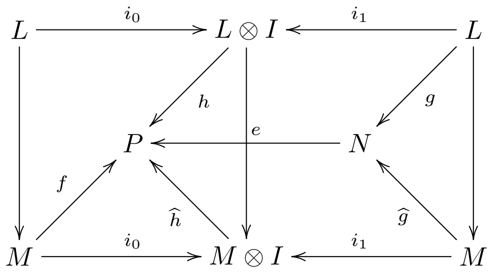
$$\mathbb{Q} \xrightarrow{\delta} \mathbb{Q} \oplus \mathbb{Q}$$

with a free $\mathbb{Q}$-generator $[I]$ in degree -1, two free $\mathbb{Q}$-generators $[0], [1]$ in degree 0 and $\delta[I] = [0] - [1]$. We have two inclusions $i_0, i_1 : \mathbb{Q} \to I$ sending 1 to $[0], [1]$ respectively.

For M a dg A-module, we let $CM = Cone(id_M)$. Notice that the cone CM is the quotient module $M \otimes (I/\mathbb{Q}[1])$.

Using the same idea of proof as in [12], we can show the following theorems.

Theorem 3.5. *(HELP) Let L be a cell A-submodule of a cell A-module M. Let $e : N \to P$ be a quasi-isomorphism of dg A-modules. Then given maps $f : M \to P, g : L \to N$, and $h : L \otimes I \to P$ such that $f|_L = h \circ i_0$ and $e \circ g = h \circ i_1$, there are maps $\widehat{g}, \widehat{h}$ that make the following diagram commute.*



Proof. By induction on the filtration on the index set J and pullback along cells not in L, we may assume that $M \cong C(A[n] \otimes W)$ and $L \cong A[n] \otimes W$. Then using the semi-simplicity of the category of G-representations, we can further assume that W is an irreducible G-representation. Let's denote the generator of W by w^n.

Let $u = w^n \otimes [0]$ and $v = w^n \otimes [I]$ be the generators of $C(A[n] \otimes W)$. By definition, we have $d(v) = (-1)^n u$. We also have: $e \circ g(w^n) = h(w^n \otimes [1])$ and $f(u) = h(u)$. Therefore

$$\begin{aligned} &d(h(w^n \otimes [I]) - f(v)) = hd(w^n \otimes [I]) - f(dv) \\ =&h(d(w^n) \otimes [I] + (-1)^n h(w^n \otimes ([0] - [1]))) - (-1)^n f(u) \\ =&(-1)^n h(w^n \otimes [0]) + (-1)^{n+1} h(w^n \otimes [1]) - (-1)^n f(u) \\ =&(-1)^{n+1} h(w^n \otimes [1]) = (-1)^{n+1} e \circ g(w^n). \end{aligned}$$

Because $e \circ g(w^n)$ is a coboundary and e induces a quasi-isomorphism, we know that $g(w^n)$ is also a coboundary, i.e., there exist $\tilde{n} \in N^{n-1}$ such that $d(\tilde{n}) = g(w^n)$. Then $p = e(\tilde{n}) + h(w^n \otimes [I]) - f(v)$ is a cocycle. Then using the quasi-isomorphism at $n-1$, there exist a cocycle $n \in N$ and a chain $q \in P$ such that $d(q) = p - e(n)$. We define $\widehat{g}(j) = (-1)^n(\tilde{n} - n)$ and $\widehat{h}(j \otimes [I]) = q$. □

Theorem 3.6. *(Whitehead) If M is a cell A-module and $e : N \to P$ is a quasi-isomorphism of A-modules, then*

$$e_* : Hom_{\mathcal{K}_A^G}(M, N) \to Hom_{\mathcal{K}_A^G}(M, P)$$

is an isomorphism. So a quasi-isomorphism between cell A-modules is a homotopy equivalence.

Proof. The surjectivity is coming from Theorem 3.5, when we take $L = 0$. The injectivity can be checked when we replace M and L by $M \otimes_{\mathbb{Q}} I$ and $M \otimes_{\mathbb{Q}} (\partial I)$ respectively. When N, P are both cell A-modules, taking $M = P$, we get a map $f : P \to N$ which corresponds to id_P. From the functoriality, f is the homotopy inverse of e. □

Corollary 3.7. *Let M, N be two dg A-modules, and $f : M \to N$ be a morphism between dg A-modules. Let $\widehat{M}$ and $\widehat{N}$ be two cell A-modules such that $\widehat{M} \xrightarrow{r_M} M$ and $\widehat{N} \xrightarrow{r_N} N$ are quasi-isomphisms. Then there exists a morphism between cell A-modules $\widehat{f} : \widehat{M} \to \widehat{N}$ lifting f.*

Proof. From Theorem 3.6, we know that $r_{N*} : Hom_{\mathcal{K}_A^G}(\widehat{M}, \widehat{N}) \to Hom_{\mathcal{K}_A^G}(\widehat{M}, N)$ is an isomorphism. Therefore $f \circ r_M \in \mathcal{K}_A(\widehat{M}, N)$ have a preimage, which is just $\widehat{f}$. □

Remark 3.8. From the above proof, we also know that: Given a cell module M and an arbitrary dg A-module N, we have: $Hom_{\mathcal{D}_A^G}(M, N) \cong Hom_{\mathcal{K}_A^G}(M, \widehat{N}) \cong Hom_{\mathcal{K}_A^G}(M, N)$, where $\widehat{N}$ is a cell module and $\widehat{N} \to N$ is a quasi-isomorphism between dg A-modules.

Theorem 3.9. *(Approximation by cell modules) For any dg A-module M, there is a cell A-module N and a quasi-isomorphism $e : N \to M$.*

Proof. We will construct a sequential filtration N_n and compatible maps $e_n : N_n \to M$ inductively. More precisely, we need to construct cell modules N_n, whose index set is denoted by J_n, satisfy the condition 2 in the definition of cell modules. For every pair (q,r), we decompose $H^q(M)|r\rangle \cong \oplus_i V_i$ as the direct sum of irreducible G-representations V_i with the Adams degree r. Choosing a splitting of $Ker(M^q|r\rangle \xrightarrow{d} M^{q+1}|r\rangle) \twoheadrightarrow H^q(M)|r\rangle$, we think V_i as sub G-representations in $M^q|r\rangle$, because of the semi-simplicity of the category of G-representations. Then we take $N_1 = \oplus_{(q,r)} \oplus_i A[-q] \otimes V_i$ with trivial differential. There is a morphism between dg A-modules: $N_1 \to M$, which is epimorphism on the cohomologies. Inductively, assume that $e_n : N_n \to M$ has been constructed. Consider the set of the pair of cocycles consisting the pairs of unequal cohomology classes on N_n and mapping under $(e_n)^*$ to the same element of $H^*(M)$. Choose a pair $W_1^q|r\rangle$ and $W_2^q|r\rangle$ that live in the bidegree (q,r) satisfies above condition, i.e., we can view $W_1^q|r\rangle \oplus W_2^q|r\rangle$ as the kernel of the morphism e_{n*} on the cohomology of bidegree (q,r). (Here one need to take a sign for the second component.) Simply denote $W_1^q|r\rangle \oplus W_2^q|r\rangle$ by W_1. There is a morphism between dg A-modules $A[-q] \otimes W_1$ to N_n extending the map between G-representations $W_1 \to H^q(N)(r)$. Take N_{n+1} to be the pushout of N_n and $A[-q] \otimes W_1 \oplus A[-q] \otimes W_1[1]$ over $A[-q] \otimes W_1$. Then we have $0 \to W_1 \to H^q(N_n)(r) \to H^q(N_{n+1})(r) \to 0$. We get N_{n+1} by attaching N_n with a generalized sphere dg A-module $A[-q] \otimes W_1[1]$, which implies N_{n+1} is a cell A-module. It is easy to see the differentials on N_{n+1} satisfy the condition 2 in the definition of cell modules. Now we have a distinguish triangle of dg A-modules: $A[-q] \otimes W_1 \xrightarrow{i} N_n \to N_{n+1} \to (A[-q] \otimes W_1)[1]$.

Notice that: $Hom_A(A[-q] \otimes W_1, M) \cong Hom_{D(G)}(W_1, M[q]) \cong Hom_G(W_1, H^q(M))$. Therefore we have:

$$Hom_A(N_{n+1}, M) \to Hom_A(N_n, M) \xrightarrow{i} Hom_A(A[-q] \otimes W_1, M) \cong Hom_G(W_1, H^q(M)).$$

Because W_1 as a G-representation maps to zero in the cohomology group of $H^q(M)|r\rangle$, which implies $i(e_n) = 0$ in $Hom_G(W_1, H^q(M))$, one may find $e_{n+1} \in Hom_A(N_{n+1}, M)$, which extends e_n. Let N be the direct limit of the N_n. Then N is a cell module and the morphism $N \to M$ is a quasi-isomorphism by the construction. □

Putting together with all previous results, we get:

Theorem 3.10. *Let A be a cdga over G. Then the functor*

$$\mathcal{KCM}_A^G \to \mathcal{D}_A^G$$

is an equivalence of triangulated categories.

Definition 3.11. We define $\mathcal{D}_A^{G,f}$ to be the full subcategory of $\mathcal{D}_A^G$ whose objects are quasi-isomorphic to some finite cell A-module in $\mathcal{D}_A^G$.

Remark 3.12. From the proof of Theorem 3.10, we can also know that $\mathcal{KCM}_A^{G,f} \to \mathcal{D}_A^{G,f}$ is an equivalence of triangulated categories.

Example 3.13. Let $A = \mathbb{Q}$, then $\mathcal{KCM}_A^{G,f}$ is just the bounded derived category of the category of rational representations of G, denoted by $D^b(G)$.

Definition 3.14. The full triangulated subcategory of $\mathcal{D}_A^G$ generated by effective cell modules is denoted by $\mathcal{D}_A^{G,eff}$. The full triangulated subcategory of $\mathcal{D}_A^G$ generated by a family of objects $\{A\langle r\rangle[n] | r \geq 0, n \in \mathbb{Z}\}$ is denoted by $\mathcal{T}_A^G$.

Recall the definition of the idempotent completion of a dg category. Given $\mathcal{C}$ a dg category, then its idempotent completion $\mathcal{C}^\natural$ has the objects (M,p) with $p : M \to M$ an idempotent endomorphism in $Z^0\mathcal{C}$ and the hom complex given by

$$\mathcal{H}om_{\mathcal{C}^\natural}((M,p),(N,q))^* = p^* q_* \mathcal{H}om_{\mathcal{C}}(M,N).$$

Remark 3.15. In [1], Balmer and Schlichting have shown that, for $\mathcal{A}$ a triangulated category, $\mathcal{A}^\natural$ has a canonical structure of a triangulated category which makes the natural functor $\mathcal{A} \to \mathcal{A}^\natural$ exact. The same holds for the triangulated tensor categories.

4 The weight filtration for dg modules

In this section, we assume that A is an Adams connected cdga over G.

Definition 4.1. A dg A-module M is called almost free, if there exists a family of irreducible G-representations $\{V_j\}_{j\in J}$ and morphisms of dg A-modules $\phi_j : A\otimes V_j \to M$, such that the induced morphism:

$$\oplus_{j\in J} A\otimes V_j \xrightarrow{\oplus\phi_j} M$$

is an isomorphism of graded A-modules, which means that, forgetting the differentials, this is an isomorphism between G-representations. We call such $\{V_j, \phi_j\}_{j\in J}$ the generating data for M.

Example 4.2. All cell A-modules are almost free. Conversely, any cell A-module is obtained from the generating data together with suitable differentials.

The reason for introducing the notion of almost free is that we can define the weight filtration on these data. Assume that $d(\phi_j(V_j)) \subset \oplus_{i\in I}\phi_i(A\otimes V_i)$. Here we restrict ϕ_j to $A^*|0\rangle \otimes V_j \cong V_j$. The left hand side has the Adams degree $|V_j|$ and the Adams degree of right hand side is larger than or equal to $|V_i|$. So we get $|V_i| \leq |V_j|$ if $d\phi_j \neq 0$. Hence we have the subcomplex

$$W_n^J M = \oplus_{\{j,|V_j|\leq n\}}\phi_j(A\otimes V_j)$$

of M.

Remark 4.3. The subcomplex of $W_n^J M$ is independent of the choice of the family $\{\phi_j\}_{j\in J}$. This is because if we choose another family $\{\phi_{j'}\}$, then the same process as above shows that $\phi_{j'}(V_{j'}) \in W_n^J M$ and hence $W_n^{J'} M \subset W_n^J M$. By symmetry, we get the result. So we delete the J in the definition.

This gives us the increasing filtration as a dg A-module

$$W_*M : \cdots \subset W_nM \subset W_{n+1}M \subset \cdots \subset M$$

with $M = \cup_n W_nM$.

In the same way, we can define $W_{n/n'}M$ as the cokernel of the inclusion $W_{n'}M \to W_nM$ for $n \geq n'$. Write gr_n^W for $W_{n/n-1}$ and $W^{>n}$ for $W_{\infty/n}$.

W_n defines an endofunctor in $\mathcal{CM}_A^G$. Furthermore, $\{W_n\}_{n\in\mathbb{Z}}$ form a functorial tower of endofunctors on $\mathcal{KCM}_A^G$:

$$\cdots \to W_n \to W_{n+1} \to \cdots \to id.$$

Remark 4.4.

- The endofunctor W_n is exact for all n.
- For $m \leq n \leq \infty$, the sequence of endofunctors $W_m \to W_n \to W_{n/m}$ can extend to a distinguish triangle of endofunctors, i.e., for any $M \in \mathcal{KCM}_A^G$, we have a distinguish triangle $W_mM \to W_nM \to W_{n/m}M \to$ in $\mathcal{KCM}_A^G$.

Remark 4.5. Using the isomorphism of categories between $\mathcal{KCM}_A^G$ and $\mathcal{D}_A^G$, we could define the tower of exact endofunctors on $\mathcal{D}_A^G$

$$\cdots \to W_n \to W_{n+1} \to \cdots \to id.$$

Similarly we define $W_{n/n'}$, gr_n^W and $W^{>n}$ on $\mathcal{D}_A^G$.

5 Tensor structure

Recall that the Hom functor $\mathcal{H}om_A(M,N)$ defines a bi-exact bi-functor:

$$\mathcal{H}om_A : (\mathcal{KCM}_A^G)^{op} \otimes \mathcal{KCM}_A^G \to \mathcal{D}_A^G,$$

which gives a well-defined derived functor of $\mathcal{H}om_A$ between the derived categories of dg A-modules (also the derived categories of finite cell modules) by Proposition 3.10:

$$R\mathcal{H}om_A : (\mathcal{D}_A^G)^{op} \otimes \mathcal{D}_A^G \to \mathcal{D}_A^G.$$

In this section, we use these constructions to define the tensor structure on $\mathcal{D}_A^G$.

Firstly, we define a tensor structure on $\mathcal{T}_A^G$. It is defined on the generator $\{A\langle r\rangle[n]\}$ by:

$$A\langle r\rangle[n] \otimes_A A\langle s\rangle[m] = A\langle r+s\rangle[m+n].$$

Using the approximation theorem, we get a derived tensor product $\otimes^{\mathbb{L}}$ on $\mathcal{T}_A^G$. Then it will induce a tensor structure on $(\mathcal{T}_A^G)^\natural$. More precisely, take $(M,p),(N,q) \in (\mathcal{T}_A^G)^\natural$, then

$$(M,p) \otimes_A^{\mathbb{L}} (N,q) = (M \otimes_A^{\mathbb{L}} N, p \otimes q).$$

From above construction, we have a functor:

$$\otimes_A^{\mathbb{L}} : (\mathcal{T}_A^G)^\natural \otimes (\mathcal{T}_A^G)^\natural \to (\mathcal{T}_A^G)^\natural.$$

Next we want to extend the above tensor structure on $(\mathcal{T}_A^G)^\natural$ to $\mathcal{D}_A^G$.

Notice that given a generalized sphere modules $A \otimes W$, where W is a rational G-representation, there exists a positive integer n big enough, such that $A \otimes W(n)$ is in $(\mathcal{T}_A^G)^\natural$. Here (1) means the determinant representation $\wedge^n \mathbf{F}$ and $n = \dim \mathbf{F}$.

Definition 5.1. Given two generalized sphere modules $A \otimes W_i, i = 1,2$, and $n_i \in \mathbb{Z}_{\geq 0}$, such that $A \otimes W_i(n_i)$ is effective, then we define:

$$(A \otimes W_1) \otimes_A^{\mathbb{L}} (A \otimes W_2) = \mathcal{H}om_A(A(n_1+n_2), (A \otimes W_1(n_1)) \otimes_A^{\mathbb{L}} (A \otimes W_2(n_2))).$$

By the definition of the internal hom, it's easy to see that the definition is independent of the choice of n_i.

By Theorem 3.9, we get a well-defined derived functor of $\otimes_A$:

$$\otimes_A^{\mathbb{L}} : \mathcal{D}_A^G \otimes \mathcal{D}_A^G \to \mathcal{D}_A^G.$$

Remark 5.2.

- These bi-functors are adjoint, i.e.,

$$R\mathcal{H}om_A(M \otimes_A^{\mathbb{L}} N, K) \cong R\mathcal{H}om_A(M, R\mathcal{H}om_A(N,K)).$$

- The derived tensor product makes $\mathcal{D}_A^G$ into a triangulated tensor category with unit A and $\mathcal{D}_A^{G,f}$ are triangulated tensor subcategories.

These properties allow us to apply the category duality theory in [14].

Convention 5.3. Denote $M^\vee = R\mathcal{H}om_A(M, A)$.

Definition 5.4. An object $M \in \mathcal{D}_A^G$ is called rigid, if there exists an $N \in \mathcal{D}_A^G$ and morphisms $\delta : A \to M \otimes_A^{\mathbb{L}} N$ and $\epsilon : N \otimes_A^{\mathbb{L}} M \to A$ such that:

$$(id_M \otimes \epsilon) \circ (\delta \otimes id_M) = id_M$$

$$(id_N \otimes \delta) \circ (\epsilon \otimes id_N) = id_N$$

Definition 5.5. An object $M \in \mathcal{D}_A^G$ is finite if there exists a coevaluation map $\tilde{\eta} : A \to M \otimes^{\mathbb{L}} M^{\vee}$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\quad\tilde{\eta}\quad} & M \otimes^{\mathbb{L}} M^{\vee} \\ \Big\downarrow{\scriptstyle\eta} & & \Big\downarrow{\scriptstyle\gamma} \\ R\mathcal{H}om_A(M,M) & \xleftarrow{\quad\mu\quad} & M^{\vee} \otimes^{\mathbb{L}} M \end{array}$$

commutes. Here η and μ are given by the adjunction. γ changes the places of these two modules.

Remark 5.6. By Theorem 1.6 of [14], M is rigid if and only if the function

$$\epsilon_* : Hom_{\mathcal{D}_A^G}(W, Z \otimes_A^{\mathbb{L}} N) \to Hom_{\mathcal{D}_A^G}(W \otimes_A^{\mathbb{L}} M, Z)$$

is a bijection for all W and Z, where $\epsilon_*(f)$ is the composite

$$W \otimes_A^{\mathbb{L}} M \xrightarrow{f \otimes 1} Z \otimes_A^{\mathbb{L}} N \otimes_A^{\mathbb{L}} M \xrightarrow{1 \otimes \epsilon} Z \otimes_A^{\mathbb{L}} A \cong Z.$$

These conditions are also equivalent to saying that M is finite.

In the following, we will discuss the relations between finite objects in $\mathcal{D}_A^G$ and finite cell modules.

Definition 5.7. We say that a cell module N is a summand of a cell module M in $\mathcal{D}_A^G$ if there is a homotopy equivalence of A-modules between M and $N \oplus N^{'}$ for some cell A-module $N^{'}$.

Following the same proof as Theorem 5.7 in Part III of [12], we can get:

Lemma 5.8. *A cell module M is rigid if and only if it is a summand of a finite cell module in $\mathcal{D}_A^G$.*

Remark 5.9. Let $\mathcal{FCM}_A^G$ be the full subcategory of $\mathcal{CM}_A^G$ whose objects are the direct summands up to homotopy of finite cell A-modules. Then the homotopy category $\mathcal{KFCM}_A^G$ is the idempotent completion of $\mathcal{D}_A^{G,f}$. The above lemma implies that $\mathcal{KFCM}_A^G$ is the largest rigid tensor subcategory of the derived category $\mathcal{D}_A^G$. See section 5 in Part III of [12]. In particular, $\mathcal{D}_A^{G,f}$ is a rigid tensor subcategory of $\mathcal{KFCM}_A^G$.

Theorem 5.10. *Let A be an Adams connected cdga over G. Then $M \in \mathcal{D}_A^G$ is rigid if and only if $M \in \mathcal{D}_A^{G,f}$, which implies that there is an equivalence between $\mathcal{D}_A^{G,f}$ and $\mathcal{KFCM}_A^G$.*

Proof. It depends on the following lemma.

Lemma 5.11. *Assume that A is an Adams connected cdga over G. Let M be a finite cell A-module. Suppose N is a summand of M in $\mathcal{D}_A^G$. Then there is a finite A-cell module $M^{'}$ with $N \cong M^{'}$ in $\mathcal{D}_A^G$.*

Proof. By Theorem 3.9, we can assume that N is a cell module. By our assumption, we have $M = N \oplus N^{'}$ in $\mathcal{KCM}_A^G$. Since M is finite, there is a minimal n such that $W_n M \neq 0$. Thus $W_{n-1}N$ is homotopy equivalent to zero. We may assume that $W_{n-1}N = 0$ in $\mathcal{CM}_A^G$. Similarly, we may assume that $M = W_{n+r}M$ and $N = W_{n+r}N$ in $\mathcal{CM}_A^G$ for some $r \geq 0$. Then we proceed by induction on r.

Choose generating data $\{V_j, \phi_j\}_{j \in J}$ for $W_n M$. Let us prove that $W_n M = A \otimes V$ for a finite complex of G-representations V. Indeed, by the definition of the weight functor and $W_{n-1}M = 0$, we can get an isomorphism:

$$W_n M = \oplus_{|V_j|=n} \phi_j (A \otimes V_j).$$

Notice that $d(\phi_j(V_j)) \subset \oplus_i \phi_i(A \otimes V_i)$ and all these $|V_i|$'s have the same value. Using $A^*|0\rangle = \mathbb{Q}$, we get $d(\phi_j(V_j)) \subset \oplus \phi_i(V_i)$. Set $V = \oplus_{j \in J} \phi_j(V_j)$, which is a complex of G-representations. So we have $W_n M = A \otimes V$ as dg A-modules. Because the category of G-representations is semisimple, we can assume that all differentials of V are zero.

Let $p : M \to M$ be the composition of the projection $M \to N$ and the inclusion $N \to M$. Then we can see $W_n p = id \otimes q$, where $q : V \to V$ is an idempotent of V. V is a direct sum of G-representations with some shifts. Thus $W_n N \cong A \otimes im(q)$. We finish the case of $r = 0$.

Using the distinguished triangle

$$W_n N \to N \to W_{n+r/n} N \to W_n N[1],$$

we can replace N with the shifted cone of the map $W_{n+r/n} N \to A \otimes im(q)[1]$. Since $W_{n+r/n} N$ is a summand of $W_{n+r/n} M$, by induction, we get that $W_{n+r/n}$ is homotopy equivalent to a finite cell module. So the cone of $W_{n+r/n} N \to A \otimes im(q)$ is also homotopy equivalent to a finite cell module. □

Corollary 5.12. *Assume A is an Adams connected cdga over G. Then $\mathcal{D}_A^{G,f}$ is idempotent complete.*

6 Base change

Lemma 6.1. *Let N be a cell module. Then the functor $M \otimes_A N$ preserves exact sequences and quasi-isomorphisms in the variable M.*

Proof. Because N is a cell module, N has a sequential filtration $\{N_n\}$, and N_{n+1} is given by the extension of N_n by the generalized sphere modules. Notice that Lemma 6.1 is true for generalized sphere modules. By the induction of the filtration and passage to the colimits, we get the result for the general case. □

If $\phi : A \to B$ is a homomorphism of cdgas over G, we have the functor

$$\otimes_A B : \mathcal{M}_A^G \to \mathcal{M}_B^G$$

which induces a functor on cell modules and the homotopy category

$$\phi_* : \mathcal{KCM}_A^G \to \mathcal{KCM}_B^G.$$

So we have a base change functor on the derived categories level,

$$\phi_* : \mathcal{D}_A^G \to \mathcal{D}_B^G.$$

Remark 6.2. The restriction of ϕ_* on finite objects gives the functor on the bounded case.

Proposition 6.3. *If ϕ is a quasi-isomorphism, then ϕ_* is an equivalence of tensor triangulated categories.*

Proof. Firstly, there is an isomorphism:

$$Hom_{\mathcal{M}_B^G}(B \otimes_A M, N) \cong Hom_{\mathcal{M}_A^G}(M, \phi^* N),$$

for $M \in \mathcal{M}_A^G$ and $N \in \mathcal{M}_B^G$. Here ϕ^* is the pullback functor, which means that, for a given dg B-module, there is a natural dg A-module structure.

Then we have: $Hom_{\mathcal{K}_B^G}(B \otimes_A M, N) \cong Hom_{\mathcal{K}_A^G}(M, \phi^* N)$. Using Remark 3.8, we get:

$$\begin{aligned} Hom_{\mathcal{D}_B^G}(B \otimes_A M, N) &\cong Hom_{\mathcal{K}_B^G}(B \otimes_A \widehat{M}, N) \\ \cong& Hom_{\mathcal{K}_A^G}(B \otimes_A \widehat{M}, \phi^* N) \cong Hom_{\mathcal{D}_A^G}(B \otimes_A M, \phi^* N), \end{aligned}$$

where $\widehat{M}$ is a cell A-module quasi-isomorphic to M.

Next, we will check that the unit of the adjunction and the counit are both quasi-isomorphisms.

For the unit of the adjunction, if M is a cell dg A-module, then

$$\phi \otimes Id : M \cong A \otimes_A M \to \phi^*(B \otimes_A M)$$

is a quasi-isomorphism of A-modules. Firstly, assume that $M = B$. By assumption, we know that $\phi^* B$ is quasi-isomorphic to A as a dg A-module. Then assume $M = A[n] \otimes W$ for W a G-representation. $\phi^*(B \otimes_A A[n] \otimes W)$ is the same as $\phi^*(B[n] \otimes W)$. The latter is naturally quasi-isomorphic to $A[n] \otimes W$. If M is cell module, using the induction on the length of its sequential filtration, we can get the desired quasi-isomorphism.

For the counit part, given N a dg B-module, and choosing a quasi-isomorphim of dg B-module $\widehat{N} \to N$, where $\widehat{N}$ is cell B-module, then we have:

$$B \otimes_A \widehat{N} \to B \otimes_B N \cong N,$$

which is also a quasi-isomorphism. □

Corollary 6.4. *Assume that A and B are Adams connected cdgas over G. If ϕ is a quasi-isomorphism, then*

$$\phi_* : \mathcal{D}_A^{G,f} \to \mathcal{D}_B^{G,f}$$

is an equivalence of triangulated tensor categories.

Proof. Notice that an equivalence between tensor triangulated categories induces an equivalence on the subcategories of rigid objects. By Proposition 6.3, we know that $\mathcal{D}_A^G$ and $\mathcal{D}_B^G$ are equivalent. Then by Theorem 5.10, we know that ϕ induces an equivalence between $\mathcal{D}_A^{G,f}$ and $\mathcal{D}_B^{G,f}$. □

Remark 6.5. For any cdga A over G, we have a morphism $\delta : \mathbb{Q} \to A$, which sends $A^*|0\rangle$ to A. Then, for any $M \in \mathcal{M}_{\mathbb{Q}}^G$ and $N \in \mathcal{M}_A^G$, we have:

$$Hom_{\mathcal{D}_A^G}(A \otimes M, N) \cong Hom_{\mathcal{D}_{\mathbb{Q}}^G}(M, \delta^* N).$$

Here δ^* is the forgetful functor, which forgets the A-module structure.

7 Minimal models

In the rest of this chapter, we always assume that the cdgas are Adams connected.

Definition 7.1. A cdga A over G is said to be generalized nilpotent if:

- A is a free commutative graded algebra over G, i.e, $A = Sym^* E$ for some $\mathbb{Z}_{>0}$-graded G-representations E. (Or a complex of G-representations concentrated in position degrees and with zero differentials).
- For $n \geq 0$, let $A\langle n\rangle \subset A$ be the subalgebra generated by the elements of degree $\leq n$. Set $A\langle n+1, 0\rangle = A\langle n\rangle$ and for $q \geq 0$ define $A\langle n+1, q+1\rangle$ inductively as the subalgebra generated by $A\langle n\rangle$ and
$$A\langle n+1, q+1\rangle^{n+1} = \{x \in A\langle n+1\rangle | dx \in A\langle n+1, q\rangle\}.$$
Then for all $n \geq 0$, $A\langle n+1\rangle = \cup_{q \geq 0} A\langle n+1, q\rangle$.

A cdga A over G is called nilpotent, if for each $n \geq 1$, there is a $q_n \in \mathbb{Z}_{\geq 0}$ such that $A\langle n\rangle = A\langle n, q_n\rangle$ in the second condition above.

Definition 7.2. A connected cdga A over G is minimal if it is a free commutative graded algebra over G with decomposable differential: $d(A) \subset (IA)^2$. IA is the fundamental ideal, i.e., $IA = Ker(A \to \mathbb{Q} \cong A^0|0\rangle)$.

Convention 7.3. For a cdga A over G, we let QA be $IA/(IA \cdot IA)$.

Proposition 7.4. *If a connected cdga A over G is generalized nilpotent, then it is minimal. Conversely, if A is a minimal connected cdga over G and $A^q|r\rangle = 0$ unless $2r \geq q$, then A is generalized nilpotent.*

Proof. The proof is the same as Proposition 2.3 in Part of IV of [12]. If A is generalized nilpotent, then $d(A) \subset (IA)^2$ is the consequence of the double induction on n and q. Assume A is minimal. Suppose that A is not generalized nilpotent and let n be minimal such that there is an element a which does not belong to any $A\langle n, q\rangle$. Assume that a is also the element which has the minimal Adams degree. Consider any summand bc of the decomposable element da. One can assume that $0 < deg(b) \leq deg(c)$. Then $bc \in A\langle n-1\rangle$ or $deg(b) = 1$. Since $A^q|r\rangle = 0$ unless $2r \geq q$, b and c have strictly lower Adams grading than a. So b, c are in some $A\langle n, q\rangle$. Therefore da is in some $A\langle n, q\rangle$, so is a. □

Definition 7.5. Let A be a cdga over G. Given a positive integer n, an n-minimal model of A over G is a map of cdgas over G:

$$s : A\{n\} \longrightarrow A,$$

with $A\{n\}$ generalized nilpotent and generated as an algebra in degrees $\leq n$, such that s induces an isomorphism on H^m for $1 \leq m \leq n$ and an injection on H^{n+1}.

Proposition 7.6. *Let A be a cohomologically connected cdga A over G. Then for each $n = 1, 2, \cdots, \infty$, there is an n-minimal model $A\{n\}$ over G: $A\{n\} \to A$.*

Proof. Follow the idea of Proposition 2.4.9 in [13]. Because A is cohomologically connected, we have a canonical decomposition $A = \mathbb{Q} \oplus IA$. Let $E_{10}(1) \subset I^1|1\rangle$ be the G-representation $H^1(I)|1\rangle$, and we need to think it as a sub-module of $I^1|1\rangle$. We give it cohomological degree 1 and Adams degree 1. Then we have a natural inclusion $E_{10}(1) \to A$, which extends to $Sym^* E_{10}(1) \to A$ using the algebra structure of A. In fact, this is a map between cdgas over G and induces an isomorphism on $H^1(-)|1\rangle$.

Then one can adjoint elements in cohomological degree 1 and Adams degree 1 to kill elements in the kernel of the map on $H^2(-)|1\rangle$. So we have a $\mathbb{Z}$-graded G-representations $E_1(1)$, of Adams degree 1 and cohomological degree 1, a generalized nilpotent cdga $A_{1,1} = Sym^* E_1(1)$ over G and a map of cdgas over G: $A_{1,1} \to A$, which induces an isomorphism on $H^1(-)|1\rangle$ and an injection on $H^2(-)|1\rangle$.

We also have a canonical decomposition of $A_{1,1} = \mathbb{Q} \oplus I_{1,1}$.

Notice that $H^p(I_{1,1}|r\rangle) = 0$ for $r > 1, p \leq 1$. This is because that the lowest degree of cohomology of $I_{1,1}|r\rangle$ is coming from $Sym^r E_1(1)$ and all the elements of $E_1(1)$ have cohomological degree 1. Iterating this process, one can construct the Adams degree ≤ 1 part of the n-minimal model in case $n > 1$. This gives us a generalized nilpotent cdga over G,

$$A_{1,n} = Sym^* E_n(1),$$

with $E_n(1)$ in Adams degree 1 and cohomological degrees $1, 2, \cdots, n$ together with a map over G: $A_{1,n} \to A$, which induces an isomorphism on $H^i(-)|1\rangle$ for $1 \leq i \leq n$ and an injection for $i = n+1$. In addition, letting $A_{1,n} = \mathbb{Q} \oplus I_{1,n}$, we have $H^p(I_{1,n}|r\rangle) = 0$ for $r > 1, p \leq 1$.

Suppose we have constructed $\mathbb{Z}$-graded G-representations:

$$E_n(1) \subset E_n(2) \subset \cdots \subset E_n(m)$$

---

n is also allowed to be ∞. In this case, we mean that there is a map of cdgas over G: $A\{\infty\} \xrightarrow{s} A$ with $A\{\infty\}$ generalized nilpotent and s a quasi-isomorphism.

where $E_n(j)$ have Adams degrees $1, \cdots, j$ and cohomological degrees $1, \cdots, n$, a differential on $A_{n,m} = Sym^* E_n(m)$ making $A_{m,n}$ a generalized nilpotent cdga over G, and a map $A_{m,n} \to A$ of cdgas over G that is an isomorphism on $H^i(-)|j\rangle$ for $1 \le i \le n, j \le m$, and an injection for $i = n+1, j \le m$.

If $A_{m,n} = \mathbb{Q} \oplus I_{m,n}$, then $H^p(I_{m,n}|r\rangle) = 0$ for $r > m, p \le 1$. Extending $E_n(m)$ to $E_n(m+1)$ by repeating the construction for $E_n(1)$ above. Then one can check the above condition sill hold. The induction goes through.

Taking $E_n = \cup_m E_n(m)$, we have a differential on $A\{n\} = Sym^* E_n$ making $A\{n\}$ a generalized nilpotent cdga over G, and a map $A\{n\} \to A$ of cdgas over G that is an isomorphism on $H^i(-)$ for $1 \le i \le n$ and an injection for $i = n+1$. □

Remark 7.7. If $f : A \to B$ is a quasi-isomorphism of cdgas over G, and $s : A\{n\} \to A,\ t : B\{n\} \to B$ are n-minimal models, then there is an isomorphism of cdgas over G: $g : A\{n\} \to B\{n\}$ such that $g \circ s$ is homotopic to $t \circ f$. The proof is the same as the usual case in Chapter 4 of [4].

8 The t-structure of $\mathcal{D}_A^{G,f}$

The aim of this section is to define a t-structure on $\mathcal{D}_A^{G,f}$ for A which is a cohomologically connected cdga over G.

There is a canonical augmentation $\epsilon : A \to \mathbb{Q}$, given by the projection onto $A^0|0\rangle = \mathbb{Q}$. So we have a functor:

$$q = \epsilon_* : \mathcal{CM}_A^G \to \mathcal{M}_{\mathbb{Q}}^G, \qquad q(M) = M \otimes_A \mathbb{Q}$$

and an exact tensor functor: $q : \mathcal{D}_A^G \to \mathcal{D}_{\mathbb{Q}}^G$.

Remark 8.1. We recall that $\mathcal{D}_{\mathbb{Q}}^{G,f}$ is the derived category of finite dimensional G-representations. There is a canonical t-structure for $\mathcal{D}_{\mathbb{Q}}^{G,f}$. We want to use q to get the induced t-structure for $\mathcal{D}_A^{G,f}$ when A is a cohomologically connected cdga over G. This is reasonable because the following general fact:

Let $\phi : A \to B$ be a map of cohomologically connected cdgas over G. Then $\phi_* : \mathcal{D}_A^{G,f} \to \mathcal{D}_B^{G,f}$ is conservative, i.e., $\phi_*(M) \cong 0$ implies $M \cong 0$.

Proof. Take a non-zero object $M \in \mathcal{D}_A^{G,f}$. Then we can find a cell module P and a quasi-isomorphism $P \to M$ such that $W_{n-1}P = 0$, but $W_n P$ is not acyclic. We choose generating data $\{V_j, \phi_j\}_{j \in J}$ for P, such that $|V_j| \ge n$ for $j \in J$. Because n is the minimal integer of the possible Adams degree, (like the proof of Lemma 5.11), we can know that $W_n P \otimes_A \mathbb{Q}$ is not acyclic. Notice that $W_n(P \otimes_A B) = W_n P \otimes_A B$ and $W_n P \otimes_A \mathbb{Q} = (W_n P \otimes_A B) \otimes_B \mathbb{Q}$. Therefore $P \otimes_A B$ is not isomorphic to zero in $\mathcal{KCM}_B^G$ and $\phi_* M$ is non-zero in $\mathcal{D}_B^{G,f}$. □

Remark 8.2. The inclusion $\mathbb{Q} \to A$ splits ϵ. Then the functor q defined above can be identified with the functor $gr_*^W = \prod_{n \in \mathbb{Z}} gr_n^W$. One can prove it by using the decomposition of the differential $d = d^0 + d^+$, which is described in Lemma 8.4 below and comparing two functors directly.

Define full subcategories $\mathcal{D}_A^{G,f,\le 0}, \mathcal{D}_A^{G,f,\ge 0}$ and $\mathcal{H}_A^{G,f}$ of $\mathcal{D}_A^{G,f}$.

$$\mathcal{D}_A^{G,f,\le 0} = \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \ for \ n > 0\}$$

$$\mathcal{D}_A^{G,f,\ge 0} = \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \ for \ n < 0\}$$

$$\mathcal{H}_A^{G,f} \quad = \{M \in \mathcal{D}_A^{G,f} | H^n(qM) = 0 \ for \ n \ne 0\}.$$

Then we have the following theorem as in [12, 13].

Theorem 8.3. *Suppose A is cohomologically connected. Then*

$$(\mathcal{D}_A^{G,f,\leq 0}, \mathcal{D}_A^{G,f,\geq 0})$$

is a non-degenerate t-structure on $\mathcal{D}_A^{G,f}$ with heart $\mathcal{H}_A^{G,f}$.

Proof. We can assume that A is connected after replacing by its minimal model. The proof will divide into the following lemmas.

Lemma 8.4. *Suppose that A is connected. Let $M \in \mathcal{D}_A^{G,f,\leq 0}$ (resp. $M \in \mathcal{D}_A^{G,f,\geq 0}$). Then there is a cell A-module $P \in \mathcal{CM}_A^{G,f}$ with generating data $\{V_j, \phi_j\}_{j\in J}$ such that $deg(\phi_j) \leq 0$ for all $j \in J$(resp. $deg(\phi_j) \geq 0$ for all $j \in J$), and a quasi-isomorphism $P \to M$.*

Proof. We prove the case $M \in \mathcal{D}_A^{G,f,\leq 0}$ only.

We choose a quasi-isomorphism $Q \to M$ with $Q \in \mathcal{CM}_A^{G,f}$. Let $\{V_j, \phi_j\}_{j\in J}$ be generating data for Q. We can decompose d_Q with two parts d_Q^0 and d_Q^+, where d_Q^0 maps $\phi_j(V_j)$ to the submodule whose generating data (ϕ_i, V_i) have the Adams degree $|V_j|$ and d_Q^+ map to the complement part. After choosing suitable generating data, we may assume the collection S_0 of (V_j, ϕ_j) with $deg(\phi_j) = 0$ and $d_Q^0(\phi_j(V_j)) = 0$ forms a basis of

$$ker(d^0 : \oplus_{deg(\phi_j)=0}\phi_j(V_j) \to \oplus_{deg(\phi_i)=1}\phi_i(V_i)).$$

Let $\tau^{\leq 0}Q$ be the sub A-module of Q with the generating data of $S = \{(V_j, \phi_j)|deg(\phi_j) < 0\} \bigcup S_0$.

Claim: $\tau^{\leq 0}Q$ is a subcomplex of Q. Consider

$$d_Q(\phi_\alpha(V_\alpha)) = d_Q^0(\phi_\alpha(V_\alpha)) \oplus d_Q^+(\phi_\alpha(V_\alpha)).$$

Using the connected condition of A, we can know that:

1. If $\phi_\beta(V_\beta) \subset d_Q^+(\phi_\alpha(V_\alpha))$, then $deg(\phi_\beta) \leq deg(\phi_\alpha)$.

Or 2. If $\phi_\beta(V_\beta) \subset d_Q^0(\phi_\alpha(V_\alpha))$, then $deg(\phi_\beta) = deg(\phi_\alpha) + 1$.

Consider $(V_\alpha, \phi_\alpha) \in S$ with $deg(\phi_\alpha) \leq -1$. Because $(d_Q^0)^2 = 0$, every summand of $d_Q^0(\phi_\alpha(V_\alpha))$ lies in S_0. We only need to consider elements in S_0. Let $(V_\alpha, \phi_\alpha) \in S_0$. Then we have:

$$d_Q(\phi_\alpha(V_\alpha)) \subset \bigoplus_{deg(\phi_\beta)=0} \phi_\beta(A \otimes V_\beta) \oplus \bigoplus_{deg(\phi_\gamma)\leq -1} \phi_\gamma(A \otimes V_\gamma).$$

Using $d_Q^2(\phi_\alpha(V_\alpha)) = 0$, we can know that $d^0(\phi_\beta(V_\beta)) = 0$ for $deg(\phi_\beta) = 0$. This means that $d_Q(\phi_\alpha(V_\alpha)) \subset \tau^{\leq 0}Q$.

Next we can see that $\tau^{\leq 0}Q \to Q$ is a quasi-isomorphism. Using Remark 8.1 , we need only to check that $q\tau^{\leq 0}Q \to qQ$ is a quasi-isomorphism. This is clear because of $qQ \cong qM$ and $M \in \mathcal{D}_A^{f,\leq 0}$.

For the case $M \in \mathcal{D}_A^{G,f,\geq 0}$, we need to check the proof of Theorem 3.9 carefully, where we can add the extra conditions on the degrees of the generating datum. See Lemma 1.6.2 in [13]. □

Lemma 8.5. *Suppose that A is connected. Then $Hom_{\mathcal{D}_A^{G,f}}(M, N[-1]) = 0$ for $M \in \mathcal{D}_A^{G,f,\leq 0}$ and $N \in \mathcal{D}_A^{G,f,\geq 0}$.*

Proof. By Lemma 8.4, we may assume that M and $N[-1]$ are cell A-modules with the generating datum $\{(V_\alpha, \phi_\alpha)\}_{deg(\phi_\alpha)\leq 0}$ and $\{(V_\beta, \phi_\beta)\}_{deg(f_\beta)\geq 1}$. Recall we have:

$$Hom_{\mathcal{D}_A^{G,f}}(M, N[-1]) = Hom_{\mathcal{KCM}_A^{G,f}}(M, N[-1]).$$

If $\phi : M \to N[-1]$, then $deg(\phi) = 0$, which is impossible when we compute the degrees of both sides. □

Lemma 8.6. *Suppose that A is connected. For $M \in \mathcal{D}_A^{G,f}$, there is a distinguished triangle*

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1]$$

with $M^{\leq 0} \in \mathcal{D}^{G,f,\leq 0}$ and $M^{>0} \in \mathcal{D}^{G,f,\geq 0}[-1]$.

Proof. Following the same proof of Lemma 8.4, we can get a sub cell A-module $\tau^{\leq 0}M$ of M such that:

- $\tau^{\leq 0}M$ have generating data $\{(V_\alpha, \phi_\alpha)\}_{deg(\phi_\alpha)\leq 0}$.
- The map $q\tau^{\leq 0}M \to qM$ gives an isomorphism on H^n for $n \leq 0$.

Let $M^{\leq 0} = \tau^{\leq 0}M$ and let $M^{>0}$ be the cone of $\tau^{\leq 0}M \to M$. This gives us the distinguished triangle in $\mathcal{D}_A^G$:

$$M^{\leq 0} \to M \to M^{>0} \to M^{\leq 0}[1].$$

Because $M \in \mathcal{D}_A^{G,f}$, then $gr_n^W M \in D_{\mathbb{Q}}^{G,f}$ for all n and is isomorphic to zero for all but finitely many n. This means $M^{\leq 0}$ and $M^{>0}$ all satisfy these two conditions by using the long exact sequence. Recall the distinguished triangle given by the weight filtration $gr_n^W M \to M \to M^{>n} \to gr_n^W M[1]$. By induction, we can show that $M^{\leq 0}$ and $M^{>0}$ are all in $\mathcal{D}_A^{G,f}$. After applying the functor q, we can know that $M^{\leq 0} \in \mathcal{D}^{G,f,\leq 0}$ and $M^{>0} \in \mathcal{D}^{G,f,\geq 0}[-1]$.

The only thing need to check is non-degenerate for the t-structure. If we take $M \in \bigcap_{n\leq 0} \mathcal{D}^{\leq n}$, then $H^n(qM) = 0$ for all n, i.e., $qM \cong 0$ in $\mathcal{D}_{\mathbb{Q}}^{G,f}$. By the conservative property of the functor q, we know that $A \cong 0$ in $\mathcal{D}_A^{G,f}$. Another case is similar. □

Proposition 8.7. *$\mathcal{H}_A^{G,f}$ is a neutral Tannakian category over $\mathbb{Q}$.*

Proof. The derived tensor product makes $\mathcal{H}_A^{G,f}$ into an abelian tensor category. First we give a description about $\mathcal{H}_A^{G,f}$.

Lemma 8.8. *$\mathcal{H}_A^{G,f}$ is the smallest abelian subcategory of $\mathcal{H}_A^{G,f}$ containing the objects $A \otimes V$, where V is any rational G-representation, and closed under extensions in $\mathcal{H}_A^{G,f}$.*

Proof. (Induction on the weight filtration.) Let $\mathcal{H}_A^{G,T}$ be the full abelian subcategory containing all the objects $A \otimes V$, where V is any rational G-representation, and closed under extensions in $\mathcal{H}_A^{G,f}$. Let $M \in \mathcal{H}_A^{G,f}$ and $N = min\{n|W_n M \neq 0\}$. Then we have an exact sequence:

$$0 \to gr_N^W M \to M \to W^{>N} M \to 0.$$

By Lemma 5.11, we have $gr_N^W M \cong A \otimes C$, where C is in $D^b(G)$. Because the category of representations of G is semisimple, we can think C as a direct sum of rational G-representations with some shifts. Assume there exists a summand $W[i]$ of C with shift $i \neq 0$. Then applying q, we get that $0 \neq H^i(q(gr_N^W M)) \subset H^i(qM)$, which is a contradiction of our choice of $M \in \mathcal{H}_A^{G,f}$. This implies that $gr_N^W M \in \mathcal{H}_A^{G,T}$. By induction on the length of the weight filtration, $W^{>N}M$ is in $\mathcal{H}_A^{G,T}$. So $M \in \mathcal{H}_A^{G,T}$ and $\mathcal{H}_A^{G,T} = \mathcal{H}_A^{G,f}$. □

Since $(A \otimes V)^\vee = A \otimes V^\vee$, where $V^\vee$ is the dual representation of V, it follows from the above description that $M \to M^\vee$ restricts from $\mathcal{D}_A^{G,f}$ to an exact involution on $\mathcal{H}_A^{G,f}$. $\mathcal{H}_A^{G,f}$ is rigid because $\mathcal{D}_A^{G,f}$ is rigid.

The identity for the tensor product is A and $\mathcal{H}_A^{G,f}$ is $\mathbb{Q}$-linear.

We have a rigid tensor functor $q : \mathcal{H}_A^{G,f} \to \mathcal{H}_{\mathbb{Q}}^{G,f}$. Notice that $\mathcal{H}_{\mathbb{Q}}^{G,f}$ is equivalent to the category of rational representations of G. So there is a faithful forgetful functor $w : \mathcal{H}_{\mathbb{Q}}^{G,f} \to Vec_{\mathbb{Q}}$. We need only to see that q is faithful.

Recall we can identify q with $gr_*^W = \oplus gr_n^W$. Let $f : M \to N$ be a map in $\mathcal{H}_A^f$ such that $gr_n^W(f) = 0$ for all n. We need to show that $f = 0$. Again do the induction on the length of the weight filtration. We may assume that $W^n f = 0$, where n is the minimal integer such that $W_n M \oplus W_n N \neq 0$. Thus f is given by a map $\tilde{f} : W^{>n} M \to gr_n^W N$.

Claim: $\tilde{f} = 0$. Using the induction on the weight filtration, we need to show the following statement:

Given V and W pure weight rational G-representations such that $|V| > |W|$, then we have:

$$Hom_{\mathcal{H}_A^{G,f}}(A \otimes V, A \otimes W) \cong 0.$$

Firstly we have:

$$\begin{aligned} Hom_{\mathcal{H}_A^{G,f}}(A \otimes V, A \otimes W) &\cong Hom_{\mathcal{D}_A^{G,f}}(A \otimes V, A \otimes W) \\ \cong &Hom_{D(G)}(V, A \otimes W) \cong Hom_G(\mathbb{Q}, H^0(A \otimes W \otimes V^\vee)). \end{aligned}$$

Because A is connected, $H^0(A \otimes W \otimes V^\vee) \cong W \otimes V^\vee$, which is a rational representation of G with Adams degree strictly smaller than zero. This implies that:

$$Hom_G(\mathbb{Q}, H^0(A \otimes W \otimes V^\vee)) \cong 0.$$

Therefore we get that q is faithful. □

9 The bar construction

Let A be a cdga over G and let M, N be two dg A- modules. Then we define:

$$T^G(N, A, M) = N \otimes T(A) \otimes M,$$

where $T(A) = \mathbb{Q} \oplus A \oplus (A \otimes A) \oplus \cdots = \oplus_{r \geq 0} T^r(A)$ is the tensor algebra. It is spanned by the elements of the form $n[a_1|\cdots|a_r]m$. Notice that $T^G(N, A, M)$ is a simplicial graded abelian group with $N \otimes T^r(A) \otimes M$ in degree r, whose face maps are:

$$\delta_0(n[a_1|\cdots|a_r]m) = na_1[a_2|\cdots|a_r]m,$$

$$\delta_i(n[a_1|\cdots|a_r]m) = na_1[a_2|\cdots|a_i a_{i+1}|\cdots|a_r]m, \ 1 \leq i \leq r-1$$
$$\delta_r(n[a_1|\cdots|a_{r-1}|a_r]m) = n[a_1|\cdots|a_{r-1}]a_r m,$$

and degeneracies are:

$$s_i(n[a_1|\cdots|a_r]m) = n[a_1|\cdots|a_{i-1}|1|a_i|\cdots|a_r]m.$$

Define:

$$\delta = \sum_{0 \leq i \leq r} (-1)^i \delta_i : N \otimes T^r(A) \otimes M \to N \otimes T^{r-1}(A) \otimes M$$

Let $D^G(N, A, M)$ be the degenerate elements, those elements are spanned by the images of the s_i for every i.

Definition 9.1. Define the bar complex of M and N to be:

$$B^G(N, A, M) = T^G(N, A, M)/D^G(N, A, M).$$

Note that $B^G(N, A, M)$ is a bicomplex. The total differential is defined by

$$\begin{aligned}&d(n[a_1|\cdots|a_r]m)\\=&\partial(n[a_1|\cdots|a_r]m)) + (-1)^{deg(n)+deg(m)+\sum deg(a_i)}\delta(n[a_1|\cdots|a_r]m),\end{aligned}$$

where ∂ denotes the usual differential of A. We will consider the following special case that $M = N = \mathbb{Q}$, which is denoted by $\bar{B}^G(A)$, called the reduced bar construction. We collect formal properties of $\bar{B}^G(A)$.

- (Shuffle product) $\cup : \bar{B}^G(A) \otimes \bar{B}^G(A) \to \bar{B}^G(A)$

  $$[a_1|\cdots|a_p] \cup [a_{p+1}|\cdots|a_{p+q}] = \sum sgn(\sigma)[x_{\sigma(1)}|\cdots|x_{\sigma(p+q)}]$$

  where the sum is over all (p, q) shuffles $\sigma \in \Sigma_{p+q}$ (recall Σ_{p+q} is the symmetric group on $p+q$ letters) and the sign of σ is taking into account the degree of a_i.

- (Coproduct) $\Delta : \bar{B}^G(A) \to \bar{B}^G(A) \otimes \bar{B}^G(A)$

  $$\Delta([a_1|\cdots|a_n]) = \sum_{i=0}^{n}(-1)^{i(deg(a_{i+1})+\cdots+deg(a_n))}[a_1|\cdots|a_i] \otimes [a_{i+1}|\cdots|a_n].$$

- (involution) $\iota : \bar{B}^G(A) \to \bar{B}^G(A)$

  $$\iota([a_1|\cdots|a_n]) = (-1)^m[a_n|a_{n-1}|\cdots|a_1], m = \sum_{1\le i<j\le n} deg(a_i)deg(a_j).$$

These properties make $\bar{B}^G(A)$ a graded-commutative differential graded Hopf algebra in the category of G-representations. So $H^0(\bar{B}^G(A))$ is a commutative Hopf algebra over G. If consider the Adams grading structure, we can see that $\bar{B}^G(A)$ also has the Adams grading structure, and $\chi_A = H^0(\bar{B}^G(A))$ is an Adams graded Hopf algebra over G (or a graded Hopf algebra object in $\mathbf{Rep}_G$).

Definition 9.2. Define $\gamma_A = I_{\chi_A}/(I_{\chi_A})^2$, where I_{χ_A} is the augmentation ideal of χ_A.

Lemma 9.3. *γ_A determines a structure of a cdga over G.*

Remark 9.4. Recall the definition of co-Lie algebras firstly. A co-Lie algebra is a k-module γ with a cobracket map $\gamma \to \gamma \otimes \gamma$ such that the dual $\gamma^\vee$ is a Lie algebra via the dual homomorphism. Sullivan showed the following statement (Lemma 2.7 in [12] or p.279 in [16]):

A co-Lie algebra γ determines and is determined by a structure of DGA on $\wedge(\gamma[-1])$.

Proof. The coproduct Δ of $\bar{B}^G(A)$ induce a coproduct on χ_A, denoted also by Δ, satisfying:

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

for $x \in I_{\chi_A}$. (And also the involution.) Then $\Delta - \iota\Delta$ gives the cobracket on χ_A. So is γ_A and the quotient map $I_{\chi_A} \to \gamma_A$ is a map of co-Lie algebras. By the remark above, we know γ_A determines a cdga structure $\wedge(\gamma_A[-1])$. This structure is compatible with the G-representation structure. Notice that I_{χ_A} does not have the Adams degree 0 part. Then $\wedge(\gamma_A[-1])|0\rangle = \mathbb{Q}$. So $\wedge(\gamma_A[-1])$ is a cdga over G. □

Lemma 9.5. *Let A be a cdga over G. Then $H^*(\bar{B}^G(A))$ and χ_A is functorial in A and is a quasi-isomorphism invariant in A.*

Proof. Use the Eilenberg-Moore spectral sequence. See Lemma 2.21 in [3]. □

Theorem 9.6. *Let A be a cohomologically connected cdga over G. Then the 1-minimal model $A\{1\}$ of A is isomorphic to $\wedge(\gamma_A[-1])$.*

Proof. Follow the proof of [3]. From Lemma 9.5, we can assume that A is a generalized nilpotent cdga over G. A generalize nilpotent cdga A over G is a direct limit (A_α) of nilpotent cdga's. So we can assume that A is a nilpotent cdga over G with a free generator E which is a complex of G-representations. We need to use the following lemma, whose proof is totally the same as Lemma 2.32 in [3].

Lemma 9.7. *Let A be as above with free generator G-representation V. Fix an integer $s > 0$. Consider the decreasing filtration on $\bar{B}^G(A)$,*

$$F^k\bar{B}^G(A) = \langle x_{11}\cdots x_{1n_1}\otimes\cdots\otimes x_{m1}\cdots x_{mn_m}|$$

$$s\sum degx_{ij} + s\sum n_j - (2s-1)m \geq k\rangle.$$

Then, for a sufficiently large s depending on A, the resulting spectral sequence satisfies

$$\wedge(V[1]) \cong E_{2s} \cong E_\infty \cong Gr_F H^*(\bar{B}^G(A)).$$

Assuming this lemma, the projection map $\bar{B}^G(A) \to A$ induces a map $\phi : QH^*(\bar{B}^G(A)) \to (QA)[1]$. This is because that the boundaries and decomposable elements map to decomposable elements. In fact, boundaries are the sum of the form $\partial(a)$ and $a_1 \cdot a_2$. Using that the generalized nilpotence implies the minimal property (Proposition 7.4), both of these elements are decomposable. By the above lemma, this is an isomorphism. Furthermore, ϕ is an isomorphism between co-Lie algebras. Restricting ϕ to the degree 0 part, we know that $\wedge(\gamma_A[-1]) = \wedge(QH^0(\bar{B}^G(A))[-1])$ is isomorphic to 1-minimal model of A. □

10 Alternative identifications of the category $\mathcal{H}_A^f$

In Proposition 7.4, we show that the connected generalized nilpotent cdga over G can be recognized as a connected minimal cdga over G. Similarly we can also define the minimal cell A-module.

Definition 10.1. An Adams degree bounded below cell A-module is minimal if it is almost free and $d(M) \subset (IA)M$.

Definition 10.2. Let M be an Adams degree bounded below A-module. We define the nilpotent filtration $\{F_tM\}$ by letting $F_0M = 0$ and inductively letting F_tM be the sub A-module generated by $F_{t-1}M \cup \{m|dm \in F_{t-1}M\}$.

Remark 10.3. The minimal cell modules have the similar properties as the connected minimal cdgas. We can also define the generalized nilpotent A-modules. Because the proof of the following properties is the same as Part IV, section 3 in [12], we only list the main properties.

- A bounded below A-module M is generalized nilpotent if and only if it is a minimal cell A module.
- Let N be a dg A-module. Then there is a quasi-isomorphism $e : M \to N$, where M is a minimal A-module. This is unique up to the homotopy.

Next, we want to use another way to describe cell A-modules, which is called the connection matrix. See [12](namely the twisting matrix) or [13].

Definition 10.4. Let (M, d_M) be a complex of G-representations. An A-connection for M is a map $\Gamma : M \to IA \otimes M$ of G representations and cohomological degree 1. We say Γ is flat if $d\Gamma + \Gamma^2 = 0$. Here $d\Gamma = d_{IA\otimes M} \circ \Gamma + \Gamma \circ d_M$ and we extend Γ to $\Gamma : IA \otimes M \to IA \otimes M$ by the Leibniz rule.

Remark 10.5. Given a connection $\Gamma : M \to IA \otimes M$, we define

$$d_0 : M \to A \otimes M = M \oplus IA \otimes M, m \to d_M m \oplus \Gamma m$$

and extend d_0 to $d_\Gamma : A \otimes M \to A \otimes M$ by the Leibniz rule. The above equation is equivalent to saying that $d_\Gamma^2 = 0$.

Definition 10.6. We call an A-connection Γ for M nilpotent if M admits a filtration by complexes of G-representations:

$$0 = M_{-1} \subset M_0 \subset \cdots \subset M_n \subset \cdots \subset M$$

such that $M = \cup_n M_n$ and such that $d_M(M_n) \subset M_{n-1}$ and $\Gamma(M_n) \subset IA \otimes M_{n-1}$ for every $n \geq 0$.

Remark 10.7. Let $\Gamma : M \to IA \otimes M$ be a flat nilpotent connection. Then the dg A-module $(A \otimes M, d_\Gamma)$ is a cell module.

Lemma 10.8. *Let $\Gamma : M \to IA \otimes M$ be a flat connection. Suppose there is an integer r_0 such that $|m| \geq r_0$ for all $m \in M$. Then Γ is nilpotent.*

Proof. The proof is the same as Lemma 1.13.3 in [13]. □

Definition 10.9. A morphism $f : (M, d_M, \Gamma_M) \to (N, d_N, \Gamma_N)$ is a map of complexes of G-representations:

$$f = f_0 + f^+ : M \to A \otimes N = N \oplus IA \otimes N$$

such that $d_{\Gamma_N} f = f d_{\Gamma_M}$.

Definition 10.10. We denote the category of flat nilpotent connections over A by $Conn_A^G$ and denote the full subcategory of flat nilpotent connections on M with M a bounded complex of rational G-representations by $Conn_A^{G,f}$.

We can define a tensor operation on $Conn_A$ by

$$(M, \Gamma) \otimes (M', \Gamma') = (M \otimes M', \Gamma \otimes id + id \otimes \Gamma').$$

Complexes of $\mathbb{Q}$-vector spaces act on $Conn_A$ by:

$$(M, \Gamma) \otimes K = (M, \Gamma) \otimes (K, 0).$$

We recall that I is the complex $\mathbb{Q} \xrightarrow{\delta} \mathbb{Q} \oplus \mathbb{Q}$ with $\mathbb{Q}$ in degree -1 and with connection 0. We have the two inclusions $i_0, i_1 : \mathbb{Q} \to I$.

Definition 10.11. Two maps $f, g : (M, \Gamma) \to (M', \Gamma')$ are homotopic if there is a map $h : (M, \Gamma) \otimes I \to (M', \Gamma')$ satisfying $f = h \circ (id \otimes i_0), g = h \circ (id \otimes i_1)$.

Definition 10.12. Denote the homotopy category of $Conn_A^G$ by $\mathcal{H}Conn_A^G$, which has the same objects as $Conn_A^G$ and morphisms are homotopy classes of maps in $Conn_A^G$.

Remark 10.13. When we pass to homotopy classes and given a cell A-module M, it is totally determined by the underlying G-representation M_0, i.e., $M_0 = M \otimes_A \mathbb{Q}$.

We list the main properties of flat connections. The proof is the same as in Section 1.14 in [13].

- The category of A-cell modules is equivalent to the category of flat nilpotent A-connections.
- The above equivalence passes to an equivalence of $\mathcal{H}Conn_A^G$ with the homotopy category $\mathcal{KCM}_A^G$ as triangulated tensor categories.
- Suppose that A is connected. The second equivalence defines an equivalence of Tannakian categories $\mathcal{H}_A^{G,f}$ and the category of flat connections on G-representations $Conn_A^{G,f}$.

Given A a cohomologically connected cdga over G, by Theorem 9.6, we know that: $\gamma_A = A\{1\}^1$. Assume that A is a generalized nilpotent dga over G, which implies that $A\{1\} \cong A$. Then the co-Lie algebra structure of γ_A is given by the restriction of d to A^1. Notice that, by the minimal property (Proposition 7.4), d factors through: $d : A^1 \to \wedge^2 A^1 \subset A^2$. Let M be a complex of rational G-representations and $\Gamma : M \to IA \otimes M$ is a flat connection. In fact, Γ is a map $\Gamma; M \to A^1 \otimes M$ and the flatness is just saying that Γ makes M into an Adams graded co-module for the co-Lie algebra γ_A over G. In fact, we have equivalences between categories:

$$\mathcal{H}_A^{G,f} \cong Conn_A^{G,f} \cong co-rep^{G,f}(\gamma_A).$$

11 The main theorem

Lemma 11.1. *Let $\mathcal{D}$ be a triangulated category with t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. We denote its heart by $\mathcal{H}$. Assume that there is a triangulated functor $\rho : D^b(\mathcal{H}) \to \mathcal{D}$ such that:*

- *$\rho|_{\mathcal{H}[i]}$ is an inclusion for any $i \in \mathbb{Z}$;*
- *$\mathcal{D}$ is bounded, i.e., for any $M \in \mathcal{D}$, there exist $a \leq b \in \mathbb{Z}$ satisfying $M \in \mathcal{D}^{[a,b]} = \mathcal{D}_{\geq a} \cap \mathcal{D}_{\leq b}$.*
- *For any $M, N \in \mathcal{H}$ and $n \in \mathbb{Z}$, ρ induces an isomorphism*

$$Hom_{D^b(\mathcal{H})}(M, N[n]) \xrightarrow{\sim} Hom_{\mathcal{D}}(\rho(M), \rho(N)[n]).$$

Then ρ is an equivalence between triangulated categories.

Proof. We do the induction on the length of the object. Given an object A in $\mathcal{D}$, there exist the minimal a and maximal b such that $A \in \mathcal{D}_{\geq a} \cap \mathcal{D}_{\leq b}$. Then we define the length of A to be $b - a$. Firstly, we prove the following:

For any $A, B \in D^b(\mathcal{H})$ and $n \in \mathbb{Z}$, we have:

$$Hom_{D^b(\mathcal{H})}(M, N[n]) \xrightarrow{\sim} Hom_{\mathcal{D}}(\rho(M), \rho(N)[n]). \tag{11.1}$$

star

By induction, we assume that, for any $A \in D^b(\mathcal{H})^{a,b}$, $B \in D^b(\mathcal{H})^{c,d}$ and $max\{b-a, d-c\} \leq m-1$, the above is true. Take any A with length smaller than m, and B with length $m = b - a$. There is a distinguished triangle:

$$\tau_{\geq a}\tau_{\leq b-1}B \to B \to \tau_{\geq b}\tau_{\leq b}B \to \tau_{\geq a}\tau_{\leq b-1}B[1] \to .$$

Then we have a long exact sequence:

$$Hom(A, \tau_{\geq a}\tau_{\leq b-1}B) \to Hom(A, B) \to Hom(A, \tau_{\geq b}\tau_{\leq b}B)$$
$$\to Hom(A, \tau_{\geq a}\tau_{\leq b-1}B[1]) \to \cdots .$$

Using functoriality to compare the above sequence with:

$$Hom(\rho(A), \rho(\tau_{\geq a}\tau_{\leq b-1}B)) \to Hom(\rho(A), \rho(B)) \to Hom(\rho(A), \rho(\tau_{\geq b}\tau_{\leq b}B))$$
$$\to Hom(\rho(A), \rho(\tau_{\geq a}\tau_{\leq b-1}B[1])) \to \cdots .$$

We know (11.1) is holding for A, B by the five lemma and induction. Then we assume both A and B have length m. Using the similar method and induction again, we can know that (11.1) is holding, i.e., ρ is fully faithful.

Next we want to use induction to show that ρ is essentially surjective. It is enough to show that, for any object $B \in \mathcal{D}$, there exists $A \in D^b(\mathcal{H})$ such that $\rho(A) \cong B$. Take any element $B \in \mathcal{D}$ with length m. Then we have:

$$\tau_{\geq a}\tau_{\leq b-1}B \to B \to \tau_{\geq b}\tau_{\leq b}B \to \tau_{\geq a}\tau_{\leq b-1}B[1] \to .$$

In other words, we have:

$$\tau_{\geq b}\tau_{\leq b}B[-1] \xrightarrow{f} \tau_{\geq a}\tau_{\leq b-1}B \to B \to \tau_{\geq b}\tau_{\leq b}B \to .$$

By assumption, we have A_1 and $A_2 \in D^b(\mathcal{H})$ map to $\tau_{\geq b}\tau_{\leq b}B[-1]$ and $\tau_{\geq a}\tau_{\leq b-1}B$ respectively. By (11.1), we know that there exists a map g from A_1 to A_2, whose image under ρ is just f. We take $A = cone(g)$. Then by the axiom of triangulated categories, there exists a map $\rho(A) \to B$. By the five lemma and Yoneda lemma, applying the functor of type $Hom(\tilde{B},)$, where $\tilde{B} \in \mathcal{D}$, we know that $\rho(A) \cong B$. □

Theorem 11.2. *Let A be a cohomologically connected cdga over G. Then*

- *There is a functor:*
$$\rho : D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}.$$
- *The functor ρ constructed above is an equivalence of triangulated categories if and only if A is 1-minimal.*

Proof. We first need to construct a functor ρ. Consider a bounded complex

$$M^* = \{M^n, \delta^n : M^n \to M^{n+1}\}$$

in $\mathcal{H}_A^{G,f}$. Assume that each M^n is minimal. Furthermore, we assume that it is given by the generating data $\{V_{j^n}, \phi_{j^n}\}_{j^n \in J^n}$ and the connection matrix Γ^n. Then we define ρM^* with its generating data $\{V_{j^n}, \phi_{j^n}[n]\}_{j^n \in J^n}$ and its differential given by:

$$d|_{\phi_{j^n}[n](V_{j^n})} = \Gamma^n[n] + \delta^n[n].$$

If $f^* : M^* \to N^*$ is a quasi-isomorphism of chain complexes, then $\rho(f^*)$ is a quasi-isomorphism of A-modules.

Let's prove the second statement now. We assume that A is 1-minimal, i.e., $A \cong \wedge^*(\gamma[-1])$, where γ is the co-Lie algebra consisted by the indecomposable elements of $H^0(\bar{B}^G(A))$.

In order to apply the above result to our case ($\mathcal{D} = \mathcal{D}_A^{G,f}$ and $\mathcal{H} = \mathcal{H}_A^{G,f}$), we need to check the conditions in Lemma 11.1. The first and second condition are automatic. Let's check the third condition.

Notice that $\mathcal{H}_A^{G,f}$ can be identified with the category of co-representations of γ in the category of G-representations. In fact, given a finite dimensional co-representation V, we can associate it with a cell module $A \otimes V$.

We recall the following basic facts. (Lemma 23.1, Example 1(p.315) and p.319, 320 in[8].) Given a differential graded Lie algebra L, we have:

$$Ext^n_L(\mathbb{Q}, \mathbb{Q}) \cong Ext^n_{UL}(\mathbb{Q}, \mathbb{Q}) \cong H^n((\wedge^*(L[-1]))^\vee),$$

and

$$Ext^n_L(\mathbb{Q}, V) \cong Ext^n_{UL}(\mathbb{Q}, V) \cong H^n((\wedge^*(L[-1]))^\vee \otimes V),$$

where $\vee$ means taking the dual, UL is the universal enveloping Lie algbera of L and V is any L-module. $L[-1]_k = L_{k-1}$.

Applying to the co-Lie algbebra γ, we get:

$$Ext^n_\gamma(\mathbb{Q}, \mathbb{Q}) \cong H^n(\wedge^*(\gamma[-1])).$$

In fact, the proof of this isomorphism can be extended to the following case.

Given a co-Lie algebra γ over G and a γ co-representation V, we have:

$$Ext^n_\gamma(\mathbb{Q}, V) \cong H^n(\wedge^*(\gamma[-1]) \otimes V)|0\rangle.$$

Notice that the left hand side computes the extension groups in the category of γ-representations. We have:

$$H^n(\wedge^*(\gamma[-1])\otimes V)|0\rangle = Hom_G(\mathbb{Q}, H^n(\wedge^*(\gamma[-1])\otimes V)).$$

Therefore we get:

$$\begin{aligned}
&Hom_{D^b(\mathcal{H}_A^{G,f})}(\wedge^*(\gamma[-1])\otimes V, \wedge^*\gamma[-1]\otimes W[n])\\
\cong& Ext^n_{\mathcal{H}_A^{G,f}}(\wedge^*(\gamma[-1])\otimes V, \wedge^*(\gamma[-1])\otimes W) \cong Ext^n_\gamma(\mathbb{Q}, V^\vee\otimes W)\\
\cong& H^n(\wedge^*(\gamma[-1])\otimes V^\vee\otimes W)|0\rangle \cong H^0(\wedge^*(\gamma[-1])\otimes V^\vee \otimes W[n])|0\rangle\\
\cong& Hom_{\mathcal{D}_A^{G,f}}(\mathbb{Q}, \wedge^*(\gamma[-1])\otimes V^\vee\otimes W[n]) \cong Hom_{\mathcal{D}_A^{G,f}}(\wedge^*(\gamma[-1])\otimes V, \wedge^*(\gamma[-1])\otimes W[n]).
\end{aligned}$$

One can check that the composition of these isomorphisms is given by $\rho: D^b(\mathcal{H}_A^{G,f}) \longrightarrow \mathcal{D}_A^{G,f}$.

Conversely, we assume that ρ is an equivalence. Without loss of generality, we assume that A is generalized nilpotent. The above computation tells us that $H^n(A\otimes V)|0\rangle \cong Ext^n_\gamma(\mathbb{Q}, V)$, where γ is the co-Lie algebra consisting of the indecomposable elements in $H^0(\bar{B}(A))$. Let us consider the map $A\to \wedge^*(\gamma[-1])$, Applying the functor $Hom_G(V[n],\cdot)$ for any $n\in\mathbb{Z}$ and any G-representation V, we get:

$$Hom_G(V[n], A) \cong H^n(A\otimes V)|0\rangle \cong Ext^n_\gamma(\mathbb{Q}, V) \cong Hom_G(V[n], \wedge^*(\gamma[-1])).$$

This implies that, under the level of the G-representations, the above map is a quasi-isomorphism. Therefore A is 1-minimal. □

Corollary 11.3. *Let A be a cohomologically connected cdga over G. Then*

- *There is a functor:*

  $$\rho: D^b(co-rep^{G,f}_{\mathbb{Q}}(\chi_A)) \longrightarrow \mathcal{D}_A^{G,f}.$$

  Furthermore, ρ induces a functor on the hearts:

  $$\mathcal{H}(\rho): co-rep^{G,f}(\chi_A) \to \mathcal{H}_A^{G,f},$$

  which is an equivalence of Tannakian categories.

- *The functor ρ is an equivalence of triangulated categories if and only if A is 1-minimal.*

12 Relation with the weighted completion

Suppose that Γ is an abstract group, that G as before and that $\rho:\Gamma\to G(\mathbb{Q})$ is a homomorphism with Zariski dense image. We recall the definition of the weighted completion of Γ relative to ρ and a central cocharacter $\omega:\mathbb{G}_m\to R$ in [11].

Definition 12.1. A weighted Γ-module with respect to ρ and w is a finite dimensional $\mathbb{Q}$ vector space with Γ-action together with a weight filtration

$$\cdots \subset W_{m-1}M \subset W_mM \subset W_{m+1}M \subset \cdots$$

by Γ invariant subspaces. And satisfy:

- the intersection of the W_mM is 0 and their union is M,
- for each m, the representation $\Gamma \to Aut\ Gr^W_m M$ should factor through ρ and a homomorphism $\phi_m: G\to Aut\ Gr^W_m M$,

- $Gr^W_m M$ has weight m viewed as a $\mathbb{G}_m$-module via

$$\mathbb{G}_m \xrightarrow{w} G \xrightarrow{\phi_m} Aut\ Gr^W_m M.$$

Remark 12.2. Weighted Γ-modules together with the Γ-equivariant morphisms between weighted Γ-modules form the category of weighted Γ-modules, denoted by $W - Mod^G(\Gamma)$. Notice that the category of weighted Γ-modules is Tannakian.

Definition 12.3. The weighted completion of Γ with respect to ρ and w is the tannakian fundamental group of the category of weighted Γ-modules with respect to ρ and w. We denote it by $\mathcal{G}$ for simplicity. Then we have a short exact sequence:

$$1 \to \mathcal{U} \to \mathcal{G} \to G \to 1,$$

where $\mathcal{U}$ is the prounipotent radical of $\mathcal{G}$. We denote the Lie algebra of $\mathcal{U}$ by $\mathfrak{u}$.

Notice that, given V any rational G-representation, we can view V as a right Γ-module by ρ. In the following, we assume that $H^i(\Gamma, V)$ is finite dimensional for any finite dimensional rational G-representation and $i \in \mathbb{Z}_{\geq 0}$. Let $E^\bullet$ be any cohomologically connected cdga over G. Applying Theorem 8.3, we get the heart $\mathcal{H}^{G,f}_{E^\bullet}$ of $\mathcal{D}^{G,f}_{E^\bullet}$. Because we are interested in the abelian category only, we may also assume that $E^\bullet$ is 1-minimal. There is a canonical functor Φ from $\mathcal{H}^{G,f}_{E^\bullet}$ to $W - Mod^G(\Gamma)$ by sending $M \in \mathcal{H}^{G,f}_{E^\bullet}$ to itself. We denote the forgetful functor from $\mathcal{H}^{G,f}_{E^\bullet}$ (resp. $W - Mod^G(\Gamma)$) to the category of Γ modules by F_H (resp. F_W). Notice that the composition of Φ with F_W is F_H.

Remark 12.4. There is another equivalent definition of the weighted completion by universal properties. See Section 7 in [11] for example. The above canonical functor is just induced by the canonical map $\mathcal{G} \to \pi_1(\mathcal{H}^{G,f}_{E^\bullet}, F)$, where $\pi_1(\mathcal{H}^{G,f}_{E^\bullet}, F)$ is the tannakian fundamental group of $\mathcal{H}^{G,f}_{E^\bullet}$ and F is the forgetful functor to the category of $\mathbb{Q}$-vector spaces.

Assumption: For $i \in \mathbb{Z}_{>0}$, there exists a map:

$$\varphi_i : \bigoplus_\alpha H^i(\Gamma, (V_\alpha)^\vee) \otimes V_\alpha \to H^i(E^\bullet), \tag{12.1}$$

extension1

where the index set runs through all irreducible representations with non-positive weights. Furthermore, we require that $\{\varphi_i\}_{i\in\mathbb{Z}_{\geq 0}}$ is multiplicative, in other words, the multiplicative structure:

$$H^i(\Gamma, (V_\alpha)^\vee) \otimes H^j(\Gamma, (V_\beta)^\vee) \to H^{i+j}(\Gamma, (V_\alpha \otimes V_\beta)^\vee)$$

is compatible with the multiplicative structure on $H^*(E^\bullet)$ induced by the multiplication of $E^\bullet$.

In the next step, we want to use maps (12.1) to construct a functor $\Psi : W - Mod^G(\Gamma) \to \mathcal{H}^{G,f}_{E^\bullet}$. In fact, after rewriting the maps (12.1) and using 1-minimal property of $E^\bullet$, we get the following maps:

$$Ext^i_{\mathbb{Q}[\Gamma]-Mod}(\mathbb{Q}, (V_\alpha)^\vee) \xrightarrow{\varphi} Ext^i_{\mathcal{H}^{G,f}_{E^\bullet}}(E^\bullet, E^\bullet \otimes (V_\alpha)^\vee).$$

Furthermore, we have maps:

$$Ext^i_{\mathbb{Q}[\Gamma]-Mod}(V_1, V_2) \to Ext^i_{\mathcal{H}^{G,f}_{E^\bullet}}(E^\bullet \otimes V_1, E^\bullet \otimes V_2), \tag{12.2}$$

extension2

for any rational representations V_1, V_2 over G such that $V_1 \otimes (V_2)^\vee$ is a representation with non-positive weight.

Definition 12.5. For any $M \in W - Mod^G(\Gamma)$, we let l_M be the integer such that $W_{l_M} M \neq 0$ and $W_i M = 0$ for $i < l_M$ and we let u_M be the least integer such that $W_{u_m} M = M$. Then we define the length of M to be $u_M - l_M + 1$ and denote it by $l(M)$.

If $M \in W-Mod^G(\Gamma)$ has length one, then M is just a rational G-representation with a pure weight. We define $\Psi(M) = E^\bullet \otimes M$. In general, if $M \in W-Mod^G(\Gamma)$ has length bigger than one, then M can be considered as an element ξ in $Ext^{n-1}_{\mathbb{Q}[\Gamma]-Mod}(Gr^W_{u_M}M, Gr^W_{l_M}M)$. Then we define $\Psi(M)$ to be the element in $\mathcal{H}^{G,f}_{E^\bullet}$ corresponding to the image of ξ under maps (12.2), which is a $(n-1)$-step Yoneda extension of $E^\bullet \otimes Gr^W_{l_M}M$ by $E^\bullet \otimes Gr^W_{u_M}M$.

Theorem 12.6. *Let $E^\bullet$ be a cdga over G which satisfies the above assumption. In maps (12.1), we assume further that φ_1 is an G-equivariant isomorphism and φ_2 is an G-equivariant injection. Then Ψ is an equivalence between Tannakian categories.*

Proof. Recall in section 10, we could identify $\mathcal{H}^{G,f}_{E^\bullet}$ with the category of $(\gamma_{E^\bullet})^\vee$-representations in the category of rational G-representations. Here $(\gamma_{E^\bullet})^\vee$ is the dual of $\gamma_{E^\bullet}$ (Definition 9.2), i.e., a Lie algebra in the category of rational G-representations. Similarly, $W-Mod^G(\Gamma)$ can be identified with the category of representations over $\mathfrak{u}$ in the category of rational G-representations. Then Ψ induces a G-equivariant morphism between pronilpotent Lie algebras $\varphi : (\gamma_{E^\bullet})^\vee \to \mathfrak{u}$.

The requirement of φ_1, φ_2 will imply that: there exists a G-equivariant isomorphism:

$$\bigoplus_\alpha H^1(\Gamma, V_\alpha) \otimes (V_\alpha)^\vee \cong H^1((\gamma_{E^\bullet})^\vee).$$

and a G-equivariant injection:

$$\bigoplus_\alpha H^2(\Gamma, V_\alpha) \otimes (V_\alpha)^\vee \hookrightarrow H^2((\gamma_{E^\bullet})^\vee).$$

where the index set α runs all the irreducible G- representations with positive Adams weights.

Using Corollary 8.2. in [11], we get that:

1. There is a natural G-equivariant isomorphism: $\varphi^* : H^1(\mathfrak{u}) \xrightarrow{\cong} H^1((\gamma_{E^\bullet})^\vee)$.
2. There is a natural G-equivariant injection: $\varphi^* : H^2(\mathfrak{u}) \hookrightarrow H^2((\gamma_{E^\bullet})^\vee)$.

Then by Proposition 2.1. in [10], we know that $\varphi : (\gamma_{E^\bullet})^\vee \to \mathfrak{u}$ is an isomorphism of Lie algebras in the category of G-represenations. Therefore, the functor Ψ from $\mathcal{H}^{G,f}_{E^\bullet}$ to $W-Mod^G(\Gamma)$ is an equivalence. □

13 Split Tannakian categories over a reductive group

In this section, we describe a special kind of Tannakian categories. Let R be any reductive group over $\mathbb{Q}$.

Definition 13.1. We say that C is a neutral Tannakian category over R if C is a neutral Tannakian category over $\mathbb{Q}$ and there exists an exact faithful $\mathbb{Q}$-linear tensor functor $\widetilde{\omega} : C \to Rep(R)$, whose composition with the forgetful functor $F : Rep(R) \to Vec_{\mathbb{Q}}$ is the fiber functor $\omega : C \to Vec_{\mathbb{Q}}$.

Example 13.2. Assume that R satisfies Convection 2.1 and that A is a cohomologically connected cdga over R. From Theorem 8.3, we know the existence of the heart $\mathcal{H}^{R,f}_A$ of $\mathcal{D}^{R,f}_A$, which is a neutral Tannakian category over R.

Example 13.3. Another typical example is the relative completion $\mathcal{G}$ with respect a given map from a finite generated group $\Gamma \xrightarrow{\rho} R(\mathbb{Q})$. For the definition of the relative completion, we refer to [9]. Roughly speaking, the relative completion is just the weighted completion without considering the weight. In [10], Hain shows that the relative completion $\mathcal{G}$ can be expressed as a semi-product of a prounipotent algebraic group and R. This implies that there exists a functor from the category of finite dimensional representations over $\mathcal{G}$, which is a Tannakian category, to the category of rational representations over R. It's easy to check this functor is an exact faithful tensor functor.

Motivated by the above examples, we define:

Definition 13.4. A neutral Tannakian category C over R is split if the full subcategory of C consisting of semi-simple objects is isomorphic to $Rep(R)$.

Remark 13.5. The Tannakian fundamental group of a neutral Tannakian category C with a tensor generator is isomorphic to a linear proalgebraic group. See Proposition 2.20 in [7]. If we assume further that this Tannakian category is split over G and its Tannakian fundamental group is connected, then it will be the form $U \rtimes R$, where U is a prounipotent algebraic group. Example 13.2 and Example 13.3 satisfy these conditions.

In the end, we want to use the method of framed objects (Section 6 of [3] for example) to give a description of the coordinate ring of the Tannakian fundamental group of any split neautral Tannkian category C with a tensor generator. We assume that the Tannakian fundamental group is connected. As explained in Remark 13.5, it's enough to determine the coordinate ring of the prounipotent radical as a Hopf algebra object in $Rep(R)$.

Definition 13.6. A framed object in C is an object X in C together with an element $u \in Hom_R(V, X)$ (called the frame vector) and an element of $v \in Hom_R(X, \mathbb{Q})$ (called the frame covector), where V is an irreducible G-representation. We denote this object by (X, u, v).

Definition 13.7. Two framed objects $X, Y \in C$ are identified if there is a mapping $X \to Y$ compatible with the framing.

Notice that such pairs X, Y define a relation $\mathcal{R}$ on the set of all framed objects. Then χ_C is defined as the set of equivalence classes of the smallest equivalence relation containing $\mathcal{R}$. By definition, χ_C is graded over the irreducible rational representations of G. For a given rational R-representation V, we denote the V graded piece of χ_C by $\chi_C(V)$.

Claim 13.8. χ_C is a Hopf algebra in $Rep(R)$.

In order to explain our claim, first we recall a fundamental result – Proposition 3.1 in [9]. If $(V_\alpha)_\alpha$ is a set of irreducible rational R-representations, viewed right R-modules, then, as an (R, R) bimodule, $\mathcal{O}(R)$ is canonically isomorphic to $\bigoplus_\alpha (V_\alpha)^\vee \boxtimes V_\alpha$. In other words, there exists a Hopf algebraic structure on $\bigoplus_\alpha (V_\alpha)^\vee \boxtimes V_\alpha$. If we denote the generator of $(V_\alpha)^\vee \boxtimes V_\alpha$ by $v_\alpha^\vee \boxtimes v_\alpha$, then we have:

(1). $m((v_\alpha^\vee \boxtimes v_\alpha) \otimes (v_\beta^\vee \boxtimes v_\beta)) = \oplus_\gamma (v_\gamma^\vee \boxtimes v_\gamma)$, where m is the multiplication of the Hopf algebra and the index set runs through the irreducible representations V_γ appearing in the tensor product of $V_\alpha \otimes V_\beta$.

(2). Let Δ be the coproduct map of the Hopf algebra $\bigoplus_\alpha (V_\alpha)^\vee \boxtimes V_\alpha$. Then we can express

$$\Delta(v_\alpha^\vee \boxtimes v_\alpha) = \oplus_{\beta,\gamma} m_\alpha^{\beta,\gamma} ((v_\beta)^\vee \boxtimes v_\beta) \otimes ((v_\gamma)^\vee \boxtimes v_\gamma),$$

where $m_\alpha^{\beta,\gamma}$ is the corresponding multiplicity.

Now we move to our claim.

- The sum on χ_C is the Baer sum.
- The product is defined by the tensor product of underlying objects together with the tensor product of the framings. Let $(X_1, u_1, v_1), (X_2, u_2, V_2)$ be two framed objects, which represent two classes $[(X_1, u_1, v_1)]$ and $[(X_2, u_2, v_2)]$ in $\chi_C(V_\alpha)$ and $\chi_C(V_\beta)$ respectively. Then we define:

$$[(X_1, u_1, v_1)] \cdot [(X_2, u_2, v_2)] = [(X_1 \otimes X_2, v_1 \otimes v_2, v_1 \otimes v_2)] \in \chi_C(V_\alpha \otimes V_\beta),$$

  Then one can project this element into pieces $\chi_C(V_\gamma)$, where the irreducible representations V_γ appearing in the tensor product of $V_\alpha \otimes V_\beta$.

- The coproduct

$$\psi = \oplus_\alpha \psi_\alpha;$$
$$\psi_\alpha = \oplus \psi_\alpha^{\beta,\gamma};$$
$$\psi_\alpha^{\beta,\gamma} : \chi_C(V_\alpha) \to \chi_C(V_\beta) \otimes \chi_C(V_\gamma),$$

where the first index set α runs over all irreducible representations V_α and the second index set is the same as the index set appearing in the coproduct (13), is defined as follows. We let $[(M,u,v)] \in \chi_C(V_\alpha)$ and let

$$\sum x_i \otimes y_i = 1 \in Hom_R(H, V_\gamma) \otimes Hom_R(V_\gamma, H).$$

Here 1 corresponds the identity map under the isomorphism:

$$Hom_R(H, V_\gamma) \otimes Hom_R(V_\gamma, H) \cong End(Hom_R(V_\gamma, H)).$$

Let M_{x_i} be M with frame vector u and frame covector x_i and let M_{y_i} be M with frame vector y_i and frame covector v. Then we put

$$\psi_\alpha^{\beta,\gamma}([(M,u,v)]) = \sum [(M,u,x_i)] \otimes [(M,y_i,v)].$$

Proposition 13.9. *Let C be a split neutral Tannakian category with a tensor generator and the Tannakian fundamental group of C is connected. Then C is equivalent to the category of finite dimensional χ_C-comodules in $Rep(R)$.*

Proof. Step 1. As we state in Remark 13.5, C is isomorphic to the category of representations over a linear proalgebraic group $\mathcal{G}$, which is a semi-product of a prounipotent algebraic group $\mathcal{U}$ with R. Recall the following basic properties about extension groups in C or the category of finite dimensional $\mathcal{G}$-representations (Section 5.1 in [11]):

A. The category of $\mathcal{G}$-modules has enough injectives, therefore the cohomology groups can be identified with the extension groups:

$$H^i(\mathcal{G}, V) = Ext^i_{\mathcal{G}-mod}(\mathbb{Q}, V)$$

for $V \in \mathcal{G} - mod$ and $\mathbb{Q}$ is the trivial object or the trivial representation. If V is the trivial representation $\mathbb{Q}$, we will denote $H^i(\mathcal{G}, V)$ by $H^i(\mathcal{G})$ for short. When $V \in C$, we have:

$$H^i(\mathcal{G}, V) = Ext^i_C(\mathbb{Q}, V),$$

because each element in the extension class can be replaces by an equivalent extension consisting of only finite dimensional representations.

B. We consider $V \in Rep(R)$ as a $\mathcal{G}$-representation. Then we have isomorphism in $Rep(R)$ (Theorem 5.3 in [11]):

$$H^i(\mathcal{U}) \cong \bigoplus_\alpha H^i(\mathcal{G}, (V_\alpha)^\vee) \otimes V_\alpha \cong \bigoplus_\alpha Ext^i_C(1, (V_\alpha)^\vee) \otimes V_\alpha, \qquad (13.19)$$

where the index set runs over all irreducible R-representations.

Step 2. Notice that the coproduct on χ_C is conilpotent. For the definition of a conilpotent coproduct, we refer to Section 3.8 in [6]. Using Theorem 3.9.1 in [6], we know that $\mathcal{U}_C$, which is defined to be $Spec(\chi_C)$, is a prounipotent algebraic group over $\mathbb{Q}$.

Step 3. Now we want to construct a functor from C, which can be identified as the category of representations over $\mathcal{G}$, to the category of finite dimensional χ_C-comodules (the same as $\mathcal{U}_C$ modules) in $Rep(R)$. Because $Rep(R)$ is a full subcategory of C as we assume, we also consider $V \in Rep(R)$ as an element in C. Let $M \in C$. Notice that $M \cong \bigoplus_\alpha Hom_C(V_\alpha, M) \otimes V_\alpha$. Under this isomorphism, any $m \in M$ can be decomposed as the sum of the form $u_\alpha \otimes m_\alpha \in Hom_R(V_\alpha, M) \otimes V_\alpha$. Then we define a χ_C-comodule structure on M by:

$$M \xrightarrow{\varphi_M} \chi_C \otimes M : m \to \oplus_\alpha [(M, u_\alpha, 0)] \otimes m_\alpha. \tag{13.2}$$

We denote the functor from C to the category of χ_C-comodules in $Rep(R)$ by Φ. Then under Tannakian duality, Φ induces a group homomorphism from $\mathcal{U}_C$ to $\mathcal{U}$.
Step 4. By the definition of χ_C, we may check:

- $H^1(\mathcal{U}_C) \cong \bigoplus_\alpha Ext^1_C(1, (V_\alpha)^\vee) \otimes V_\alpha$ and $H^2(\mathcal{U}_C) \cong \bigoplus_\alpha Ext^2_C(1, (V_\alpha)^\vee) \otimes V_\alpha$.
- Furthermore, Φ induces a map from $H^i(\mathcal{U}) \to H^i(\mathcal{U}_C)$, which are coincide with (13.1).

If we denote the corresponding Lie algebra of $\mathcal{U}_C$ and $\mathcal{U}$ by $\mathfrak{u}_C$ and $\mathfrak{u}$, using Proposition 2.1 in [11] again, we know that: $\Phi : \mathfrak{u}_C \to \mathfrak{u}$ is a R-equivariant isomorphism between pronilpotent Lie algebras. Therefore, Φ induces an equivalence of Tannkian categories between C and the category of finite dimensional χ_C-comodules in $Rep(R)$. □

Example 13.10. Let F be a field finitely generated over a prime field and let l be a prime number. A mixed Tate representation of the absolute Galois group $Gal(\bar{F}/F)$ over F is a filtered representation such that the odd graded pieces are 0 and the $(2r)$-th graded piece is a sum of copies of $\mathbb{Q}_l(-r)$. Then the category of mixed Tate representations of $Gal(\bar{F}/F)$, denoted by $MTM_{F,l}$, is a split neutral Tannakian category over $\mathbb{G}_m$. Using Proposition 13.9, we know that $MTM_{l,F}$ is isomorphic to the category of finite dimensional $\chi_{MTM_{l,F}}$ comodules in $Rep(\mathbb{G}_m)$. The latter fact has been shown in [2].

Example 13.11. Let F be a number field, then the abelian category $MTM(F, \mathbb{Q})$ of mixed Tate motives with rational coefficients over F exists. See [13] for example. In fact, $MTM(F, \mathbb{Q})$ is also a split neutral Tannakian category over $\mathbb{G}_m$. One may use Proposition 13.9 to describe such category. Moreover, the full rigid tensor subcategory of the abelian category of mixed motives generated by the motive of a fixed smooth projective variety is conjecturally a split neutral Tannakian categories over some reductive group.

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