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ABSTRACT

The multi-scale zero relaxation singular limit for gas dynamics in thermal non-equilibrium with multiple non-equilibrium modes in multi-dimensions with physical boundaries from non-equilibrium to thermal equilibrium of compressible Euler flow is proved in this paper for analytical data by establishing the uniform local-in-time estimates. A cancellation mechanism is utilized to deal with the nonlinear singular terms that cause the increase in both time and space derivatives in energy estimates. The rates of the relaxations corresponding to different non-equilibrium modes tending to zero discussed in this paper can be arbitrarily different.

I. INTRODUCTION

A. Motivation and background

The purpose of this paper is to investigate the singular limit of multi-dimensional gas dynamics in thermal non-equilibrium with several non-equilibrium modes as the relaxation times of non-equilibrium modes tend to zero in different scales in the presence of physical boundaries and the analytic setting.

In local thermodynamic equilibrium, it is infinitely rapid for all molecular processes to take place within a gas, which thus can instantaneously adjust to changes in its environment, and the motion in local thermodynamic equilibrium is described by the celebrated compressible Euler equations. In some cases, e.g., for air at room temperature, this assumption is reasonable. However, for the increased temperature, the internal structure of the molecules has to be taken into consideration and the molecular processes take a considerable time; the assumption of the local thermodynamic equilibrium state becomes inadequate.[1–3]

The thermal non-equilibrium modes for gas dynamics describing deviations from local equilibrium are governed by rate equations with local relaxation times, which characterize the time scales for the nonequilibrium modes to relax to the equilibrium state.[1–3] It is important, from both the physical and mathematical points of view, to investigate the asymptotic behavior of gas flow when relaxation times tend to zero. This amounts to understand the physical process from the thermal non-equilibrium to equilibrium. In the partial differential equation (PDE) theory, this is a singular limit problem, which poses a significant challenge. It is, in particular, so in the presence of physical boundaries and/or initial layers. When physical boundaries present, the behaviors of solutions are, in general, very singular near boundaries.

For the system of gas dynamics in thermal non-equilibrium with several non-equilibrium modes in one dimension, the zero relaxation limit is studied in Ref. 4 under the assumption that all the relaxations for the different non-equilibrium modes are equal. This is quite a restrictive assumption from both physical and mathematical considerations. Indeed, the analysis for the case when all the relaxations for the different non-equilibrium modes are equal is very similar to the case with only one non-equilibrium mode since one may sum the rate equations of non-equilibrium modes in this case, and the sum satisfies an equation, which is almost the same as the rate equation for the case with only one non-equilibrium mode. Thus, it is extremely difficult to extend the analysis in Ref. 4 to the case with different relaxations for different non-equilibrium modes. In particular, when the relaxations tend to zero at different rates, it poses a multi-scale singular limit problem with multiple small physical parameters that tend to zero in possible different orders. It should be noted that only few results of rigorous justification of such multi-scale singular limits for fluid equations are available, especially, in the presence of physical boundaries in multi-dimensions.

The purpose of our program is to extend the analysis in Ref. 4 in the following aspects: investigate the asymptotic behavior of gas flow in thermal non-equilibrium with several non-equilibrium modes in multiple space dimensions when relaxations of different non-equilibrium modes tend to zero at different rates (i.e., in different orders) in the presence of physical boundaries.

It is very interesting to consider the singular small relaxation asymptotic behavior for thermal non-equlibrium gas flow with multiple non-equilibrium modes. Physically, the number of non-equilibrium modes depends on the temperature range for diatomic gases. The interaction of rotation and vibration is significant for gases with high temperature so that the upper vibrational states of the molecules are to be populated considerably. Mathematically, multi-scale singular limit problems for fluid equations are fantastic topics. Indeed, there are few results available for rigorous justification of more than two scale singular limit, particularly, when physical boundaries present.

B. The problem, main theorem, and related works

In this paper, we study the gas dynamics with several thermal nonequilibrium modes in $\mathbb{R}^2_+$. (The same arguments apply for $\mathbb{R}^d_+$ with $d \geq 2$, we present the results for $d = 2$ for simplicity.) A real gas has non-equilibrium molecular processes, such as vibration, chemical composition, and rotation. To characterize the physical non-equilibrium process of a real gas, we assume that the gas has $n-1, n \geq 2$ non-equilibrium modes. Then, the motion of the gas is described by the following multi-scale relaxation system in $(t, x, y) \in \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+$:

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ (\rho \mathscr{E})_t + \operatorname{div}(\rho \mathscr{E} u + pu) = 0, \\ (\rho e_i)_t + \operatorname{div}(\rho e_i u) = \rho \dfrac{E_i - e_i}{\tau_i}, \quad 2 \leq i \leq n, \end{cases} \tag{1.1}$$

where $\rho, u = (u_1, u_2)^t, p, \mathscr{E}, e_i, E_i$, and τ_i are the gas density, velocity, pressure, specific total energy, specific energy for the ith non-equilibrium mode, local equilibrium value of the corresponding specific vibrational energy, and local relaxation time for the ith mode, respectively. Here, the relaxation time scale $\tau_i = \tau_i(\varepsilon)$ satisfies $\tau_i(\varepsilon) \to 0$ as $\varepsilon \to 0$.

The specific total energy $\mathscr{E}$ is the combination of internal energy and kinetic energy,

$$\mathscr{E} = e + \frac{1}{2}|u|^2. \tag{1.2}$$

It is assumed that the gas flow is in instantaneous translational equilibrium everywhere. The subscript 1 is used to denote thermodynamic quantities related to the translational mode, while $i, 2 \leq i \leq n$ are for those related to the non-equilibrium modes. The internal energy e is the sum of the equilibrium energy e_1 and the non-equilibrium energy $e_i, 2 \leq i \leq n$,

$$e = \sum_{i=1}^{n} e_i. \tag{1.3}$$

The total specific entropy s reads

$$s = \sum_{i=1}^{n} s_i, \tag{1.4}$$

where s_1 is the specific translational entropy of the molecules and $s_i, 2 \leq i \leq n$ are the entropy of the ith mode. Moreover, we use $T_i, 1 \leq i \leq n$ to denote the temperature associated with the ith mode of the molecules. For a non-equilibrium state, there must exist some $i, 2 \leq i \leq n$ such that $T_i \neq T_1$. That is, a state in equilibrium is equivalent to

$$T_i = T_1, \quad 2 \leq i \leq n. \tag{1.5}$$

With these notations, the thermodynamic equations for the translational mode and the non-equilibrium modes read

$$\begin{aligned} de_1 &= T_1(s_1, v)ds_1 - p(s_1, v)dv, \\ de_i &= T_i(s_i)ds_i, \quad 2 \le i \le n. \end{aligned} \tag{1.6}$$

Here and in the following:

$$v = \frac{1}{\rho} \tag{1.7}$$

is the specific volume (in the rest of this paper, we will always work on variable v instead of ρ). Combining the above two equations, we obtain the thermodynamic equation for the nonequilibrium gas flows as a whole,

$$de = T_1(s_1, v)ds + \sum_{i=2}^{n} [T_i(s_i) - T_1(s_1, v)]ds_i - p(s_1, v)dv. \tag{1.8}$$

For more physical background, one can refer to Refs. 1–3 and 5.

For the equilibrium mode, any two thermodynamic variables can be chosen as independent ones. Here, we use the notations

$$p = p(e_1, v) = \tilde{p}(v, T_1), \quad T_1 = T_1(e_1, v), \quad s_1 = s_1(e_1, v). \tag{1.9}$$

In addition, the local equilibrium values E_i for the non-equilibrium energy e_i are given functions of v and e_1,

$$E_i = E_i(e_1, v), \ 2 \le i \le n. \tag{1.10}$$

Hence, if one expresses e_i as a function of T_i,

$$e_i = \omega_i(T_i), \quad 2 \le i \le n, \tag{1.11}$$

then

$$E_i = \omega_i(T_1), \quad 2 \le i \le n. \tag{1.12}$$

We now state the physical assumption throughout this paper,

$$\begin{aligned} &\tilde{p}_v = \frac{\partial}{\partial v}\tilde{p}(v, T_1) < 0, \quad T_{1e_1} = \frac{\partial}{\partial e_1}T_1(v, e_1) > 0, \\ &\omega_i'(T_1) > 0, \quad p_{e_1} = \frac{\partial}{\partial e_1}p(v, e_1) \ne 0. \end{aligned} \tag{1.13}$$

In this paper, for the simplicity of presentation, we assume that the degree of freedom adjusting instantaneously in the internal energy is α, and the relation with T_1 is given, while β_i of the degrees of freedom are related to the ith nonequilibrium modes. That is,

$$e_1 = \frac{\alpha}{2}pv = \frac{\alpha}{2}RT_1, \quad E_i = \frac{\beta_i}{2}pv, \tag{1.14}$$

where α, R, and β_i are positive constants, we assume $R = 1$ for simplicity. It should be noted that the same analysis in this paper applies to the general equations of state. Formally, if the relaxation time $\tau_i(\varepsilon)$ is taken so short that

$$e_i = E_i, \ 2 \le i \le n,$$

are the adequate approximation to the rate equations, then one has the equilibrium theory,

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla p = 0, \\ (\rho \bar{\mathscr{E}})_t + \operatorname{div}(\rho \bar{\mathscr{E}} u + pu) = 0, \end{cases} \tag{1.15}$$

which is the celebrated system of compressible Euler equations,[24] where

$$\bar{\mathscr{E}} = e_1 + \sum_{i=2}^{n} E_i + \frac{|u|^2}{2}.$$

The aim of this paper is to investigate the strong multi-scale zero relaxation limit of the gas dynamics with several nonequilibrium modes (1.1) in $\mathbb{R}_+^2 = \{(x, y) : -\infty < x < \infty,\ 0 \le y < \infty\}$. We impose the solid-wall boundary condition on the velocity field for system (1.1),

$$u \cdot \nu|_{y=0} = 0, \tag{1.16}$$

where ν is the unit outer normal to the boundary $y = 0$, i.e., $\nu = (0, -1)^t$. The initial data for the non-equlibrium system (1.1) are given by

$$(\rho^\varepsilon, u^\varepsilon, p^\varepsilon, e_i^\varepsilon)(0, x, y) = (\rho_0^\varepsilon, u_0^\varepsilon, p_0^\varepsilon, (e_i)_0^\varepsilon)(x, y), (x, y) \in \mathbb{R}_+^2,\ i = 1, \dots, n, \tag{1.17}$$

satisfying

$$\Gamma^{-1} \le (\rho_0^\varepsilon, p_0^\varepsilon, (e_i)_0^\varepsilon)(x, y) \le \Gamma, i = 1, \dots, n,\ (x, y) \in \mathbb{R}_+^2 \tag{1.18}$$

for some positive constant Γ independent of ε, and the compatibility condition

$$u_0^\varepsilon \cdot \nu|_{y=0} = 0. \tag{1.19}$$

In the rest of this paper, we let

$$v^\varepsilon = \frac{1}{\rho^\varepsilon}, \quad v_0^\varepsilon = \frac{1}{\rho_0^\varepsilon} \tag{1.20}$$

to denote the specific volume.

Let $(E_i^\varepsilon)_0 = \frac{\beta_i}{\alpha}(e_1^\varepsilon)_0$ $(2 \le i \le n)$ be the initial local equilibrium value for the ith non-equilibrium mode, and set

$$(\chi_i)_0^\varepsilon = (E_i^\varepsilon)_0 - (e_i^\varepsilon)_0,\ 2 \le i \le n.$$

Furthermore, we assume that the initial data satisfy the uniform bounds in the analytic class, $2 \le i \le n$,

$$\sum_{m=0}^{\infty} \frac{\delta^{2m}}{(m!)^2} \Bigg(\sum_{j=0}^{2} \|\partial_y^m e^{\delta\langle D\rangle} \partial_t^j (v_0^\varepsilon, p_0^\varepsilon,\ u_0^\varepsilon, \frac{(\chi_i^\varepsilon)_0}{v_0^\varepsilon})\|_{H_{x,y}^N}^2 + \sum_{j=0}^{1} \|\partial_y^m e^{\delta\langle D\rangle} \partial_t^j ((\chi_i^\varepsilon)_0, (e_i^\varepsilon)_0)\|_{H_{x,y}^N}^2 \Bigg) \le M \tag{1.21}$$

for some $N \ge 5$, where $M < \infty$ is a positive constant independent of $\tau_i(\varepsilon)$ $(2 \le i \le n)$, $\|\cdot\|_{H_{x,y}^N}$ is the standard Sobolev norm, and $\delta > 0$ is a positive constant representing the analytic bandwidth, which can be taken as 2 (i.e., $\delta = 2$) without loss of generality. Note that $\partial_t^j(\cdot)$ is defined through (1.1) and (2.3).

Now, we state our main result as follows:

Theorem I.1. *For the initial data $(v_0^\varepsilon, u_0^\varepsilon, p_0^\varepsilon, (e_2^\varepsilon)_0, \dots, (e_n^\varepsilon)_0)$ given by (1.17) with (1.18)–(1.21), there exists $T > 0$ independent of ε such that for every $\varepsilon \in (0, 1)$, there exists a unique analytic solution $(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \cdots, e_n^\varepsilon)$ to the initial boundary value problem (1.1), (1.16), and (1.17) on $[0, T]$ and an analytic function (v^e, u^e, p^e) defined on $\mathbb{R}_+^2 \times [0, T]$ such that $(\rho^e = \frac{1}{v^e}, u^e, p^e)$ solves the compressible Euler equations (1.15) with the boundary condition $u^e . \nu = 0$ satisfying*

$$\sup_{[0,T]} \|(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \dots, e_n^\varepsilon) - (v^e, u^e, p^e, E_2, \dots, E_n)\|_{L^2} \to 0,$$
$$\sup_{[0,T]} \|(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \dots, e_n^\varepsilon) - (v^e, u^e, p^e, E_2, \dots, E_n)\|_{L^\infty} \to 0,$$

as $\varepsilon \to 0$, where $E_i = \frac{\beta_i}{2} p^e v^e$ $(i = 2, \dots, n)$.

There have been few rigorous justifications of multi-scale singular limits with multiple small physical parameters that tend to zero at possible different rates, in particular, in the presence of physical boundaries. For a problem in the entire $\mathbb{R}^d$-space or torus $\mathbb{T}^d$, a singular problem for the first-order hyperbolic system is considered in Ref. 6 with two small parameters. A typical physical example for this is the compressible MHD equations involving a small Mach number and Alfvén number. It should be noted that the singular terms considered in Ref. 6 are linear and the rates of small parameters tending to zero are restricted on the ratio of two small parameters. Moreover, the problem considered in Ref. 6 does not involve physical boundaries. The problem studied in the present paper is for a new type of multi-scale singular limit for which the singular terms are nonlinear, and the rates of small parameters tending to zero are arbitrary, without any restrictions on their orders. Moreover, we consider the problem in the presence of physical boundaries. In fact, for another singular limit problem, the low Mach number limit of compressible fluids, important progress has been made for the cases of the whole space, e.g., in Refs. 7–9. However, for problems in the presence of physical boundaries, it becomes much more challenging.

The singular terms of system (1.1) contribute to increasing the derivatives in normal direction in the higher order energy estimates. We consider to introduce the analytic settings in the normal direction. However, the singular terms lead to the increase in t-derivatives in some sense from the rate equations in (1.1), which cannot be overcome by our analytic tools. Meanwhile, the relaxation (singular) terms χ_i/τ_i are closely related to $p_t, (\chi_i)_t$ ($\chi_i = E_i - e_i$) from the observation of (2.3). Hence, the uniform estimates of the relevant t-derivatives are also crucial to the uniform estimates. To control the strong multi-scale singularity, we will utilize the cancellation mechanism behind the non-equilibrium system in analytic functional space.

We should note that the vanishing-viscosity limit is studied in analytic settings for Navier–Stokes equations in the presence of physical boundaries in Refs. 10–13. The studies of the initial boundary value problems for various hyperbolic systems with stiff relaxations can be found in Refs. 14–22, mainly for the Broadwell model or Jin-Xin model in 1D.

A very useful tool in the study of the asymptotic behavior of solutions to fluid equations as some physical parameters such as viscosity tend to zero in the presence of physical boundaries is the matched asymptotic expansions. However, for the problem studied in this paper, the asymptotic expansion near the boundary is quite unclear even at the formal level, when the rates that the different relaxations tend to zero are different. Therefore, we adopt the approach of the compactness arguments by establishing uniform estimates, as done for the vanishing viscosity problem for Navier–Stokes equations under the Navier-type slip boundary condition in Ref. 23.

In Refs. 4 and 24, the zero-relaxation limit for the outer pressure problem of gas dynamics in thermal non-equilibrium with several non-equilibrium modes (1.1) in one space dimension is considered, with the boundary condition that the pressure p is prescribed as a given positive constant. This outer pressure problem is a free boundary problem in Eulerian coordinates, and it is reduced to a fixed boundary problem in Lagrangian coordinates. It should be emphasized that the case considered in Ref. 4 is that all the relaxation parameters corresponding to the non-equilibrium modes are the same, i.e., $\tau_i = \tau$ $(2 \le i \le n)$, not only with the same convergence rate to zero. This assumption and the additional assumption that the quantity $\sum_{i=2}^{n}\chi_i$ is zero on the boundary initially are crucial to establishing the uniform estimates with respect to the relaxation parameters in Sobolev space so that the zero relaxation limit can be achieved, where $\chi_i = E_i - e_i$ $(2 \le i \le n)$. In fact, let $\chi = \sum_{i=2}^{n}\chi_i$, one can show that if $\chi = 0$ initially on the boundary, then it keeps being zero on the boundary in the time interval of the existence of solutions. Using this condition, one can derive the boundary conditions for the second spatial derivatives of the pressure $p_{xx} = 0$ and the first derivative of the velocity $u_x = 0$ on the boundary. Those conditions are crucial to establishing the uniform regularity of solutions for the problem considered in Ref. 4. Indeed, the quantity $\chi = \sum_{i=2}^{n}(\chi_i)$ is used in Ref. 4 as a unknown, which vanishes on the boundary. The case considered in Ref. 4 that all the relaxation parameters corresponding to the non-equilibrium modes are the same makes the problem similar to the problem with a single non-equilibrium mode. In the case when the relaxation parameters of the different non-equilibrium modes tend to zero with the different rates, the above-mentioned approach fails, and the loss of derivatives occurs.

In this paper, we consider the solid wall problem for the multi-dimensional model of gas dynamics in thermal non-equilibrium with different relaxation rates. Compared with Ref. 4, we cannot obtain the uniform estimates for the multidimensional problem in Sobolev space even for the case when the rates of the relaxation parameters are the same, since the structure of multi-dimensional problem also causes difficulties in eliminating the boundary effects and the singularities in the energy estimates. To overcome the difficulty of the boundary effect in energy estimates, which corresponds to the loss of space derivatives in the energy estimates, we naturally consider the problem in the analytic settings. Moreover, we can derive the uniform estimates for the multi-dimensional model involving different relaxation rates by virtue of a cancellation mechanism in analytic setting in this paper.

Mathematical analysis of the global existence and large time behavior for the equations of thermal non-equilibrium gas dynamics for Cauchy problems or initial boundary value problems in 1D can be found in Refs. 25–28. The analysis for the Green functions and pointwise behavior for the linearized system around an equilibrium constant state is done in Ref. 5 in 3D.

The arrangement of the rest of this paper is as follows: We will prepare some essential preliminaries in Sec. II, including the equivalent nonequilibrium system and some useful estimates in analytic functional space whose proofs can be found in the Appendix. Section III is devoted to proving the uniform estimates and our main theorem.

II. PRELIMINARIES

A. Equivalent nonequilibrium system

Let us denote

$$\chi_i = E_i - e_i, \quad 2 \le i \le n. \tag{2.1}$$

By using (1.6), (1.7), and (1.14), we may rewrite (1.1) as the following equivalent forms:

$$\begin{cases} v_t + u^t \nabla v - v divu = 0, \\ u_t + u^t \nabla u + v \nabla p = 0, \\ (e_1)_t + p v divu + u^t \nabla e_1 = -\sum_{i=2}^n \dfrac{\chi_i}{\tau_i}, \\ (e_i)_t + u^t \nabla e_i = \dfrac{\chi_i}{\tau_i}, \quad 2 \le i \le n \end{cases} \tag{2.2}$$

and

$$\begin{cases} p_t + u^t \nabla p + \dfrac{\alpha+2}{\alpha} p \mathrm{div} u = -\dfrac{2}{\alpha v} \sum_{i=2}^n \dfrac{\chi_i}{\tau_i}, \\ u_t + u^t \nabla u + v \nabla p = 0, \\ (\chi_i)_t + \dfrac{\beta_i}{\alpha} p v divu + u^t \nabla \chi_i = -\dfrac{\chi_i}{\tau_i} - \dfrac{\beta_i}{\alpha} \sum_{i=2}^n \dfrac{\chi_i}{\tau_i}, \\ s_t + u^t \nabla s = \sum_{i=2}^n \left(\dfrac{1}{T_i} - \dfrac{1}{T_1} \right) \dfrac{\chi_i}{\tau_i}. \end{cases} \tag{2.3}$$

In the following analysis, we will work on systems (1.1), (2.2), and (2.3) alternatively for our convenience.

B. Analytic functional space

We will investigate the multi-scale singular limit problem in the analytic functional space. For this, we introduce some notations. As usual, the Sobolev space $H^s, s \in \mathbb{N}$ is defined by

$$H^s = \left\{ u \in L^2_{x,y}(\mathbb{R}^2_+) : \|u\|_{H^s} = \left(\sum_{\ell+k \le s} \|\partial_y^\ell \partial_x^k u\|^2_{L^2_{x,y}} \right)^{\frac{1}{2}} < \infty \right\}.$$

For a function f compactly supported in Fourier space in x variable, let us define

$$f_\sigma(x,y) \stackrel{\text{def}}{=} e^{\sigma(t)\langle D_x \rangle} f(x,y) = \mathscr{F}^{-1}_{\xi \to x}(e^{\sigma(t)\langle \xi \rangle} \mathscr{F}_{x \to \xi} f(\xi, y)),$$

where $\sigma(t) \in [1,2]$ is a decreasing function and $\langle \xi \rangle = \sqrt{1+|\xi|^2}$. In the sequel, we take

$$\sigma(t) = 2 - \lambda t$$

for some positive λ, which will be determined later.

Now, we introduce some analytic norms,

$$\|u\|^2_{\mathscr{H}^s} \stackrel{def}{=} \sum_{m=0}^\infty \frac{\sigma(t)^{2m}}{(m!)^2} \|\partial_y^m u_\sigma\|^2_{H^s},$$

$$\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}} \stackrel{def}{=} \sum_{m=1}^\infty \frac{\sigma(t)^{2m-1} m}{(m!)^2} \|\partial_y^m u_\sigma\|^2_{H^s} + \sum_{m=0}^\infty \frac{\sigma(t)^{2m}}{(m!)^2} \|\partial_y^m \langle D_x \rangle^{\frac{1}{2}} u_\sigma\|^2_{H^s}.$$

We state some useful lemmas in analytic functional space before discussing the multi-scale singular limit. Note that C is a general positive constant independent of ε and $\langle \cdot, \cdot \rangle_{H^s}$ denotes the H^s inner product throughout this paper.

Lemma II.1. Let $s \ge 5$ and $\iota \in [0,1]$. It holds that

$$\begin{aligned} &\|\langle D_x \rangle^{-\iota} (fg)_\sigma\|_{H^s} \\ &\le C\big((\|\langle D_x \rangle^{-\iota} f_\sigma\|_{H^s} + \|\langle D_x \rangle^{-\iota} \partial_y f_\sigma\|_{H^s}) \|g_\sigma\|_{H^{s-2}} \big) \\ &+ C\big((\|f_\sigma\|_{H^{s-2}} + \|f_\sigma\|_{H^{s-1}}) \|\langle D_x \rangle^{-\iota} g_\sigma\|_{H^s} \big), \end{aligned} \tag{2.4}$$

$$\|\langle D_x\rangle^{-\iota}(fg)_\sigma\|_{H^s} \le C\big((\|g_\sigma\|_{H^{s-2}} + \|g_\sigma\|_{H^{s-1}})\|\langle D_x\rangle^{-\iota} f_\sigma\|_{H^s}\big) + C\big((\|f_\sigma\|_{H^{s-2}} + \|f_\sigma\|_{H^{s-1}})\|\langle D_x\rangle^{-\iota} g_\sigma\|_{H^s}\big), \tag{2.5}$$

where f, g are any two functions in $H^s(\mathbb{R}^2_+)$.

To deal with the nonlinear terms $f\partial_x g$ in analytic settings, we introduce the following lemma. In the sequel, $f(t,x,y), g(t,x,y), h(t,x,y)$ are general functions in analytic functional space.

Lemma II.2. Let $s \ge 5$. Then, we have

$$\sum_{m=0}^{\infty} \frac{\sigma(t)^{2m}}{(m!)^2}\langle \partial_y^m (f\partial_x g)_\sigma, \partial_y^m h_\sigma\rangle_{H^s} \le C\big(\|f\|^2_{\mathscr{H}^{s-2}} + \|f\|^2_{\mathscr{H}^{s-1}}\big)\|g\|^2_{\mathscr{H}^{s,\frac12}} + C\big(\|g\|^2_{\mathscr{H}^{s-1}} + \|g\|^2_{\mathscr{H}^{s}}\big)\|f\|^2_{\mathscr{H}^s} + C\|h\|^2_{\mathscr{H}^{s,\frac12}}. \tag{2.6}$$

As for the nonlinear term $f\partial_y g$, we can apply the following lemma:

Lemma II.3. Let $s \ge 5$. It holds that

$$\sum_{m=0}^{\infty} \frac{\sigma(t)^{2m}}{(m!)^2}\langle \partial_y^m (f\partial_y g)_\sigma, \partial_y^m h_\sigma\rangle_{H^s} \le C\big(\|f\|^2_{\mathscr{H}^{s-2}} + \|f\|^2_{\mathscr{H}^{s-1}}\big)\|g\|^2_{\mathscr{H}^{s,\frac12}} + C\big(\|g\|^2_{\mathscr{H}^{s-1}} + \|g\|^2_{\mathscr{H}^{s}}\big)\|f\|^2_{\mathscr{H}^s} + C\|h\|^2_{\mathscr{H}^{s,\frac12}} \tag{2.7}$$

and

$$\sum_{m=0}^{\infty} \frac{\sigma(t)^{2m}}{(m!)^2}\langle \partial_y^m (f\partial_y g)_\sigma, \partial_y^m h_\sigma\rangle_{H^s} \le C\Big(\|f\|^2_{\mathscr{H}^{s}} + \|f\|^2_{\mathscr{H}^{s,\frac12}}\Big)\|g\|^2_{\mathscr{H}^{s-1}} + C\big(\|f\|^2_{\mathscr{H}^{s-2}} + \|f\|^2_{\mathscr{H}^{s-1}}\big)\|g\|^2_{\mathscr{H}^{s,\frac12}} + C\|h\|^2_{\mathscr{H}^{s,\frac12}}. \tag{2.8}$$

The following lemma is a direct consequence of Lemma II.1.

Lemma II.4. Let $s \ge 5$. Then, we have that

$$\|fg\|^2_{\mathscr{H}^s} \le C\|f\|^2_{\mathscr{H}^s}(\|g\|^2_{\mathscr{H}^{s-2}} + \|g\|^2_{\mathscr{H}^{s-1}}).$$

III. UNIFORM ESTIMATES AND RELAXATION LIMIT

We prove our main convergence result by the compactness argument, which has been applied in studying the limit behavior of solutions in conormal Sobolev space or Sobolev space, such as[4,23] by establishing uniform high order estimates of the solution to the non-equlibrium system (1.1) stated as follows:

Proposition III.1. Let s be an integer satisfying $s \ge 5$. For the initial data $(v_0^\varepsilon, u_0^\varepsilon, p_0^\varepsilon, (e_2^\varepsilon)_0, \ldots, (e_n^\varepsilon)_0)$ given by (1.17) with (1.18)–(1.21), there exists $T > 0$ independent of ε such that for every $\varepsilon \in (0,1)$, there exists a unique analytic solution $(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon)$ to the initial boundary value problem (1.1), (1.16), and (1.17) on $[0,T]$. Moreover, there exists a constant C independent of ε such that

$$\begin{aligned}
&\sup_{[0,T]}\left(\sum_{j=0}^{2}\|\partial_t^j(v^\varepsilon, p^\varepsilon, u^\varepsilon, \frac{\chi_2^\varepsilon}{v^\varepsilon}, \ldots, \frac{\chi_n^\varepsilon}{v^\varepsilon})\|^2_{\mathscr{H}^s}\right.\\
&\left.+\sum_{j=0}^{1}\|\partial_t^j(\chi_2^\varepsilon, \ldots, \chi_n^\varepsilon, e_1^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon,)\|^2_{\mathscr{H}^s}\right)\\
&+\lambda\int_0^T\left(\sum_{j=0}^{2}\|\partial_t^j(v^\varepsilon, p^\varepsilon, u^\varepsilon, \frac{\chi_2^\varepsilon}{v^\varepsilon}, \ldots, \frac{\chi_n^\varepsilon}{v^\varepsilon})\|^2_{\mathscr{H}^{s,\frac12}}\right.\\
&\left.+\sum_{j=0}^{1}\|\partial_t^j(\chi_2^\varepsilon, \ldots, \chi_n^\varepsilon, e_1^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon,)\|^2_{\mathscr{H}^{s,\frac12}}\right)\\
&+\sum_{i=2}^{n}\int_0^T \frac{4}{\alpha\beta_i\tau_i(\varepsilon)}\|\frac{\chi_i^\varepsilon}{v^\varepsilon}\|^2_{\mathscr{H}^s} + \frac{1}{\tau_i(\varepsilon)}\|\chi_i^\varepsilon\|^2_{\mathscr{H}^s} \le C.
\end{aligned}$$

In fact, we can verify that the above proposition can be deduced from the following proposition by a continuous argument, which will be given later. In the sequel, we will omit the superscript ε for simplicity.

Proposition III.2. For integer $s \geq 5$, assume that the initial data are given as in Proposition III.1. *Then, there exists a constant $C > 0$ independent of $\varepsilon \in (0,1)$ such that the analytic solutions $(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon)$ on $[0,T]$ to the initial boundary value problem* (1.1), (1.16), *and* (1.17) *in $\mathbb{R}_+^2$ satisfy*

$$\frac{1}{2}\frac{d}{dt}N(t) + \lambda F(t) + G(t) \leq CN(t) + C(1 + \mathscr{P}(N(t)))F(t), \tag{3.1}$$

where

$$N(t) = \sum_{j=0}^{2} \|\partial_t^j (v, p, u, \frac{\chi_2}{v}, \ldots, \frac{\chi_n}{v})\|_{\mathscr{H}^s}^2 + \sum_{j=0}^{1} \|\partial_t^j (\chi_2, \ldots, \chi_n, e_1, e_2, \ldots, e_n)\|_{\mathscr{H}^s}^2,$$

$$F(t) = \sum_{j=0}^{2} \|\partial_t^j (v, p, u, \frac{\chi_2}{v}, \ldots, \frac{\chi_n}{v})\|_{\mathscr{H}^{s,\frac{1}{2}}}^2 + \sum_{j=0}^{1} \|\partial_t^j (\chi_2, \ldots, \chi_n, e_1, e_2, \ldots, e_n)\|_{\mathscr{H}^{s,\frac{1}{2}}}^2,$$

$$G(t) = \sum_{i=2}^{n} \left(\frac{4}{\alpha\beta_i\tau_i(\varepsilon)} \|\frac{\chi_i^\varepsilon}{v^\varepsilon}\|_{\mathscr{H}^s}^2 + \frac{1}{\tau_i(\varepsilon)} \|\chi_i^\varepsilon\|_{\mathscr{H}^s}^2 \right),$$

and $\mathscr{P}(\cdot)$ is a polynomial with degree larger than or equal to 2.

The Proof of Proposition III.2 will follow from the following lemmas.

Let

$$\eta_i = \rho(E_i - e_i) = \frac{1}{v}\chi_i, \quad 2 \leq i \leq n.$$

Then, one can rewrite the equation of χ_i in (2.3) as

$$(\eta_i)_t + \eta_i \mathrm{div} u + \frac{\beta_i}{\alpha} p \mathrm{div} u + u^t \nabla \eta_i = -\frac{\eta_i}{\tau_i} - \frac{\beta_i}{\alpha} \sum_{i=2}^{n} \frac{\eta_i}{\tau_i}. \tag{3.2}$$

Combining (2.3) and (3.2), we have

$$\begin{cases} v_t + u^t \nabla v - v \mathrm{div} u = 0, \\ p_t + u^t \nabla p + \dfrac{\alpha+2}{\alpha} p \mathrm{div} u = -\dfrac{2}{\alpha} \sum_{i=2}^{n} \dfrac{\eta_i}{\tau_i}, \\ u_t + u^t \nabla u + v \nabla p = 0, \\ \left(p - \dfrac{2}{\beta_i}\eta_i\right)_t + u^t \nabla p + \dfrac{\alpha+4}{\alpha} p \mathrm{div} u \\ -\dfrac{2}{\beta_i}(\eta_i \mathrm{div} u + u^t \nabla \eta_i) = \dfrac{2}{\beta_i}\dfrac{\eta_i}{\tau_i}, \quad 2 \leq i \leq n. \end{cases} \tag{3.3}$$

Lemma III.3. Let $s \geq 5$. Then, we have the estimate,

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\sum_{j=0}^{2} \left(\|\partial_t^j (v,u,p)\|_{\mathscr{H}^s}^2 + \sum_{i=2}^{n} b_i \|\partial_t^j p - \frac{2}{\beta_i}\partial_t^j \eta_i\|_{\mathscr{H}^s}^2 \right) \\ &+ \lambda \sum_{j=0}^{2} \left(\|\partial_t^j (v,u,p)\|_{\mathscr{H}^{s,\frac{1}{2}}}^2 + \sum_{i=2}^{n} b_i \|\partial_t^j p - \frac{2}{\beta_i}\partial_t^j \eta_i\|_{\mathscr{H}^{s,\frac{1}{2}}}^2 \right) \\ &+ \sum_{j=0}^{2}\sum_{i=2}^{n} \frac{4}{\alpha\beta_i\tau_i} \|\partial_t^j \eta_i\|_{\mathscr{H}^s}^2 \\ &\leq C \sum_{j=0}^{2} (1 + \|\partial_t^j (v,u,p)\|_{\mathscr{H}^s}^2 + \sum_{i=2}^{m} \|\partial_t^j \eta_i\|_{\mathscr{H}^s}^2) \\ &\times (\|\partial_t^j (v,u,p)\|_{\mathscr{H}^{s,\frac{1}{2}}}^2 + \sum_{i=2}^{n} \|\partial_t^j \eta_i\|_{\mathscr{H}^{s,\frac{1}{2}}}^2). \end{aligned} \tag{3.4}$$

Proof. We first act $e^{\sigma(t)\langle D_x\rangle}$ on (3.3) to get

$$\begin{cases} \partial_t v_\sigma + \lambda\langle D_x\rangle v_\sigma + (u^t\nabla v - v divu)_\sigma = 0, \\ \partial_t p_\sigma + \lambda\langle D_x\rangle p_\sigma + (u^t\nabla p + \frac{\alpha+2}{\alpha} p\text{div}u)_\sigma \\ \quad = -\frac{2}{\alpha}\sum_{i=2}^{n}\frac{(\eta_i)_\sigma}{\tau_i}, \\ \partial_t u_\sigma + \lambda\langle D_x\rangle u_\sigma + (u^t\nabla u + v\nabla p)_\sigma = 0, \\ \partial_t(p - \frac{2}{\beta_i}\eta_i)_\sigma + \lambda\langle D_x\rangle(p - \frac{2}{\beta_i}\eta_i)_\sigma \\ +(u^t\nabla p + \frac{\alpha+4}{\alpha}p\text{div}u)_\sigma - \frac{2}{\beta_i}(\eta_i\text{div}u + u^t\nabla\eta_i)_\sigma \\ \quad = \frac{2}{\beta_i}\frac{(\eta_i)_\sigma}{\tau_i}, \quad 2\le i\le n. \end{cases} \tag{3.5}$$

Take $\frac{\sigma(t)}{m!}\partial_y^m$ on each equation of (3.5), and then, take the H^s inner product with $\frac{\sigma(t)}{m!}\partial_y^m v_\sigma$, $\frac{\sigma(t)}{m!}\partial_y^m p_\sigma$, $\frac{\sigma(t)}{m!}\partial_y^m u_\sigma$, $\frac{\sigma(t)}{m!} b_i\partial_y^m(p - \frac{2}{\beta_i}\eta_i)_\sigma$, respectively. Summing them up for all i and m, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|(v,u,p)\|_{\mathscr{H}^s}^2 + \sum_{i=2}^{n} b_i\|p - \frac{2}{\beta_i}\eta_i\|_{\mathscr{H}^s}^2\right) \\ &+\lambda\left(\|(v,u,p)\|_{\mathscr{H}^{s,\frac{1}{2}}}^2 + \sum_{i=2}^{n} b_i\|p - \frac{2}{\beta_i}\eta_i\|_{\mathscr{H}^{s,\frac{1}{2}}}^2\right) \\ &+\sum_{i=2}^{n}\frac{4}{\alpha\beta_i\tau_i}\|\eta_i\|_{\mathscr{H}^s}^2 \\ &= \sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\langle\partial_y^m(-u^t\nabla v + v divu)_\sigma, \partial_y^m v_\sigma\rangle_{H^s} \\ &+\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\langle\partial_y^m(u^t\nabla p + \frac{\alpha+2}{\alpha}p\text{div}u)_\sigma, \partial_y^m p_\sigma\rangle_{H^s} \\ &+\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\langle\partial_y^m(u^t\nabla u + v\nabla p)_\sigma, \partial_y^m u_\sigma\rangle_{H^s} \\ &+\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\sum_{i=2}^{n} b_i\langle(u^t\nabla p + \frac{\alpha+4}{\alpha}p\text{div}u, \partial_y^m(p - \frac{2}{\beta_i}\eta_i)_\sigma\rangle_{H^s} \\ &-\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\sum_{i=2}^{n} b_i\langle\frac{2}{\beta_i}(\eta_i\text{div}u + u^t\nabla\eta_i)_\sigma, \partial_y^m(p - \frac{2}{\beta_i}\eta_i)_\sigma\rangle_{H^s} \\ &\le \sum_{j=1}^{5}|I_j|, \end{aligned}$$

where $b_i = \frac{\beta_i}{\alpha}$. Note that we have used the cancellation,

$$\begin{aligned} &\langle -\frac{2}{\alpha}\partial_y^m\sum_{i=2}^{n}(\frac{\eta_i}{\tau_i})_\sigma, \partial_y^m p_\sigma\rangle_{H^s} + \sum_{i=2}^{n} b_i\langle\frac{2}{\beta_i}\partial_y^m(\frac{\eta_i}{\tau_i})_\sigma, \partial_y^m(p - \frac{2}{\beta_i}\eta_i)_\sigma\rangle_{H^s} \\ &= -\sum_{i=2}^{n}\frac{4}{\alpha\beta_i\tau_i}\langle\partial_y^m(\eta_i)_\sigma, \partial_y^m(\eta_i)_\sigma\rangle_{H^s}. \end{aligned} \tag{3.6}$$

Now, we estimate I_j, $1\le j\le 5$ using Lemmas II.2 and II.3,

$$
\begin{aligned}
|I_1| &\le |\sum_{m=0}^{\infty} \frac{\sigma(t)^2}{(m!)^2} \langle \partial_y^m(-u_1\partial_x v - u_2\partial_y v + v\partial_x u_1 + v\partial_y u_2)_\sigma, \partial_y^m v_\sigma\rangle_{H^s}| \\
&\le C(\|u_1\|^2_{\mathscr{H}^{s-2}} + \|u_1\|^2_{\mathscr{H}^{s-1}})\|v\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C(\|v\|^2_{\mathscr{H}^{s-1}} + \|v\|^2_{\mathscr{H}^{s}})\|u_1\|^2_{\mathscr{H}^{s}} \\
&+ C(\|u_2\|^2_{\mathscr{H}^{s-2}} + \|u_2\|^2_{\mathscr{H}^{s-1}})\|v\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C(\|v\|^2_{\mathscr{H}^{s-1}} + \|v\|^2_{\mathscr{H}^{s}})\|u_2\|^2_{\mathscr{H}^{s}} \\
&+ C(\|v\|^2_{\mathscr{H}^{s-2}} + \|v\|^2_{\mathscr{H}^{s-1}})\|u_1\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C(\|u_1\|^2_{\mathscr{H}^{s-1}} + \|u_1\|^2_{\mathscr{H}^{s}})\|v\|^2_{\mathscr{H}^{s}} \\
&+ C(\|v\|^2_{\mathscr{H}^{s-2}} + \|v\|^2_{\mathscr{H}^{s-1}})\|u_2\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C(\|u_2\|^2_{\mathscr{H}^{s-1}} + \|u_2\|^2_{\mathscr{H}^{s}})\|v\|^2_{\mathscr{H}^{s}} \\
&+ C\|v\|^2_{\mathscr{H}^{s,\frac{1}{2}}} \\
&\le C\|u\|^2_{\mathscr{H}^{s}}\|v\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C\|v\|^2_{\mathscr{H}^{s}}\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}} \\
&+ C\|u\|^2_{\mathscr{H}^{s}}\|v\|^2_{\mathscr{H}^{s}} + C\|v\|^2_{\mathscr{H}^{s,\frac{1}{2}}},
\end{aligned}
$$

where we have used the fact

$$
\sum_{m=0}^{\infty} \frac{\sigma(t)^2}{(m!)^2} \|\partial_y^m f\|^2_{\mathscr{H}^{k-1}} \le \|f\|^2_{\mathscr{H}^{k}} \le \|f\|^2_{\mathscr{H}^{k,\frac{1}{2}}}, \quad k \ge 1. \tag{3.7}
$$

Similarly, we obtain

$$
\begin{aligned}
|I_2| &\le C\|u\|^2_{\mathscr{H}^{s}}\|p\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C\|p\|^2_{\mathscr{H}^{s}}\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}} \\
&+ C\|u\|^2_{\mathscr{H}^{s}}\|p\|^2_{\mathscr{H}^{s}} + C\|p\|^2_{\mathscr{H}^{s,\frac{1}{2}}}, \\
|I_3| &\le C\|v\|^2_{\mathscr{H}^{s}}\|p\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C\|v\|^2_{\mathscr{H}^{s}}\|p\|^2_{\mathscr{H}^{s}} \\
&+ C\|u\|^2_{\mathscr{H}^{s}}\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C\|u\|^2_{\mathscr{H}^{s}}\|u\|^2_{\mathscr{H}^{s}} + C\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}},
\end{aligned}
$$

and

$$
\begin{aligned}
|I_4| + |I_5| &\le C\|u\|^2_{\mathscr{H}^{s}}\|p\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C\|u\|^2_{\mathscr{H}^{s}}\|p\|^2_{\mathscr{H}^{s}} \\
&+ C\|p\|^2_{\mathscr{H}^{s}}\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C\sum_{i=2}^{n}\|u\|^2_{\mathscr{H}^{s}}\|\eta_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}} \\
&+ C\sum_{i=2}^{n}\|\eta_i\|^2_{\mathscr{H}^{s}}\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + C\sum_{i=2}^{n}\|u\|^2_{\mathscr{H}^{s}}\|\eta_i\|^2_{\mathscr{H}^{s}} \\
&+ C\sum_{m=0}^{\infty}\|p - \frac{2}{\beta_i}\eta_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}.
\end{aligned}
$$

Summarizing the above to get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|(v,u,p)\|^2_{\mathscr{H}^{s}} + \sum_{i=2}^{n} b_i\|p - \frac{2}{\beta_i}\eta_i\|^2_{\mathscr{H}^{s}}\right) \\
&+ \lambda\left(\|(v,u,p)\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + \sum_{i=2}^{n} b_i\|p - \frac{2}{\beta_i}\eta_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}\right) \\
&+ \sum_{i=2}^{n}\frac{4}{\alpha\beta_i\tau_i}\|\eta_i\|^2_{\mathscr{H}^{s}} \\
&\le C(1 + \|(v,u,p)\|^2_{\mathscr{H}^{s}} + \sum_{i=2}^{m}\|\eta_i\|^2_{\mathscr{H}^{s}}) \\
&\times (\|(v,u,p)\|^2_{\mathscr{H}^{s,\frac{1}{2}}} + \sum_{i=2}^{n}\|\eta_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}).
\end{aligned} \tag{3.8}
$$

We can also obtain the similar estimates of t-derivatives for ∂_t (3.5) and ∂_{tt} (3.5) as the progress to deduce (3.8). That is,

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg(\|(v_t,u_t,p_t)\|^2_{\mathscr{H}^s}+\sum_{i=2}^n b_i\|p_t-\frac{2}{\beta_i}(\eta_i)_t\|^2_{\mathscr{H}^s}\bigg)\\
&+\lambda\bigg(\|(v_t,u_t,p_t)\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\sum_{i=2}^n b_i\|p_t-\frac{2}{\beta_i}(\eta_i)_t\|^2_{\mathscr{H}^{s,\frac{1}{2}}}\bigg)\\
&+\sum_{i=2}^n\frac{4}{\alpha\beta_i\tau_i}\|(\eta_i)_t\|^2_{\mathscr{H}^s}\\
&\le C(1+\|(v_t,u_t,p_t)\|^2_{\mathscr{H}^s}+\sum_{i=2}^m\|(\eta_i)_t\|^2_{\mathscr{H}^s})\\
&\times(\|(v_t,u_t,p_t)\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\sum_{i=2}^n\|(\eta_i)_t\|^2_{\mathscr{H}^{s,\frac{1}{2}}})
\end{aligned}\tag{3.9}$$

and

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\bigg(\|\partial_t^2(v,u,p)\|^2_{\mathscr{H}^s}+\sum_{i=2}^n b_i\|\partial_t^2p-\frac{2}{\beta_i}\partial_t^2\eta_i\|^2_{\mathscr{H}^s}\bigg)\\
&+\lambda\bigg(\|\partial_t^2(v,u,p)\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\sum_{i=2}^n b_i\|\partial_t^2p-\frac{2}{\beta_i}\partial_t^2\eta_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}\bigg)\\
&+\sum_{i=2}^n\frac{4}{\alpha\beta_i\tau_i}\|\partial_t^2\eta_i\|^2_{\mathscr{H}^s}\\
&\le C(1+\|\partial_t^2(v,u,p)\|^2_{\mathscr{H}^s}+\sum_{i=2}^m\|\partial_t^2\eta_i\|^2_{\mathscr{H}^s})\\
&\times(\|\partial_t^2(v,u,p)\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\sum_{i=2}^n\|\partial_t^2\eta_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}).
\end{aligned}\tag{3.10}$$

Thus, the proof of this lemma is completed by combining (3.8)–(3.10). □

We deduce the estimates on χ_i based on the equations of p and χ_i in (2.3).

Lemma III.4. Let $s\ge 5$. Then, we have for $2\le i\le n$,

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\chi_i\|^2_{\mathscr{H}^s}+\lambda\|\chi_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\frac{1}{\tau_i}\|\chi_i\|^2_{\mathscr{H}^s}\le C\Big[&(\|p\|^2_{\mathscr{H}^s}\|v\|^2_{\mathscr{H}^s}+\|p\|^2_{\mathscr{H}^s})\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|u\|^2_{\mathscr{H}^s}\|p\|^2_{\mathscr{H}^{s,\frac{1}{2}}}\\
&+\|u\|^2_{\mathscr{H}^s}\|\chi_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|v\|^2_{\mathscr{H}^s}\|p_t\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|\chi_i\|^2_{\mathscr{H}^s}\Big]
\end{aligned}\tag{3.11}$$

and

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|(\chi_i)_t\|^2_{\mathscr{H}^s}+\lambda\|(\chi_i)_t\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\frac{1}{\tau_i}\|(\chi_i)_t\|^2_{\mathscr{H}^s}\le C\Big[&(\|p_t\|^2_{\mathscr{H}^s}\|v_t\|^2_{\mathscr{H}^s}+\|p_t\|^2_{\mathscr{H}^s})\|u_t\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|u_t\|^2_{\mathscr{H}^s}\|p_t\|^2_{\mathscr{H}^{s,\frac{1}{2}}}\\
&+\|u_t\|^2_{\mathscr{H}^s}\|(\chi_i)_t\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|v_t\|^2_{\mathscr{H}^s}\|\partial_t^2p\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|(\chi_i)_t\|^2_{\mathscr{H}^s}\Big].
\end{aligned}\tag{3.12}$$

Proof. It follows from (2.3) that

$$(\chi_i)_t+\frac{\beta_i}{\alpha}pvdivu+u^t\nabla\chi_i-\frac{\beta_i}{2}v(p_t+u^t\nabla p+\frac{\alpha+2}{\alpha}p\mathrm{div}u)=-\frac{\chi_i}{\tau_i}.\tag{3.13}$$

Note that p_t appears in (3.13), which will lead to the increase in t-derivatives when we perform the energy estimate on χ_i.

Taking $e^{\sigma(t)\langle D_x\rangle}$ on both sides of Eq. (3.13) and then acting $\frac{\sigma(t)^m}{m!}\partial_y^m$ on both sides, we finally take H^s the inner product with $\frac{\sigma(t)^m}{m!}\partial_y^m\chi_i$ to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\chi_i\|^2_{\mathscr{H}^s}+\lambda\|\chi_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\frac{1}{\tau_i}\|\chi_i\|^2_{\mathscr{H}^s}\le C\Big(&\|pv\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|u\|^2_{\mathscr{H}^s}\|\chi_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}\\
&+\|p\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|u\|^2_{\mathscr{H}^s}\|p\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+\|vp_t\|^2_{\mathscr{H}^s}+\|\chi_i\|^2_{\mathscr{H}^s}+\|\chi_i\|^2_{\mathscr{H}^{s,\frac{1}{2}}}\Big),
\end{aligned}$$

where we have used Lemmas II.2 and II.3.

Using Lemma II.4 and (3.7), we can deduce estimate (3.11) directly. Similarly, estimate (3.12) can be proved after taking t-derivative on both sides of (3.13). Thus, the proof of this lemma is completed. □

From the system (2.3), we have, for $2 \le i \le n$,

$$\frac{\chi_i}{\tau_i} = \frac{\beta_i}{2} v[p_t + u^t \nabla p + \frac{\alpha+2}{\alpha} p \mathrm{div} u] - (\chi_i)_t - \frac{\beta_i}{\alpha} p v div u - u^t \nabla \chi_i. \tag{3.14}$$

Then, we have

$$\begin{aligned}(e_i)_t + u^t \nabla e_i &= \frac{\beta_i}{2} v[p_t + u^t \nabla p + \frac{\alpha+2}{\alpha} p \mathrm{div} u] \\ &\quad - (\chi_i)_t - \frac{\beta_i}{\alpha} p v div u - u^t \nabla \chi_i \\ &= \frac{\beta_i}{2} v[p_t + u^t \nabla p + \frac{\alpha+2}{\alpha} p \mathrm{div} u] - v(\eta_i)_t - \eta_i v_t \\ &\quad - \frac{\beta_i}{\alpha} p v div u - v u^t \nabla \eta_i - \eta_i u^t \nabla v.\end{aligned} \tag{3.15}$$

Based on (3.15), we can deduce the following uniform estimate on $e_i, 2 \le i \le n$:

Lemma III.5. Let $s \ge 5$. Then, it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_i\|^2_{\mathscr{H}^s} + \lambda \|e_i\|^2_{\mathscr{H}^{s,\frac12}} &\le C\Big(\|u\|^2_{\mathscr{H}^s}\|e_i\|^2_{\mathscr{H}^{s,\frac12}} + \|v\|^2_{\mathscr{H}^s}\|p_t\|^2_{\mathscr{H}^{s,\frac12}} \\ &\quad + \|v\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^s}\|p\|^2_{\mathscr{H}^{s,\frac12}} + \|v\|^2_{\mathscr{H}^s}\|p\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^{s,\frac12}} \\ &\quad + \|v\|^2_{\mathscr{H}^s}\|(\eta_i)_t\|^2_{\mathscr{H}^s} + \|\eta_i\|^2_{\mathscr{H}^s}\|v_t\|^2_{\mathscr{H}^s} + \|e_i\|^2_{\mathscr{H}^s} \\ &\quad + \|v\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^s}\|\eta_i\|^2_{\mathscr{H}^{s,\frac12}} + \|\eta_i\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^s}\|v\|^2_{\mathscr{H}^{s,\frac12}} + \|e_i\|^2_{\mathscr{H}^{s,\frac12}}\Big)\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(e_i)_t\|^2_{\mathscr{H}^s} + \lambda \|(e_i)_t\|^2_{\mathscr{H}^{s,\frac12}} &\le C\Big(\|u_t\|^2_{\mathscr{H}^s}\|(e_i)_t\|^2_{\mathscr{H}^{s,\frac12}} + \|v_t\|^2_{\mathscr{H}^s}\|p_{tt}\|^2_{\mathscr{H}^{s,\frac12}} \\ &\quad + \|v_t\|^2_{\mathscr{H}^s}\|u_t\|^2_{\mathscr{H}^s}\|p_t\|^2_{\mathscr{H}^{s,\frac12}} + \|v_t\|^2_{\mathscr{H}^s}\|p_t\|^2_{\mathscr{H}^s}\|u_t\|^2_{\mathscr{H}^{s,\frac12}} \\ &\quad + \|v_t\|^2_{\mathscr{H}^s}\|(\eta_i)_{tt}\|^2_{\mathscr{H}^s} + \|(\eta_i)_t\|^2_{\mathscr{H}^s}\|v_{tt}\|^2_{\mathscr{H}^s} + \|(e_i)_t\|^2_{\mathscr{H}^s} \\ &\quad + \|v_t\|^2_{\mathscr{H}^s}\|u_t\|^2_{\mathscr{H}^s}\|(\eta_i)_t\|^2_{\mathscr{H}^{s,\frac12}} \\ &\quad + \|(\eta_i)_t\|^2_{\mathscr{H}^s}\|u_t\|^2_{\mathscr{H}^s}\|v_t\|^2_{\mathscr{H}^{s,\frac12}} + \|(e_i)_t\|^2_{\mathscr{H}^{s,\frac12}}\Big).\end{aligned} \tag{3.17}$$

Proof. The proof is similar to that of Lemma III.4. The key point is to control e_i by v_t and $(\eta_i)_t$ instead of $(\chi_i)_t$. Here, we omit it for simplicity. □

Now, we state the estimates for e_1 in the following lemma:

Lemma III.6. Let $s \ge 5$. Then, it holds that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|e_1\|^2_{\mathscr{H}^s} + \lambda \|e_1\|^2_{\mathscr{H}^{s,\frac12}} &\le C\Big(\|p\|^2_{\mathscr{H}^s}\|v\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^{s,\frac12}} + \|u\|^2_{\mathscr{H}^s}\|e_1\|^2_{\mathscr{H}^{s,\frac12}} + \|v\|^2_{\mathscr{H}^s}\|p_t\|^2_{\mathscr{H}^s} \\ &\quad + \|v\|^2_{\mathscr{H}^s}\|u\|^2_{\mathscr{H}^s}\|p\|^2_{\mathscr{H}^{s,\frac12}} + \|e_1\|^2_{\mathscr{H}^s} + \|e_1\|^2_{\mathscr{H}^{s,\frac12}}\Big)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|(e_1)_t\|^2_{\mathscr{H}^s} + \lambda \|(e_1)_t\|^2_{\mathscr{H}^{s,\frac12}} &\le C\Big(\|p_t\|^2_{\mathscr{H}^s}\|v_t\|^2_{\mathscr{H}^s}\|u_t\|^2_{\mathscr{H}^{s,\frac12}} + \|u_t\|^2_{\mathscr{H}^s}\|(e_1)_t\|^2_{\mathscr{H}^{s,\frac12}} \\ &\quad + \|v_t\|^2_{\mathscr{H}^s}\|p_{tt}\|^2_{\mathscr{H}^s} + \|v_t\|^2_{\mathscr{H}^s}\|u_t\|^2_{\mathscr{H}^s}\|p_t\|^2_{\mathscr{H}^{s,\frac12}} \\ &\quad + \|(e_1)_t\|^2_{\mathscr{H}^s} + \|(e_1)_t\|^2_{\mathscr{H}^{s,\frac12}}\Big).\end{aligned} \tag{3.19}$$

Proof. From (3.3), we have

$$(e_1)_t + p v div u + u^t \nabla e_1 = -v\sum_{i=2}^{n}\frac{\eta_i}{\tau_i} = \frac{\alpha}{2} v[p_t + u^t \nabla p + \frac{\alpha+2}{\alpha} p \mathrm{div} u]. \tag{3.20}$$

Then, just following the approach of proving Lemma III.4, we can obtain (3.18) and (3.19) directly.

In addition, we can obtain the uniform estimate for e_1 though that of p and v, since we have

$$\begin{aligned} &\|e_1\|_{\mathscr{H}^s}^2 \le C\|p\|_{\mathscr{H}^s}^2\|v\|_{\mathscr{H}^s}^2, \\ &\|(e_1)_t\|_{\mathscr{H}^s}^2 \le C(\|p\|_{\mathscr{H}^s}^2\|v_t\|_{\mathscr{H}^s}^2 + \|v\|_{\mathscr{H}^s}^2\|p_t\|_{\mathscr{H}^s}^2). \end{aligned} \tag{3.21}$$

Hence, the proof of this lemma is completed. □

Proof of Proposition III.1. It is clear that Proposition III.2 is a direct consequence of Lemmas III.3–III.6. Once the Proposition III.2 holds, one can choose $\lambda = 4CC_1^K$ with $C_1 = \max\{1, C_1\}$, where K is the order of the polynomial $\mathscr{P}$ and T_1 such that

$$\sigma(t) \ge 1 \quad \text{for} \quad t \in [0, T_1],$$

and then,

$$\frac{1}{2}\frac{d}{dt}N(t) + 2CC_1^K F(t) + G(t) \le CN(t).$$

Then, there exist $0 < T \le T_1$ and a constant C depending on C_1 but independent of ε such that the solution $(v, u, p, e_2, \ldots, e_n)$ of (1.1) or (2.2) exists on $[0, T]$ and satisfy the uniform estimates in Proposition III.1 by a continuous argument.

Thus, the Proof of Proposition III.1 is completed. □

Proof of Theorem I.1. Based on the uniform estimates in Proposition III.1, we have

$$\begin{aligned} N(t) &= \sum_{j=0}^{2}\|\partial_t^j(v^\varepsilon, p^\varepsilon, u^\varepsilon, \frac{\chi_2^\varepsilon}{v^\varepsilon}, \ldots, \frac{\chi_n^\varepsilon}{v^\varepsilon})\|_{\mathscr{H}^s}^2 \\ &\quad + \sum_{j=0}^{1}\|\partial_t^j(\chi_2^\varepsilon, \ldots, \chi_n^\varepsilon, e_1^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon)\|_{\mathscr{H}^s}^2 \le C \end{aligned}$$

for any $t \in [0, T]$. In particular, one has that

$$\sup_{[0,T]}\left(\|(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon)\|_{H^s}^2 + \|\partial_t(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon)\|_{H^s}^2\right) \le C,$$

with $s \ge 5$. Then, by virtue of the standard compactness argument, we obtain the multiple scale singular limit,

$$\begin{aligned} &\sup_{[0,T]}\|(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon) - (v^e, u^e, p^e, E_2, \ldots, E_n)\|_{L^2} \to 0, \\ &\sup_{[0,T]}\|(v^\varepsilon, u^\varepsilon, p^\varepsilon, e_2^\varepsilon, \ldots, e_n^\varepsilon) - (v^e, u^e, p^e, E_2, \ldots, E_n)\|_{L^\infty} \to 0, \end{aligned}$$

as $\varepsilon \to 0$. One can also refer to Refs. 4, 18, and 23 for the details. The Proof of Theorem I.1 is completed. □

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APPENDIX: ESTIMATES IN ANALYTIC SETTINGS

In this appendix, we give the proofs of Lemmas II.1–II.3 in Sec. II.

1. Proof of Lemma II.1

Proof. We first consider the special case of $\sigma = 0$. From the definition of H^s, we directly have

$$\|\langle D_x\rangle^{-\iota}(fg)\|_{H^s} \le C\sum_{\ell+k\le s}\|\partial_y^\ell\partial_x\langle D_x\rangle^{-\iota}(fg)\|_{L^2_{x,y}}.$$

For simplicity, here, we only consider the typical case of $\ell = s$. The other cases can be handled in a similar way. By virtue of the fact that

$$\|\langle D_x\rangle^{-\iota}(fg)\|_{L^2_x} \le C\|\langle D_x\rangle^{-\iota} f\|_{L^2_x}\|g\|_{H^1_x},$$

which is given in Ref. 13, we have

$$\begin{aligned}
\|\partial_y^s\langle D_x\rangle^{-\iota}(fg)\|_{L^2_{x,y}} &\le C\sup_{\substack{s_1+s_2=s,\\ s_1\ge s_2}}\|\langle D_x\rangle^{-\iota}(\partial_y^{s_1}f\partial_y^{s_2}g)\|_{L^2_{x,y}}\\
&+C\sup_{\substack{s_1+s_2=s,\\ s_1\le s_2}}\|\langle D_x\rangle^{-\iota}(\partial_y^{s_1}f\partial_y^{s_2}g)\|_{L^2_{x,y}}\\
&\le C\sup_{\substack{s_1+s_2=s,\\ s_2\le[\frac{s}{2}]-1}}\|\langle D_x\rangle^{-\iota}\partial_y^{s_1}f\|_{L^\infty_y(L^2_x)}\|\partial_y^{s_2}g\|_{L^2_y(H^1_x)}\\
&+C\sup_{\substack{s_1+s_2=s,\\ s_1\le[\frac{s}{2}]-1}}\|\langle D_x\rangle^{-\iota}\partial_y^{s_2}g\|_{L^2_y(L^2_x)}\|\partial_y^{s_1}f\|_{L^\infty_y(H^1_x)}\\
&+C\|\langle D_x\rangle^{-\iota}\partial_y^{s-[\frac{s}{2}]}f\|_{L^\infty_y(L^2_x)}\|\partial_y^{[\frac{s}{2}]}g\|_{L^2_y(H^1_x)}\\
&+C\|\langle D_x\rangle^{-\iota}\partial_y^{s-[\frac{s}{2}]}g\|_{L^2_y(L^2_x)}\|\partial_y^{[\frac{s}{2}]}f\|_{L^\infty_y(H^1_x)}\\
&\le C\sum_{\ell\le s}\|\langle D_x\rangle^{-\iota}\partial_y^{\ell}f\|_{L^\infty_y(L^2_x)}\|g\|_{H^{s-2}}\\
&+C\sum_{\ell\le s-3}\|\partial_y^{\ell}f\|_{L^\infty_y(H^1_x)}\|\langle D_x\rangle^{-\iota}g\|_{H^s}\\
&\le C\big((\|\langle D_x\rangle^{-\iota}f\|_{H^s}+\|\langle D_x\rangle^{-\iota}\partial_y f\|_{H^s})\|g\|_{H^{s-2}}\big)\\
&+C\big((\|f\|_{H^{s-2}}+\|f\|_{H^{s-1}})\|\langle D_x\rangle^{-\iota}g\|_{H^s}\big).
\end{aligned}$$

Noting that, here, $s \ge 5$ guarantees that

$$[\frac{s}{2}] + 1 \le s - 2.$$

The general case of $\sigma \in [1, 2]$ can be verified as Lemma 4.1 in Ref. 13. Thus, we have proved the first inequality of this lemma. We can prove the second one similarly. □

2. Proof of Lemma II.2

Proof. It follows from Leibniz's rule that

$$\langle\partial_y^m(f\partial_x g)_\sigma, \partial_y^m h_\sigma\rangle_{H^s} \le \sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\|\langle D_x\rangle^{-\frac{1}{2}}(\partial_y^{m_1}f\partial_y^{m_2}\partial_x g)_\sigma\|_{H^s}\|\langle D_x\rangle^{\frac{1}{2}}\partial_y^m h_\sigma\|_{H^s}.$$

Using the second inequality of Lemma II.1, we obtain

$$\begin{aligned}
&\langle\partial_y^m(f\partial_x g)_\sigma, \partial_y^m h_\sigma\rangle_{H^s}\\
&\le C\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\Big[(\|\partial_y^{m_2}\partial_x g_\sigma\|_{H^{s-2}}+\|\partial_y^{m_2}\partial_x g_\sigma\|_{H^{s-1}})\|\langle D_x\rangle^{-\frac{1}{2}}\partial_y^{m_1}f_\sigma\|_{H^s}\\
&\quad+(\|\partial_y^{m_1}f_\sigma\|_{H^{s-2}}+\|\partial_y^{m_1}f_\sigma\|_{H^{s-1}})\|\langle D_x\rangle^{-\frac{1}{2}}\partial_y^{m_2}\partial_x g_\sigma\|_{H^s}\Big]\|\langle D_x\rangle^{\frac{1}{2}}\partial_y^m h_\sigma\|_{H^s}\\
&\le C\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\Big[(\|\partial_y^{m_2}g_\sigma\|_{H^{s-1}}+\|\partial_y^{m_2}g_\sigma\|_{H^s})\|\partial_y^{m_1}f_\sigma\|_{H^s}\\
&\quad+(\|\partial_y^{m_1}f_\sigma\|_{H^{s-2}}+\|\partial_y^{m_1}f_\sigma\|_{H^{s-1}})\|\langle D_x\rangle^{\frac{1}{2}}\partial_y^{m_2}g_\sigma\|_{H^s}\Big]\|\langle D_x\rangle^{\frac{1}{2}}\partial_y^m h_\sigma\|_{H^s}.
\end{aligned}$$

Then, by Cauchy's inequality, we have

$$\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\langle\partial_y^m(f\partial_x g)_\sigma,\partial_y^m h_\sigma\rangle_{H^s}\le C\sum_{m=0}^{\infty}\left(\sum_{m_1+m_2=m}\frac{\sigma(t)^m}{m_1!m_2!}(\|\partial_y^{m_2}g_\sigma\|_{H^{s-1}}+\|\partial_y^{m_2}g_\sigma\|_{H^s})\|\partial_y^{m_1}f_\sigma\|_{H^s}\right)^2$$
$$+C\sum_{m=0}^{\infty}\left(\sum_{m_1+m_2=m}\frac{\sigma(t)^m}{m_1!m_2!}\left(\|\partial_y^{m_1}f_\sigma\|_{H^{s-2}}+\|\partial_y^{m_1}f_\sigma\|_{H^{s-1}}\right)\|\langle D_x\rangle^{\frac{1}{2}}\partial_y^{m_2}g_\sigma\|_{H^s}\right)^2+C\|h_\sigma\|^2_{\mathscr{H}^{s,\frac{1}{2}}}$$
$$\le C\left(\sum_{m=0}^{\infty}\frac{\sigma(t)^2}{(m!)^2}(\|\partial_y^m g_\sigma\|_{H^{s-1}}+\|\partial_y^m g_\sigma\|_{H^s})^2\right)\sum_{m=0}^{\infty}\frac{\sigma(t)^2}{(m!)^2}\|\partial_y^m f_\sigma\|^2_{H^s}$$
$$+C\left(\sum_{m=0}^{\infty}\frac{\sigma(t)^2}{(m!)^2}(\|\partial_y^m f_\sigma\|_{H^{s-2}}+\|\partial_y^m f_\sigma\|_{H^{s-1}})^2\right)\sum_{m=0}^{\infty}\frac{\sigma(t)^2}{(m!)^2}\|\langle D_x\rangle^{\frac{1}{2}}\partial_y^m g_\sigma\|^2_{H^s}+C\|h_\sigma\|^2_{\mathscr{H}^{s,\frac{1}{2}}}$$
$$\le C\left(\|g_\sigma\|^2_{\mathscr{H}^{s-1}}+\|g_\sigma\|^2_{\mathscr{H}^s}\right)\|f_\sigma\|^2_{\mathscr{H}^s}$$
$$+C\left(\|f_\sigma\|^2_{\mathscr{H}^{s-2}}+\|f_\sigma\|^2_{\mathscr{H}^{s-1}}\right)\|g_\sigma\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+C\|h_\sigma\|^2_{\mathscr{H}^{s,\frac{1}{2}}}.$$

The proof of this lemma is completed. □

3. Proof of Lemma II.3

Proof. Using Leibniz's rule and the second inequality in Lemma II.1, we have

$$\langle\partial_y^m(f\partial_y g)_\sigma,\partial_y^m h_\sigma\rangle_{H^s}\le\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}\|(\partial_y^{m_1}f\partial_y^{m_2+1}g)_\sigma\|_{H^s}\|\partial_y^m h_\sigma\|_{H^s}$$
$$\le C\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}(\|\partial_y^{m_2+1}g_\sigma\|_{H^{s-2}}+\|\partial_y^{m_2+1}g_\sigma\|_{H^{s-1}})\times\|\partial_y^{m_1}f_\sigma\|_{H^s}\|\partial_y^m h_\sigma\|_{H^s}$$
$$+C\sum_{m_1+m_2=m}\frac{m!}{m_1!m_2!}(\|\partial_y^{m_1}f_\sigma\|_{H^{s-2}}+\|\partial_y^{m_1}f_\sigma\|_{H^{s-1}})\|\times\partial_y^{m_2+1}g_\sigma\|_{H^s}\|\partial_y^m h_\sigma\|_{H^s}.$$

Then, it follows from Young's inequality that

$$\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\langle\partial_y^m(f\partial_y g)_\sigma,\partial_y^m h_\sigma\rangle_{H^s}$$
$$\le C\sum_{m=1}^{\infty}\left(\sum_{m_1+m_2=m}\frac{\sigma(t)}{\sqrt{m}m_1!m_2!}\|\partial_y^{m_2+1}g_\sigma\|_{H^{s-2}}\|\partial_y^{m_1}f_\sigma\|_{H^s}\right)^2$$
$$+C\sum_{m=1}^{\infty}\left(\sum_{m_1+m_2=m}\frac{\sigma(t)}{\sqrt{m}m_1!m_2!}\|\partial_y^{m_2+1}g_\sigma\|_{H^{s-1}}\|\partial_y^{m_1}f_\sigma\|_{H^s}\right)^2$$
$$+C\sum_{m=1}^{\infty}\left(\sum_{m_1+m_2=m}\frac{\sigma(t)}{\sqrt{m}m_1!m_2!}\|\partial_y^{m_1}f_\sigma\|_{H^{s-2}}\|\partial_y^{m_2+1}g_\sigma\|_{H^s}\right)^2$$
$$+C\sum_{m=1}^{\infty}\left(\sum_{m_1+m_2=m}\frac{\sigma(t)}{\sqrt{m}m_1!m_2!}\|\partial_y^{m_1}f_\sigma\|_{H^{s-1}}\|\partial_y^{m_2+1}g_\sigma\|_{H^s}\right)^2$$
$$+C\|h\|^2_{\mathscr{H}^{s,\frac{1}{2}}}$$
$$\le C\sum_{m=0}^{\infty}\left(\frac{\sigma(t)^m}{m!}(\|\partial_y^m g_\sigma\|_{H^{s-1}}+\|\partial_y^m g_\sigma\|_{H^s})\right)^2\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\|\partial_y^m f_\sigma\|^2_{H^s}$$
$$+C\sum_{m=0}^{\infty}\left(\frac{\sigma(t)^m}{m!}(\|\partial_y^m f_\sigma\|_{H^{s-2}}+\|\partial_y^m f_\sigma\|_{H^{s-1}})\right)^2$$
$$\times\sum_{m=1}^{\infty}\frac{\sigma(t)^{2m-1}m}{(m!)^2}\|\partial_y^m g_\sigma\|^2_{H^s}+C\|h\|^2_{\mathscr{H}^{s,\frac{1}{2}}}.$$

One has, by the definition of the analytic norms,

$$\sum_{m=0}^{\infty}\frac{\sigma(t)^{2m}}{(m!)^2}\langle\partial_y^m(f\partial_y g)_\sigma,\partial_y^m h_\sigma\rangle_{H^s}$$
$$\le C\Big(\|g\|^2_{\mathscr{H}^{s-1}}+\|g\|^2_{\mathscr{H}^s}\Big)\|f\|^2_{\mathscr{H}^s}$$
$$+C\Big(\|f\|^2_{\mathscr{H}^{s-2}}+\|f\|^2_{\mathscr{H}^{s-1}}\Big)\|g\|^2_{\mathscr{H}^{s,\frac{1}{2}}}+C\|h\|^2_{\mathscr{H}^{s,\frac{1}{2}}}.$$

Thus, the first inequality has been proved.

Similarly, the inequality (2.8) can be proved by using (2.4). □

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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